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Monotone quantities of p-harmonic functions and their applications

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Abstract: We derive local and global monotonic quantities associated to p-harmonic functions on manifolds with nonnegative scalar curvature. As applications, we obtain inequalities relating the mass of asymptotically flat 3-manifolds, the p-capacity and the Willmore functional of the boundary. As $p \to 1$, one of the results retrieves a classic relation that the ADM mass dominates the Hawking mass if the surface is area outer-minimizing.

1. Introduction and statement of results

The Riemannian Penrose inequality (RPI) in 3-dimension relates the mass of an asymptotically flat manifold to the area of its boundary if the boundary is the outermost minimal surface in the sense that it is not enclosed by another minimal surface. The inequality was proved by Huisken-Ilmanen [23] in the case of connected boundary via a weak formulation of the inverse mean curvature flow. The general case was proved by Bray [8] using a conformal flow of metrics. The higher dimensional inequality was proved by Bray and Lee [9] in dimensions less than eight.

Recently, Agostiniani-Mantegazza-Mazzieri-Oronzio [3] gave a new proof of the 3-dimensional RPI in the case of connected boundary. In their approach, a related inequality was first established via p-harmonic functions with $p > 1$. The RPI was obtained by letting $p \to 1$ using [17, Theorem 1.2] on the limiting behavior of p-capacity. A main ingredient in the approach was a generalization to p-harmonic functions of a monotonic property of harmonic functions found by Agostiniani-Mazzieri-Oronzio in [2].

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In [34], some distinct monotone properties of harmonic functions were found by the second author. It is natural to ask if those properties of harmonic functions can be generalized to p-harmonic functions, and if so, what applications such a generalization may give.

This paper is motivated by the above questions. Among other things, we retrieve in Corollary 5.1 that

$$\sqrt{\frac{|\partial M|}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\partial M} H^2\right)\leq \mathfrak{m}, \quad \text{if } \partial M \text{ is area outer-minimizing.} \tag{1.1}$$

Here area outer-minimizing means that every surface Σ enclosing ∂M has larger area. (1.1) represents a well-known relation

$$\text{Hawking mass} \leq \text{ADM mass},$$

provided ∂M is area outer-minimizing and M has simple topology. This was first proved by Huisken and Ilmanen [23] via the method of weak inverse mean curvature flow. In Corollary 5.1, we show (1.1) can be derived from p-harmonic functions.

We recall that a complete Riemannian 3-manifold (M, g) is asymptotically flat (AF) if there is a compact set K such that $M \setminus K$ is diffeomorphic to the exterior of a ball in $\mathbb{R}^3$, such that if δ_{ij} is the Euclidean metric, then for $0 \leq l \leq 2$

$$|g_{ij}-\delta_{ij}| = O_2(r^{-\sigma}) \tag{1.2}$$

as $r \to \infty$, with $\sigma > \frac{1}{2}$, where r is the Euclidean distance from a fixed point. This means $|D^l(g_{ij}-\delta_{ij})| = O(r^{-l-\sigma})$ for $0 \leq l \leq 2$. Suppose the scalar curvature $\mathcal{S}$ of M is integrable, then the ADM mass $\mathfrak{m}$ introduced in [4] is well-defined [5, 13]:

$$\mathfrak{m} = \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r}(g_{ij,i}-g_{ii,j})\nu_e^j d\sigma_e, \tag{1.3}$$

where $S_r = \{|x| = r\}$, ν_e is the unit outward normal to S_r and $d\sigma_e$ is the area element on S_r both with respect to the Euclidean metric g_e. In this work, we prove the following:

Theorem 1.1. *Let (M^3, g) be a complete, orientable, asymptotically flat 3-manifold with boundary ∂M. Suppose ∂M is connected and $H_2(M, \partial M) = 0$.*

For $1 < p \leq 2$, let u be the p-harmonic function on M with $u = 0$ on ∂M, and $u \to 1$ at infinity. If g has nonnegative scalar curvature, then

$$4\pi + \int_{\partial M} |\nabla u| H \geq a^{-2}(1+2a) \int_{\partial M} |\nabla u|^2, \tag{1.4}$$

$$c^{\frac{1}{a}}\left(8\pi - a^{-1}\int_{\partial M} |\nabla u| H\right) \leq 4\pi(5-p)\mathfrak{m}, \tag{1.5}$$

$$c^{\frac{1}{a}}\left(4\pi - a^{-2}\int_{\partial M} |\nabla u|^2\right) \leq 4\pi(3-p)\mathfrak{m}. \tag{1.6}$$

Here $a = \frac{3-p}{p-1}$, $c = a^{-1}(\frac{C_p}{4\pi})^{\frac{1}{p-1}}$, C_p is the p-capacity of ∂M in (M, g) and H is the mean curvature of ∂M. If equality holds in any of the above inequalities, then (M, g) is isometric to $\mathbb{R}^3$ outside a round ball.

For simplicity, we only consider orientable manifolds in this work. By taking two-fold cover, some results are also true for non-orientable manifolds. Besides retrieving (1.1), other applications of Theorem 1.1 include sufficient conditions via C^0-data of regions separating the boundary and the infinity, which imply the positivity of the mass. See Corollary 5.2 for details.

We prove Theorem 1.1 by exhibiting a family of monotonic quantities for p-harmonic functions and analyzing their asymptotic behavior.

First, let us review the concepts of p-harmonic functions and p-capacity, also see Section 1.1 in [3]. Let (M, g) denote a complete Riemannian 3-manifold with boundary ∂M, which is assumed to be compact throughout this work. Given any $p \in (1, 3)$, a function u is called a p-harmonic function such that u vanishes at ∂M and $u \to 1$ at infinity if $u \in W^{1,p}_{loc}(M)$ and satisfies

$$\begin{cases} \mathbf{div}(|\nabla u|^{p-2}\nabla u) = 0, & \text{in } M \text{ in the weak sense;} \\ u = 0 & \text{on } \partial M; \\ u(x) \to 1 & \text{as } x \to \infty. \end{cases} \tag{1.7}$$

In case of asymptotically flat manifold, such a u exists in $W^{1,p}_{loc}$ and on any precompact set u is in $C^{1,\beta}$ for some $\beta > 0$. Moreover, u is smooth whenever $|\nabla u| > 0$, see [14, Theorems 1,2], see also [15]. the maximum principle holds for u, see [20, Lemma 3.18 and Theorem 6.5], which implies $0 \leq u < 1$ and $\partial M = \{u = 0\}$. Moreover, the Hopf lemma holds, see [40, Section 2], which shows $|\nabla u| > 0$ at ∂M. As $x \to \infty$, u has an asymptotic expansion of

$$u = 1 - cr^{-a} + o_2(r^{-a}), \quad r = |x|. \tag{1.8}$$

That is $|D^\ell u - D^\ell(1-cr^{-a})| = o(r^{-a-\ell})$ for $\ell = 1, 2$. See [6, Theorem 3.1], where $a = \frac{3-p}{p-1}$, $c = a^{-1}(\frac{C_p}{4\pi})^{\frac{1}{p-1}}$, and C_p is the p-capacity of ∂M in (M, g) given by

$$C_p = \inf\left\{\int_M |\nabla\phi|^p\right\},$$

where the infimum is taken over all Lipschitz functions ϕ with compact support such that $\phi = 1$ at ∂M. C_p is related to u by

$$C_p = \int_M |\nabla u|^p = \int_{\{u=\tau\}} |\nabla u|^{p-1}$$

if τ is a regular value of u, see [7] for instance. Here we omit the volume element and the area element for simplicity.

We want to study quantities related to the mass of an AF manifold. It was know in [16, Lemma 2.2] (also see [24, Proposition A.2]) that, as $r \to \infty$,

$$4\pi r - \int_{S_r} H + \frac{A(r)}{r} = 8\pi\mathfrak{m} + o(1). \tag{1.9}$$

Here H is the mean curvature and $A(r)$ is the area of S_r, respectively. From this, one can check that any function $f(t)$ with $f'(t) > 0$, and if $v(x) = f(r(x))$, then

$$\begin{aligned} &4\pi r - f'(r)^{-1}\int_{\{v=f(r)\}} |\nabla v|H + f'(r)^{-2}r^{-1}\int_{\{v=f(r)\}} |\nabla v|^2 \\ &\quad = 8\pi\mathfrak{m} + o(1). \end{aligned} \tag{1.10}$$

Given the p-harmonic function u, motivated by (1.8), let $f(t) = 1 - ct^{-a}$, if $f(t)$ is a regular value of u, then the above expression becomes:

$$F(t) = 4\pi t - (ca)^{-1}t^{a+1}\int_{\{u=f(t)\}} |\nabla u|H + (ca)^{-2}t^{2a+1}\int_{\{u=f(t)\}} |\nabla u|^2, \tag{1.11}$$

which is the quantity considered in [2, 3].

Motivated by [34], two other quantities may be constructed from u and f. We define

$$\begin{cases} \mathcal{B}(t) = 4\pi t - (f')^{-2}t^{-1}\int_{\{u=f(t)\}} |\nabla u|^2 \\ \qquad\;\; = 4\pi t - (ca)^{-2}t^{2a+1}\int_{\{u=f(t)\}} |\nabla u|^2, \\ \mathcal{A}(t) = 8\pi t - (f')^{-1}\int_{\{u=f(t)\}} |\nabla u|H \\ \qquad\;\; = 8\pi t - (ca)^{-1}t^{a+1}\int_{\{u=f(t)\}} |\nabla u|H. \end{cases} \tag{1.12}$$

$F(t), \mathcal{A}(t), \mathcal{B}(t)$ are related to the Hawking mass of the level surface, see Appendix C.

Similar to $\Psi(s)$ in [34, Section 3], we also define

$$
\begin{aligned}
D(t) &= t^{-a}\mathcal{B}'(t) \\
&= 4\pi t^{-a} + c^{-1}\int_{\{u=f(t)\}} |\nabla u| H \\
&\quad - (ca)^{-2}(1+2a)t^{a}\int_{\{u=f(t)\}} |\nabla u|^2.
\end{aligned} \tag{1.13}
$$

On the other hand, if (M, g) is a complete manifold without boundary and G is the positive p-harmonic Green's function with pole at x_0 and approaching 0 at infinity (provided it exists), then define

$$
\mathcal{G}(\tau) = -4a^2\pi\tau + \tau^{-1}\int_{\{G=\tau\}} |\nabla G|^2. \tag{1.14}
$$

This was studied in [36, 11]. We will see in Section 2 that $\mathcal{A}(t)$, $\mathcal{B}(t)$, $D(t)$, $F(t)$ and $\mathcal{G}(\tau)$ and their monotone properties are closely related.

The monotone property of F for p-harmonic functions was proved in [2, 3] and the monotone properties of $\mathcal{A}$, $\mathcal{B}$, D for harmonic functions were obtained in [34]. In this work, we prove the following:

Theorem 1.2. *For $1 < p \le 2$, suppose u is a p-harmonic function on a complete, orientable Riemannian 3-manifold (M, g) with compact boundary ∂M such that $u = 0$ on ∂M and $u \to 1$ at infinity. Suppose ∂M is connected, $H_2(M, \partial M) = 0$, and g has non-negative scalar curvature. Let $f(t)$, $\mathcal{A}(t)$, $\mathcal{B}(t)$, $D(t)$ be given as above. Let $\Sigma(t) = \{u = f(t)\}$. Suppose $0 < t_1 < t_2$ such that $\Sigma(t_1)$ and $\Sigma(t_2)$ are regular. Then*

(i) (Local monotonicity) $D(t_1) \ge D(t_2)$. Moreover, if $D(t_1) = D(t_2)$, then $\{f(t_1) < u < f(t_2)\}$ is isometric to an annulus in $\mathbb{R}^3$.

(ii) (Global positivity) If (M, g) is asymptotically flat, then $D(t) \ge 0$ and $(1 + 2a)\mathcal{B}(t) - a\mathcal{A}(t) \ge 0$ whenever $\Sigma(t)$ is regular. Moreover, equality holds if and only if $\{u > f(t)\}$ is isometric to $\mathbb{R}^3$ outside a round ball.

(iii) (Global monotonicity) If (M, g) is asymptotically flat, then

$$
\mathcal{B}(t_2) \ge \mathcal{B}(t_1) \quad \textit{and} \quad \mathcal{A}(t_2) \ge \mathcal{A}(t_1).
$$

Moreover, if $\mathcal{B}(t_1) = \mathcal{B}(t_2)$ or $\mathcal{A}(t_1) = \mathcal{A}(t_2)$, then $\{u > f(t_1)\}$ is isometric to $\mathbb{R}^3$ outside a round ball.

Theorem 1.2, together with the asymptotically behaviors of $\mathcal{A}(t)$ and $\mathcal{B}(t)$, i.e. Proposition 4.1, will imply Theorem 1.1.

Next, we want to give a unified treatment on the monotone properties regarding $\mathcal{A}(t)$, $\mathcal{B}(t)$, $D(t)$, $F(t)$ and $\mathcal{G}(\tau)$ at least if no critical points are present. First we summarise the results we want to consider in the following table:

<table>
<tr><th></th><th>type</th><th>$p = 2$</th><th>$p \in (1, 2)$</th><th>IMCF $(p = 1)$</th></tr>
<tr><td>F</td><td>local</td><td>Agostiani –Mazzieri –Oronzio [2]</td><td>Agostiani –Mantegazza –Mazzieri-Oronzio [3]</td><td rowspan="3">Geroch [18]
Jang–Wald [25]
Huisken–Ilmanen [23]</td></tr>
<tr><td>D
$\mathcal{A}, \mathcal{B}$</td><td>local
non-local</td><td>Miao [34]</td><td>Theorem 1.2 in this work</td></tr>
<tr><td>$\mathcal{G}$</td><td>non-local</td><td>Munteanu –Wang [36]</td><td>Chan–Chu –Lee–Tsang [11]</td></tr>
</table>

Here non-locality in this table means either one made use of the asymptotically flatness of the manifold, or one used the asymptotically behavior near the pole of the Green's function. The above monotone properties are consequences of a single formula described in the following theorem, assuming there is no critical point.

Theorem 1.3. *Let (M, g) be a compact, 3-dimensional Riemannian manifold with boundary ∂M. Suppose ∂M consists of two connected components $\partial_+ M$, $\partial_- M$. Let $\alpha \in [-1, 1]$ and $\beta = 0$ or $\frac{2}{1-\alpha}$ if $|\alpha| < 1$. Suppose u is a solution with $|\nabla u| > 0$ to the boundary value problem*

$$\begin{cases} \Delta u = \alpha \nabla^2 u(\nu, \nu) + \frac{2|\nabla u|^2}{u}, & \text{in } M \\ u = c_+, & \text{at } \partial M_+ \\ u = c_-, & \text{at } \partial M_- \end{cases} \tag{1.15}$$

for two positive constants $c_- < c_+$. Here $\nabla^2 u$ denotes the Hessian of u and $\nu = \frac{\nabla u}{|\nabla u|}$. Then the following equality holds:

(1.16)

$$\begin{aligned} &\frac{1}{u^\beta}\left(\mathcal{R}_\alpha(u) - \mathcal{S}^t |\nabla u|\right) \\ &\quad = 2\,\mathbf{div}\left[\frac{1}{u^\beta}\left(\nabla|\nabla u| - \Delta u \frac{\nabla u}{|\nabla u|} + \frac{2\beta - 1}{\beta - 1}\frac{|\nabla u|\nabla u}{u}\right)\right] =: \mathbf{div}(u^{-\beta} X), \end{aligned}$$

where $\mathcal{S}^t = 2K$ denotes the scalar curvature of the level sets $\Sigma_t = \{u = t\}$, where K is the Gaussian curvature, and

$$\mathcal{R}_\alpha(u) = \mathcal{S}|\nabla u| + |\nabla u|^{-1}|T|^2 - \alpha^2|\nabla u|^{-1}u_{\nu\nu}^2$$

in which $\mathcal{S}$ is the scalar curvature of M, $u_{\nu\nu} = \nabla^2 u(\nu,\nu)$, and

$$T = \nabla^2 u + u^{-1}(\nabla u \otimes \nabla u - |\nabla u|^2 g).$$

As a result, upon integration,

$$\begin{aligned}
&\frac{1}{2}\int_M \frac{1}{u^\beta}\mathcal{R}_\alpha(u)dV \\
&\quad = 2\pi\chi\int_{c_-}^{c_+}\frac{1}{t^\beta}dt \\
&\qquad + \left(\int_{\partial_+ M} - \int_{\partial_- M}\right)\frac{1}{u^b}\left(-H|\nabla u| + \frac{2\beta-1}{\beta-1}\frac{|\nabla u|^2}{u}\right)dA,
\end{aligned} \tag{1.17}$$

where $H = H_u$ is the mean curvature of the boundary with respect to $\nabla u/|\nabla u|$ and χ is the Euler characteristic of $\partial_- M$ and hence of every level set of u.

A motivation to (1.15) is that it gives the equation $|x|$ satisfies in the setting of $\mathbb{R}^3$, where $|x|^{-a}$ is a p-harmonic function and $2\log|x|$ is a solution to the inverse mean curvature flow. See the discussions following Corollary 2.1.

We want to add that all monotonic properties of p-harmonic functions mentioned above have a model space of the exterior of a round ball in $\mathbb{R}^3$. In [37], Oronzio obtains monotonicity formulae modeled on Schwarzschild manifolds.

The organization of the paper is as follows. In Section 2, we will prove Theorem 1.3 and study the relation between $\mathcal{A}, \mathcal{B}, D, F, \mathcal{G}$. In Section 3, we will prove Theorem 1.2, and in Section 4, we will study the asymptotical behavior of $\mathcal{A}, \mathcal{B}$ and prove Theorem 1.1. We will give applications in Section 5 and list some facts in the appendices.

We would like to acknowledge that some results in the paper were known to Man-Chun Lee and Tin-Yau Tsang. We thank for their communications [28]. We are also grateful for the anonymous referees whose comments improved the paper.

2. Relating $\mathcal{A}$, $\mathcal{B}$, D, F and $\mathcal{G}$

We start this section with a proof of Theorem 1.3.

Proof of Theorem 1.3. Since $|\nabla u| > 0$, we have $c_- < u < c_+$ in the interior of M. The following computations are from [10, (4.8)] and [39]. By Bochner's identity and the Gauss equation,

$$\begin{aligned}2\Delta|\nabla u| &= 2|\nabla u|^{-1}\big(|\nabla u|^2\operatorname{Ric}(\nu,\nu) + |\nabla^2 u|^2 + \langle\nabla\Delta u, \nabla u\rangle - |\nabla|\nabla u||^2\big)\\ &= 2|\nabla u|^{-1}\big(|\nabla^2 u|^2 + \langle\nabla\Delta u, \nabla u\rangle - |\nabla|\nabla u||^2\big)\\ &\quad + |\nabla u|(\mathcal{S} - \mathcal{S}^t + H^2 - |A|^2)\end{aligned}$$

where H is the mean curvature and A is the second fundamental form of the level sets Σ_t. By Lemma A.1:

$$|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 = |\nabla u|^2|A|^2 - u_{\nu\nu}^2,$$

and

$$H = |\nabla u|^{-1}(\Delta u - u_{\nu\nu}).$$

Replacing $H^2 - |A|^2$ with u and its derivatives, one has

$$\begin{aligned}2\Delta|\nabla u| = |\nabla u|^{-1}\big(&|\nabla^2 u|^2 + 2\langle\nabla\Delta u, \nabla u\rangle\\ &+ (\Delta u)^2 - 2u_{\nu\nu}\Delta u\big) + |\nabla u|(\mathcal{S} - \mathcal{S}^t).\end{aligned}$$

As a result, one has the following formula, see [10, (4.8)]:

$$\begin{aligned}&2\,\mathbf{div}\Big(\nabla|\nabla u| - \Delta u\frac{\nabla u}{|\nabla u|}\Big)\\ =&\frac{|\nabla^2 u|^2}{|\nabla u|} + (\mathcal{S} - \mathcal{S}^t)|\nabla u| - \frac{(\Delta u)^2}{|\nabla u|}\\ =&\frac{1}{|\nabla u|}\Big(|\nabla^2 u|^2 - \alpha^2 u_{\nu\nu}^2 - 4\alpha u_{\nu\nu}\frac{|\nabla u|^2}{u} - \frac{4|\nabla u|^4}{u^2}\Big)\\ &+ (\mathcal{S} - \mathcal{S}^t)|\nabla u|.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbf{div}\Big(\frac{|\nabla u|\nabla u}{u}\Big) &= \frac{|\nabla u|\Delta u}{u} + \frac{\langle\nabla|\nabla u|, \nabla u\rangle}{u} - \frac{|\nabla u|^3}{u^2}\\ &= \frac{|\nabla u|}{u}\Big((\alpha+1)u_{\nu\nu} + \frac{|\nabla u|^2}{u}\Big).\end{aligned}$$

Since

$$\frac{1}{2}X = \nabla|\nabla u| - \Delta u\frac{\nabla u}{|\nabla u|} + \frac{2\beta - 1}{\beta - 1}\frac{|\nabla u|\nabla u}{u},$$

hence

$$\begin{aligned}\langle \nabla u^{-\beta}, X\rangle =& -2\beta u^{-\beta-1}\Big(\langle \nabla|\nabla u|, \nabla u\rangle - |\nabla u|\Delta u + \frac{2\beta-1}{\beta-1}\frac{|\nabla u|^3}{u}\Big)\\ =& -2\beta u^{-\beta-1}\Big((1-\alpha)|\nabla u|u_{\nu\nu} + \frac{1}{\beta-1}\frac{|\nabla u|^3}{u}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}& u^\beta\,\mathbf{div}(u^{-\beta}X) - (\mathcal{S}-\mathcal{S}^t)|\nabla u|\\ =&\frac{1}{|\nabla u|}\Big(|\nabla^2 u|^2 - \alpha^2 u_{\nu\nu}^2 - 4\alpha u_{\nu\nu}\frac{|\nabla u|^2}{u} - \frac{4|\nabla u|^4}{u^2}\Big)\\ &+\frac{2(2\beta-1)}{\beta-1}\frac{|\nabla u|}{u}\Big((\alpha+1)u_{\nu\nu} + \frac{|\nabla u|^2}{u}\Big)\\ &-2\beta\frac{|\nabla u|}{u}\Big((1-\alpha)u_{\nu\nu} + \frac{1}{\beta-1}\frac{|\nabla u|^2}{u}\Big)\\ =&\frac{1}{|\nabla u|}\Big[|\nabla^2 u|^2 - \alpha^2 u_{\nu\nu}^2 + 2(1-\alpha)u_{\nu\nu}\frac{|\nabla u|^2}{u} - \frac{2|\nabla u|^4}{u^2}\Big]\\ =&\frac{|\nabla^2 u + u^{-1}(\nabla u\otimes\nabla u - |\nabla u|^2 g)|^2}{|\nabla u|} - \frac{\alpha^2 u_{\nu\nu}^2}{|\nabla u|}.\end{aligned}$$

Note that $T(\nu,\nu) = u_{\nu\nu}$ because $u_\nu = |\nabla u|$. This verifies (1.16).

Upon integration, one has

$$\begin{aligned}&\int_M \frac{1}{u^\beta}\mathcal{R}_\alpha(u)dV - \int_{c_-}^{c_+}\frac{1}{t^b}\Big(\int_{\Sigma_t}\mathcal{S}^t\Big)dt\\ =&\int_{\partial M_+}\Big\langle u^{-\beta}X, \frac{\nabla u}{|\nabla u|}\Big\rangle dA - \int_{\partial M_-}\Big\langle u^{-\beta}X, \frac{\nabla u}{|\nabla u|}\Big\rangle dA,\end{aligned}$$

as the unit outward normal to ∂M is ν at $\partial_+ M$ and $-\nu$ at $\partial_- M$. Using the identity $|\nabla u|H_u = \Delta u - \frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|}$, one obtains

$$\begin{aligned}\Big\langle X, \frac{\nabla u}{|\nabla u|}\Big\rangle =& 2\Big(\frac{\langle\nabla|\nabla u|,\nabla u\rangle}{|\nabla u|} - \Delta u + \frac{2\beta-1}{\beta-1}\frac{|\nabla u|^2}{u}\Big)\\ =& 2\Big(-H_u|\nabla u| + \frac{2\beta-1}{\beta-1}\frac{|\nabla u|^2}{u}\Big)\end{aligned}$$

which implies equation (1.17) by the Gauss-Bonnet theorem. □

Since $\alpha^2 \le 1$,

$$\mathcal{R}_\alpha(u) = \mathcal{S}|\nabla u| + |\nabla u|^{-1}(|T|^2 - T^2(\nu,\nu)) + (1-\alpha^2)|\nabla u|^{-1}u_{\nu\nu}^2 \ge \mathcal{S}|\nabla u|$$

we have:

Corollary 2.1. *Suppose $\mathcal{S} \ge 0$ in Theorem 1.3, then*

$$2\pi\chi \int_{c_-}^{c_+} \frac{1}{t^\beta} dt + \left(\int_{\partial_+ M} - \int_{\partial_- M}\right) \frac{1}{t^\beta}\left(-H_u|\nabla u| + \frac{2\beta-1}{\beta-1}\frac{|\nabla u|^2}{u}\right) dA \ge 0, \tag{2.1}$$

where χ is the Euler characteristic of $\partial_- M$. If the equality holds and $\alpha < 1$, (M^3, g) is isometric to an annulus in $\mathbb{R}^3$ and $u = C\rho$ where ρ is the Euclidean distance to the center of the annulus.

Remark 2.1. One can classify the rigidity case of $\alpha = 1$ too. As it is not needed in the main results, we include it in Appendix B.

Proof. (2.1) follows from Theorem 1.3. If equality holds, then $\mathcal{S} = 0$ and

$$|T|^2 - T^2(\nu,\nu) + (1 - |\alpha|^2)u_{\nu\nu}^2 = 0. \tag{2.2}$$

Let $v = u^2$, $t_0^2 = c_-$, $t_1^2 = c_2$ so that $t_0 \le v \le t_1$, then $T = \frac{1}{u}(\nabla^2 v - \frac{|\nabla v|^2}{2v} g)$. Since $|\alpha| \le 1$, we have $T(X,Y) = 0, T(X,\nu) = 0$, for any X, Y tangent to the level set of v, and

$$\nabla^2 v(X,Y) = |\nabla v| A(X,Y);\ \nabla^2 v(X,\nu) = X(|\nabla v|), \tag{2.3}$$

where $\nu = \nabla v/|\nabla v|$. Hence $|\nabla v| =: \eta$ is constant on each level set, and on $\{v = t\}$,

$$A(X,Y) = \frac{|\nabla v|}{2v} g(X,Y), \quad H = \frac{|\nabla v|}{v} = \frac{\eta}{t}.$$

Therefore, the level sets are umbilical and

$$g = \eta^{-2}(t)dt^2 + \gamma_t,$$

where $\eta(t) = |\nabla v|$ depends only on t and γ_t is the induced metric on $\{v = t\}$. Now $\partial_t \gamma_t = 2\eta^{-1}(t)A_t = t^{-1}\gamma_t$ where A_t is the level set $\Sigma_t = \{v = t\}$. So $\gamma_t = t t_0^{-1}\gamma_{t_0}$. It remains to find η.

If $|\alpha| < 1$, then $u_{\nu\nu} = 0$. So

$$\eta' = \frac{1}{|\nabla v|} v_{\nu\nu} = \frac{2u_\nu^2}{|\nabla v|} = \frac{|\nabla v|}{2v} = \frac{1}{2t}\eta.$$

If $\alpha = -1$, then

$$\frac{\eta^2}{t} = |\nabla v| H = \Delta v - v_{\nu\nu} = -2v_{\nu\nu} + \frac{2\eta^2}{t} = -2\eta\eta' + \frac{2\eta^2}{t}.$$

We still have $\eta' = \frac{1}{2t}\eta$. Hence, in either case, $t^{-\frac{1}{2}}\eta(t) = t_0^{-\frac{1}{2}}\eta(t_0)$. So if we let $t = r^2$, then

$$\begin{aligned} g &= t_0 \eta^{-2}(t_0) t^{-1} dt^2 + t_0^{-1} t \gamma_{t_0} \\ &= t_0^{-1}\big((n-1)^2 H^{-2}(t_0) dr^2 + r^2 \gamma_{t_0}\big). \end{aligned}$$

To find γ_{t_0}, by Lemma A.1 and the facts that $|\nabla v|$ is constant on the level set and the level set is umbilical, we have

$$\eta(t) \frac{\partial}{\partial t} H = -\frac{1}{2}\left(\frac{3}{2}H^2 - 2K\right).$$

On the other hand

$$\frac{\partial}{\partial t} H = \frac{\partial}{\partial t}\left(\frac{\eta}{t}\right) = -\frac{\eta}{2t^2}.$$

So $K = \frac{1}{4}H^2$ which is a positive constant. Thus, each level set is a sphere, and

$$g = 4t_0^{-1} H^{-2}(t_0)\big(dr^2 + r^2 \sigma_0\big),$$

where σ_0 is the standard unit sphere in $\mathbb{R}^3$. This completes the proof. □

Now we illustrate how Corollary 2.1 relates to the previously mentioned *local* monotone quantities. It should be emphasized that Corollary 2.1 assumed $|\nabla u| > 0$. Such an assumption was not necessary in the corresponding results below.

(I) *Inverse mean curvature flow*: Take $\alpha = 1$ and $\beta = 0$. The function $U = 2\log u$ satisfies $\Delta U = \nabla_{\nu\nu} U + |\nabla U|^2$. This means the level sets $\{\Sigma_U\}$ of U flow by inverse mean curvature flow and $|\Sigma_U| = |\Sigma_0| e^U$. In this case,

$$0 \le 4\pi \int_{c_-}^{c_+} 1\, dt + \left(\int_{\partial_+ M} - \int_{\partial_- M}\right)\left(-2H|\nabla u| + 2\frac{|\nabla u|^2}{u}\right)$$

$$= 8\pi(u(\Sigma_+) - u(\Sigma_-) - \frac{1}{2}u(\Sigma_+)\int_{\Sigma_+} H^2 + \frac{1}{2}u(\Sigma_-)\int_{\Sigma_+} H^2$$

which implies $m_H(\Sigma_+) \geq m_H(\Sigma_-)$, where m_H is the Hawking energy [19]:

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\Big(1 - \frac{1}{16\pi}\int_\Sigma H^2\Big).$$

The monotonic property for the Hawking energy under inverse mean curvature flowed was first proved by Georch [18] in case there are no critical points and by Huisken-Ilmanen [23] in general under some topological assumptions.

(II) *p-harmonic functions and the monotonicity formulas*: Let $1 > U > 0$ be a positive p-harmonic function. In [2, 3] Agostiniani-Mantegazza-Mazzieri-Oronzio showed the following monotonicity formula: For $0 < t_1 < t_2$

$$F(t_2) \geq F(t_1)$$

where $F(t)$ is given by (1.11) for U. In case of $|\nabla U| > 0$, this can also be derived from Corollary 2.1. In fact, let $u = (1-U)^{-\frac{1}{a}}$. Then u satisfies (1.15) for $\alpha = 2-p$ by Lemma B.1. Moreover, $U = 1-ct^{-a}$ if and only if $u = c^{-\frac{1}{a}}t$. Moreover at $\{U = f(t)\}$, $|\nabla u| = c^{-\frac{1}{a}}(ca)^{-1}t^{a+1}|\nabla U|$ and $\nabla u/|\nabla u| = \nabla U/|\nabla U|$. Hence apply Corollary 2.1 to u with $\beta = 0$, we obtain the monotonicity of $F(t)$ for U.

(III) *Harmonic functions and the monotonicity of $D(t)$*: Let U be a positive harmonic function. In [34, Lemma 3.1] the second author showed

$$D(t_1) \geq D(t_2)$$

for $0 < t_1 < t_2$. As in **(II)** in case $|\nabla U| > 0$, let $u = (1-U)^{-1}$, the above result is also a consequence of Corollary 2.1 with $\beta = 2$.

Remark 2.2. Conceptually, it should not come as a surprise that Theorem 1.3 implies various other monotonicity formulas which rely on Gauss-Bonnet's theorem. Informally speaking, this requires the Gaussian curvature term appearing in a divergence identity such as equation (1.16) to be of the form $f(u)|\nabla u|K$ for some function f. In order to ensure that the remaining terms on the left hand side of (1.16) are non-negative, the freedom of choosing f is drastically restricted.

Remark 2.3. For $\beta = 0$, Theorem 1.3 also generalizes to initial data sets (M, g, k) satisfying the dominant energy condition. In this case the equation

$$\Delta u = \alpha \nabla^2 u(\nu, \nu) + \frac{2|\nabla u|^2}{u}$$

needs to be replaced by the system

$$\begin{aligned}
\Delta u &= -\operatorname{tr}_g(k)|\nabla u| + \alpha \nabla^2 u(\eta, \eta) + \alpha k(\eta, \eta)|\nabla u| + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v} \\
\Delta v &= \operatorname{tr}_g(k)|\nabla v| + \alpha \nabla^2 v(\eta, \eta) - \alpha k(\eta, \eta)|\nabla v| + \frac{3|\nabla u||\nabla v| + \langle \nabla u, \nabla v\rangle}{u+v}
\end{aligned}$$

where $\eta = \frac{\nabla u|\nabla v| + \nabla v|\nabla u|}{|\nabla u|\nabla v| + \nabla v|\nabla u||}$. Note that in case $k = 0$, we can set $u = v$ and the above system reduces to equation (2.3). For more details see Theorem 1.1. and Corollary 1.2 in [21]. We believe that for $\beta = \frac{2}{1-\alpha}$ there is no analogue for initial data sets satisfying the dominant energy condition.

Let (M^3, g) denote a complete manifold with compact boundary ∂M. As mentioned in the introduction, $\mathcal{A}, \mathcal{B}, D, F, \mathcal{G}$ are closely related. Let u be a p-harmonic function satisfying $u \to 1$ at ∞ and $u < 1$ on M. To facilitate a comparison with the monotonicity for p-harmonic Green's function from [11, 36], we define, for the given u,

$$\mathcal{G}(t) = -4a^2\pi c t^{-a} + \left(c t^{-a}\right)^{-1} \int_{\{u = f(t)\}} |\nabla u|^2. \tag{2.4}$$

Note that if (M, g) is complete without boundary and G is the positive p-harmonic Green's function with pole at x_0 and approaching 0 at infinity, then choosing $u = 1 - G$ and $f(t) = 1 - t^{-a}$, we have

$$\begin{aligned}
\mathcal{G}(t) = &\ -4a^2\pi t^{-a} + \left(t^{-a}\right)^{-1} \int_{\{1-G = 1 - t^{-a}\}} |\nabla u|^2 \\
= &\ \mathcal{G}(\tau), \qquad \text{upon a substitution } \tau = t^{-a},
\end{aligned} \tag{2.5}$$

the quantity given in (1.14). We have:

Lemma 2.1. *Suppose $|\nabla u| > 0$. Then*

(i) $D(t) = t^{-a-1}[(1+2a)\mathcal{B}(t) - a\mathcal{A}(t)]$. $F(t) = \mathcal{A}(t) - \mathcal{B}(t)$.
(ii) $D'(t) = -at^{-a-1}F'(t)$. *Hence* $D' \leq 0$ *if and only if* $F' \geq 0$.
(iii) $\mathcal{G}(t) = -ca^2 t^{-a-1}\mathcal{B}(t)$.
(iv) $\mathcal{G}'(t) = ca^3 t^{-a-2} F(t)$. *Hence* $\mathcal{G}' \geq 0$ *if and only if* $F \geq 0$.

(v) $\mathcal{B}' \geq 0$ *if and only if* $D \geq 0$.

Proof. (v) follows from the definition of D.

(i) $F(t) = \mathcal{A}(t) - \mathcal{B}(t)$ follows from their definitions. By (1.12), (1.11), and Lemma A.2

$$
\begin{aligned}
&\mathcal{B}'(t) \\
&= 4\pi - (ca)^{-2}(1+2a)t^{2a}\int_{u=f(t)} |\nabla u|^2 - (ca)^{-1}t^a \int_{u=f(t)} (2\Delta u - |\nabla u| H) \\
&= 4\pi - (ca)^{-2}(1+2a)t^{2a}\int_{u=f(t)} |\nabla u|^2 + (ca)^{-1}at^a \int_{u=f(t)} (|\nabla u| H \\
&= t^{-1}\big((1+2a)\mathcal{B}(t) - a\mathcal{A}(t)\big),
\end{aligned}
$$

because u is p-harmonic so that $\Delta u = \frac{2-p}{1-p}|\nabla u|H$. By the definition of D in (1.13), (i) follows. To prove (ii), by (i) we have

$$
\begin{aligned}
D'(t) &= -(a+1)t^{-1}D(t) + t^{-a}\big((1+2a)\mathcal{B}'(t) - a\mathcal{A}'(t)\big) \\
&= -at^{-a}F'(t).
\end{aligned}
$$

(iii) follows from (2.4), (1.12) and (1.11). To prove (iv):

$$
\begin{aligned}
\mathcal{G}'(t) &= ca^2(1+a)t^{-a-2}\mathcal{B}(t) - ca^2 t^{-1}D(t) \\
&= ca^2(1+a)t^{-a-2}\mathcal{B}(t) - ca^2 t^{-a-2}\big[(1+2a)\mathcal{B}(t) - a\mathcal{A}(t)\big] \\
&= ca^3 t^{-a-2}F(t).
\end{aligned}
$$

□

Proposition 2.1. *Let (M^3, g) be a complete Riemannian manifold with non-negative scalar curvature, with compact boundary ∂M. For $1 < p \leq 3$, suppose u is a p-harmonic function on M with $u = 0$ at ∂M and $u \to 1$ at infinity. Suppose ∂M is connected and $|\nabla u| > 0$. Then the following monotonicity holds:*

(i) (Local monotonicity) $D(t_1) \geq D(t_2)$ and $F(t_2) \geq F(t_1)$, $\forall\ t_1 < t_2$. Moreover, if equality holds, then $\{f(t_1) < u < f(t_2)\}$ is isometric to an annulus in $\mathbb{R}^3$.

(ii) (Global monotonicity) If (M, g) is asymptotically flat, then

- *$D(t) \geq 0$ and $(1+2a)\mathcal{B}(t) - a\mathcal{A}(t) \geq 0$;*
- *$\mathcal{A}(t)$ and $\mathcal{B}(t)$ are monotone non-degreasing. If $\mathcal{B}(t_1) = \mathcal{B}(t_2)$ or $\mathcal{A}(t_1) = \mathcal{A}(t_2)$, then $\{u > f(t_1)\}$ is isometric to $\mathbb{R}^3$ outside a round ball.*

Proof. (i) We have seen $F'(t) \geq 0$ in Case (**II**) after Corollary 2.1. It in turns shows (i) by Lemma 2.1 (ii). Or we can prove $D(t_1) \geq D(t_2)$ directly as follows. Let $w = (\frac{1-u}{c})^{-\frac{1}{a}}$, where $a = (3-p)/(p-1)$. Let $\alpha = 2-p$. Then w is a positive solution to

$$\Delta w = \alpha w_{\nu\nu} + 2w^{-1}|\nabla w|^2.$$

Choose $\beta = 2/(1-\alpha) = 2/(p-1)$, then $1-\beta = -a$. Given $t_1 < t_2$, by Corollary 2.1,

$$\begin{aligned}
0 &\leq 4\pi \int_{t_1}^{t_2} \frac{1}{t^\beta} + \left(\int_{w=t_2} - \int_{w=t_1}\right) \frac{1}{t^\beta}\left(-H_w|\nabla w| + \frac{2\beta-1}{\beta-1}\frac{|\nabla w|^2}{w}\right) \\
&= -4\pi a^{-1}(t_2^{-a} - t_1^{-a}) \\
&\quad + \int_{\{u=f(t_2)\}} (-H_u(ca)^{-1}|\nabla u| + (1+2a)a^{-1}(ca)^{-2}t_2^a|\nabla u|^2) \\
&\quad - \int_{\{u=f(t_1)\}} (-H_u(ca)^{-1}|\nabla u| + (1+2a)a^{-1}(ca)^{-2}t_1^a|\nabla u|^2) \\
&= a^{-1}(D(t_1) - D(t_2)),
\end{aligned}$$

which proves (i). (The equality case follows from Corollary 2.1.) Note that, by Lemma 2.1 (ii), this implies $F'(t) \geq 0$.

(ii) Since (M,g) is asymptotically flat, the asymptotical behavior of u in (1.8) implies, as $t \to \infty$,

$$\int_{u=f(t)} |\nabla u|^2 = O(t^{-2a}), \quad \int_{u=f(t)} H|\nabla u| = O(t^{-a}).$$

Therefore, $D(t) = O(t^{-a})$ by (1.13). In particular, $D(t) \to 0$ as $t \to \infty$. By (i), we have $D(t) \geq 0$. As a result, by Lemma 2.1 (i), $(1+2a)\mathcal{B}(t) \geq \mathcal{A}(t)$. Also recall $D(t) = t^{-a}\mathcal{B}'(t)$ from (1.13). Hence, $D(t) \geq 0$ shows $\mathcal{B}'(t) \geq 0$, i.e. $\mathcal{B}(t)$ is monotone non-decreasing. Since $\mathcal{A}(t) = \mathcal{B}(t) + F(t)$, we see $\mathcal{A}(t)$ is monotone non-decreasing as well. □

We may also obtain global monotone property for the Green's function using Lemma 2.1. Suppose $G > 0$ is the p-harmonic Green's function with pole at x_0 and $G \to 0$ at infinity. Let $u = 1 - G$ on (M,g). The behavior of $G(x)$ at the pole (see [26, 33]) implies $F(t) \to 0$ as $t \to 0^+$. By Case (**II**) or Proposition 2.1 (i), one knows $F'(t) \geq 0$. Therefore, $F(t) \geq 0$. By Lemma 2.1 (iv), $\mathcal{G}'(t) \geq 0$, which is equivalent to $\mathcal{G}'(\tau) \leq 0$. This corresponds to the monotone property of G in [36, 11]. Hence we have:

Proposition 2.2 (Global monotonicity)**.** *Let (M^3, g) be a complete Riemannian manifold. Suppose M supports a positive p-harmonic Green's function G with pole at x_0 and $G \to 0$ at infinity. Assume that $|\nabla G| > 0$ on $M \setminus \{x_0\}$. Then $\mathcal{G}'(\tau) \leq 0$, where $\mathcal{G}(\tau)$ is as in (1.14).*

Remark 2.4. It is worthy of writing $\mathcal{B}(t)$, $\mathcal{A}(t)$ directly via the level set $\{u = s\}$ and the parameter s. Let $t = (\frac{c}{1-s})^{\frac{1}{a}}$, $\mathcal{B}(t)$ and $\mathcal{A}(t)$ take the form of

$$\mathcal{B}(s) = \left(\frac{c}{1-s}\right)^{\frac{1}{a}} \left[4\pi - \frac{1}{a^2(1-s)^2} \int_{u=s} |\nabla u|^2\right] \tag{2.6}$$

and

$$\mathcal{A}(s) = \left(\frac{c}{1-s}\right)^{\frac{1}{a}} \left[8\pi - \frac{1}{a(1-s)} \int_{u=s} |\nabla u| H\right]. \tag{2.7}$$

If $p = 2$, then $a = 1$, these were given in [34]. A feature of such expressions is that they pinpoint a fact

$$u^{(s)} := \frac{u - s}{1 - s}$$

is the p-harmonic function, vanishing on $\{u = s\}$ and approaching 1 at ∞. This indicates $\mathcal{B}(\cdot)$ and $\mathcal{A}(\cdot)$ are suitable scalings of

$$4\pi - a^{-2} \int_{\partial M_s} |\nabla u^{(s)}|^2 \quad \text{and} \quad 8\pi - a^{-1} \int_{\partial M_s} |\nabla u^{(s)}| H$$

on $M_s = \{u \geq s\}$.

3. Monotonicity of $\mathcal{A}(t), \mathcal{B}(t), D(t)$ via regularization

We will prove Theorem 1.2 stated in the introduction. First, we point out that it suffices to establish the monotonicity of the quantities involved. Once the monotonicity is shown, the equality case will follow by applying Corollary 2.1, starting from the boundary $\partial M = \{u = 0\}$ which is a regular level set of u.

We recall the setting of Theorem 1.2: (M^3, g) is a complete three manifold with smooth compact boundary ∂M such that

(i) ∂M is connected;
(ii) $H_2(M, \partial M) = \{0\}$;
(iii) M has one end which is asymptotically flat; and
(iv) the scalar curvature $\mathcal{S}$ of M is nonnegative.

Let u be the solution of the p-harmonic equation so that $u = 0$ on ∂M and $u \to 1$ near infinity. As before, we denote $f(t) = f_0(t)$ in the theorem by

$$f_0(t) = 1 - ct^{-a},$$

where $a = (3-p)/(p-1)$, $c = a^{-1}(C_p/(4\pi))^{\frac{1}{p-1}}$, and C_p is the p-capacity of ∂M in M. Given any $T > 0$ with $0 < f_0(T) < 1$, the level set $\{u = f_0(T)\}$ is compact and will not intersect ∂M by strong maximum principle [20, Theorem 6.5]. Assume this level set is a regular level set of u. We approximate u by smooth functions. Following [3], for any $\varepsilon > 0$, let $v = v_\varepsilon$ be the solution of

$$\begin{cases} \mathbf{div}(|\nabla v|_\varepsilon^{p-2}\nabla v) = 0, & \text{in } M(T); \\ v = 0 & \text{on } \partial M; \\ v = f_0(T) & \text{on } \Sigma(T); \end{cases} \tag{3.1}$$

where $M(T) = \{0 < u < f_0(T)\}$, $\Sigma(t) = \{u = f_0(t)\}$, and for any $\eta > 0$,

$$|\nabla v|_\eta = \sqrt{|\nabla v|^2 + \eta^2}. \tag{3.2}$$

Then v_ε is smooth. As $\varepsilon \to 0$, $v_\varepsilon \to u$ in $C^{1,\beta}$ norm for some $\beta > 0$, and $v_\varepsilon \to u$ in C^∞ norm near the points where $|\nabla u| > 0$ by [14, 15]. Define

$$\begin{cases} C_{p,\varepsilon} = \int_{\partial M} |\nabla v|_\varepsilon^{p-2}|\nabla v|; \\ c_\varepsilon = a^{-1}(\frac{C_{p,\varepsilon}}{4\pi})^{\frac{1}{p-1}}. \end{cases} \tag{3.3}$$

Note that as $\varepsilon \to 0$, $c_\varepsilon \to c$, $C_{p,\varepsilon} \to C_p$. For $0 < t < T$, let $f_\varepsilon(t) = 1 - c_\varepsilon t^{-a}$ and let $\Sigma(\varepsilon, t) = \{v = v_\varepsilon = f_\varepsilon(t)\}$, which will be in the interior of $M(T)$, provided ε is small enough. Observe that $C_{p,\varepsilon} = \int_{\Sigma(\varepsilon,t)} |\nabla v|_\varepsilon^{p-2}|\nabla v|$ whenever $\Sigma(\varepsilon, t)$ is regular. If $\Sigma(\varepsilon, t)$ is regular, define corresponding $D_\varepsilon(t)$ and $\mathcal{B}_\varepsilon(t)$ as follows:

$$D_\varepsilon(t) = 4\pi t^{-a} - (c_\varepsilon a)^{-2}(1+2a)t^a \int_{\Sigma(\varepsilon,t)} |\nabla v|^2 + c_\varepsilon^{-1} \int_{\Sigma(\varepsilon,t)} |\nabla v| H, \tag{3.4}$$

$$\mathcal{B}_\varepsilon(t) = 4\pi t - (c_\varepsilon a)^{-2} t^{2a+1} \int_{\Sigma(\varepsilon,t)} |\nabla v_\varepsilon|^2. \tag{3.5}$$

Here $H = H_v$ is the mean curvature with respect to $\nu = \nabla v/|\nabla v|$. The corresponding $f, \mathcal{B}, D$ for u will also be denoted by $f_0, \mathcal{B}_0, D_0$ etc. By the proof of [3, Lemma 1.3], the following fact is true:

Lemma 3.1. *Suppose $\{u = f_0(t)\}$ is regular with $0 < t < T$ which implies that $0 < f_0(t) < f_0(T)$. Then for $\varepsilon > 0$ small enough, $\Sigma(\varepsilon, t) = \{v_\varepsilon = f_\varepsilon(t)\}$ is also regular. Moreover,*

$$\lim_{\varepsilon\to 0} D_\varepsilon(t) = D_0(t);\ \lim_{\varepsilon\to 0} \mathcal{B}_\varepsilon(t) = \mathcal{B}_0(t).$$

To simplify notation, in what follows, whenever there is no confusion, we will suppress the index ε. For example, we denote c_ε by c and the original c, which is the limit of c_ε as $\varepsilon \to 0$, will be denoted by c_0 instead. Direct computations give:

Lemma 3.2. *Suppose $\Sigma(\varepsilon, t)$ is regular, then at this level set:*

$$\begin{cases} \Delta v = (2-p)\frac{|\nabla v|^2}{|\nabla v|_\varepsilon^2} v_{\nu\nu} \\ H = \frac{1}{|\nabla v|}(\Delta v - v_{\nu\nu}) = -\frac{1}{|\nabla v|}\frac{(p-1)|\nabla v|^2+\varepsilon^2}{|\nabla v|_\varepsilon^2} v_{\nu\nu} \end{cases}$$

where

$$v_{\nu\nu} = \nabla^2 v(\nu,\nu) = \frac{\langle \nabla|\nabla v|, \nabla v\rangle}{|\nabla v|} = \langle \nabla|\nabla v|, \nu\rangle$$

and $\nu = \nabla v/|\nabla v|$.

Hence if $\Sigma(\varepsilon, t)$ is regular, then

$$D_\varepsilon(t) = c^{-1}\int_{\Sigma(\varepsilon,t)} \big(4\pi C_{p,\varepsilon}^{-1}|\nabla v|_\varepsilon^{p-2}|\nabla v|(1-v) - a^{-2}(1+2a)(1-v)^{-1}|\nabla v|^2 + \Delta v - v_{\nu\nu}\big). \tag{3.6}$$

Let

$$X = X_\varepsilon = W - U + V \tag{3.7}$$

where

$$\begin{cases} W = 4\pi C_{p,\varepsilon}^{-1}(1-v)|\nabla v|_\varepsilon^{p-2}\nabla v; \\ U = a^{-2}(1+2a)(1-v)^{-1}|\nabla v|\nabla v; \\ V = \frac{\Delta v}{|\nabla v|}\nabla v - \nabla|\nabla v|. \end{cases}$$

Since U or V may not be defined or smooth if $|\nabla v| = 0$, we further regularize these functions as follows. For, $\delta > 0$, let

$$\begin{cases} U_\delta = a^{-2}(1+2a)(1-v)^{-1}|\nabla v|_\delta \nabla v, \\ V_\delta = \frac{\Delta v}{|\nabla v|_\delta}\nabla v - \nabla|\nabla v|_\delta. \end{cases}$$

Then W, U_δ, V_δ are smooth. Suppose $0 < t_1 < t_2 < T$ so that $\Sigma(\varepsilon, t_1), \Sigma(,\varepsilon, t_2)$ are regular. One can see that

$$D_\varepsilon(t_2) - D_\varepsilon(t_1) = \lim_{\delta\to 0}\int_{\{f_\varepsilon(t_1)<v<f_\varepsilon(t_2)\}} \mathbf{div}(W - U_\delta + V_\delta). \tag{3.8}$$

Lemma 3.3.

(i)

$$\mathbf{div}\, W = -4\pi C_{p,\varepsilon}^{-1}|\nabla v|_\varepsilon^{p-2}|\nabla v|^2.$$

(ii) At $|\nabla v| = 0$,

$$\mathbf{div}\, U_\delta = \delta a^{-2}(1+2a)(1-v)^{-1}\Delta v.$$

At $|\nabla v| > 0$,

$$\begin{aligned}\mathbf{div}\, U_\delta &= a^{-2}(1+2a)\Big((2-p)\frac{|\nabla v|_\delta|\nabla v|^2}{|\nabla v|_\varepsilon^2} + \frac{|\nabla v|^2}{|\nabla v|_\delta}\Big)(1-v)^{-1}v_{\nu\nu}\\ &\quad + a^{-2}(1+2a)(1-v)^{-2}|\nabla v|^2|\nabla v|_\delta.\end{aligned}$$

(iii) At $|\nabla v| = 0$, $\mathbf{div}\, V_\delta \le 0$. *At* $|\nabla v| > 0$,

$$\begin{aligned}\mathbf{div}\, V_\delta \le \Big(&\frac{(2-p)^2|\nabla v|^4}{|\nabla v|_\varepsilon^4|\nabla v|_\delta} - \frac{(2-p)|\nabla v|^4}{|\nabla v|_\varepsilon^2|\nabla v|_\delta^3}\Big)v_{\nu\nu}^2\\ &- |\nabla v|_\delta^{-1}|\nabla v|^2\big(|A|^2 + \mathrm{Ric}(\nu,\nu)\big),\end{aligned}$$

where A the second fundamental form of the level surface with respect to the unit normal ν.

Proof. (i) Since v satisfies (3.1)

$$\begin{aligned}\mathbf{div}\, W &= \mathbf{div}\big((1-v)|\nabla v|_\varepsilon^{p-2}\nabla v\big)\\ &= 4\pi C_{p,\varepsilon}^{-1}\big[(1-v)\,\mathbf{div}\big(|\nabla v|_\varepsilon^{p-2}\nabla v\big) - \big\langle \nabla v, |\nabla v|_\varepsilon^{p-2}\nabla v\big\rangle\big]\\ &= -4\pi C_{p,\varepsilon}^{-1}|\nabla v|_\varepsilon^{p-2}|\nabla v|^2.\end{aligned} \tag{3.9}$$

(ii) At $|\nabla v| = 0$, it is easy to see that

$$\mathbf{div}\, U_\delta = \delta a^{-2}(1+2a)(1-v)^{-1}\Delta v.$$

At $|\nabla v| > 0$, by Lemma 3.2, we have

$$\begin{aligned}\mathbf{div}\, U_\delta =& a^{-2}(1+2a)\Big((1-v)^{-1}|\nabla v|_\delta \Delta v + (1-v)^{-2}|\nabla v|^2|\nabla v|_\delta \\ &+ (1-v)^{-1}\frac{|\nabla v|}{|\nabla v|_\delta}\langle \nabla|\nabla v|, \nabla v\rangle\Big) \\ =& a^{-2}(1+2a)\Big((1-v)^{-1}|\nabla v|_\delta \Delta v + (1-v)^{-2}|\nabla v|^2|\nabla v|_\delta \\ &+ (1-v)^{-1}\frac{|\nabla v|^2}{|\nabla v|_\delta}\nabla^2 v(\nu,\nu)\Big) \\ =& a^{-2}(1+2a)\Big((2-p)(1-v)^{-1}\frac{|\nabla v|_\delta|\nabla v|^2}{|\nabla v|_\varepsilon^2}v_{\nu\nu} \\ &+ (1-v)^{-2}|\nabla v|^2|\nabla v|_\delta + (1-v)^{-1}\frac{|\nabla v|^2}{|\nabla v|_\delta}v_{\nu\nu}\Big).\end{aligned}$$

(iii) To compute $\mathbf{div}\, V_\delta$, by Lemma 3.2 we have

$$\begin{aligned}\mathbf{div}\Big(\frac{\Delta v}{|\nabla v|_\delta}\nabla v\Big) =& \frac{(\Delta v)^2}{|\nabla v|_\delta} + \frac{\langle \nabla \Delta v, \nabla v\rangle}{|\nabla v|_\delta} - \frac{1}{2}\frac{\Delta v}{|\nabla v|_\delta^3}\cdot\langle \nabla\big(|\nabla v|^2\big), \nabla v\rangle \\ =& (2-p)^2\frac{|\nabla v|^4}{|\nabla v|_\varepsilon^4|\nabla v|_\delta}v_{\nu\nu}^2 - (2-p)\frac{|\nabla v|^4}{|\nabla v|_\varepsilon^2|\nabla v|_\delta^3}v_{\nu\nu}^2 \\ &+ \frac{\langle \nabla \Delta v, \nabla v\rangle}{|\nabla v|_\delta},\end{aligned}$$

which is zero at the points where $|\nabla v| = 0$ because Δv will be zero at this point by (3.1).

$$\begin{aligned}\mathbf{div}(\nabla|\nabla v|_\delta) =& \Delta|\nabla v|_\delta \\ =& \frac{1}{2}|\nabla v|_\delta^{-1}\big(\Delta|\nabla v|_\delta^2 - 2|\nabla|\nabla v|_\delta|^2\big) \\ =& |\nabla v|_\delta^{-1}\big(|\nabla^2 v|^2 + \mathrm{Ric}(\nabla v, \nabla v) + \langle \nabla v, \nabla \Delta v\rangle - |\nabla|\nabla v|_\delta|^2\big) \\ =& |\nabla v|_\delta^{-1}\big(|\nabla^2 v|^2 + \mathrm{Ric}(\nabla v, \nabla v)\big) \\ &+ \frac{\langle \nabla \Delta v, \nabla v\rangle}{|\nabla v|_\delta} - \frac{1}{4}\frac{1}{|\nabla v|_\delta^3}|\nabla\big(|\nabla v|^2\big)|^2,\end{aligned}$$

which is nonnegative at the points $|\nabla v| = 0$ because $\Delta|\nabla v|_\delta \geq 0$.

Hence if $|\nabla v| = 0$, then $\mathbf{div}\, V_\delta \le 0$. If $|\nabla v| > 0$, then

$$\begin{aligned}\mathbf{div}\, V_\delta =&(2-p)^2 \frac{|\nabla v|^4}{|\nabla v|_\varepsilon^4 |\nabla v|_\delta} v_{\nu\nu}^2 - (2-p)\frac{|\nabla v|^4}{|\nabla v|_\varepsilon^2 |\nabla v|_\delta^3} v_{\nu\nu}^2 + \frac{|\nabla v|^2}{|\nabla v|_\delta^3}|\nabla |\nabla v||^2\\ &- |\nabla v|_\delta^{-1}\big(|\nabla^2 v|^2 + \mathrm{Ric}(\nabla v, \nabla v)\big).\end{aligned}$$

For $|\nabla v| > 0$, $|\nabla|\nabla v||^2 = (|\widetilde{\nabla}|\nabla v||^2 + v_{\nu\nu}^2)$, where $\widetilde{\nabla}$ is the derivative on the level set. Hence if $|\nabla v| > 0$, then

$$\begin{aligned}&\mathbf{div}\, V_\delta\\ =&(2-p)^2 \frac{|\nabla v|^4}{|\nabla v|_\varepsilon^4 |\nabla v|_\delta} v_{\nu\nu}^2 - (2-p)\frac{|\nabla v|^4}{|\nabla v|_\varepsilon^2 |\nabla v|_\delta^3} v_{\nu\nu}^2\\ &+ \frac{|\nabla v|^2}{|\nabla v|_\delta^3}\big(|\widetilde{\nabla}|\nabla v||^2 + v_{\nu\nu}^2\big)\\ &- |\nabla v|_\delta^{-1}\big(|\nabla v|^2|A|^2 + 2|\widetilde{\nabla}|\nabla v||^2 + v_{\nu\nu}^2 + |\nabla v|^2\,\mathrm{Ric}(\nu,\nu)\big)\\ \le&\left((2-p)^2 \frac{|\nabla v|^4}{|\nabla v|_\varepsilon^4 |\nabla v|_\delta} - (2-p)\frac{|\nabla v|^4}{|\nabla v|_\varepsilon^2 |\nabla v|_\delta^3}\right) v_{\nu\nu}^2\\ &- |\nabla v|_\delta^{-1}|\nabla v|^2\big(|A|^2 + \mathrm{Ric}(\nu,\nu)\big).\end{aligned} \tag{3.10}$$

This completes the proof of the lemma. □

Lemma 3.4. *With the assumptions and notation as in Lemma 3.3, assume $1 < p \le 2$, and $0 < t_1 < t_2 < T$ so that $\{u = f_0(t_1)\}, \{u = f_0(t_2)\}$ are regular. We have the following:*

(i) For $\varepsilon > 0$ small enough, so that $\Sigma(\varepsilon, t_1)$, $\Sigma(\varepsilon, t_2)$ are regular, we have

$$D_\varepsilon(t_2) - D_\varepsilon(t_1) \le \int_{\{f_\varepsilon(t_1) < v < f_\varepsilon(t_2)\}} E(\varepsilon)$$

where $E(\varepsilon) \ge 0$ is a continuous function which is uniformly bounded independent of ε and $E(\varepsilon) \to 0$ everywhere as $\varepsilon \to 0$ in $M(T)$.

(ii) There is a constant C independent of ε such that

$$\int_{\{f_\varepsilon(t_1) < v < f_\varepsilon(t_2), |\nabla v| > 0\}} \frac{|\Delta v|}{|\nabla v|} \le C\varepsilon^{-1}.$$

Proof. (i) We remark that since $v = v_\varepsilon$ converges in $C^{1,\beta}(M(T))$ to u, $\Sigma(\varepsilon, t_1)$, $\Sigma(\varepsilon, t_2)$ are regular surfaces in the interior of $M(T)$, provided $\varepsilon > 0$ is small enough.

In the following, we denote f_ε by f for simplicity. By (3.8),

$$D_\varepsilon(t_2) - D_\varepsilon(t_1) = \lim_{\delta\to 0}\int_{\{f(t_1)<v<f(t_2)\}} \mathbf{div}(W - U_\delta + V_\delta). \tag{3.11}$$

To prove (i), we need to estimate the above limit. By Lemma 3.3, for any $\delta > 0$,

$$\begin{aligned}
&\int_{\{f(t_1)<v<f(t_2)\}} \mathbf{div}(W - U_\delta + V_\delta)\\
&\le \int_{\{f(t_1)<v<f(t_2)\}} \mathbf{div}\, W + C\delta\\
&\quad - \int_{\{f(t_1)<v<f(t_2)\}} \mathbf{1}_{\{|\nabla v|>0\}}\,\mathbf{div}\, U_\delta + \int_{\{f(t_1)<v<f(t_2)\}} \mathbf{1}_{\{|\nabla v|>0\}}(\mathrm{I}_\delta - \mathrm{II}_\delta),
\end{aligned}$$

for some $C > 0$ independent of δ, where $\mathbf{1}_Y$ is the characteristic function of the set Y, and

$$\begin{cases}
\mathrm{I}_\delta = ((2-p)^2\frac{|\nabla v|^4}{|\nabla v|_\varepsilon^4|\nabla v|_\delta} - (2-p)\frac{|\nabla v|^4}{|\nabla v|_\varepsilon^2|\nabla v|_\delta^3})v_{\nu\nu}^2 - |\nabla v|_\delta^{-1}|\nabla v|^2\,\mathrm{Ric}(\nu,\nu)\\
\mathrm{II}_\delta = |\nabla v|_\delta^{-1}|\nabla v|^2|A|^2.
\end{cases}$$

Since v is smooth, $v_{\nu\nu}$ is uniformly bounded on $\mathbf{1}_{\{|\nabla v|>0\}}$ and so $\mathbf{1}_{\{|\nabla v|>0\}}\,\mathbf{div}\, U_\delta$ is uniformly bounded. Similarly, $\mathbf{1}_{\{|\nabla v|>0\}}\mathrm{I}_\delta$ is uniformly bounded because Ricci curvature is bounded. Also II_δ is nonnegative and is nondecreasing as $\delta\to 0$, by dominated and monotone convergence theorems, we have:

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{f(t_1)<v<f(t_2)} \mathbf{div}(W - U_\delta + V_\delta)\\
&\le \int_{\{f(t_1)<v<f(t_2)\}} \mathbf{div}\, W\\
&\quad + \int_{\{f(t_1)<v<f(t_2)\}} \mathbf{1}_{\{|\nabla v|>0\}}(-\,\mathbf{div}\, U + \mathrm{I}_0 - \mathrm{II}_0)\\
&=:\mathrm{III}.
\end{aligned} \tag{3.12}$$

Here in $\{|\nabla v| > 0\}$,

$$\mathrm{I}_0 = \lim_{\delta\to 0}\mathrm{I}_\delta;\quad \mathrm{II}_0 = \lim_{\delta\to 0}\mathrm{II}_\delta.$$

In particular, since the LHS in (3.11) is finite, the integrals of $\mathbf{div}\, W$ and

$\mathbf{1}_{\{|\nabla v|>0\}}(-\operatorname{\mathbf{div}} U + \mathrm{I}_0)$ are finite, we have

$$\int_{\{f(t_1)<v<f(t_2)\}} \mathbf{1}_{\{|\nabla v|>0\}} |\nabla v||A|^2 < \infty. \tag{3.13}$$

Let

$$\tau_1 = f(t_1), \tau_2 = f(t_2), \Sigma_\tau = \{v = \tau\}.$$

By the co-area formula and the Morse-Sard's theorem, which implies the set of critical values of v is of measure zero in $[\tau_1, \tau_2]$, for any L^1 function h in $\{\tau_1 < v < \tau_2\}$

$$\int_{\{\tau_1<v<\tau_2\}} |\nabla v| h = \int_{\{\tau_1<\tau<\tau_2, \tau\in R\}} \left(\int_{\Sigma_\tau} h \right) d\tau$$

where R is the set of regular values of v. We now apply this to different terms in III.

$$\begin{aligned}
\int_{\{f(t_1)<v<f(t_2)\}} \operatorname{\mathbf{div}} W &= \int_{\{f(t_1)<v<f(t_2)\}} (-4\pi C_{p,\varepsilon}^{-1} |\nabla v|^2 |\nabla v|_\varepsilon^{p-2}) \\
&= \int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} \int_{\Sigma_\tau} (-4\pi C_{p,\varepsilon}^{-1} |\nabla v||\nabla v|_\varepsilon^{p-2}) d\tau \\
&= -\int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} 4\pi d\tau
\end{aligned}$$

On the other hand, in the set $\{|\nabla v| > 0\}$,

$$|\nabla v|^{-1} \operatorname{\mathbf{div}} U = a^{-2}(1+2a)\left[\left((2-p)\frac{|\nabla v|^2}{|\nabla v|_\varepsilon^2}+1\right)(1-v)^{-1} v_{\nu\nu} + (1-v)^{-2}|\nabla v|^2\right].$$

So $\mathbf{1}_{\{|\nabla v|>0\}} |\nabla v|^{-1} \operatorname{\mathbf{div}} U$ is uniformly bounded because v is smooth and $v \leq f(t_2) \leq f(T) < 1$. Hence

$$\int_{\{f(t_1)<v<f(t_2)\}} \mathbf{1}_{\{|\nabla v|>0\}} \operatorname{\mathbf{div}} U = \int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} \int_{\Sigma_\tau} \mathbf{1}_{\{|\nabla v|>0\}} |\nabla v|^{-1} \operatorname{\mathbf{div}} U.$$

Similarly, we also have

$$\int_{\{f(t_1)<v<f(t_2)\}} \mathbf{1}_{\{|\nabla v|>0\}} \mathrm{I}_0 = \int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} \int_{\Sigma_\tau} \mathbf{1}_{\{|\nabla v|>0\}} |\nabla v|^{-1} \mathrm{I}_0.$$

Let $\phi =: |A|^2 \mathbf{1}_{\{|\nabla v|>0\}}$. Since $|\nabla v|$ is continuous, $|A|^2$ is smooth in $\{|\nabla v| > 0\}$, ϕ is measurable. Since $\phi \geq 0$, one can apply co-area formula to $\phi_k = \min\{\phi, k\}$

for $k \in \mathbb{N}$ so that

$$\int_{\{f(t_1)<v<f(t_2)\}} |\nabla v|\phi_k = \int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} \int_{\Sigma_\tau} \phi_k d\tau.$$

Since $\phi_k \uparrow \phi$, one can apply monotone convergence theorem to both sides to conclude that

$$\int_{\{f(t_1)<v<f(t_2)\}} |\nabla v|\phi = \int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} \int_{\Sigma_\tau} \phi d\tau.$$

However, by (3.13), $|\nabla v|\phi = |\nabla v||A|^2\mathbf{1}_{\{|\nabla v|>0\}} = \mathrm{II}_0$ is integrable, we have

$$\int_{\{f(t_1)<v<f(t_2)\}} \mathbf{1}_{\{|\nabla v|>0\}}\mathrm{II}_0 = \int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} \int_{\Sigma_\tau} \mathbf{1}_{\{|\nabla v|>0\}}|\nabla v|^{-1}\mathrm{II}_0$$

which is finite.

Hence

(3.14)

$$\mathrm{III} = \int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} \left[-4\pi + \int_{\Sigma_\tau} (-|\nabla v|^{-1}\,\mathbf{div}\,U + |\nabla v|^{-1}(\mathrm{I}_0 - \mathrm{II}_0))\right] d\tau.$$

Here we have used the fact that $\mathbf{1}_{\{|\nabla v|>0\}} = 1$ on Σ_τ for $\tau \in R$. Recall that, for $1 < p \le 2$, if $|\nabla v| > 0$, then

$$v_{\nu\nu} = -\frac{|\nabla v|\,|\nabla v|_\varepsilon^2}{(p-1)|\nabla v|^2 + \varepsilon^2} H.$$

By (3), let $\lambda > 0$ be a positive function to be determined later, if $|\nabla v| > 0$, we have

$$\begin{aligned}
(3.15)\quad & |\nabla v|^{-1}\,\mathbf{div}\,U \\
=& a^{-2}(1+2a)\Big[-\Big((2-p)\frac{|\nabla v|^2}{|\nabla v|_\varepsilon^2} + 1\Big)(1-v)^{-1}\frac{|\nabla v|\,|\nabla v|_\varepsilon^2}{(p-1)|\nabla v|^2+\varepsilon^2} H \\
&+ (1-v)^{-2}|\nabla v|^2\Big] \\
=& a^{-2}(1+2a)\Big[-\Big(\frac{(3-p)|\nabla v|^2+\varepsilon^2}{(p-1)|\nabla v|^2+\varepsilon^2}\Big)(1-v)^{-1}|\nabla v| H
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-v)^{-2}|\nabla v|^2\Big] \\
&\geq -\frac{1}{2}(1+2a)\lambda H^2 \\
&\quad + a^{-2}(1+2a)\left[1-\frac{1}{2}\lambda^{-1}\left(\frac{|\nabla v|^2+(3-p)^{-1}\varepsilon^2}{|\nabla v|^2+(p-1)^{-1}\varepsilon^2}\right)^2\right](1-v)^{-2}|\nabla v|^2 \\
&= -\frac{1}{2}(1+2a)\lambda H^2 + a^{-2}(1+2a)(1-Q)(1-v)^{-2}|\nabla v|^2
\end{aligned}
$$

where

$$
Q =: \frac{1}{2}\lambda^{-1}\left(\frac{|\nabla v|^2+(3-p)^{-1}\varepsilon^2}{|\nabla v|^2+(p-1)^{-1}\varepsilon^2}\right)^2. \tag{3.16}
$$

At the points where $|\nabla v| > 0$, for $1 < p \leq 2$, we have

$$
\begin{aligned}
&|\nabla v|^{-1}(\mathrm{I}_0 - \mathrm{II}_0) \\
=&|\nabla v|^{-2}\left[(2-p)^2\frac{|\nabla v|^4}{|\nabla v|_\varepsilon^4} - (2-p)\frac{|\nabla v|^2}{|\nabla v|_\varepsilon^2}\right]v_{\nu\nu}^2 - \left(|A|^2 + \mathrm{Ric}(\nu,\nu)\right) \\
\leq&(2-p)(1-p)\frac{|\nabla v|^2}{|\nabla v|_\varepsilon^4}v_{\nu\nu}^2 - \frac{3}{4}H^2 + K \\
=&-\left[\frac{(2-p)(p-1)|\nabla v|^4}{((p-1)|\nabla v|^2+\varepsilon^2)^2} + \frac{3}{4}\right]H^2 + K. \\
=:&-PH^2 + K,
\end{aligned} \tag{3.17}
$$

where K is the Gaussian curvature of the level set and $P > 0$ is the function inside the square bracket. Here we have used the facts that $\mathcal{S} \geq 0$,

$$
\mathrm{Ric}(\nu,\nu) = \frac{1}{2}(\mathcal{S} - 2K + H^2 - |A|^2) \geq -K + \frac{1}{2}(H^2 - |A|^2)
$$

and that

$$
|A|^2 = |\mathring{A}|^2 + \frac{1}{2}H^2 \geq \frac{1}{2}H^2,
$$

where $\mathring{A}$ is the traceless part of A.

Since $H_2(M, \partial M) = \{0\}$ and since ∂M is connected, the level-set Σ_τ is also connected according to [3, pages 9-10]. Consequently, we have $\int_{\Sigma_\tau} K \leq 4\pi$

for all regular τ. Hence for $\tau \in R$, if we choose $\lambda = 2(1+2a)^{-1}P$, we have

$$\begin{aligned} -4\pi + &\int_{\Sigma_\tau} (-|\nabla v|^{-1} \,\mathbf{div}\, U + |\nabla v|^{-1}(\mathrm{I}_0 - \mathrm{II}_0) \\ &\leq \int_{\Sigma_\tau} \left(\frac{1}{2}\lambda(1+2a) - P\right) H^2 - a^{-2}(1+2a)(1-v)^{-2}|\nabla v|^2(1-Q) \\ &= -\int_{\Sigma_\tau} a^{-2}(1+2a)(1-v)^{-2}|\nabla v|^2(1-Q). \end{aligned}$$

Combining with (3.14), using co-area formula and Moser-Sard's theorem again (Q is uniformly bounded, see below):

$$\begin{aligned} \mathrm{III} &\leq -\int_{\{\tau_1 < v < \tau_2, \tau \in R\}} \int_{\Sigma_\tau} a^{-2}(1+2a)(1-v)^{-2}|\nabla v|^2(1-Q) \\ &= -\int_{\{f(t_1) < v < f(t_2)\}} a^{-2}(1+2a)(1-v)^{-2}|\nabla v|^3(1-Q), \end{aligned} \tag{3.18}$$

Since $P \geq \frac{3}{4}$ and $1 < p \leq 2$ so that

$$\left(1 + \frac{3-p}{p-1}\right)^{-1} \leq \frac{|\nabla v|^2 + (3-p)^{-1}\varepsilon^2}{|\nabla v|^2 + (p-1)^{-1}\varepsilon^2} \leq 1,$$

from the definition of Q in (3.16), we conclude that Q is uniformly bounded independent of ε. Since $v \leq f_0(T) < 1$, $v \to u$ in $C^{1,\beta}$ norm for some $\beta > 0$, we have $|\nabla v| \to |\nabla u|$ as $\varepsilon \to 0$. At the points where $|\nabla u| = 0$, then one conclude that

$$(1-v)^{-2}|\nabla v|^3(1-Q) \to 0$$

as $\varepsilon \to 0$. At the points where $|\nabla u| > 0$, as $\varepsilon \to 0$ we have

$$Q = \frac{1}{4}(1+2a)\left[\frac{(2-p)(p-1)|\nabla v|^4}{((p-1)|\nabla v|^2 + \varepsilon^2)^2} + \frac{3}{4}\right]^{-1} \left(\frac{|\nabla v|^2 + (3-p)^{-1}\varepsilon^2}{|\nabla v|^2 + (p-1)^{-1}\varepsilon^2}\right)^2 \to 1.$$

Hence we also have:

$$(1-v)^{-2}|\nabla v|^3(1-Q) \to 0.$$

From this, the fact that $v < f(T) < 1$, v converges in $C^{1,\beta}$ to u, (3.8), (3.12) and (3.18) we conclude that (i) is true with

$$E(\varepsilon) = a^{-2}(1+2a)(1-v)^{-2}|\nabla v|^3|1-Q|.$$

To prove (ii), if $p = 2$, then $\Delta v = 0$ and the result is obvious. Observe that as $\varepsilon \to 0$,

$$\begin{aligned}\int_{\{f(t_1)<v<f(t_2)\}} \mathbf{div}\, V_\delta &= \int_{\Sigma(t_2)} \left\langle V_\delta, \frac{\nabla v}{|\nabla v|} \right\rangle - \int_{\Sigma(t_1)} \left\langle V_\delta, \frac{\nabla v}{|\nabla v|} \right\rangle \\ &\to \int_{\Sigma(t_2)} (\Delta v - v_{\nu\nu}) - \int_{\Sigma(t_1)} (\Delta v - v_{\nu\nu}) \\ &\geq -C_1\end{aligned}$$

for some constant $C_1 > 0$ independent of ε, because $\{u = f_0(t_1)\}, \{u = f_0(t_2)\}$ are regular and $v = v_\varepsilon$ converge in C^∞ norm near these two level sets [3]. For $1 < p < 2$, by Lemma 3.3(iii), for $|\nabla v| = 0$, $\mathbf{div}\, V_\delta \leq 0$, so

$$\begin{aligned}&\int_{\{f(t_1)<v<f(t_2)\}} \mathbf{div}\, V_\delta \\ &\leq \int_{\{f(t_1)<v<f(t_2),|\nabla v|>0\}} \bigg[\left(\frac{(2-p)^2|\nabla v|^4}{|\nabla v|_\varepsilon^4|\nabla v|_\delta} - \frac{(2-p)|\nabla v|^4}{|\nabla v|_\varepsilon^2|\nabla v|_\delta^3}\right) v_{\nu\nu}^2 \\ &\quad - |\nabla v|_\delta^{-1}|\nabla v|^2 \operatorname{Ric}(\nu,\nu)\bigg].\end{aligned}$$

On the other hand,

$$\left||\nabla v|_\delta^{-1}|\nabla v|^2 \operatorname{Ric}(\nu,\nu)\right| \leq C_2$$

by a constant C_2 independent of $0 < \delta, \varepsilon \leq 1$. Hence, letting $\delta \to 0$, by the monotone convergence theorem, we have

$$\begin{aligned}-C_3 &\leq \int_{\{f(t_1)<v<f(t_2),|\nabla v|>0\}} \left(\frac{(2-p)^2|\nabla v|^3}{|\nabla v|_\varepsilon^4} - \frac{(2-p)|\nabla v|}{|\nabla v|_\varepsilon^2}\right) v_{\nu\nu}^2 \\ &\leq (2-p)(1-p) \int_{\{f(t_1)<v<f(t_2),|\nabla v|>0\}} \frac{|\nabla v|}{|\nabla v|_\varepsilon^2} v_{\nu\nu}^2\end{aligned}$$

for some C_3 independent of ε. Hence

$$\int_{\{f(t_1)<v<f(t_2),|\nabla v|>0\}} \frac{|\nabla v|}{|\nabla v|_\varepsilon^2} v_{\nu\nu}^2 \leq (2-p)^{-1}(p-1)^{-1} C_3,$$

and

$$\int_{\{f(t_1)<v<f(t_2),|\nabla v|>0\}} \frac{|\Delta v|}{|\nabla v|}$$

$$
\begin{aligned}
&= \int_{\{f(t_1)<v<f(t_2),|\nabla v|>0\}} (2-p)\frac{|\nabla v|}{|\nabla v|_\varepsilon^2}|v_{\nu\nu}| \\
&\le (2-p)\varepsilon^{-1}\int_{\{f(t_1)<v<f(t_2),|\nabla v|>0\}} \frac{|\nabla v|}{|\nabla v|_\varepsilon}|v_{\nu\nu}| \\
&\le \frac{1}{2}(2-p)\varepsilon^{-1}\int_{\{f(t_1)<v<f(t_2),|\nabla v|>0\}} \frac{|\nabla v|}{|\nabla v|_\varepsilon}\left(\frac{1}{|\nabla v|_\varepsilon}v_{\nu\nu}^2 + |\nabla v|_\varepsilon\right) \\
&\le C\varepsilon^{-1}
\end{aligned}
$$

for some constant $C > 0$ independent of ε. Here we have used the fact that $|\nabla v|_\varepsilon \ge \varepsilon$, and $|\nabla v| \to |\nabla u|$. □

We are ready to prove Theorem 1.2. Before we prove the theorem, let us fix some notation. Let u be as in the theorem, then $|\nabla u| > 0$ outside a compact set by the asymptotic behavior (1.8) of u. Let $0 < t_1 < t_2$ be such that such that $\Sigma(t_1) = \{u = f(t_1)\}, \Sigma(t_2) = \{u = f(t_2)\}$ are regular. Recall that $f(t) = 1 - ct^{-a}$, $a = (3-p)/(p-1)$, $1 < p \le 2$. Fix $T > t_2$ so that $\Sigma(T)$ is regular. For any $\varepsilon > 0$, let v_ε be the solution of (3.1). Let $f_\varepsilon(t) = 1 - c_\varepsilon t^{-a}$ and c_ε be as in (3.3). Let $D_\varepsilon(t)$ be as in (3.4) whenever $\Sigma(\varepsilon, t) = \{v_\varepsilon = f_\varepsilon(t)\}$ is regular. By [3], for $\varepsilon > 0$ small enough, $\Sigma(\varepsilon, t_1), \Sigma(\varepsilon, t_2)$ are regular.

Proof of Theorem 1.2. (i) With the above setting, by Lemma 3.4, for ε small enough, we have

$$
D_\varepsilon(t_2) - D_\varepsilon(t_1) \le \int_{\{f_\varepsilon(t_1)<v<f_\varepsilon(t_2)\}} E(\varepsilon).
$$

where $E(\varepsilon) \ge 0$ is uniformly bounded independent of ε and converges to zero everywhere in $M(T) = \{0 < u < f(T)\}$. Let $\varepsilon \to 0$, we conclude by Lemma 3.1

$$
D(t_2) - D(t_1) \le 0.
$$

(ii) If M is AF, then $\Sigma(t_2)$ is regular for $t_2 >> 1$ and $D(t_2) \to 0$ as $t_2 \to \infty$ by the proof of Proposition 2.1. By (i) we have $D(t) \ge 0$ whenever $\Sigma(t)$ is regular. Since

$$
D = (2a+1)\mathcal{B} - a\mathcal{A},
$$

we conclude that (ii) is true.

To prove (iii), recall

$$
\mathcal{B}_\varepsilon(t) = 4\pi t - (c_\varepsilon a)^{-2} t^{2a+1}\int_{\Sigma(\varepsilon,t)} |\nabla v_\varepsilon|^2
$$

whenever $\Sigma(\varepsilon, t)$ is regular.

$$\begin{aligned}\mathcal{B}_\varepsilon(t) =& c_\varepsilon^{\frac{1}{a}} \int_{\Sigma(\varepsilon,t)} (c_\varepsilon a)^{1-p}(1-v_\varepsilon)^{-\frac{1}{a}}|\nabla v_\varepsilon|_\varepsilon^{p-2}|\nabla v_\varepsilon| - a^{-2}(1-v_\varepsilon)^{-(2+\frac{1}{a})}|\nabla v_\varepsilon|^2 \\ =& c^{\frac{1}{a}} \int_{\Sigma(\varepsilon,t)} Q_\varepsilon\end{aligned}$$

Try to find X_ε so that at $\Sigma(\varepsilon, t)$, $\langle X_\varepsilon, \frac{\nabla v_\varepsilon}{|\nabla v|_\varepsilon}\rangle = Q_\varepsilon$. So let

$$\begin{aligned}X_\varepsilon &= (c_\varepsilon a)^{1-p}(1-v_\varepsilon)^{-\frac{1}{a}}|\nabla v_\varepsilon|_\varepsilon^{p-2}\nabla v_\varepsilon - a^{-2}(1-v_\varepsilon)^{-(2+\frac{1}{a})}|\nabla v_\varepsilon|\,\nabla v_\varepsilon \\ &=: Z_\varepsilon - Y_\varepsilon.\end{aligned} \tag{3.19}$$

For any $\delta > 0$, let

$$Y_{\varepsilon,\delta} = a^{-2}(1-v_\varepsilon)^{-(2+\frac{1}{a})}\sqrt{|\nabla v_\varepsilon|^2 + \delta^2}\nabla v_\varepsilon.$$

Then $Z_\varepsilon - Y_{\delta,\varepsilon}$ is a smooth vector field. Since $\Sigma(t_1), \Sigma(t_2)$ are regular, we can conclude that $\Sigma(\varepsilon, t_1), \Sigma(\varepsilon, t_2)$ are regular for small ε and so:

$$c_\varepsilon^{\frac{1}{a}} \lim_{\delta\to 0} \int_{\{f_\varepsilon(t_1)<v_\varepsilon<f_\varepsilon(t_2)\}} \mathbf{div}(Z_\varepsilon - Y_{\delta,\varepsilon}) = \mathcal{B}_\varepsilon(t_2) - \mathcal{B}_\varepsilon(t_1). \tag{3.20}$$

Now

$$\mathbf{div}\, Z_\varepsilon = (c_\varepsilon a)^{1-p}\frac{1}{a}(1-v_\varepsilon)^{-(\frac{1}{a}+1)}|\nabla v_\varepsilon|_\varepsilon^{p-2}|\nabla v_\varepsilon|^2, \tag{3.21}$$

which is zero if $|\nabla v_\varepsilon| = 0$. On the other hand, if $|\nabla v_\varepsilon| = 0$, then

$$\mathbf{div}\, Y_{\delta,\varepsilon} = a^{-2}\delta(1-v_\varepsilon)^{-(2+\frac{1}{a})}\Delta v_\varepsilon.$$

If $|\nabla v_\varepsilon| > 0$, then

$$\begin{aligned}\mathbf{div}\, Y_{\delta,\varepsilon} =& a^{-2}(1-v_\varepsilon)^{-(2+\frac{1}{a})}\Bigg[\sqrt{|\nabla v_\varepsilon|^2+\delta^2}\Delta v_\varepsilon + \frac{|\nabla v_\varepsilon|\langle\nabla|\nabla v_\varepsilon|, \nabla v_\varepsilon\rangle}{\sqrt{|\nabla v_\varepsilon|^2+\delta^2}} \\ &+ \left(2+\frac{1}{a}\right)(1-v_\varepsilon)^{-1}\sqrt{|\nabla v_\varepsilon|^2+\delta^2}|\nabla v_\varepsilon|^2\Bigg]\end{aligned}$$

Note that for $|\nabla v_\varepsilon| > 0$, $\langle\nabla|\nabla v_\varepsilon|, \nabla v_\varepsilon\rangle = |\nabla v_\varepsilon|(v_\varepsilon)_{\nu\nu}$. Since v_ε is smooth, one can see that $\mathbf{div}\, Z_\varepsilon$ is bounded and $\mathbf{div}\, Y_{\delta,\varepsilon}$ is uniformly bounded independent of δ. Moreover, $\mathbf{div}\, Y_{\delta,\varepsilon} \to \mathbf{div}\, Y_\varepsilon$ if $|\nabla v_\varepsilon| > 0$ as $\delta \to 0$ and $\mathbf{div}\, Y_{\delta,\varepsilon} \to 0$

if $|\nabla v_\varepsilon| = 0$. By Lebesgue's dominated convergence theorem, let $\delta \to 0$ we have:

$$c^{-\frac{1}{a}}\big(\mathcal{B}_\varepsilon(t_2) - \mathcal{B}_\varepsilon(t_1)\big) = \int_{\{f_\varepsilon(t_1)<v_\varepsilon<f_\varepsilon(t_2)\}} \mathbf{1}_{\{|\nabla v_\varepsilon|>0\}}\, \mathbf{div}(Z_\varepsilon - Y_\varepsilon).$$

At $|\nabla v_\varepsilon| > 0$,

$$\begin{aligned} &|\nabla v_\varepsilon|^{-1}\, \mathbf{div}(Z_\varepsilon - Y_\varepsilon) \\ =& a^{-1}(1-v_\varepsilon)^{-(\frac{1}{a}+2)}\Big[\frac{4\pi}{C_{p,\varepsilon}}(1-v_\varepsilon)|\nabla v_\varepsilon|_\varepsilon^{p-2}|\nabla v_\varepsilon| - a^{-1}\big(\Delta v_\varepsilon + (v_\varepsilon)_{\nu\nu}\big) \\ &- a^{-2}(1+2a)(1-v_\varepsilon)^{-1}|\nabla v_\varepsilon|^2\Big]. \end{aligned} \tag{3.22}$$

Since $1 - v_\varepsilon \geq C > 0$ for $f_\varepsilon(t_1) < v_\varepsilon < f_\varepsilon(t_1) < 1$ for some $C > 0$, and v_ε is smooth, $|\nabla v_\varepsilon|^{-1}\mathbf{1}_{\{|\nabla v_\varepsilon|>0\}}\, \mathbf{div}(Z_\varepsilon - Y_\varepsilon)$ is in L^1. Let $\tau = f_\varepsilon(t)$, $\tau_1 = f_\varepsilon(t_1), \tau_2 = f_\varepsilon(t_2)$ and let R be the set of regular values of v_ε. By the co-area formula and the Morse-Sard's theorem, using (3.4), (3.22) and the fact that $\tau = f_\varepsilon(t)$,

$$\begin{aligned} &c_\varepsilon^{-\frac{1}{a}}\big(\mathcal{B}_\varepsilon(t_2) - \mathcal{B}_\varepsilon(t_1)\big) \\ =& \int_{\{f_\varepsilon(t_1)<v_\varepsilon<f_\varepsilon(t_2)\}} \mathbf{1}_{\{|\nabla v_\varepsilon|>0\})}\, \mathbf{div}(Z_\varepsilon - Y_\varepsilon)\mathbf{1}_{\{|\nabla v_\varepsilon|>0\})} \\ =& \int_{\tau_1}^{\tau_2}\int_{\{v_\varepsilon=\tau\}} |\nabla v_\varepsilon|^{-1}\mathbf{1}_{\{|\nabla v_\varepsilon|>0\})}\, \mathbf{div}(Z_\varepsilon - Y_\varepsilon)d\tau \\ =& \int_{\{\tau_1<\tau<\tau_2, \tau\in R\}}\int_{\{v_\varepsilon=\tau\}} |\nabla v_\varepsilon|^{-1}\, \mathbf{div}(Z_\varepsilon - Y_\varepsilon)d\tau \\ =& c_\varepsilon a^{-1}\int_{\{\tau_1<\tau<\tau_2, \tau\in R\}} (1-\tau)^{-(\frac{1}{a}+2)} \\ &\cdot \big[c_\varepsilon D_\varepsilon\big(t(\tau)\big) - a^{-1}\big(\Delta v_\varepsilon + (v_\varepsilon)_{\nu\nu}\big) - \Delta v_\varepsilon - (v_\varepsilon)_{\nu\nu}\big]d\tau. \end{aligned}$$

On the other hand,

$$\begin{aligned} &a^{-1}\big(\Delta v_\varepsilon + (v_\varepsilon)_{\nu\nu}\big) + \Delta v_\varepsilon - (v_\varepsilon)_{\nu\nu} \\ =& \Big[\frac{p-1}{3-p}\frac{(3-p)|\nabla v_\varepsilon|^2 + \varepsilon^2}{|\nabla v_\varepsilon|_\varepsilon^2} + \frac{(1-p)|\nabla v|^2 - \varepsilon^2}{|\nabla v_\varepsilon|_\varepsilon^2}\Big](v_\varepsilon)_{\nu\nu} \\ =& \frac{2(p-2)\varepsilon^2}{3-p}\frac{\Delta v_\varepsilon}{|\nabla v_\varepsilon|^2}. \end{aligned}$$

Hence

$$\begin{aligned}
&c^{-\frac{1}{a}}(\mathcal{B}_\varepsilon(t_2)-\mathcal{B}_\varepsilon(t_1))\\
=&\int_{\tau_1<\tau<\tau_2,\tau\in R} a^{-1}c_\varepsilon(1-\tau)^{-2-\frac{1}{a}}D_\varepsilon(\tau)d\tau\\
&+\int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} a^{-1}(1-\tau)^{-2-\frac{1}{a}}\int_{\Sigma_{\varepsilon,\tau}}\frac{2(p-2)\varepsilon^2}{3-p}\frac{\Delta v_\varepsilon}{|\nabla v_\varepsilon|^2}d\tau\\
=&\int_{\{\tau_1<\tau<\tau_2,\tau\in R\}} a^{-1}c_\varepsilon(1-\tau)^{-2-\frac{1}{a}}D_\varepsilon(\tau)d\tau\\
&+\int_{\{\tau_1<v_\varepsilon<\tau_2\}} a^{-1}\mathbf{1}_{\{|\nabla v_\varepsilon|>0\}}(1-v_\varepsilon)^{-2-\frac{1}{a}}\frac{2(p-2)\varepsilon^2}{3-p}\frac{\Delta v_\varepsilon}{|\nabla v_\varepsilon|}d\tau\\
=&(1)+(2).
\end{aligned}$$

By Lemma 3.4(ii),

$$(2)\geq -C_1\varepsilon$$

for some constant $C_1>0$ independent of ε. By Lemma 3.4(i),

$$(1)\geq C_2D_\varepsilon(t_2)+o(1)$$

as $\varepsilon\to 0$ for some constant $C_2>0$ independent of ε. Let $\varepsilon\to 0$, we conclude that

as $\varepsilon\to 0$. Let $\varepsilon\to 0$ by Lemma 3.1 we have

$$c^{-\frac{1}{a}}(\mathcal{B}(t_2)-\mathcal{B}(t_1))\geq C_2D(t_2)\geq 0,$$

by part (ii).

In [3], it was shown that $F(t_2)\geq F(t_1)$. Since F, $\mathcal{A}$ and $\mathcal{B}$ are related by $F(t)=\mathcal{A}(t)-\mathcal{B}(t)$, the monotonicity of $\mathcal{A}(t_2)\geq\mathcal{A}(t_1)$.

This finishes the proof of Theorem 1.2. □

4. Asymptotical behavior

Let (M^3,g), $\mathcal{A}(t)$ and $\mathcal{B}(t)$ be given as in Theorem 1.2. By (1.8), the level set $\Sigma(t)=\{u=f(t)\}$ is regular when t is large. By Theorem 1.2, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are nondecreasing in t for those t with $\Sigma(t)$ being regular.

If $p=2$, it is known, as $x\to\infty$, the harmonic function u has an asymptotic expansion

$$u=1-c|x|^{-1}+O_2(|x|^{-1-\sigma}), \tag{4.1}$$

see [32, Lemma A.2] for instance. Here c is a positive constant and $\sigma \in (\frac{1}{2}, 1)$ is a decay rate of g_{ij} in (1.2). With the help of (4.1), it was shown in [34] that $\mathcal{A}(t)$ converges to $12\pi\mathfrak{m}$ and $\mathcal{B}(t)$ converges to $4\pi\mathfrak{m}$, respectively, as $t \to \infty$. (Note that $\mathcal{A}$ and $\mathcal{B}$ here differ with those in [34] by a factor of $\frac{1}{4\pi}C_2$.)

If $p \in (1, 3)$ and $p \neq 2$, unlike harmonic functions, near infinity we only have

$$u = 1 - cr^{-a} + o_2(r^{-a}) \tag{4.2}$$

(to the authors' knowledge). Note that this in particular implies that level-sets of u are regular near ∞. Nevertheless, a Hawking mass estimate

$$\limsup_{t\to\infty} \frac{t}{2}\left(1 - \frac{1}{16\pi}\int_{\Sigma(t)} H^2\right) \leq \mathfrak{m} \tag{4.3}$$

was proved in [3, Lemma 2.5]. (4.3) can be viewed as a Hawking mass estimate because the ratio between t and the area-radius of $\{\Sigma(t)\}$ tends to 1 by (4.2). From this, it was shown in [3] that

$$\limsup_{t\to\infty} F(t) \leq 8\pi\mathfrak{m}.$$

As a corollary of (4.3) proved in [3], one has

Proposition 4.1.

(i) $\limsup_{t\to\infty} \mathcal{A}(t) \leq 4\pi(5-p)\mathfrak{m}$.
(ii) $\limsup_{t\to\infty} \mathcal{B}(t) \leq 4\pi(3-p)\mathfrak{m}$.

Proof. Let c be as in the proof of Theorem 1.2. Namely, for $1 < p < 3$,

$$ca = \left(\frac{C_p}{4\pi}\right)^{\frac{1}{p-1}}$$

where $a = (3-p)/(p-1)$ and C_p is the p-capacity of ∂M. Let $\tau = f(t)$. Then

$$t = \left(\frac{c}{1-\tau}\right)^{\frac{1}{a}}.$$

Let $\Sigma_\tau = \{u = \tau\}$. Suppose τ is a regular value of u. By Lemma A.2,

$$\begin{aligned} \frac{d}{d\tau}\int_{\Sigma_\tau} |\nabla u| H &= \int_{\Sigma_\tau}\left(K - \frac{3}{4}H^2 + H\frac{\Delta u}{|\nabla u|}\right) - E(\tau) \\ &\leq 4\pi - \int_{\Sigma_\tau} \frac{5-p}{4(p-1)}H^2 \end{aligned} \tag{4.4}$$

where

$$E(\tau) = \int_{\Sigma_\tau} \left(\frac{|\widetilde{\nabla}|\nabla u|^2|}{|\nabla u|^2} + \frac{1}{2}(\mathcal{S} + |\mathring{A}|^2) \right) \geq 0.$$

Given any $\tilde{\mathfrak{m}} > \mathfrak{m}$, by (4.3),

$$-\int_{\Sigma_\tau} H^2 \leq 32\pi c^{-\frac{1}{a}} (1-\tau)^{\frac{1}{a}} \tilde{\mathfrak{m}} - 16\pi \tag{4.5}$$

for τ close to 1. By (4.2), $\int_{\Sigma_\tau} |\nabla u| H \to 0$ as $\tau \to 1$. Integrating (4.4) and using (4.5), we have

$$\begin{aligned} -\int_{\Sigma_\tau} |\nabla u| H \leq & -8\pi a(1-\tau) \\ & + 8\pi \cdot \frac{5-p}{p-1} c^{-\frac{1}{a}} \frac{a}{1+a} (1-\tau)^{1+\frac{1}{a}} \tilde{\mathfrak{m}}. \end{aligned} \tag{4.6}$$

Therefore,

$$\begin{aligned} \mathcal{A}(t) = \mathcal{A}(t(\tau)) &= \left(\frac{c}{1-\tau} \right)^{\frac{1}{a}} \left(8\pi - \frac{1}{a(1-\tau)} \int_{\Sigma_\tau} |\nabla u| H \right) \\ &\leq 8\pi \frac{5-p}{p-1} \frac{1}{1+a} \tilde{\mathfrak{m}} \\ &= 4\pi(5-p)\tilde{\mathfrak{m}}. \end{aligned}$$

As $\tilde{\mathfrak{m}} > \mathfrak{m}$ is arbitrary, from this (i) follows.

To show (ii), by (4.6),

$$\begin{aligned} \frac{d}{d\tau} \int_{\Sigma_\tau} |\nabla u|^2 &= -a \int_{u=\tau} |\nabla u| H \\ &\leq a \left[-8\pi a(1-\tau) + 8\pi \cdot \frac{5-p}{p-1} c^{-\frac{1}{a}} \frac{a}{1+a} (1-\tau)^{1+\frac{1}{a}} \tilde{\mathfrak{m}} \right]. \end{aligned}$$

By (4.2) and the fact $|\Sigma(t)| = O(t^2)$, we have $\int_{\Sigma_\tau} |\nabla u|^2 \to 0$ as $\tau \to 1$. Hence,

$$\begin{aligned} -\int_{\Sigma_\tau} |\nabla u|^2 \leq & -4\pi a^2 (1-\tau)^2 \\ & + 8\pi \cdot \frac{5-p}{p-1} c^{-\frac{1}{a}} \frac{a^3}{(1+a)(1+2a)} (1-\tau)^{2+\frac{1}{a}} \tilde{\mathfrak{m}}. \end{aligned}$$

This implies

$$\mathcal{B}(t) = \mathcal{B}(t(\tau)) = \left(\frac{c}{1-\tau}\right)^{\frac{1}{a}} \left(4\pi - a^{-2}(1-\tau)^{-2} \int_{\Sigma_\tau} |\nabla u|^2\right)$$
$$\leq 4\pi(3-p)\tilde{\mathfrak{m}}.$$

From this (ii) follows. □

5. Applications

We give applications of results in the previous sections. First recall the definitions of some quantities. Consider a complete, orientable, asymptotically flat 3-manifold (M^3, g) with smooth boundary ∂M. For $1 < p \leq 2$, denote

$$\begin{cases} C_p = p\text{-capacity of } \partial M \text{ in } (M,g); \\ a = \frac{3-p}{p-1} \\ c = a^{-1}(\frac{C_p}{4\pi})^{\frac{1}{p-1}} \end{cases} \tag{5.1}$$

By Theorem 1.2 and Proposition 4.1, we have:

Theorem 5.1. *Let (M^3, g) be a complete, orientable, asymptotically flat 3-manifold with smooth boundary ∂M. Suppose ∂M is connected and $H_2(M, \partial M) = 0$. For $1 < p \leq 2$, let u be the p-harmonic function on M with $u = 0$ on ∂M, and $u \to 1$ at infinity. If g has nonnegative scalar curvature, then*

$$(1-s)4\pi + \int_{u=s} |\nabla u| H - a^{-2}(1+2a)(1-s)^{-1} \int_{u=s} |\nabla u|^2 \geq 0,$$
$$c^{\frac{1}{a}}(1-s)^{-\frac{1}{a}} \left(8\pi - \frac{1}{a(1-s)} \int_{u=s} |\nabla u| H\right) \leq 4\pi(5-p)\mathfrak{m},$$

and

$$c^{\frac{1}{a}}(1-s)^{-\frac{1}{a}} \left(4\pi - a^{-2}(1-s)^{-2} \int_{u=s} |\nabla u|^2\right) \leq 4\pi(3-p)\mathfrak{m},$$

whenever s is a regular value of u. In particular, at ∂M,

$$4\pi + \int_{\partial M} |\nabla u| H \geq a^{-2}(1+2a) \int_{\partial M} |\nabla u|^2, \tag{5.2}$$

$$c^{\frac{1}{a}} \left(8\pi - a^{-1} \int_{\partial M} |\nabla u| H\right) \leq 4\pi(5-p)\mathfrak{m}, \tag{5.3}$$

and

$$c^{-\frac{1}{a}}\left(4\pi - a^{-2}\int_{\partial M}|\nabla u|^2\right) \le 4\pi(3-p)\mathfrak{m}. \tag{5.4}$$

Here H is the mean curvature of ∂M with respect to $\nu = \nabla u/|\nabla u|$. Moreover, if equality holds in any of (5.2)–(5.4), then (M, g) is isometric to $\mathbb{R}^3$ outside a round ball.

If $p = 2$, Theorem 5.1 reduces to inequalities in [34, Theorems 3.1, 3.2]. Similar to the $p = 2$ case, we note (5.2) + (5.4) $\Longrightarrow$ (5.3).

If $H = 0$, (5.3) reduces to $2c^{\frac{1}{a}}(5-p)^{-1} \le \mathfrak{m}$, i.e.

$$\frac{2}{5-p}\left(\frac{3-p}{p-1}\right)^{\frac{1-p}{3-p}}\left(\frac{C_p}{4\pi}\right)^{\frac{1}{3-p}} \le \mathfrak{m}. \tag{5.5}$$

Similar to the applications in [3], by [17, Theorem 1.2] letting $p \to 1$ in (5.5), one retrieves the Riemannian Penrose inequality in the case that ∂M is a connected, outer minimizing surface.

Next, we give a corollary of (5.2) and (5.4).

Corollary 5.1. *Let (M^3, g) be a complete, orientable, asymptotically flat 3-manifold with smooth boundary ∂M. Suppose ∂M is connected and $H_2(M, \partial M) = 0$. Let $W = \frac{1}{16\pi}\int_{\partial M} H^2$, where H is the mean curvature of ∂M. For $1 < p \le 2$, if g has nonnegative scalar curvature, then*

$$1 \le a^{\frac{1}{a}}\left(\frac{4\pi}{C_p}\right)^{\frac{1}{3-p}}(3-p)\mathfrak{m} + \frac{a^2}{(1+2a)^2}\left(\sqrt{W} + \sqrt{W + \frac{1+2a}{a^2}}\right)^2. \tag{5.6}$$

If equality holds, then (M, g) is isometric to $\mathbb{R}^3$ minus a round ball.

As a result of (5.6), if ∂M is area outer-minimizing in (M, g), then

$$\sqrt{\frac{|\partial M|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\partial M} H^2\right) \le \mathfrak{m}. \tag{5.7}$$

Proof. Let u be the p-harmonic function on M with $u = 0$ on ∂M and $u \to 1$ at infinity. By (5.2) and Hölder's inequality,

$$4\pi + \sqrt{16\pi W}\left(\int_{\partial M}|\nabla u|^2\right)^{\frac{1}{2}} \ge a^{-2}(1+2a)\int_{\partial M}|\nabla u|^2.$$

This implies

$$a^{-2}\int_{\partial M}|\nabla u|^2 \le 4\pi\frac{a^2}{(1+2a)^2}\left(\sqrt{W}+\sqrt{W+\frac{1+2a}{a^2}}\right)^2 \tag{5.8}$$

by elementary reasons. (5.6) follows from (5.4) and (5.8).

Letting $p\to 1$ in (5.6) and using the fact $\{C_p\}_{1<p\le 2}$ is bounded (which can be seen by choosing a fixed test function in the variational definition of C_p), we have

$$\sqrt{\frac{\limsup_{p\to 1} C_p}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\partial M}H^2\right)\le \mathfrak{m}. \tag{5.9}$$

If ∂M is area outer-minimizing in (M,g), it was shown in [17] (also see [1, 3]) that $\lim_{p\to 1} C_p = |\partial M|$. Hence, (5.7) follows from (5.9). $\square$

Remark 5.1. For each fixed p, (5.6) implies the 3-dimensional Riemannian positive mass theorem. For instance, suppose M is topologically $\mathbb{R}^3$ and apply (5.6) to the complement of a small geodesic ball $B_r(x)$ with radius r in (M,g). Since C_p remains bounded and $W\to 1$, as $r\to 0$, we see $\mathfrak{m}\ge 0$.

Remark 5.2. In [35], Moser gave another proof of the existence of weak inverse mean curvature flow (IMCF) by constructing p-harmonic functions for $p>1$ and letting $p\to 1$. In [41], Xiao used IMCF to obtain estimates on p-capacity and commented on their limit as $p\to 1$. We do not use IMCF in this work. Our approach is along the same line as in [3]. We would like to thank Jie Xiao for bring our attention to Remark 1.2 in the work [41].

Theorem 5.2. *Let (M,g) be a complete, orientable, asymptotically flat 3-manifold with nonnegative scalar curvature, with smooth boundary ∂M. Suppose ∂M is connected and $H_2(M,\partial M)=0$. Let $H_{\max}$ denote the maximum of the mean curvature H of ∂M. Suppose $H_{\max}\ge 0$. Then for $1<p\le 2$,*

$$\begin{aligned}2\le{}& a^{\frac{1}{a}}\left(\frac{4\pi}{C_p}\right)^{\frac{1}{3-p}}(5-p)\mathfrak{m}\\ &+H_{\max}a^{\frac{1-p}{3-p}}\left(\frac{C_p}{4\pi}\right)^{\frac{1}{3-p}}\left[\frac{a(\sqrt{W}+\sqrt{W+\frac{1+2a}{a^2}})}{(1+2a)}\right]^{\frac{2(2-p)}{3-p}}.\end{aligned}\tag{5.10}$$

Consequently, if $\Omega\subset M$ is a bounded region separating ∂M and ∞, meaning that $\partial\Omega$ has two connected components S_0 and S_1, where S_0 encloses ∂M

(and is allowed to coincide with ∂M) and S_1 encloses S_0, then

$$H_{\max}\left[\frac{a(\sqrt{W}+\sqrt{W+\frac{1+2a}{a^2}})}{(1+2a)}\right]^{\frac{2(2-p)}{3-p}} \le 2\left(\frac{4\pi}{C_p(\Omega)}\right)^{\frac{1}{3-p}} a^{\frac{p-1}{3-p}} \implies \mathfrak{m} > 0.$$

Proof. Since $p \le 2$,

$$\begin{aligned}\int_{\partial M} |\nabla u| &= \int_{\partial M} |\nabla u|^{\frac{p-1}{3-p}} |\nabla u|^{\frac{2(2-p)}{3-p}} \\ &\le \left(\int_{\partial M} |\nabla u|^{p-1}\right)^{\frac{1}{3-p}} \left(\int_{\partial M} |\nabla u|^2\right)^{\frac{2-p}{3-p}}.\end{aligned}$$

Recall the p-capacity of Σ in (M, g) is given by

$$C_p = \int_{\partial M} |\nabla u|^{p-1}.$$

Thus,

$$\int_{\partial M} |\nabla u| \le C_p^{\frac{1}{3-p}} \left(\int_{\partial M} |\nabla u|^2\right)^{\frac{2-p}{3-p}}. \tag{5.11}$$

As $H_{\max} \ge 0$, (5.10) follows from (5.3), (5.11) and (5.8).

Suppose $\Omega \subset M$ is a bounded region separating ∂M and ∞. Let u_Ω be the p-harmonic function with $u_\Omega|_{S_0} = 0$ and $u_\Omega|_{S_1} = 1$. Let

$$C_p(\Omega) = \int_\Omega |\nabla u_\Omega|^p = \int_{S_0} |\nabla u_\Omega|^{p-1}.$$

Then $C_p < C_p(\Omega)$ by the variational nature of the p-capacity. Hence, (5.10) holds with C_p replaced by $C_p(\Omega)$ and the inequality becomes strict. This implies the rest claim. $\square$

Remark 5.3. One can have a rough estimate of $C_p(\Omega)$ in terms of $\mathrm{Vol}(\Omega)$, the volume of (Ω, g), and L, the distance between S_0 and S_1. Let $f(x)$ be a test function so that $f = 0$ on the region enclosed by S_0 with ∂M, $f(x) = L^{-1}d(x)$ for x outside S_0 with $d(x) \le L$ and $f(x) = 1$ if $d(x) \ge L$. Here $d(x)$ denote the distance from x to S_0. Then

$$C_p(\Omega) \le \int_\Omega |\nabla f|^p \le L^{-p}\,\mathrm{Vol}(\Omega).$$

Thus, Theorem 5.2 implies the following: If

$$H_{\max}\left[\frac{a(\sqrt{W}+\sqrt{W+\frac{1+2a}{a^2}})}{(1+2a)}\right]^{\frac{2(2-p)}{3-p}} \leq 2\left(\frac{4\pi L^p}{\mathrm{Vol}(\Omega)}\right)^{\frac{1}{3-p}} a^{\frac{p-1}{3-p}}, \tag{5.12}$$

then $\mathfrak{m} > 0$. Note the right side of (5.12) does not depend on ∂M. If $p = 2$, (5.12) becomes $H_{\max} \leq 8\pi L^2 \mathrm{Vol}(\Omega)^{-1}$, which is exactly the condition in [34, Equation (5.2)].

Recently, there are results on positive mass theorems on complete manifolds with arbitrary ends, see [30, 29, 42] for instances. We apply Theorem 5.1 to obtain a special case of those results.

Proposition 5.1. *Let (M^3, g) be a complete noncompact manifold with nonnegative scalar curvature with two ends $E, \widetilde{E}$ such that M is diffeomorphic to $\mathbb{R}^3 \setminus \{0\}$ and such that E is AF. Suppose there is a harmonic function u such that $u \to 1$ at the infinity of E and $u \to 0$ at the infinity of $\widetilde{E}$ and suppose the Ricci curvature of M is bounded below. Then the ADM mass of E satisfies:*

$$8\pi\mathfrak{m} \geq C_2 > 0$$

where C_2 is the capacity of u. That is $C_2 = \int_{\{u=\tau\}} |\nabla u|$, where $0 < \tau < 1$ is any constant so that $\{u = \tau\}$ is regular.

Proof. Assume that the origin 0 corresponds to the infinity of $\widetilde{E}$. Let $\rho_i \downarrow 0$ and let v_i be the harmonic function such that $v_i = 0$ at $\partial B_0(\rho_i)$ and $v_i \to 1$ at infinity of E. Here $B_0(\rho)$ is the Euclidean ball of radius ρ. Then v_i will converge to u in C^∞ norm on compact sets.

Fix $0 < \tau < 1$ so that τ is a regular value of u. Hence if i is large enough, then τ is also a regular value of v_i. By [34] or Theorem 5.1 with $p = 2$, we have

$$8\pi\mathfrak{m} \geq \frac{C_2(i)}{4\pi}(1-\tau)^{-1}\left(4\pi - (1-\tau)^{-2}\int_{\{v_i=\tau\}} |\nabla v_i|^2\right).$$

Here $C_2(i)$ is the 2-capacity of v_i. Since $v_i \to u$, by the maximum principle, there is a compact set Q in M such that $B_x(1) \subset Q$ for all $x \in \{v_i = \tau\}$. Here $B_x(1)$ is the geodesic ball centered as x with radius 1. Since the Ricci curvature of M is bounded from below, by the gradient estimate for positive harmonic functions by Cheng-Yau [12], see also [31, Theorem 6.1], we conclude that there is a constant β depending only on the lower bound of the Ricci curvature so that

$$|\nabla v_i(x)| \leq \beta v_i(x) = \beta\tau$$

for all $x \in \{v_i = \tau\}$ for all i, if i is large enough. Hence we have:

$$\begin{aligned}8\pi\mathfrak{m} \geq& \frac{C_2(i)}{4\pi}(1-\tau)^{-1}\left(4\pi - \beta\tau(1-\tau)^{-2}\int_{\{v_i=\tau\}}|\nabla v_i|\right)\\ =&\frac{C_2(i)}{4\pi}(1-\tau)^{-1}(4\pi - \beta C_2(i)\tau(1-\tau)^{-2}).\end{aligned}$$

Let $i \to \infty$, since $C_2(i) \to C_2$, we have

$$8\pi\mathfrak{m} \geq \frac{C_2(i)}{4\pi}(1-\tau)^{-1}(4\pi - \beta C_2\tau(1-\tau)^{-2}).$$

Let $\tau \to 0$, the result follows. □

Appendix A. Basic computations

Let (M^n, g) be a Riemannian manifold and u is a smooth function on M. Let $\Sigma_\tau = \{u = \tau\}$. Assume $|\nabla u| > 0$ on Σ_τ. We have the following:

Lemma A.1. *Let $\nu = \frac{\nabla u}{|\nabla u|}$. Let H be the mean curvature of Σ_τ with respect to ν. Then*

$$\begin{cases}\frac{\partial}{\partial\tau} = \frac{1}{|\nabla u|}\nu = \frac{\nabla u}{|\nabla u|^2};\\ H = \frac{1}{|\nabla u|}(\Delta u - u_{\nu\nu});\\ \frac{\partial}{\partial\tau}|\nabla u|^2 = 2(\Delta u - H|\nabla u|);\\ \frac{\partial}{\partial\tau}d\sigma = \frac{1}{|\nabla u|}Hd\sigma;\\ |\nabla u|\frac{\partial}{\partial\tau}H = -\frac{1}{2}(\frac{n}{n-1}H^2 - \mathcal{S}^\tau) - (|\nabla u|\widetilde{\Delta}\frac{1}{|\nabla u|} + \frac{1}{2}(\mathcal{S} + |\mathring{A}|^2));\end{cases}$$

where $u_{\nu\nu} = \nabla^2 u(\nu,\nu)$, $\nu = \nabla u/|\nabla u|$; $d\sigma$ is the area element of Σ_τ; $\mathcal{S}$ is the scalar curvature of M; $\mathcal{S}^\tau$ is the scalar curvature of Σ_τ; $\mathring{A}$ is the traceless part of the second fundamental form A of Σ_τ with respect to ν; and $\widetilde{\Delta}$ is the Laplacian operator of Σ_τ with respect to the induced metric.

Moreover, if X, Y are tangential to Σ_τ, then $\nabla^2 u(X,Y) = |\nabla u|^2 A(X,Y)$, $\nabla^2 u(X,\nu) = X(|\nabla u|)$.

Lemma A.2. *Let Σ_τ be as in the previous lemma. Then*

$$\frac{d}{d\tau}\int_{\Sigma_\tau}|\nabla u|H = \int_{\Sigma_\tau}\left(\frac{1}{2}\left(\mathcal{S}^\tau - \frac{n}{n-1}H^2\right) + H\frac{\Delta u}{|\nabla u|}\right) - E(\tau)$$

where

$$E(\tau) = \int_{\Sigma_\tau} \Big(\frac{|\widetilde{\nabla}|\nabla u|^2|}{|\nabla u|^2} + \frac{1}{2}(\mathcal{S} + |\mathring{A}|^2) \Big).$$

and

$$\frac{d}{d\tau} \int_{\Sigma_\tau} |\nabla u|^2 = \int_{\Sigma_\tau} (2\Delta u - |\nabla u| H).$$

Appendix B. On equation (1.15)

In n-dimension, (1.15) takes the form:

$$\Delta u = \alpha u_{\nu\nu} + \frac{(n-1)|\nabla u|^2}{u}, \tag{B.1}$$

where $2-n \leq \alpha \leq 1$. We assume $u > 0$ and $|\nabla u| > 0$.

Lemma B.1.

(i) If $2-n < \alpha < 1$, then u is a solution to (B.1) if and only if $U = u^{-a}$ is a p-harmonic function. Here $p = 2-\alpha$ so that $1 < p < n$ and $a = (n-p)/(p-1)$.

(ii) If $\alpha = 1$, then u is a solution to (B.1) if and only if the level sets of $U = (n-1)\log u$ is a solution to the inverse mean curvature flow. Namely

$$\mathbf{div}\Big(\frac{\nabla U}{|\nabla U|}\Big) = |\nabla U|.$$

(iii) If $\alpha = 2-n$, then u is a solution to (B.1) if and only $U = \log u$ is n-harmonic.

Proof. This follows from direct computations. We only prove (i). Let U be p-harmonic in (M^n, g) with $1 < p < n$, so that

$$\Delta U = (2-p)U_{\nu\nu},$$

with $\nu = \nabla U/|\nabla U|$. Here we assume that $U > 0, |\nabla U| > 0$. Let $u = U^{-\frac{1}{a}}$ where $a = (n-p)/(p-1)$. Then

$$\begin{aligned}\Delta u &= -\frac{1}{a}U^{-1-\frac{1}{a}}\Delta U + \frac{1}{a}\Big(1+\frac{1}{a}\Big)U^{-2-\frac{1}{a}}|\nabla U|^2 \\ &= -\frac{1}{a}U^{-1-\frac{1}{a}}(2-p)U_{\nu\nu} + \frac{1}{a}\Big(1+\frac{1}{a}\Big)U^{-2-\frac{1}{a}}|\nabla U|^2\end{aligned}$$

Since $\widetilde{\nu} =: \nabla u/|\nabla u| = -\nu$,

$$u_{\nu\nu} = -\frac{1}{a}U^{-1-\frac{1}{a}}U_{\nu\nu} + \frac{1}{a}\Big(1+\frac{1}{a}\Big)U^{-2-\frac{1}{a}}|\nabla U|^2.$$

Hence

$$\begin{aligned}\Delta u - (2-p)u_{\nu\nu} &= (p-1)\frac{1}{a}\Big(1+\frac{1}{a}\Big)U^{-2-\frac{1}{a}}|\nabla U|^2 \\ &= (n-1)a^{-2}U^{-2-\frac{1}{a}}|\nabla U|^2 \\ &= (n-1)\frac{|\nabla u|^2}{u},\end{aligned}$$

because $|\nabla u|^2 = a^{-2}U^{-2-\frac{2}{a}}|\nabla U|^2$. Hence U is p-harmonic. The converse can be proved similarly. ☐

Lemma B.2. *Suppose the equality holds in Corollary 2.1 and $\alpha = 1$. Then g can be written as*

$$g = \frac{1}{\frac{1}{2}\chi - m_H\rho^{-1}}d\rho^2 + \rho^2\gamma_0$$

where

$$m_H = \Big(\frac{|\partial_- M|}{4\pi}\Big)^{\frac{1}{2}}\Big(2\pi\chi - \frac{1}{4}\int_{\partial_- M} H^2\Big),$$

and γ_0 is a metric of constant curvature and with area 4π. Moreover $u = C\rho$ for some constant C.

Proof. In the proof of Corollary 2.1, if $\alpha = 1$, then $\beta = 0$ and by Theorem 1.3,

$$2\pi\chi\tau + \int_{\{u=\tau\}}(|\nabla u|^2 - H|\nabla u|) = 2\pi\chi c_- + \int_{\partial_- M}(|\nabla u|^2 - H|\nabla u|)$$

for all $c_- \le \tau \le c_+$. In terms of v, we have

$$t^{\frac{1}{2}}\Big(2\pi\chi - \frac{1}{4}\int_{\{v=t\}} H^2\Big) = t_0^{\frac{1}{2}}\Big(2\pi\chi - \frac{1}{4}\int_{\{v=t_0\}} H^2\Big) =: m$$

because $H = \frac{|\nabla v|}{v} = \frac{\eta}{t}$. We have

$$\eta^2(t) = 4t_0|\Sigma_{t_0}|^{-1}t(2\pi\chi - mt^{-\frac{1}{2}})$$

because $|\Sigma_t| = tt_0^{-1}|\Sigma_{t_0}|$. Hence, if we let $t = r^2$, then

$$\begin{aligned} g &= \frac{|\Sigma_{t_0}|dt^2}{4t_0(2\pi\chi - mt^{-\frac{1}{2}})} + tt_0^{-1}\gamma_{t_0} \\ &= \frac{|\Sigma_{t_0}|dr^2}{t_0(2\pi\chi - mr^{-1})} + r^2 t_0^{-1}\gamma_{t_0} \\ &= \frac{d\rho^2}{\frac{1}{2}\chi - \widetilde{m}\rho^{-1}} + \rho^2\gamma_0, \end{aligned}$$

where $\rho = (\frac{|\Sigma_{t_0}|}{4\pi t_0})^{\frac{1}{2}} r$ and $\widetilde{m} = (\frac{|\Sigma_{t_0}|}{4\pi t_0})^{\frac{1}{2}} m$. γ_0 is a metric of constant curvature and with area 4π. ☐

Appendix C. Comparing $\mathcal{A}(t), \mathcal{B}(t), F(t)$ to the Hawking mass

We compare $\mathcal{A}(t), \mathcal{B}(t), F(t)$ with the Hawking mass. Let $1 > u > 0$ be p-harmonic with $1 < p < 3$ with $|\nabla u| > 0$. Recall that

$$\mathcal{B}(t) = 4\pi t - (ca)^{-2} t^{2a+1} \int_{\{u=1-ct^{-a}\}} |\nabla u|^2.$$

Denote c by c_p because it depends on p. Let $U = (1-p)\log(1-u)$. Direct computations show that U satisfies (see [35]):

$$\mathbf{div}(|\nabla U|^{p-2}\nabla U) = |\nabla U|^p.$$

In terms of U,

$$\mathcal{B}(t) = 4\pi c_p^{\frac{1}{a}} \exp\left(\frac{\tau}{3-p}\right)\left(1 - \frac{1}{4\pi(3-p)^2}\int_{\{U=\tau\}} |\nabla U|^2\right),$$

where $\tau = (1-p)\log(c_p t^{-a})$. By [35], see also [27], U will converge to the weak solution U_1 of the inverse mean curvature flow as $p \to 1$ in some weak sense. If the convergence is C^2, then one can see that $\mathcal{B}$ will converge to a constant multiple of the Hawking mass of a level set of U_1. Similarly, $\mathcal{A}$ and F also converge to constant multiples of the Hawking mass.

On the other hand,

$$\begin{aligned} |\nabla U|^p &= \mathbf{div}\left(\frac{\nabla U}{|\nabla U|}|\nabla U|^{p-1}\right) \\ &= |\nabla U|^{p-2}(|\nabla U|H + (p-1)U_{\nu\nu}). \end{aligned}$$

Here H is the mean curvature of the level surfaces of U. Hence,

$$|\nabla U|^2 = |\nabla U|H + (p-1)U_{\nu\nu}. \tag{C.1}$$

This gives the following forms of $\mathcal{B}, \mathcal{A}$ in terms of the mean curvature, which also indicate that as $p \to 1$, $\mathcal{B}, \mathcal{A}$ will converge to multiples of the Hawking mass of the level surface of the inverse mean curvature flow.

Proposition C.1.

$$\mathcal{B}(\tau) = 4\pi c_p^{\frac{1}{a}} \exp\left(\frac{\tau}{3-p}\right)\left[1 - \frac{1}{4\pi(3-p)^2}\int_{\{U=\tau\}}\left(H + (p-1)\frac{U_{\nu\nu}}{|\nabla U|}\right)^2\right],$$

$$\mathcal{A}(\tau) = 8\pi c_p^{\frac{1}{a}} \exp\left(\frac{\tau}{3-p}\right)\left[1 - \frac{1}{8\pi(3-p)}\int_{\{U=\tau\}} H\left(H + (p-1)\frac{U_{\nu\nu}}{|\nabla U|}\right)\right].$$

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