

Analytic and Reidemeister torsions of digraphs and path complexes*

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Dedicated to our dear friend Peter Li on the occasion of his 70th birthday

Abstract: We define the notions of Reidemeister torsion and analytic torsion for directed graphs by means of the path homology theory introduced by the authors in [7, 8, 9, 11]. We prove the identity of the two notions of torsions as well as obtain formulas for torsions of Cartesian products and joins of digraphs.

Keywords: Analytic torsion, Reidemeister torsion, digraphs, path homology.

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Received February 21, 2022.

2010 Mathematics Subject Classification: Primary 05C20, 05C38; secondary 05C76, 55N35, 55U25.

*AG is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - SFB 1283/2 2021 - 317210226. YL is supported by the National Science Foundation of China (Grant No. 12071245 and 11761131002).

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1. Introduction

Let M be a compact oriented Riemannian manifold. Assume that M is triangulated by a simplicial complex K . Let ρ be a acyclic representation of $\pi_1(K)$ by orthogonal matrices, i.e., the twisted cohomology group $H^p(K; \rho)$ is trivial for all p . The Reidemeister torsion $\tau_\rho(M)$ is defined from the cochain complex of K by taking an alternating product of determinants [16, 13]. It is a manifold invariant and is used to distinguish homotopy equivalent spaces [3].

To describe the Reidemeister torsion in analytic terms, Ray and Singer [15] defined an analytic torsion $T_\rho(M)$ for any compact oriented manifold M and orthogonal representation ρ of the fundamental group $\pi_1(M)$. Their definition used the spectrum of the Hodge Laplacian on twisted forms. When ρ is orthogonal, Cheeger [2] and Müller [14] proved that $\tau_\rho(M) = T_\rho(M)$. When ρ is orthogonal but not acyclic, one can also see the definition of analytic torsion $T_\rho(M)$ in [6]. Bismut and Zhang [1] were the first to introduce the analytic torsion for any $\rho : \pi_1 \rightarrow GL(n, C)$ and without the orientation assumption, and established a comparison formula for the analytic and combinatorial torsion in this most general situation. In [4] and [5], by introducing the combinatorial vector field on CW complex, Forman defined various Zeta functions which related to the Reidemeister torsion on CW complex.

In this paper we introduce the notions of Reidemeister and analytic torsions on finite digraphs by means of the path homology theory of Grigor'yan, Lin, Muranov and Yau [7, 8, 9, 11]. Namely, we use the homology basis to construct a preferred basis of the path complex on a digraph G , which leads to the definition of the Reidemeister torsion $\tau(G)$. Next, we define the Hodge Laplace operator Δ_p acting on p -paths and use the positive eigenvalues of Δ_p in order to define the analytic torsion $T(G)$ on graphs. This definition is basically similar to the R-torsion defined from the combinatorial Laplacian on the smooth triangulation of manifolds in [15]. Although the homology groups can be nontrivial in our case, we still can prove that $\tau(G) = T(G)$ (Theorem 3.14) by using an extension of the argument of [15, Proposition 1.7]. In fact, this is the start point of Ray-Singer to make their conjecture.

Given two finite digraphs X and Y , we obtain formulas for the torsions of their Cartesian product $X \square Y$ and join $X * Y$ (Theorems 4.8 and 5.7). Our proofs rely essentially on the Künneth formulas for chain complexes of $X \square Y$ and $X * Y$ proved in [10] and [11]. The approach to the proof is borrowed from [15, Thm. 2.5] but our setting is more complicated in the following sense. The notion of torsion depends on the choice of an inner product in the chain spaces, and the cases of the Cartesian product and join require usage of different inner products. Besides, the case of join requires usage of an augmented chain complex. For that reason, the final formulas for $\tau(X \square Y)$ and $\tau(X * Y)$ stated in Corollaries 4.15 and 5.8 are more complicated than one could expect. By using the these Künneth formulas for chain complexes of $X \square Y$ and $X * Y$, one can get our results from the torsion of the tensor product of two finite dimensional complexes [12] and [17].

In Section 2 we revise the path homology theory. In Section 3 we introduce the notions of the Hodge Laplacian on an arbitrary finite-dimensional chain complex, prove the Hodge decomposition, define the notions of R-torsion and analytic torsion, and prove their identity (Theorem 3.14).

In Section 4 we revise the notion of the Cartesian product of digraphs, the Künneth formula for the Cartesian product, and use it to prove the formula for the torsion of $X \square Y$ (Theorem 4.8 and Corollaries 4.12, 4.15). In Section 5 we fulfil a similar program for the join of digraphs (Theorem 5.7 and Corollaries 5.8, 5.9).

We give numerous examples of application of our results by computing torsions of various digraphs including simplices, cubes, spheres, cycles, prism, etc.

2. Path complexes and path homology

Let us briefly revise the definition of *path complex* and *path homology* introduced by Grigor'yan, Lin, Muranov and Yau in [11] (see also [7]).

2.1. Path complex

Let V be a finite set. For any $p \geq 0$, an *elementary p -path* is any (ordered) sequence i_0, \dots, i_p of $p+1$ vertices of V that will be denoted simply by $i_0 \dots i_p$ or by $e_{i_0 \dots i_p}$. The number p is called the length of the path $i_0 \dots i_p$.

Formal \mathbb{R} -linear combinations of $e_{i_0 \dots i_p}$ are called *p -paths*. Denote by $\Lambda_p = \Lambda_p(V)$ the linear space of all p -paths; that is, the elements of Λ_p are

$$v = \sum_{i_0 i_1 \dots i_p} v^{i_0 i_1 \dots i_p} e_{i_0 i_1 \dots i_p}, \quad \text{where } v^{i_0 i_1 \dots i_p} \in \mathbb{R}.$$

Definition 2.1. For any $p \geq 0$, define the *boundary operator* $\partial : \Lambda_{p+1} \rightarrow \Lambda_p$ by

$$(2.1) \quad (\partial v)^{i_0 \dots i_p} = \sum_{k \in V} \sum_{q=0}^{p+1} (-1)^q v^{i_0 \dots i_{q-1} k i_q \dots i_p},$$

where the index k is inserted so that it is preceded by q indices $i_0 \dots i_{q-1}$. Set also $\Lambda_{-1} = \{0\}$ and define the operator $\partial : \Lambda_0 \rightarrow \Lambda_{-1}$ by setting $\partial v = 0$ for all $v \in \Lambda_0$.

It follows from (2.1) that

$$(2.2) \quad \partial e_{j_0 \dots j_{p+1}} = \sum_{q=0}^{p+1} (-1)^q e_{j_0 \dots \widehat{j_q} \dots j_{p+1}},$$

where $\widehat{}$ means omission of the index.

It is easy to show that $\partial^2 v = 0$ for any $v \in \Lambda_p$ ([11, Lemma 2.1]). Hence, the family of linear spaces $\{\Lambda_p\}$ with the boundary operator ∂ determine a chain complex that will be denoted by $\Lambda(V)$.

Definition 2.2. An elementary p -path $e_{i_0 \dots i_p}$ on a set V is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, \dots, p-1$, and *irregular* otherwise.

Let I_p be the subspace of Λ_p that is spanned by all irregular $e_{i_0 \dots i_p}$. It is easy to verify that $\partial I_p \subset I_{p-1}$ (cf. [11]). Hence, the boundary operator ∂ is

well-defined on the quotient space $\mathcal{R}_p := \Lambda_p/I_p$:

$$\partial : \mathcal{R}_p \rightarrow \mathcal{R}_{p-1}$$

for all $p \geq 0$. Clearly, \mathcal{R}_p is linearly isomorphic to the space of all regular p -paths:

$$(2.3) \quad \mathcal{R}_p \cong \text{span}\{e_{i_0 \dots i_p} : i_0 \dots i_p \text{ is regular}\}.$$

For simplicity of notation, we will identify \mathcal{R}_p with the space of all regular p -paths. With this identification, the formula (2.1) for the operator $\partial : \mathcal{R}_{p+1} \rightarrow \mathcal{R}_p$ is true only for regular paths $i_0 \dots i_p$ whereas $(\partial v)^{i_0 \dots i_p} = 0$ if $i_0 \dots i_p$ is irregular. The identity (2.2) remains true if we replace by 0 each irregular path on the right hand side.

Denote by $\mathcal{R}(V)$ the chain complex $\{\mathcal{R}_p\}$ with the boundary operator ∂ .

Definition 2.3. A *path complex* over a set V is a non-empty collection P of regular elementary paths on V with the following property:

$$(2.4) \quad \text{if } e_{i_0 \dots i_n} \in P \text{ then } e_{i_0 \dots i_{n-1}} \in P \text{ and } e_{i_1 \dots i_n} \in P.$$

When a path complex P is fixed, all the paths from P are called *allowed*, whereas the elementary paths that are not in P are called *non-allowed*. Condition (2.4) means that if we remove the first or the last element of an allowed n -path then the resulting $(n-1)$ -path is also allowed.

The set of all n -paths from P is denoted by P_n . The set P_{-1} consists of a single empty path e . The elements of P_0 (that is, allowed 0-paths) are called the *vertices* of P . Clearly, P_0 is a subset of V . By the property (2.4), if $i_0 \dots i_n \in P$ then all i_k are vertices of P . Hence, we can (and will) remove from the set V all non-vertices so that $V = P_0$.

There are two natural families/examples of path complexes. Any abstract finite simplicial complex S is a collection of subsets of a finite vertex set V that satisfies the following property:

$$\text{if } \sigma \in S \text{ then any subset of } \sigma \text{ is also in } S.$$

Let us enumerate the elements of V by distinct reals and identify any subset s of V with the elementary path that consists of the elements of s put in the (strictly) increasing order. Denote by $P(S)$ this collections of elementary paths on V that uniquely determines S . The defining property of a simplex can be restated the following:

$$(2.5) \quad \text{if } v \in P(S) \text{ then any subsequence of } v \text{ is also in } P(S).$$

Consequently, the family $P(S)$ satisfies the property (2.4) so that $P(S)$ is a path complex. The allowed n -paths in $P(S)$ are exactly the n -simplexes.

2.2. Digraphs

Another natural family of path complexes comes from digraphs.

Definition 2.4. A *digraph* $G = (V, E)$ is a couple, where V is a set, whose elements are called the *vertices*, and E is a subset of $\{V \times V \setminus \text{diag}\}$ that consists of ordered pairs of vertices called (directed) *edges* or *arrows*. The fact that a pair (x, y) is an arrow will be denoted by $x \rightarrow y$.

An elementary n -path $i_0 \dots i_n$ on the vertex set V of a digraph is called allowed if $i_{k-1} \rightarrow i_k$ for any $k = 1, \dots, n$. Denote by $P_n = P_n(G)$ the set of all allowed n -paths. In particular, we have $P_0 = V$ and $P_1 = E$. Clearly, the collection $P = \bigcup_n P_n$ of all allowed paths satisfies the condition (2.4) so that P is a path complex. This path complex is naturally associated with the digraph G and will be denoted by $P(G)$.

2.3. Path homology

Let us return to an arbitrary path complex P over V . Denote by $\mathcal{A}_p(P)$ the subspace of $\mathcal{R}_p(V)$ spanned by the allowed elementary p -paths, that is,

$$(2.6) \quad \mathcal{A}_p = \text{span}\{e_{i_0 \dots i_p} : i_0 \dots i_p \in E_p\}.$$

The elements of \mathcal{A}_p are called *allowed p -paths*.

Note that the spaces \mathcal{A}_p of allowed paths are in general *not* invariant for ∂ . Consider the following subspace of \mathcal{A}_p

$$(2.7) \quad \Omega_p \equiv \Omega_p(P) := \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}.$$

The spaces Ω_p are ∂ -invariant. Indeed, $v \in \Omega_p$ implies $\partial v \in \mathcal{A}_{p-1}$ and $\partial(\partial v) = 0 \in \mathcal{A}_{p-2}$, whence $\partial v \in \Omega_{p-1}$. The elements of Ω_p are called *∂ -invariant p -paths*.

Hence, we obtain a chain complex $\Omega = \Omega(P)$:

$$(2.8) \quad 0 \leftarrow \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \dots$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$.

Set

$$Z_p = \ker \partial|_{\Omega_p} \text{ and } B_p = \partial\Omega_{p+1}.$$

Definition 2.5. Define for all $p \geq 0$ the *path homology groups* $H_p(P)$ of the path complex P by

$$(2.9) \quad H_p(P) := H_p(\Omega(P)) = Z_p/B_p.$$

Let us note that the spaces $H_p(P)$ (as well as the spaces $\Omega_p(P)$) can be computed directly by definition using simple tools of linear algebra, in particular, those implemented in modern computational software. On the other hand, some theoretical tools for computation of homology groups, like homotopy theory and Künneth formulas, were developed in [8, 10, 11].

In particular, for any digraph G define its path homology groups by

$$H_p(G) = H_p(P(G)).$$

In what follows we are going to deal with only finite chain complexes:

$$(2.10) \quad 0 \leftarrow \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Omega_N \leftarrow 0$$

where $N \in \mathbb{N}$. Clearly, any chain complex (2.8) can be truncated to the form (2.10).

For path complexes and digraphs this means that we restrict the length of allowed paths to N . There is a large family of digraphs where the chain complex Ω is finite naturally because $\Omega_N = \{0\}$ for some N (and, hence, $\Omega_n = \{0\}$ for all $n \geq N$). All examples of digraphs that are considered in this paper have naturally finite chain complex Ω .

If this is not the case then we can choose N arbitrarily and truncate the chain complex Ω to (2.10). The number N will be referred to as the dimension of the chain complex (2.10) or that of the underlying path complex.

Some examples of chain complexes Ω and homology groups of digraphs will be given in Section 3.3.

3. Finite chain complexes

Let us fix a finite chain complex Ω (2.10) of finite dimensional linear spaces Ω_p . We are interested in chain complexes that are coming from path complexes as described above, but in this section we revise rather well known facts about general chain complexes Ω .

Let us choose arbitrarily an inner product $\langle \cdot, \cdot \rangle$ in each linear space Ω_p . In the case when Ω comes from a path complex, an inner product in Ω_p can be

taken from the ambient space \mathcal{R}_p . In this paper we use two different inner products in \mathcal{R}_p . Let $u, v \in \mathcal{R}_p$ and

$$u = \sum_{\mathbf{i}} u^{\mathbf{i}} e_{\mathbf{i}} \quad \text{and} \quad v = \sum_{\mathbf{i}} v^{\mathbf{i}} e_{\mathbf{i}}$$

where $\mathbf{i} = i_0 \dots i_p$. The first (*standard*) inner product is

$$(3.1) \quad \langle u, v \rangle = \sum_{\mathbf{i}} u^{\mathbf{i}} v^{\mathbf{i}},$$

and the second (*normalized*) inner product is

$$(3.2) \quad \langle u, v \rangle = \frac{1}{p!} \sum_{\mathbf{i}} u^{\mathbf{i}} v^{\mathbf{i}}.$$

These inner products will be used in examples and in Section 4, but in general we do not impose any restriction on the choice of inner products in the spaces Ω_p .

3.1. Hodge Laplacian

Denote by ∂_p the operator $\partial : \Omega_p \rightarrow \Omega_{p-1}$. Assuming that the inner product structure in Ω is chosen, consider the operator $\partial_p^* : \Omega_{p-1} \rightarrow \Omega_p$ that is the adjoint operator of ∂_p with respect to the inner products in Ω_p and Ω_{p-1} .

Definition 3.1. Define the *Hodge-Laplace operator* $\Delta_p : \Omega_p \rightarrow \Omega_p$ by

$$(3.3) \quad \Delta_p u = \partial_p^* \partial_p u + \partial_{p+1} \partial_{p+1}^* u.$$

We will use a shorter notation

$$\Delta_p u = \partial^* \partial u + \partial \partial^* u$$

since it is clear from this expression in which spaces Ω_p act the operators ∂ and ∂^* .

An element $u \in \Omega_p$ is called *harmonic* if $\Delta_p u = 0$.

Lemma 3.2. *An element $u \in \Omega_p$ is harmonic if and only if $\partial u = 0$ and $\partial^* u = 0$.*

Proof. If $\partial u = 0$ and $\partial^* u = 0$ then by (3.3) we have $\Delta_p u = 0$. Conversely, if $\Delta_p u = 0$ then we obtain

$$0 = \langle \Delta_p u, u \rangle = \langle \partial^* \partial u, u \rangle + \langle \partial \partial^* u, u \rangle = \langle \partial u, \partial u \rangle + \langle \partial^* u, \partial^* u \rangle,$$

whence $\|\partial u\| = \|\partial^* u\| = 0$. □

Denote by \mathcal{H}_p the set of all harmonic elements in Ω_p so that \mathcal{H}_p is a subspace of Ω_p .

Lemma 3.3. (Hodge decomposition) *The space Ω_p is an orthogonal sum of three subspaces as follows:*

$$(3.4) \quad \Omega_p = \partial \Omega_{p+1} \oplus \partial^* \Omega_{p-1} \oplus \mathcal{H}_p.$$

Proof. If $u \in \partial \Omega_{p+1}$ and $v \in \partial^* \Omega_{p-1}$ then $u = \partial u'$ and $v = \partial^* v'$, and we have

$$\langle u, v \rangle = \langle \partial u', \partial^* v' \rangle = \langle \partial^2 u', v' \rangle = 0$$

so that the subspaces $\partial \Omega_{p+1}$ and $\partial^* \Omega_{p-1}$ are orthogonal. Denote by K the orthogonal complement of $\partial \Omega_{p+1} \oplus \partial^* \Omega_{p-1}$ in Ω_p . Then we have

$$u \in K \Leftrightarrow \langle u, v \rangle = 0 \text{ for all } v \in \partial \Omega_{p+1} \text{ and } v \in \partial^* \Omega_{p-1}$$

that is,

$$\begin{aligned} u \in K &\Leftrightarrow \langle u, \partial v' \rangle = 0 \quad \forall v' \in \Omega_{p+1} \text{ and } \langle u, \partial^* v \rangle = 0 \quad \forall v \in \Omega_{p-1} \\ &\Leftrightarrow \langle \partial^* u, v' \rangle = 0 \quad \forall v' \in \Omega_{p+1} \text{ and } \langle \partial u, v \rangle = 0 \quad \forall v \in \Omega_{p-1} \\ &\Leftrightarrow \partial^* u = 0 \text{ and } \partial u = 0 \\ &\Leftrightarrow u \in \mathcal{H}_p. \end{aligned}$$

Hence, $K = \mathcal{H}_p$ which finishes the proof. □

Corollary 3.4. *There is a natural isomorphism*

$$(3.5) \quad H_p \cong \mathcal{H}_p.$$

Proof. Observe first that $Z_p := \ker \partial_p$ is the orthogonal complement of $\partial^* \Omega_{p-1}$ in Ω_p because, for any $u \in \Omega_p$,

$$u \in Z_p \Leftrightarrow \partial u = 0 \Leftrightarrow \langle u, \partial^* v \rangle = 0 \quad \forall v \in \Omega_{p-1} \Leftrightarrow u \perp \partial^* \Omega_{p-1}.$$

It follows from (3.4) that

$$(3.6) \quad Z_p = \partial\Omega_{p+1} \bigoplus \mathcal{H}_p = B_p \bigoplus \mathcal{H}_p$$

whence

$$\mathcal{H}_p \cong Z_p/B_p = H_p. \quad \square$$

Remark 3.5. It follows from this argument that \mathcal{H}_p is an orthogonal complement of B_p in Z_p and that a harmonic form $u \in \mathcal{H}_p$ that corresponds to a homology class $\omega \in H_p$, minimizes the norm $\|\cdot\|$ among all elements of ω .

3.2. R-torsion

Let Ω be a finite chain complex of finite dimensional linear spaces over \mathbb{R} :

$$0 \xleftarrow{\partial} \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Omega_N \xleftarrow{\partial} 0.$$

Denote $B_p = \partial\Omega_{p+1}$, $Z_p = \ker \partial|_{\Omega_p}$ and $H_p = Z_p/B_p$.

In any Ω_p choose a basis ω_p and a basis h_p in H_p . For each element of h_p choose its representative in Z_p and denote the resulting independent set by \tilde{h}_p .

Let b_p be any basis in B_p . For each element $w \in b_{p-1}$ choose one element $v \in \partial^{-1}w \subset \Omega_p$ so that $\partial v = w$. Let \tilde{b}_p be the collection of chosen elements v so that

$$(3.7) \quad b_{p-1} = \partial\tilde{b}_p.$$

Note that always $\tilde{b}_0 = \emptyset$. Since b_{p-1} is linearly independent, the set \tilde{b}_p is also linearly independent. Clearly, the union (b_p, \tilde{h}_p) is a basis in Z_p . Since the subspaces Z_p and $\text{span}(\tilde{b}_p)$ of Ω_p have a trivial intersection $\{0\}$, by the rank-nullity theorem we conclude that the direct sum of these subspaces is Ω_p . Hence, the union $(b_p, \tilde{h}_p, \tilde{b}_p)$ of the these sequences is a basis in Ω_p .

If U and W are two bases in an n -dimensional linear space, then denote by (U/W) the transformation matrix from W to U and set

$$[U/W] = |\det(U/W)|.$$

In the case $n = 0$ set $[U/W] = 1$.

Denote ω the collection $\{\omega_p\}$ of the bases in Ω_p and similarly let $h = \{h_p\}$ be the collection of the bases in H_p .

Definition 3.6. The *R-torsion* $\tau(\Omega, \omega, h)$ of the chain complex Ω with the preferred bases ω and h is a positive real number defined by

$$(3.8) \quad \log \tau(\Omega, \omega, h) = \sum_{p=0}^N (-1)^p \log [b_p, \tilde{h}_p, \tilde{b}_p / \omega_p].$$

We justify this definition in the following statement.

Lemma 3.7. (a) *The value of $\tau(\Omega, \omega, h)$ does not depend on the choice of the bases b_p , the representatives in \tilde{b}_p and the representatives in \tilde{h}_p (which justifies the notation $\tau(\Omega, \omega, h)$).*

(b) *If ω' and h' are other collections of bases in Ω and H respectively, then*

$$(3.9) \quad \log \tau(\Omega, \omega', h') = \log \tau(\Omega, \omega, h) + \sum_{p=0}^N (-1)^p (\log [\omega_p / \omega'_p] + \log [h'_p / h_p]).$$

The relation $[U/W] = 1$ for bases U and W is an equivalence relation, and each equivalence class determines a *volume form* in the underlying linear space. We see from (3.9) that $\tau(\Omega, \omega, h)$ depends only on the volume forms determined by ω and h in the spaces Ω_p and H_p , respectively,

Proof of Lemma 3.7. (a) Let b'_p be another basis in B_p with the corresponding set \tilde{b}'_p , and \tilde{h}'_p be another set of representatives of h_p . Let us first verify that

$$(3.10) \quad [b'_p, \tilde{h}'_p, \tilde{b}'_p / b_p, \tilde{h}_p, \tilde{b}_p] = [b'_p / b_p] [b'_{p-1} / b_{p-1}].$$

Let $\tilde{h}_p = \{u_1, u_2, \dots\}$ and $\tilde{h}'_p = \{u'_1, u'_2, \dots\}$. Since u'_i and u_i represent the same homology class, we have

$$(3.11) \quad u'_i = u_i + b_i \text{ for some } b_i \in B_p.$$

Let $\tilde{b}_p = \{v_1, v_2, \dots\}$ and $\tilde{b}'_p = \{v'_1, v'_2, \dots\}$ so that

$$b_{p-1} = \partial \tilde{b}_p = \{\partial v_1, \partial v_2, \dots\} \text{ and } b'_{p-1} = \partial \tilde{b}'_p = \{\partial v'_1, \partial v'_2, \dots\}.$$

Since b_{p-1} and b'_{p-1} are bases in the same subspace B_{p-1} , the transformation matrix $(c_{ij}) = (b'_{p-1} / b_{p-1})$ is well defined so that

$$\partial v'_i = \sum_j c_{ij} \partial v_j.$$

It follows that

$$(3.12) \quad v'_i = z_i + \sum_j c_{ij} v_j \quad \text{for some } z_i \in Z_p.$$

Since $Z_p = \text{span}(b_p, h_p)$, we obtain from (3.11) and (3.12) that

$$(b'_p, \tilde{h}'_p, \tilde{b}'_p / b_p, \tilde{h}_p, \tilde{b}_p) = \begin{pmatrix} (b'_p/b_p) & \vdots & \vdots \\ 0 & \text{id} & \vdots \\ 0 & 0 & (b'_{p-1}/b_{p-1}) \end{pmatrix}$$

where the dots \vdots denote the terms coming from b_i and z_i . Since this matrix is upper block-diagonal, we obtain (3.10).

Consequently, we have

$$\begin{aligned} [b'_p, \tilde{h}'_p, \tilde{b}'_p / \omega_p] &= [b'_p, \tilde{h}'_p, \tilde{b}'_p / b_p, \tilde{h}_p, \tilde{b}_p] [b_p, \tilde{h}_p, \tilde{b}_p / \omega_p] \\ &= [b'_p/b_p] [b'_{p-1}/b_{p-1}] [b_p, \tilde{h}_p, \tilde{b}_p / \omega_p]. \end{aligned}$$

Computing the sum in (3.8) we obtain

$$(3.13) \quad \begin{aligned} &\sum_{p=0}^N (-1)^p \log [b'_p, \tilde{h}'_p, \tilde{b}'_p / \omega_p] \\ &= \sum_{p=0}^N (-1)^p \log [b_p, \tilde{h}_p, \tilde{b}_p / \omega_p] \\ &\quad + \sum_{p=0}^N (-1)^p \log [b'_p/b_p] + \sum_{p=0}^N (-1)^p \log [b'_{p-1}/b_{p-1}]. \end{aligned}$$

It remains to observe that the expression in (3.13) vanishes because it is equal to

$$\sum_{p=0}^N (-1)^p \log [b'_p/b_p] + \sum_{q=-1}^{N-1} (-1)^{q+1} \log [b'_q/b_q] = (-1)^N \log [b'_N/b_N] = 0.$$

(b) Let $h_p = \{\eta_1, \eta_2, \dots\}$ and $h'_p = \{\eta'_1, \eta'_2, \dots\}$ so that

$$\eta'_i = \sum_j c_{ij} \eta_j$$

where $(c_{ij}) = (h'_p/h_p)$. For the representatives $u_i \in \tilde{h}_p$ of η_i and $u'_i \in \tilde{h}'_p$ of η'_i in Z_p we have then

$$u'_i = \sum_j c_{ij} u_{ij} + b_i \text{ for some } b_i \in B_p.$$

It follows that

$$(b_p, \tilde{h}'_p, \tilde{b}_p / b_p, \tilde{h}_p, \tilde{b}_p) = \begin{pmatrix} \text{id} & \vdots & 0 \\ 0 & (h'_p/h_p) & 0 \\ 0 & 0 & \text{id} \end{pmatrix}$$

where the dots \vdots denote the terms coming from b_i . Hence, we obtain

$$\begin{aligned} [b_p, \tilde{h}'_p, \tilde{b}_p / \omega'_p] &= [b_p, \tilde{h}'_p, \tilde{b}_p / b_p, \tilde{h}_p, \tilde{b}_p][b_p, \tilde{h}_p, \tilde{b}_p / \omega_p][\omega_p / \omega'_p] \\ &= [h'_p/h_p][b_p, \tilde{h}_p, \tilde{b}_p / \omega_p][\omega_p / \omega'_p], \end{aligned}$$

whence (3.9) follows. □

Let us fix an inner product in each space Ω_p and denote by ι the *inner product structure* in Ω , that is, the collection of all inner products for $p = 0, \dots, N$. Then we have the induced inner product in the subspaces B_p, Z_p and \mathcal{H}_p . Using the isomorphism $H_p \cong \mathcal{H}_p$ we transfer the inner product to H_p . Hence, in this case we have a canonical choice of volume forms ω in Ω_* and h in H_* as we prefer orthonormal bases ω_p in Ω_p and h_p in H_p . In fact, we can identify h_p with an orthonormal basis in \mathcal{H}_p and set $\tilde{h}_p = h_p$. With this choice of ω and h , we define the R-torsion of (Ω, ι) by

$$\tau(\Omega, \iota) = \tau(\Omega, \omega, h).$$

By (3.9) the right hand side does not depend on the choice of orthonormal bases ω and h .

Corollary 3.8. *Let ι and ι' be two inner product structures in Ω . Assume that there are positive reals $c_p, p = 0, \dots, N$, such that, for all $u, v \in \Omega_p$,*

$$\iota'(u, v) = c_p \iota(u, v).$$

Then

$$(3.14) \quad \tau(\Omega, \iota') = \tau(\Omega, \iota) \prod_{p=0}^N c_p^{\frac{1}{2}(-1)^p(\dim \Omega_p - \dim H_p)}.$$

In particular, if all $c_p = c$ are equal, then we obtain

$$\tau(\Omega, \iota') = \tau(\Omega, \iota)$$

because

$$\sum_{p=0}^N (-1)^p \dim \Omega_p = \sum_{p=0}^N (-1)^p \dim H_p = \chi(\Omega),$$

where $\chi(\Omega)$ is the Euler characteristic of Ω .

Proof. Since the notion of orthogonality is the same for ι and ι' , the space \mathcal{H}_p is also the same. If ω_p and h_p are ι -orthonormal bases in Ω_p and \mathcal{H}_p , respectively, then $\omega'_p = \frac{1}{\sqrt{c_p}}\omega_p$ and $h'_p = \frac{1}{\sqrt{c_p}}h_p$ are ι' -orthonormal bases. Since

$$[\omega_p/\omega'_p] = c_p^{\dim \Omega_p/2} \quad \text{and} \quad [h'_p/h_p] = c_p^{-\dim H_p/2},$$

we obtain from (3.9)

$$\log \tau(\Omega, \iota') = \log \tau(\Omega, \iota) + \frac{1}{2} \sum_{p=0}^N (-1)^p (\dim \Omega_p - \dim H_p) \log c_p,$$

whence (3.14) follows. □

Let Ω be a chain complex that comes from a path complex P . Then the inner product in Ω_p can be taken from the ambient space \mathcal{R}_p . So the obtained R-torsion $\tau(\Omega(P), \iota)$ will also be denoted by $\tau(P, \iota)$.

Let ι be the standard inner product (3.1) in \mathcal{R}_p and ι' be the normalized inner product (3.2) in \mathcal{R}_p . We set

$$\tau(P) = \tau(P, \iota) \quad \text{and} \quad \tau'(P) = \tau(P, \iota').$$

In this case $c_p = \frac{1}{p!}$ and we obtain from (3.14)

$$(3.15) \quad \tau'(P) = \tau(P) \prod_{p=0}^N (p!)^{\frac{1}{2}(-1)^{p+1}(\dim \Omega_p - \dim H_p)}.$$

Denoting

$$r_p = \dim \Omega_p - \dim H_p$$

and observing that

$$\sum_{p=0}^N (-1)^p r_p = 0$$

and

$$\begin{aligned}
 \sum_{p=2}^N (-1)^{p+1} r_p \log(p!) &= \sum_{p=2}^N (-1)^{p+1} r_p \sum_{k=2}^p \log k \\
 &= \sum_{k=2}^N \log k \sum_{p=k}^N (-1)^{p+1} r_p \\
 &= \sum_{k=2}^N \log k \sum_{p=0}^{k-1} (-1)^p r_p \\
 &= \sum_{k=2}^N \log k^{r_0 - r_1 + \dots + (-1)^{k-1} r_{k-1}},
 \end{aligned}$$

we obtain from (3.15)

$$(3.16) \quad \tau'(P) = \tau(P) (2^{r_0 - r_1} \cdot 3^{r_0 - r_1 + r_2} \cdot 4^{r_0 - r_1 + r_2 - r_3} \dots)^{1/2}.$$

3.3. Examples

Let us give some examples of computation of R-torsion by definition.

Example 3.9. Consider a *line* digraph $G = (V, E)$ that consists of m vertices $V = \{0, 1, \dots, m - 1\}$ and $m - 1$ arrows having the form either $i \rightarrow i + 1$ or $i + 1 \rightarrow i$, for $i = 0, \dots, m - 2$. An example of a line digraph is shown in Fig. 1.

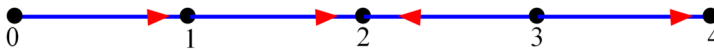


Figure 1: A line digraph with $m = 5$.

Denote

$$(3.17) \quad \bar{e}_{i(i+1)} = \begin{cases} e_{i(i+1)} & \text{if } i \rightarrow i + 1 \\ e_{(i+1)i} & \text{if } i + 1 \rightarrow i \end{cases}$$

so that $\bar{e}_{i(i+1)} \in \Omega_1$, and set

$$(3.18) \quad \sigma_i = \begin{cases} 1, & \text{if } i \rightarrow i + 1 \\ -1, & \text{if } i + 1 \rightarrow i \end{cases}$$

so that

$$\partial \bar{e}_{i(i+1)} = \sigma_i(e_{i+1} - e_i).$$

Choose the following ι -orthonormal bases in Ω_0 and Ω_1 :

$$\omega_0 = \{e_i : i = 0, \dots, m - 1\}$$

and

$$\omega_1 = \{\bar{e}_{i(i+1)} : i = 0, \dots, m - 2\}.$$

Clearly, $\Omega_p = \{0\}$ for $p \geq 2$. In particular, we have $\chi(G) = 1$. Since $\dim H_0 = 1$ (as for any connected graph) and $\dim H_p = 0$ for $p \geq 2$, it follows that $\dim H_1 = 0$.

Since $B_1 = \partial\Omega_2 = \{0\}$, it follows that also $Z_1 = \{0\}$. We have $Z_0 = \Omega_0$ and, hence, $\dim B_0 = m - 1$. Choose in $B_0 = \partial\Omega_1$ the basis

$$b_0 = \{\sigma_i(e_{i+1} - e_i), \quad i = 0, \dots, m - 2\}$$

and set, respectively,

$$\tilde{b}_1 = \{\bar{e}_{i(i+1)}, \quad i = 0, \dots, m - 2\}.$$

The orthogonal complement of B_0 in Z_0 is one-dimensional:

$$\mathcal{H}_0 = \text{span}\{e_0 + \dots + e_{m-1}\},$$

so that

$$h_0 = \left\{ \frac{1}{\sqrt{m}}(e_0 + \dots + e_{m-1}) \right\}.$$

We see that

(3.19)

$$[b_0, h_0, \tilde{b}_0 / \omega_0] = |\det| \begin{pmatrix} -\sigma_0 & & & & & & 0 & \frac{1}{\sqrt{m}} \\ \sigma_0 & -\sigma_1 & & & & & & \frac{1}{\sqrt{m}} \\ & \sigma_1 & \ddots & & & & & \vdots \\ & & \ddots & \ddots & & & & \vdots \\ & & & \ddots & \ddots & & & \vdots \\ & & & & \ddots & \ddots & & \vdots \\ & & & & & \ddots & -\sigma_{m-2} & \vdots \\ 0 & & & & & & \sigma_{m-2} & \frac{1}{\sqrt{m}} \end{pmatrix} = \sqrt{m},$$

because expanding the determinant in the last column, we obtain that is equal to

$$(-1)^{m+1} m \sigma_0 \dots \sigma_{m-2} \frac{1}{\sqrt{m}}.$$

Since $(b_1, h_1, \tilde{b}_1) = \omega_1$, it follows that

$$\tau(G) = \prod_{p=0}^1 [b_p, h_p, \tilde{b}_p / \omega_p]^{(-1)^p} = \sqrt{m}.$$

For the normalized inner product ι' we have the same value $\tau'(G) = \sqrt{m}$ since $\Omega_p = \{0\}$ for all $p \geq 2$.

Example 3.10. Consider a digraph $G = (V, E)$ with the vertex set $V = \{0, 1, 2\}$ and with the edge set $E = \{01, 12, 02\}$ (Fig. 2). This digraph is called a *triangle*.

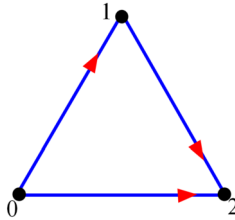


Figure 2: A triangle digraph.

We have

$$\Omega_0 = \text{span}\{e_0, e_1, e_2\}, \quad \Omega_1 = \text{span}\{e_{01}, e_{12}, e_{02}\}, \quad \Omega_2 = \{e_{012}\}$$

and $\Omega_p = \{0\}$ otherwise. Hence,

$$B_0 = \partial\Omega_1 = \text{span}\{e_1 - e_0, e_2 - e_1\}, \quad B_1 = \partial\Omega_2 = \text{span}\{e_{01} - e_{02} + e_{12}\}$$

and $B_p = \{0\}$ otherwise. Next, we have

$$Z_0 = \text{span}\{e_0, e_1, e_2\}, \quad Z_1 = \text{span}\{e_{01} - e_{02} + e_{12}\}$$

and $Z_p = \{0\}$ otherwise. It follows that $\dim H_0 = 1$ and $\dim H_p = 0$ otherwise.

We choose the following ι -orthonormal bases in Ω_p :

$$\omega_0 = \{e_0, e_1, e_2\}, \quad \omega_1 = \{e_{01}, e_{12}, e_{02}\}, \quad \omega_2 = \{e_{012}\}.$$

Choose also

$$\begin{aligned} b_0 &= \{e_1 - e_0, e_2 - e_1\}, & \tilde{b}_1 &= \{e_{01}, e_{12}\} \\ b_1 &= \{e_{01} - e_{02} + e_{12}\}, & \tilde{b}_2 &= \{e_{012}\}. \end{aligned}$$

The orthogonal complement of B_0 in Z_0 is

$$\mathcal{H}_0 = \text{span}\{e_0 + e_1 + e_2\},$$

so that

$$h_0 = \left\{ \frac{1}{\sqrt{3}}(e_0 + e_1 + e_2) \right\}.$$

We see that

$$(b_p, h_p, \tilde{b}_p) = \begin{cases} \{e_1 - e_0, e_2 - e_1, \frac{1}{\sqrt{3}}(e_0 + e_1 + e_2)\}, & p = 0 \\ \{e_{01} - e_{02} + e_{12}, e_{01}, e_{12}\}, & p = 1 \\ \{e_{012}\}, & p = 2. \end{cases}$$

It follows that

$$\begin{aligned} [b_0, h_0, \tilde{b}_0 / \omega_0] &= |\det| \begin{pmatrix} -1 & 0 & \frac{1}{\sqrt{3}} \\ 1 & -1 & \frac{1}{\sqrt{3}} \\ 0 & 1 & \frac{1}{\sqrt{3}} \end{pmatrix} = \sqrt{3}, \\ [b_1, h_1, \tilde{b}_1 / \omega_1] &= |\det| \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = 1 \end{aligned}$$

and

$$[b_2, h_2, \tilde{b}_2 / \omega_2] = |\det|(1) = 1.$$

Hence, we obtain

$$\tau(G) = \prod_{p=0}^2 [b_p, h_p, \tilde{b}_p / \omega_p]^{(-1)^p} = \sqrt{3}.$$

For the normalized inner product ι' we obtain from (3.14)

$$\tau'(G) = \tau(G) \prod_{p=0}^2 (p!)^{\frac{1}{2}(-1)^{p+1}(\dim \Omega_p - \dim H_p)} = \sqrt{3} 2^{-\frac{1}{2}} = \sqrt{3/2}.$$

Example 3.11. Consider a digraph $G = (V, E)$ with the set of vertices $V = \{0, 1, 2, 3\}$ and the set of edges $E = \{01, 02, 13, 23\}$ (Fig. 3). This digraph is called a *square*.

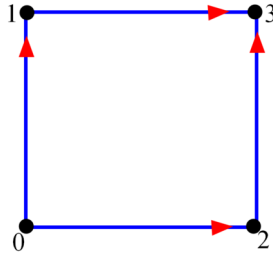


Figure 3: A square digraph.

We have

$$\begin{aligned} \Omega_0 &= \text{span}\{e_0, e_1, e_2, e_3\}, \quad \Omega_1 = \text{span}\{e_{01}, e_{02}, e_{13}, e_{23}\}, \\ \Omega_2 &= \text{span}\{e_{013} - e_{023}\}, \quad \Omega_p = \{0\} \text{ for } p \geq 3. \end{aligned}$$

Hence,

$$\begin{aligned} B_0 &= \partial\Omega_1 = \text{span}\{e_1 - e_0, e_2 - e_0, e_3 - e_1\} \\ B_1 &= \partial\Omega_2 = \text{span}\{e_{01} + e_{13} - e_{02} - e_{23}\} \end{aligned}$$

and $B_p = \{0\}$ otherwise. Next we have

$$Z_0 = \text{span}\{e_0, e_1, e_2, e_3\}, \quad Z_1 = \text{span}\{e_{01} + e_{13} - e_{02} - e_{23}\}$$

and $Z_p = \{0\}$ otherwise. Consequently, $\dim H_0 = 1$ and $\dim H_p = 0$ for $p \geq 1$.

We choose the following ι -orthonormal bases in Ω_p :

$$\omega_0 = \{e_0, e_1, e_2, e_3\}, \quad \omega_1 = \{e_{01}, e_{02}, e_{13}, e_{23}\}, \quad \omega_2 = \left\{ \frac{1}{\sqrt{2}}(e_{013} - e_{023}) \right\}.$$

Choose also

$$\begin{aligned} b_0 &= \{e_1 - e_0, e_2 - e_0, e_3 - e_1\}, & \tilde{b}_1 &= \{e_{01}, e_{02}, e_{13}\} \\ b_1 &= \{e_{01} - e_{02} + e_{13} - e_{23}\}, & \tilde{b}_2 &= \{e_{013} - e_{023}\}. \end{aligned}$$

The orthogonal complement of B_0 in Z_0 is

$$\mathcal{H}_0 = \text{span}\{e_0 + e_1 + e_2 + e_3\}$$

and we take

$$h_0 = \left\{ \frac{1}{2}(e_0 + e_1 + e_2 + e_3) \right\}.$$

It follows that

$$(b_p, h_p, \tilde{b}_p) = \begin{cases} \{e_1 - e_0, e_2 - e_0, e_3 - e_1, \frac{1}{2}(e_0 + e_1 + e_2 + e_3)\}, & p = 0 \\ \{e_{01} - e_{02} + e_{13} - e_{23}, e_{01}, e_{02}, e_{13}\}, & p = 1 \\ \{e_{013} - e_{023}\}, & p = 2. \end{cases}$$

Hence,

$$[b_0, h_0, \tilde{b}_0 / \omega_0] = |\det| \begin{pmatrix} -1 & -1 & 0 & \frac{1}{2} \\ 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} = 2,$$

$$[b_1, h_1, \tilde{b}_1 / \omega_1] = |\det| \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} = 1$$

$$[b_2, h_2, \tilde{b}_2 / \omega_2] = |\det|(\sqrt{2}) = \sqrt{2},$$

and we obtain

$$\tau(G) = \prod_{p=0}^2 [b_p, h_p, \tilde{b}_p / \omega_p]^{(-1)^p} = 2\sqrt{2}.$$

For the normalized inner product ι' we obtain from (3.14)

$$\tau'(G) = \tau(G) \prod_{p=0}^2 (p!)^{\frac{1}{2}(-1)^{p+1}(\dim \Omega_p - \dim H_p)} = 2\sqrt{2}2^{-\frac{1}{2}} = 2.$$

Note that the triangle and square digraphs have the same homology groups and are even homotopy equivalent (see [8]) but their torsions are different. Moreover, the torsion is not preserved by covering mappings between digraphs, which are surjective mappings that preserve arrows. For example, consider a mapping $\Phi : X \rightarrow Y$ of a square X on Fig. 3 onto a line digraph $Y = \{0 \rightarrow 1 \rightarrow 2\}$ such that $\Phi(0) = 0$, $\Phi(1) = \Phi(2) = 1$ and $\Phi(3) = 2$, which is obviously covering but $\tau(X) = 2\sqrt{2}$ while $\tau(Y) = \sqrt{3}$.

Example 3.12. We say that a digraph $G = (V, E)$ is *cyclic* if it is connected (as an undirected graph), every vertex had the degree 2, and there are no

double arrows. For example, the triangle from Example 3.10 and the square from Example 3.11 are cyclic.

Here we assume that G is neither triangle nor square. Some examples of such digraphs are shown on Fig. 4.

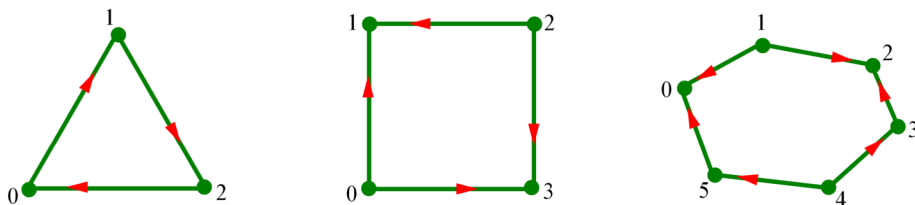


Figure 4: Three cyclic digraphs with 3, 4 and 6 vertices.

Note that a triangular digraph on Fig. 4 is not a triangle in the sense of Example 3.10 because of different orientation of the arrows, and the quadrilateral digraph here is not a square for the same reason.

For a cyclic digraph that is neither triangle nor square, it is known that $\Omega_p(G) = \{0\}$ and $H_p(G) = \{0\}$ for all $p \geq 2$, whereas

$$\dim H_0(G) = \dim H_1(G) = 1$$

and, hence, $\chi(G) = 0$ (see [7, Sect. 4.5]). Assume that G has m vertices $0, 1, \dots, m - 1$ that we identify with residues mod m . The numeration of vertices can be chosen so that all arrows have the form either $i \rightarrow i + 1$ or $i + 1 \rightarrow i$, for $i = 0, \dots, m - 1$.

Let us use notations $\bar{e}_{i(i+1)}$ from (3.17) and σ_i from (3.18) so that $\bar{e}_{i(i+1)} \in \Omega_1$ and

$$\partial \bar{e}_{i(i+1)} = \sigma_i(e_{i+1} - e_i).$$

Choose the following ι -orthonormal bases in Ω_0 and Ω_1 :

$$\omega_0 = \{e_i : i = 0, \dots, m - 1\}$$

and

$$\omega_1 = \{\bar{e}_{i(i+1)} : i = 0, \dots, m - 1\}.$$

Observe that $Z_0 = \Omega_0$ and

$$Z_1 = \ker \partial|_{\Omega_1} = \text{span} \left\{ \sum_{i=0}^{m-1} \sigma_i \bar{e}_{i(i+1)} \right\}$$

because

$$\partial \left(\sum_i \alpha_i \bar{e}_{i(i+1)} \right) = \sum_i \alpha_i \sigma_i (e_{i+1} - e_i) = \sum_i (\alpha_{i-1} \sigma_{i-1} - \alpha_i \sigma_i) e_i,$$

which vanishes if α_i is proportional to $1/\sigma_i = \sigma_i$. Then $B_0 = \partial\Omega_1$ has dimension $m - 1$ and we choose

$$b_0 = \{ \sigma_i (e_{i+1} - e_i), \quad i = 0, \dots, m - 2 \}$$

and, respectively,

$$\tilde{b}_1 = \{ \bar{e}_{i(i+1)}, \quad i = 0, \dots, m - 2 \}.$$

The orthogonal complement of B_0 in $Z_0 = \Omega_0$ is

$$\mathcal{H}_0 = \text{span}\{e_0 + \dots + e_{m-1}\},$$

so that

$$h_0 = \left\{ \frac{1}{\sqrt{m}} (e_0 + \dots + e_{m-1}) \right\}.$$

Hence, as in (3.19), we obtain

$$[b_0, h_0, \tilde{b}_0 / \omega_0] = \sqrt{m}.$$

Next, we have $B_1 = \partial\Omega_2 = \{0\}$ whence $b_1 = \emptyset$, $\mathcal{H}_1 = Z_1$ and

$$h_1 = \left\{ \frac{1}{\sqrt{m}} \sum_{i=0}^{m-1} \sigma_i \bar{e}_{i(i+1)} \right\}.$$

We see that

$$[b_1, h_1, \tilde{b}_1 / \omega_1] = \left| \det \begin{pmatrix} \frac{1}{\sqrt{m}} \sigma_0 & 1 & & & 0 \\ \frac{1}{\sqrt{m}} \sigma_1 & & 1 & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \\ \vdots & & & & & 1 \\ \frac{1}{\sqrt{m}} \sigma_{m-1} & 0 & & & & 0 \end{pmatrix} \right| = \frac{1}{\sqrt{m}}.$$

It follows that

$$\tau(G) = \prod_{p=0}^1 [b_p, h_p, \tilde{b}_p / \omega_p]^{(-1)^p} = m$$

and also $\tau'(G) = m$.

3.4. Analytic torsion

Let Ω be a chain complex as above and ι be an inner product structure on Ω . It is easy to check from (3.3) that Δ_p is a self-adjoint non-negative definite operator on Ω_p . Hence, its eigenvalues are non-negative reals, denote them by $\{\lambda_i\}_{i=1}^{\dim \Omega_p}$. The zeta function $\zeta_p(s)$ of Δ_p is defined by

$$\zeta_p(s) = \sum_{\lambda_i > 0} \frac{1}{\lambda_i^s}.$$

Definition 3.13. The *analytic torsion* $T(\Omega, \iota)$ of the chain complex Ω with an inner product structure ι is defined by

$$(3.20) \quad \log T(\Omega, \iota) = \frac{1}{2} \sum_{p=0}^N (-1)^p p \zeta'_p(0).$$

The next theorem is one of the main results of this paper.

Theorem 3.14. *We have*

$$\tau(\Omega, \iota) = T(\Omega, \iota).$$

This theorem was proved in [15, Proposition 1.7] for a special case when the homology groups H_p are trivial. We use a modification of the argument of [15] that works with arbitrary homology groups.

Proof. Observe that

$$\zeta'_p(s) = - \sum_{\lambda_i > 0} (\log \lambda_i) \lambda_i^{-s},$$

whence

$$(3.21) \quad \zeta'_p(0) = - \sum_{\lambda_i > 0} (\log \lambda_i) = - \log D_p,$$

where

$$D_p := \prod_{\lambda_i > 0} \lambda_i$$

is the determinant of Δ_p restricted on the direct sum of the eigenspaces with positive eigenvalues. In the view of (3.20) and (3.21), it suffices to prove that

$$(3.22) \quad \log \tau(\Omega, \iota) = - \frac{1}{2} \sum_{p=0}^N (-1)^p p \log D_p.$$

As before, we use notations $B_p = \text{im } \partial_{p+1} = \partial\Omega_{p+1}$ and $Z_p = \ker \partial_p$ so that $H_p = Z_p/B_p$. Since any element of $u \in B_p$ has the form $u = \partial v$ for some $v \in \Omega_{p+1}$, we have

$$(3.23) \quad \Delta_p u = \partial^* \partial \partial v + \partial \partial^* u = \partial(\partial^* u) \in B_p.$$

Hence, B_p is an invariant subspace of Δ_p . Therefore, there exists an orthonormal basis $b_p = \{b_p^i\}$ of B_p that consists of the eigenvectors of Δ_p :

$$\Delta_p b_p^i = \beta_p^i b_p^i,$$

where β_p^i are the corresponding eigenvalues. Since by (3.4) B_p is orthogonal to \mathcal{H}_p and all the eigenvectors of Δ_p with eigenvalue 0 belong to \mathcal{H}_p , we have $\beta_p^i > 0$.

By (3.23) we have $\Delta_p b_p^i = \partial \partial^* b_p^i$, whence

$$(3.24) \quad \partial \partial^* b_p^i = \beta_p^i b_p^i.$$

Set

$$\tilde{b}_p^i := \frac{1}{\beta_{p-1}^i} \partial^* b_{p-1}^i \in \Omega_p.$$

We have by (3.24)

$$\partial \tilde{b}_p^i = \frac{1}{\beta_{p-1}^i} \partial \partial^* b_{p-1}^i = \frac{1}{\beta_{p-1}^i} \cdot \beta_{p-1}^i b_{p-1}^i = b_{p-1}^i$$

so that the sequences $\tilde{b}_p = \{\tilde{b}_p^i\}$ and $b_{p-1} = \{b_{p-1}^i\}$ satisfy the identity (3.7) and, hence, can be used in the definition of R-torsion. Since also

$$\partial^* \tilde{b}_p^i = \frac{1}{\beta_{p-1}^i} \partial^* \partial^* b_{p-1}^i = 0,$$

we obtain

$$\Delta_p \tilde{b}_p^i = \partial^* \partial \tilde{b}_p^i + \partial \partial^* \tilde{b}_p^i = \partial^* b_{p-1}^i + 0 = \beta_{p-1}^i \tilde{b}_p^i.$$

Hence, \tilde{b}_p^i are the eigenvectors of Δ_p with eigenvalues β_{p-1}^i . Moreover, the sequence $\{\tilde{b}_p^i\}$ is orthogonal because by (3.24) for $i \neq j$

$$\langle \tilde{b}_p^i, \tilde{b}_p^j \rangle = \frac{1}{\beta_{p-1}^i \beta_{p-1}^j} \langle \partial^* b_{p-1}^i, \partial^* b_{p-1}^j \rangle$$

$$\begin{aligned}
 &= \frac{1}{\beta_{p-1}^i \beta_{p-1}^j} \langle \partial \partial^* b_{p-1}^i, b_{p-1}^j \rangle \\
 &= \frac{1}{\beta_{p-1}^j} \langle b_{p-1}^i, b_{p-1}^j \rangle \\
 &= 0.
 \end{aligned}$$

In the case $i = j$ we obtain similarly

$$\|\tilde{b}_p^i\|^2 = \frac{1}{\beta_{p-1}^i} \langle b_{p-1}^i, b_{p-1}^i \rangle = \frac{1}{\beta_{p-1}^i}.$$

Note also that the vectors b_p^i and \tilde{b}_p^j are necessarily orthogonal since

$$\langle b_p^i, \tilde{b}_p^j \rangle = \frac{1}{\beta_{p-1}^i} \langle b_p^i, \partial^* b_{p-1}^j \rangle = \frac{1}{\beta_{p-1}^i} \langle \partial b_p^i, b_{p-1}^j \rangle = 0.$$

Let $h_p = \{h_p^i\}$ be an orthonormal basis of \mathcal{H}_p . Then the following sequence

$$(3.25) \quad \omega_p = b_p \cup \{\sqrt{\beta_{p-1}^i} \tilde{b}_p^i\} \cup h_p$$

consists of the eigenvectors of Δ_p and is orthonormal. By construction, this sequence is a basis in Ω_p (see Section 3.2). It follows that all the positive eigenvalues of Δ_p are

$$\{\beta_{p-1}^i\} \cup \{\beta_p^i\},$$

whence

$$D_p = \prod_i \beta_{p-1}^i \prod_i \beta_p^i.$$

Setting

$$L_k := \log \prod_i \beta_k^i$$

we obtain

$$\log D_p = L_{p-1} + L_p.$$

Using that $L_N = 0$, we obtain

$$\begin{aligned}
 L_{p-1} &= (L_{p-1} + L_p) - (L_p + L_{p+1}) + \dots \\
 (3.26) \quad &= \sum_{q=p}^N (-1)^{q-p} (L_{q-1} + L_q) = \sum_{q=p}^N (-1)^{q-p} \log D_q.
 \end{aligned}$$

It follows from (3.25) that

$$[b_p, \tilde{b}_p, h_p / \omega_p] = \prod_i (\beta_{p-1}^i)^{-1/2} = (L_{p-1})^{-1/2}.$$

Using the definition of $\tau(\Omega, \iota)$ and (3.26), we obtain

$$\begin{aligned} \log \tau(\Omega, \iota) &= \sum_{p=0}^N (-1)^p \log [b_p, \tilde{b}_p, h_p / \omega_p] \\ &= -\frac{1}{2} \sum_{p=1}^N (-1)^p L_{p-1} \\ &= -\frac{1}{2} \sum_{p=1}^N (-1)^p \sum_{q=p}^N (-1)^{q-p} \log D_q \\ &= -\frac{1}{2} \sum_{q=1}^N \sum_{p=1}^q (-1)^q \log D_q \\ &= -\frac{1}{2} \sum_{q=1}^N (-1)^q q \log D_q, \end{aligned}$$

which finishes the proof of (3.22). \square

4. Cartesian product of path complexes

4.1. Product of paths

Given two finite sets X, Y , consider their Cartesian product $Z = X \times Y$. Let $z = z_0 z_1 \dots z_r$ be a regular elementary r -path on Z , where $z_k = (x_k, y_k)$ with $x_k \in X$ and $y_k \in Y$.

Definition 4.1. We say that the path z is *step-like* if, for any $k = 1, \dots, r$, either $x_{k-1} = x_k$ or $y_{k-1} = y_k$. In fact, exactly one of these conditions holds as z is regular.

Any step-like path z on Z determines by projection regular elementary paths x on X and y on Y . More precisely, x is obtained from z by taking the sequence of all X -components of the vertices of z and then by collapsing in it any subsequence of repeated vertices to one vertex. The same rule applies to y . By construction, the projections x and y are regular elementary paths

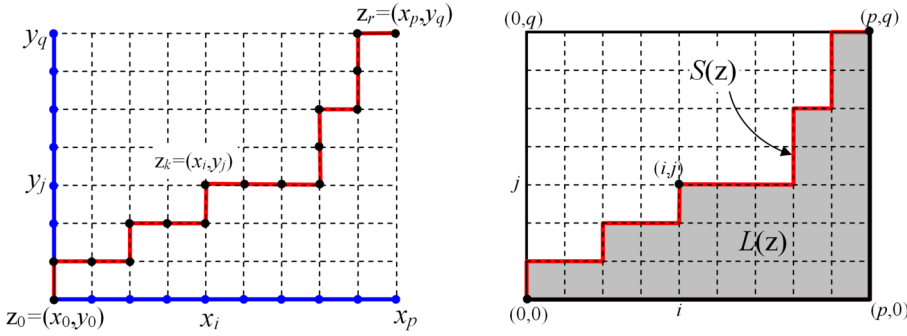


Figure 5: Left: a step-like path z and its projections x and y . Right: a staircase $S(z)$ and its elevation $L(z)$ (here $L(z) = 27$).

on X and Y , respectively. If the projections of $z = z_0 \dots z_r$ are $x = x_0 \dots x_p$ and $y = y_0 \dots y_q$ then $p + q = r$ (cf. Fig. 5(left)).

Every vertex $z_k = (x_i, y_j)$ of a step-like path z can be represented as a point (i, j) of \mathbb{Z}^2 so that the whole path z is represented by a *staircase* $S(z)$ in \mathbb{Z}^2 connecting the points $(0, 0)$ and (p, q) .

Definition 4.2. Define the *elevation* $L(z)$ of the path z as the number of cells in \mathbb{Z}_+^2 below the staircase $S(z)$ (the shaded area on Fig. 5(right)).

For given elementary regular p -path x on X and q -path y on Y , denote by $\Pi_{x,y}$ the set of all step-like paths z on Z whose projections on X and Y are x and y , respectively.

Definition 4.3. For regular elementary paths e_x on X and e_y on Y define their cross product $e_x \times e_y$ as a path on Z as follows:

$$(4.1) \quad e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z.$$

Then extend by linearity the definition of $u \times v$ to all regular paths u on X and v on Y .

Clearly, if $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ then $u \times v \in \mathcal{R}_{p+q}(Z)$. Moreover, the cross product satisfies the product rule with respect to the boundary operator ∂ :

$$(4.2) \quad \partial(u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v)$$

(see [8, Prop. 6.3]).

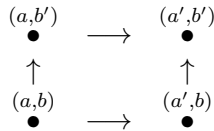
4.2. Product of path complexes and digraphs

Definition 4.4. Given two finite sets X and Y with path complexes $P(X)$ and $P(Y)$ over X and Y , respectively, define a path complex $P(Z)$ over the set $Z = X \times Y$ as follows: the elements of $P(Z)$ are step-like paths on Z whose projections on X and Y belong to $P(X)$ and $P(Y)$, respectively. The path complex $P(Z)$ is called the *Cartesian product* of the path complexes $P(X)$ and $P(Y)$ and is denoted by $P(X) \square P(Y)$.

In short: a path z on Z is allowed if it is step-like and if its projections on X and Y are allowed. In particular, if x and y are elementary allowed paths on X and Y , respectively, then all the paths $z \in \Pi_{x,y}$ are allowed on Z .

Definition 4.5. Let X and Y be digraphs. The Cartesian product $Z = X \square Y$ of the digraphs X and Y is defined as a digraph with the vertices (x, y) where $x \in X$ and $y \in Y$, and arrows $(x, y) \rightarrow (x', y')$ where either $x \rightarrow x'$ and $y = y'$ or $x = x'$ and $y \rightarrow y'$.

For example, if $a \rightarrow a'$ is an arrow in X and $b \rightarrow b'$ is an arrow in Y then they induce the following arrows in Z :



Let $P(X)$ and $P(Y)$ be the path complexes in X and Y , respectively, coming from the digraph structures. It is easy to see that

$$P(X \square Y) = P(X) \square P(Y),$$

that is, the Cartesian product of the path complexes is compatible with the Cartesian product of digraphs. The reader who is interested only in digraphs can always think of X and Y as digraphs and of Z as their Cartesian product.

For a general path complex $P(V)$ over a set V we use the short notations

$$\mathcal{A}_p(P(V)) \equiv \mathcal{A}_p(V) \quad \text{and} \quad \Omega_p(P(V)) \equiv \Omega_p(V).$$

It follows from (4.1)

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \quad \Rightarrow \quad u \times v \in \mathcal{A}_{p+q}(Z).$$

Moreover, (4.2) implies that

$$u \in \Omega_p(X) \text{ and } v \in \Omega_q(Y) \Rightarrow u \times v \in \Omega_{p+q}(Z)$$

(see [8, Prop. 6.5], [10, Prop. 4.6]). Furthermore, the following Künneth formula is true: for any $r \geq 0$,

$$(4.3) \quad \Omega_r(Z) = \bigoplus_{\{p,q \geq 0: p+q=r\}} \Omega_p(X) \otimes \Omega_q(Y),$$

where \otimes denotes the tensor product of linear spaces, and $u \otimes v$ for $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ is identified with the element $u \times v$ of $\Omega_r(Z)$ (see [8, Thm. 6.6] and [11, Thm. 6.6]).

4.3. Operators ∂^* and Δ on products

For the standard inner product ι defined by (3.1) on each of the space $\mathcal{R}(X)$, $\mathcal{R}(Y)$ and $\mathcal{R}(Z)$ the following identity is known: if $u \in \mathcal{A}_p(X)$, $v \in \mathcal{A}_q(Y)$, $\varphi \in \mathcal{A}_{p'}(X)$ and $\psi \in \mathcal{A}_{q'}(Y)$, then

$$\langle u \times v, \varphi \times \psi \rangle_\iota = \binom{p+q}{p} \langle u, \varphi \rangle_\iota \langle v, \psi \rangle_\iota$$

(see [10, Lemma 4.13]). This identity includes also the case when two paths in the inner product have different length – in this case their inner product is zero by definition. Hence, we have

$$\frac{1}{(p+q)!} \langle u \times v, \varphi \times \psi \rangle_\iota = \frac{1}{p!} \langle u, \varphi \rangle_\iota \frac{1}{q!} \langle v, \psi \rangle_\iota.$$

In the case $p' = p$ and $q' = q$ we pass to the normalized inner product ι' given by (3.2) and obtain

$$(4.4) \quad \langle u \times v, \varphi \times \psi \rangle_{\iota'} = \langle u, \varphi \rangle_{\iota'} \langle v, \psi \rangle_{\iota'}.$$

This identity is true also if $p' \neq p$ or $q' \neq q$ as in these cases the both sides vanish.

In the rest of this section we use the normalized inner product

$$\langle , \rangle = \langle , \rangle_{\iota'}$$

unless otherwise specified. In particular, we define the adjoint operator ∂^* and the Hodge Laplacian with respect to the normalized inner product and refer to them as *normalized*.

Lemma 4.6. *Let $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$. Then for the normalized adjoint operator we have*

$$(4.5) \quad \partial^*(u \times v) = \partial^*u \times v + (-1)^p u \times (\partial^*v).$$

Proof. By definition, we have, for any $w \in \Omega_{p+q+1}(Z)$

$$\langle \partial^*(u \times v), w \rangle = \langle u \times v, \partial w \rangle.$$

Any $w \in \Omega_*(Z)$ admits a representation

$$w = \sum_k \varphi_k \times \psi_k$$

where the sum is finite and

$$\varphi_k \in \Omega_{p_k}(X) \quad \text{and} \quad \psi_k \in \Omega_{q_k}(Y)$$

with $p_k + q_k = p + q + 1$ (see [8, Thm. 6.12], [10, Theorem 5.1]).

Then we have

$$\begin{aligned} \langle \partial^*(u \times v), w \rangle &= \left\langle u \times v, \sum \partial(\varphi_k \times \psi_k) \right\rangle \\ &= \left\langle u \times v, \sum (\partial\varphi_k \times \psi_k + (-1)^{p_k} \varphi_k \times \partial\psi_k) \right\rangle \\ &= \sum \langle u \times v, \partial\varphi_k \times \psi_k \rangle + (-1)^{p_k} \langle u \times v, \varphi_k \times \partial\psi_k \rangle \\ &= \sum \langle u, \partial\varphi_k \rangle \langle v, \psi_k \rangle + (-1)^{p_k} \langle u, \varphi_k \rangle \langle v, \partial\psi_k \rangle \\ &= \sum \langle \partial^*u, \varphi_k \rangle \langle v, \psi_k \rangle + (-1)^{p_k} \langle u, \varphi_k \rangle \langle \partial^*v, \psi_k \rangle. \end{aligned}$$

Note that if $p_k \neq p$ then

$$\langle u, \varphi_k \rangle = 0.$$

Hence, we can replace p_k everywhere by p and obtain

$$\begin{aligned} \langle \partial^*(u \times v), w \rangle &= \sum \langle \partial^*u, \varphi_k \rangle \langle v, \psi_k \rangle + (-1)^p \langle u, \varphi_k \rangle \langle \partial^*v, \psi_k \rangle \\ &= \sum \langle \partial^*u \times v, \varphi_k \times \psi_k \rangle + (-1)^p \langle u \times \partial^*v, \varphi_k \times \psi_k \rangle \\ &= \left\langle \partial^*u \times v + (-1)^p u \times \partial^*v, \sum \varphi_k \times \psi_k \right\rangle \\ &= \langle \partial^*u \times v + (-1)^p u \times \partial^*v, w \rangle, \end{aligned}$$

whence

$$\partial^*(u \times v) = \partial^*u \times v + (-1)^p u \times \partial^*v. \quad \square$$

Lemma 4.7. *For the normalized Hodge Laplacian we have*

$$(4.6) \quad \Delta(u \times v) = (\Delta u) \times v + u \times \Delta v.$$

Proof. Let $u \in \Omega_p(X)$ and $v \in \Omega_q(X)$. Then we have

$$\begin{aligned} \partial\partial^*(u \times v) &= \partial(\partial^*u \times v + (-1)^p u \times \partial^*v) \\ &= \partial(\partial^*u \times v) + (-1)^p \partial(u \times \partial^*v) \\ &= \partial\partial^*u \times v + (-1)^{p+1} \partial^*u \times \partial v \\ &\quad + (-1)^p (\partial u \times \partial^*v + (-1)^p u \times \partial\partial^*v) \\ &= \partial\partial^*u \times v + (-1)^{p+1} \partial^*u \times \partial v + (-1)^p \partial u \times \partial^*v + u \times \partial\partial^*v \end{aligned}$$

and

$$\begin{aligned} \partial^*\partial(u \times v) &= \partial^*(\partial u \times v + (-1)^p u \times \partial v) \\ &= \partial^*(\partial u \times v) + (-1)^p \partial^*(u \times \partial v) \\ &= \partial^*\partial u \times v + (-1)^{p-1} \partial u \times \partial^*v \\ &\quad + (-1)^p (\partial^*u \times \partial v + (-1)^p u \times \partial^*\partial v) \\ &= \partial^*\partial u \times v + (-1)^{p-1} \partial u \times \partial^*v + (-1)^p \partial^*u \times \partial v + u \times \partial^*\partial v. \end{aligned}$$

Adding up the two identities and noticing that the terms $\partial^*u \times \partial v$ and $\partial u \times \partial^*v$ cancel out, we obtain

$$\Delta(u \times v) = (\Delta u) \times v + u \times (\Delta v). \quad \square$$

4.4. Torsion of products

Let $P(V)$ be a path complex on a set V with the maximal length N . As before, let ι be the standard inner product structure on $P(V)$ given by (3.1) and ι' be the normalized inner product structure on $P(V)$ given by (3.2). Consider the corresponding standard and normalized torsions:

$$T(V) = T(\Omega(V), \iota) \quad \text{and} \quad T'(V) = T(\Omega(V), \iota').$$

In the same way we will use notation $\tau(V)$ and $\tau'(V)$ for R-torsions with respect to ι and ι' , respectively. Since $T(V) = \tau(V)$ and $T'(V) = \tau'(V)$, the relation between $T(V)$ and $T'(V)$ is given by (3.15) and (3.16).

Although the main object of interest for us is the standard torsion $T(V)$, in this section we make an essential use of $T'(V)$ as it behaves better with respect to the Cartesian product.

We need also the Euler characteristic of $\Omega(V)$:

$$\chi(V) = \chi(\Omega(V)) = \sum_{p=0}^N (-1)^p \dim \Omega_p(V) = \sum_{p=0}^N (-1)^p \dim H_p(V).$$

The next theorem is our main result about torsion on the product of path complexes.

Theorem 4.8. *If $P(Z) = P(X) \square P(Y)$ then*

$$(4.7) \quad \log T'(Z) = \chi(Y) \log T'(X) + \chi(X) \log T'(Y).$$

Before the proof of Theorem 4.8, we need to do some preparations. In the next lemmas we work with an arbitrary chain complex Ω with some inner product structure ι . Let λ be an eigenvalue of the Hodge Laplacian Δ_p on some chain complex Ω . Consider the eigenspace of λ and its subspaces:

$$\begin{aligned} E_p(\lambda) &= \{\varphi \in \Omega_p : \Delta_p \varphi = \lambda \varphi\}, \\ E'_p(\lambda) &= \{\varphi \in E_p(\lambda) : \partial \varphi = 0\}, \\ E''_p(\lambda) &= \{\varphi \in E_p(\lambda) : \partial^* \varphi = 0\}. \end{aligned}$$

In the case $\lambda = 0$ these three spaces are identical by Lemma 3.2. In the case $\lambda \neq 0$ the situation is different.

Lemma 4.9. *Assume that $\lambda \neq 0$. Then we have*

$$(4.8) \quad E'_p(\lambda) = \{\varphi \in \Omega_p : \partial \partial^* \varphi = \lambda \varphi\},$$

$$(4.9) \quad E''_p(\lambda) = \{\varphi \in \Omega_p : \partial^* \partial \varphi = \lambda \varphi\}$$

and

$$(4.10) \quad E_p(\lambda) = E'_p(\lambda) \oplus E''_p(\lambda).$$

Proof. Let us first prove (4.8). If $\varphi \in E'_p(\lambda)$ then

$$\lambda \varphi = \Delta \varphi = \partial^* \partial \varphi + \partial \partial^* \varphi = \partial \partial^* \varphi.$$

Conversely, if $\partial \partial^* \varphi = \lambda \varphi$ then

$$\partial \varphi = \frac{1}{\lambda} \partial(\partial \partial^* \varphi) = 0$$

and, hence, $\Delta\varphi = \partial\partial^*\varphi = \lambda\varphi$ so that $\varphi \in E'_p(\lambda)$. In the same way one proves (4.9).

In order to verify (4.10), observe first that the space $E'_p(\lambda)$ and $E''_p(\lambda)$ are orthogonal because for any $\varphi \in E'_p(\lambda)$ and $\psi \in E''_p(\lambda)$ we have

$$\langle \varphi, \psi \rangle = \frac{1}{\lambda^2} \langle \partial\partial^*\varphi, \partial^*\partial\psi \rangle = \frac{1}{\lambda^2} \langle \partial\partial\partial^*\varphi, \partial\psi \rangle = 0.$$

For any $\varphi \in E_p(\lambda)$ we have

$$(\partial\partial^*)^2\varphi = \partial\partial^*(\partial\partial^*\varphi + \partial^*\partial\varphi) = \partial\partial^*\Delta\varphi = \lambda\partial\partial^*\varphi,$$

which implies by (4.8) that $\partial\partial^*\varphi \in E'_p$. Similarly, we have $\partial^*\partial\varphi \in E''_p$. Finally, for any $\varphi \in E_p(\lambda)$ we have

$$\varphi = \frac{1}{\lambda}\Delta\varphi = \frac{1}{\lambda}\partial\partial^*\varphi + \frac{1}{\lambda}\partial^*\partial\varphi,$$

whence (4.10) follows. □

Lemma 4.10. *The operator $\lambda^{-1/2}\partial$ is an isometry of $E''_p(\lambda)$ onto $E'_{p-1}(\lambda)$ with the inverse $\lambda^{-1/2}\partial^*$.*

Proof. Let $\varphi \in E''_p(\lambda)$ so that $\partial^*\varphi = 0$ and $\partial^*\partial\varphi = \lambda\varphi$. For $\psi = \partial\varphi$ we have

$$\partial\partial^*\psi = \partial\partial^*\partial\varphi = \lambda\partial\varphi = \lambda\psi$$

whence $\psi \in E'_{p-1}(\lambda)$. Hence, ∂ maps $E''_p(\lambda)$ into $E'_{p-1}(\lambda)$. Let us verify that $\lambda^{-1/2}\partial$ is an isometry. For $\varphi \in E''_p(\lambda)$ and $\psi = \lambda^{-1/2}\partial\varphi$ we have

$$\langle \psi, \psi \rangle = \frac{1}{\lambda} \langle \partial\varphi, \partial\varphi \rangle = \frac{1}{\lambda} \langle \partial^*\partial\varphi, \varphi \rangle = \langle \varphi, \varphi \rangle.$$

It remains to show that the mapping $\lambda^{-1/2}\partial$ is onto and has the inverse $\lambda^{-1/2}\partial^*$. For any $\psi \in E'_{p-1}(\lambda)$ we have $\partial^*(\partial^*\psi) = 0$ and

$$\Delta_p(\partial^*\psi) = (\partial^*\partial + \partial\partial^*)\partial^*\psi = \partial^*\partial\partial^*\psi = \partial^*(\lambda\psi) = \lambda\partial^*\psi,$$

which implies $\partial^*\psi \in E''_p(\lambda)$. Since by (4.8) $\partial\partial^*\psi = \lambda\psi$, we obtain

$$\lambda^{-1/2}\partial(\lambda^{-1/2}\partial^*\psi) = \frac{1}{\lambda}\partial\partial^*\psi = \psi.$$

and we conclude that $\lambda^{-1/2}\partial$ and $\lambda^{-1/2}\partial^*$ are mutually inverse. □

Let $n_p(\lambda), n'_p(\lambda), n''_p(\lambda)$ be the dimensions of spaces $E_p(\lambda), E'_p(\lambda), E''_p(\lambda)$, respectively. It follows from Lemmas 4.9 and 4.10 that

$$\begin{aligned} n'_{p-1}(\lambda) &= n''_p(\lambda), \\ n_p(\lambda) &= n'_p(\lambda) + n''_p(\lambda). \end{aligned}$$

As it follows from the definition of the Euler characteristic $\chi(\Omega)$ and $H_p \cong \mathcal{H}_p$, we have

$$\chi(\Omega) = \sum_{p=0}^N (-1)^p n_p(0).$$

Lemma 4.11. *If $\lambda > 0$ then*

$$\sum_{p=0}^N (-1)^p n_p(\lambda) = 0.$$

Proof. We have

$$\begin{aligned} \sum_{p=0}^N (-1)^p n_p(\lambda) &= \sum_{p=0}^N (-1)^p n'_p(\lambda) + \sum_{p=0}^N (-1)^p n''_p(\lambda) \\ &= \sum_{p=0}^N (-1)^p n'_p(\lambda) + \sum_{p=1}^N (-1)^p n'_{p-1}(\lambda) + n''_0(\lambda) \\ &= (-1)^N n'_N(\lambda) + n''_0(\lambda) \\ &= 0. \end{aligned}$$

Here $n'_N(\lambda) = 0$ because for every vector $\varphi \in E'_N(\lambda)$ we have $\partial^* \varphi = 0$ and, hence,

$$\varphi = \frac{1}{\lambda} \partial \partial^* \varphi = 0,$$

and $n''_0(\lambda) = 0$ because for any $\varphi \in E''_0(\lambda)$ we have $\partial \varphi = 0$ and, hence,

$$\varphi = \frac{1}{\lambda} \partial^* \partial \varphi = 0. \quad \square$$

Now we can prove Theorem 4.8. The idea of proof is borrowed from [15, Thm. 2.5].

Proof of Theorem 4.8. The zeta function $\zeta_{p,X}(s)$ of Δ_p on X can be represented in the form

$$\zeta_{p,X}(s) = \sum_{\lambda>0} \lambda^{-s} n_p(\lambda, X),$$

where the sum is taken over all distinct positive eigenvalues λ of Δ_p and $n_p(\lambda, X)$ is the multiplicity of λ . Similar formulas hold for $\zeta_{q,Y}(s)$ and $\zeta_{r,Z}(s)$.

Let $u \in \Omega_p(X)$ be an eigenvector of Δ_p with eigenvalue λ and $v \in \Omega_q(Y)$ be an eigenvector of Δ_q with eigenvalue μ . It follows from (4.6) that $u \times v \in \Omega_{p+q}(Z)$ is an eigenvector of Δ_{p+q} with eigenvalue $\lambda + \mu$. If $\{u_i\}$ is an orthonormal basis in $\Omega_p(X)$ consisting of the eigenvectors of Δ_p and $\{v_j\}$ is an orthonormal basis in $\Omega_q(Y)$ consisting of the eigenvectors of Δ_q then the sequence $\{u_i \times v_j\}$ is orthonormal by (4.4) and, hence, forms a basis in $\Omega_p(X) \otimes \Omega_q(Y)$.

Let us fix $r \geq 0$ and recall that by the Künneth formula (4.3) $\Omega_r(Z)$ is a direct sum of the spaces $\Omega_p(X) \otimes \Omega_q(Y)$ over all pairs $p, q \geq 0$ with $p + q = r$. Hence, collecting all the bases in $\Omega_p(X) \otimes \Omega_q(Y)$ of the form $\{u_i \times v_j\}$ we obtain an orthonormal basis in $\Omega_r(Z)$. This basis consists of the eigenvectors of Δ_r in $\Omega_r(Z)$. Hence, all the eigenvalues of Δ_r in $\Omega_r(Z)$ have the form $\lambda + \mu$ where λ is an eigenvalue of Δ_p in $\Omega_p(X)$, μ is an eigenvalue of Δ_q in $\Omega_q(Y)$, and the multiplicity of $\lambda + \mu$ is $n_p(\lambda, X)n_q(\mu, Y)$.

Hence, we obtain

$$(4.11) \quad \zeta_{r,Z}(s) = \sum_{\lambda+\mu>0} \sum_{p+q=r} (\lambda + \mu)^{-s} n_p(\lambda, X) n_q(\mu, Y).$$

It follows that

$$\begin{aligned} & \sum_{r \geq 0} (-1)^r r \zeta_{r,Z}(s) \\ &= \sum_{\lambda+\mu>0} (\lambda + \mu)^{-s} \sum_{p \geq 0} \sum_{q \geq 0} (-1)^{p+q} (p + q) n_p(\lambda, X) n_q(\mu, Y) \\ (4.12) \quad &= \sum_{\lambda+\mu>0} (\lambda + \mu)^{-s} \left(\sum_{p \geq 0} (-1)^p p n_p(\lambda, X) \right) \left(\sum_{q \geq 0} (-1)^q n_q(\mu, Y) \right) \\ (4.13) \quad &+ \sum_{\lambda+\mu>0} (\lambda + \mu)^{-s} \left(\sum_{p \geq 0} (-1)^p n_p(\lambda, X) \right) \left(\sum_{q \geq 0} (-1)^q q n_q(\mu, Y) \right). \end{aligned}$$

By Lemma 4.11, if $\lambda > 0$ then

$$\sum_{p \geq 0} (-1)^p n_p(\lambda, X) = 0$$

and if $\mu > 0$ then

$$\sum_{q \geq 0} (-1)^q n_q(\mu, Y) = 0.$$

Hence, in (4.12)–(4.13) all the terms with $\lambda > 0$ and $\mu > 0$ vanish, and we obtain

$$\begin{aligned} \sum_{r \geq 0} (-1)^r r \zeta_{r,Z}(s) &= \sum_{\lambda > 0, \mu = 0} \lambda^{-s} \left(\sum_{p \geq 0} (-1)^p p n_p(\lambda, X) \right) \left(\sum_{q \geq 0} (-1)^q n_q(0, Y) \right) \\ &\quad + \sum_{\mu > 0, \lambda = 0} \mu^{-s} \left(\sum_{p \geq 0} (-1)^p n_p(0, X) \right) \left(\sum_{q \geq 0} (-1)^q q n_q(\mu, Y) \right) \\ &= \chi(Y) \sum_{p \geq 0} (-1)^p p \zeta_{p,X}(s) + \chi(X) \sum_{q \geq 0} (-1)^q q \zeta_{q,Y}(s). \end{aligned}$$

Taking derivative of the both sides at $s = 0$ and using the definition of analytic torsion, we obtain

$$(4.14) \quad \log T'(Z) = \chi(Y) \log T'(X) + \chi(X) \log T'(Y). \quad \square$$

The Künneth formula (4.3) implies that

$$\chi(Z) = \chi(X)\chi(Y),$$

which will be used in the next statement.

For any $n \geq 2$ define on the set $X^{\square n} = \underbrace{X \times \cdots \times X}_{n \text{ times}}$ the following path complex

$$P(X^{\square n}) = \underbrace{P(X) \square \cdots \square P(X)}_{n \text{ times}} = P(X)^{\square n}.$$

Corollary 4.12. *We have*

$$(4.15) \quad \log T'(X^{\square n}) = n\chi(X)^{n-1} \log T'(X).$$

Proof. Denote $\log T'(X^{\square n}) = x_n$ and $\chi(X) = a$. Then $\chi(X^n) = a^n$, and we have by (4.14)

$$x_{n+1} = ax_n + a^n x_1.$$

For $n = 1$ (4.15) is trivial. Assuming the induction hypothesis $x_n = na^{n-1}x_1$, we obtain

$$x_{n+1} = na^n x_1 + a^n x_1 = (n + 1)a^n x_1,$$

which finishes the proof by induction. □

Example 4.13. Let G be a cyclic digraph with m vertices from Example 3.12. For $n \geq 2$ the product $G^{\square n}$ can be regarded as an analogue of a torus. Since $\chi(G) = 0$, we obtain from (4.15) that, for any $n \geq 2$,

$$T'(G^{\square n}) = 1.$$

Recall for comparison that $T'(G) = T(G) = m$.

Before we can compute $T(G^{\square n})$, let us verify that

$$(4.16) \quad \dim \Omega_p(G^{\square n}) = \binom{n}{p} m^n.$$

Indeed, for $n = 1$ this is true because

$$\dim \Omega_0(G) = \dim \Omega_1(G) = m.$$

Assuming that (4.16) is true for some n , we obtain by the Künneth formula (4.3)

$$\begin{aligned} \dim \Omega_r(G^{\square(n+1)}) &= \sum_{p+q=r} \dim \Omega_p(G^{\square n}) \dim \Omega_q(G) \\ &= m \dim \Omega_r(G^{\square n}) + m \dim \Omega_{r-1}(G^{\square n}) \\ &= m \binom{n}{r} m^n + m \binom{n}{r-1} m^n \\ &= m^{n+1} \binom{n+1}{r}. \end{aligned}$$

In the same way, using that

$$\dim H_0(G) = \dim H_1(G) = 1,$$

we obtain that

$$\dim H_p(G^{\square n}) = \binom{n}{p}.$$

Hence, by (3.15) we obtain

$$\begin{aligned} T(G^{\square n}) &= T'(G^{\square n}) \prod_{p=0}^n (p!)^{\frac{1}{2}(-1)^p(\dim \Omega_p - \dim H_p)} \\ &= \prod_{p=2}^n (p!)^{\frac{1}{2}(-1)^p \binom{n}{p} (m^n - 1)}. \end{aligned}$$

In particular, we have $T(G^{\square 2}) = 2^{\frac{1}{2}(m^2-1)}$.

Example 4.14. For the interval $I = {}^0\bullet \rightarrow \bullet^1$, we have by Example 3.9 $T'(I) = \sqrt{2}$ and $\chi(I) = 1$. Consider the n -dimensional digraph *cube* $I^{\square n}$. In the case $n = 2$ it coincides with the square from Example 3.11, in the case $n = 3$ this digraph is shown on Fig. 6.

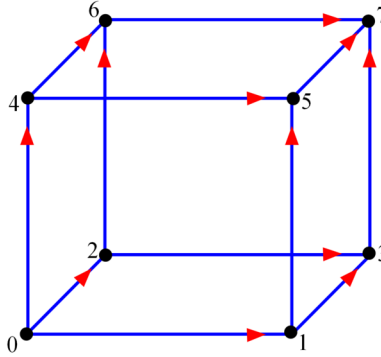


Figure 6: The cube $I^{\square 3}$.

By (4.15) we obtain

$$\log T'(I^{\square n}) = n \log \sqrt{2}$$

whence

$$T'(I^{\square n}) = 2^{n/2}.$$

Let us compute the torsion $T(I^{\square n})$ of the cube with respect to the standard inner product ι . For that let us first verify that

$$(4.17) \quad \dim \Omega_p(I^{\square n}) = 2^{n-p} \binom{n}{p}.$$

We have

$$\dim \Omega_0(I) = 2, \quad \dim \Omega_1(I) = 1 \quad \text{and} \quad \dim \Omega_p(I) = 0 \quad \text{for } p \geq 2$$

so that (4.17) holds for $n = 1$. For the inductive step from n to $n + 1$, observe that $I^{\square(n+1)} = I^{\square n} \square I$. By the Künneth formula (4.3) we have

$$\begin{aligned} \dim \Omega_r(I^{\square(n+1)}) &= \sum_{p+q=r} \dim \Omega_p(I^{\square n}) \dim \Omega_q(I) \\ &= 2 \dim \Omega_r(I^{\square n}) + \dim \Omega_{r-1}(I^{\square n}) \\ &= 2^{n+1-r} \binom{n}{r} + 2^{n-r+1} \binom{n}{r-1} \end{aligned}$$

$$= 2^{n+1-r} \binom{n+1}{r},$$

which finishes the proof of (4.17). In the same way one obtains that

$$\dim H_0(I^{\square n}) = 1 \text{ and } \dim H_p(I^{\square n}) = 0 \text{ for all } p \geq 1.$$

Hence, by (3.15) we obtain

$$\begin{aligned} T(I^{\square n}) &= T'(I^{\square n}) \prod_{p=0}^n (p!)^{\frac{1}{2}(-1)^p(\dim \Omega_p - \dim H_p)} \\ &= 2^{n/2} \prod_{p=2}^n (p!)^{\frac{1}{2}(-1)^p 2^{n-p} \binom{n}{p}}. \end{aligned}$$

For example, we have

$$T(I^{\square 2}) = 2\sqrt{2}, \quad T(I^{\square 3}) = \frac{16}{3}\sqrt{3}, \quad T(I^{\square 4}) = \frac{2048}{81}\sqrt{6}, \quad \text{etc.}$$

Corollary 4.15. *If $P(Z) = P(X) \square P(Y)$ then*

$$\begin{aligned} (4.18) \quad \log T(Z) &= \chi(Y) \log T(X) + \chi(X) \log T(Y) \\ &\quad + \frac{1}{2} \sum_{p,q \geq 1} (-1)^{p+q} \log \binom{p+q}{p} [\dim \Omega_p(X) \dim \Omega_q(Y) \\ &\quad - \dim H_p(X) \dim H_q(Y)]. \end{aligned}$$

Proof. We use the R-torsions τ and τ' defined with respect to the inner products ι and ι' , respectively. By Theorems 3.14 and 4.8 we have

$$(4.19) \quad \log \tau'(Z) = \chi(Y) \log \tau'(X) + \chi(X) \log \tau'(Y).$$

By (3.15) we have

$$\begin{aligned} \log \tau'(X) &= \log \tau(X) - \frac{1}{2} \sum_p (-1)^p \dim \Omega_p(X) \log(p!) \\ &\quad + \frac{1}{2} \sum_p (-1)^p \dim H_p(X) \log(p!), \end{aligned}$$

where summation is taken over all $p \geq 0$. Similar identities hold for Y and Z .

Substituting into (4.19), we obtain

(4.20)

$$\begin{aligned} & \log \tau(Z) - \chi(Y) \log \tau(X) - \chi(X) \log \tau(Y) \\ &= \frac{1}{2} \sum_r (-1)^r \dim \Omega_r(Z) \log(r!) - \frac{1}{2} \sum_r (-1)^r \dim H_r(Z) \log(r!) \\ & \quad - \frac{1}{2} \chi(Y) \sum_p (-1)^p \dim \Omega_p(X) \log(p!) + \frac{1}{2} \chi(Y) \sum_p (-1)^p \dim H_p(X) \log(p!) \\ & \quad - \frac{1}{2} \chi(X) \sum_q (-1)^q \dim \Omega_q(Y) \log(q!) + \frac{1}{2} \chi(X) \sum_q (-1)^q \dim H_q(Y) \log(q!) \end{aligned}$$

Denote for simplicity

$$x_p = \dim \Omega_p(X), \quad y_q = \dim \Omega_q(Y), \quad z_r = \dim \Omega_r(Z).$$

By the Künneth formula we have

$$z_r = \sum_{p+q=r} x_p y_q.$$

It follows that

$$\begin{aligned} & \sum_r (-1)^r z_r \log(r!) - \chi(Y) \sum_p (-1)^p x_p \log(p!) - \chi(X) \sum_q (-1)^q y_q \log(q!) \\ &= \sum_r (-1)^r \sum_{p+q=r} x_p y_q \log(r!) - \sum_q (-1)^q y_q \sum_p (-1)^p x_p \log(p!) \\ & \quad - \sum_p (-1)^p x_p \sum_{q \geq 0} (-1)^q y_q \log(q!) \\ &= \sum_{p,q} (-1)^{p+q} x_p y_q \log((p+q)!) - \sum_{p,q} (-1)^{p+q} x_p y_q \log(p!) \\ & \quad - \sum_{p,q} (-1)^{p+q} x_p y_q \log(q!) \\ &= \sum_{p,q} (-1)^{p+q} x_p y_q \log \binom{p+q}{p}. \end{aligned}$$

Note that the summation here can be restricted to $p, q \geq 1$ since otherwise $\log \binom{p+q}{p} = 0$. A similar formula takes place for $\dim H_p$ instead of $\dim \Omega_p$. Substituting into (4.20) we obtain (4.18). \square

Example 4.16. Let us compute the torsions of the digraph $Z = I \square Y$ where I is the interval from Example 4.14 and Y is the triangle from Example 3.10 (see Fig. 7).

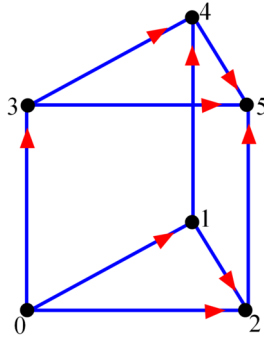


Figure 7: A prism digraph $I \square Y$.

By Examples 3.9 and 3.10, we have $\chi(I) = \chi(Y) = 1$ and

$$T'(I) = \sqrt{2}, \quad T'(Y) = \sqrt{3/2}.$$

Hence, we obtain by (4.7)

$$T'(Z) = T'(I)^{\chi(Y)} T'(Y)^{\chi(I)} = \sqrt{2} \sqrt{3/2} = \sqrt{3}.$$

Since $H_p(I)$ and $H_p(Y)$ are non-trivial only for $p = 0$, we obtain by (4.18)

$$\begin{aligned} \log T(Z) &= \chi(Y) \log T(I) + \chi(I) \log T(Y) \\ &\quad + \frac{1}{2} \sum_{p=1}^1 \sum_{q=1}^2 (-1)^{p+q} \log \binom{p+q}{p} \dim \Omega_p(I) \dim \Omega_q(Y) \\ &= \log \sqrt{2} + \log \sqrt{3} + \frac{1}{2} \log \binom{2}{1} \cdot 1 \cdot 3 - \frac{1}{2} \log \binom{3}{1} \cdot 1 \cdot 1 \\ &= \log(\sqrt{2} \sqrt{3} 2^{3/2} 3^{-1/2}) = \log 4, \end{aligned}$$

so that

$$T(Z) = 4.$$

5. Join of path complexes

5.1. Augmented chain complex

Let P be a path complex over a set V as in Section 2.1. In that section we have constructed a chain complex $\mathcal{R} = \{\mathcal{R}_p\}_{p \geq 0}$ with the boundary operator ∂ , and the space \mathcal{R}_{-1} was defined as $\{0\}$. In this section we change the definition of \mathcal{R}_{-1} as follows. For any elementary 0-paths e_i redefine ∂ by

$$\partial e_i = e$$

where e is an empty path that by definition has the length -1 . Set $\mathcal{R}_{-1} = \text{span}_{\mathbb{R}}\{e\} \cong \mathbb{R}$ and consider the *augmented* chain complex $\tilde{\mathcal{R}} = \{\mathcal{R}_p\}_{p \geq -1}$ where the operator ∂ still satisfies $\partial^2 = 0$. Consequently, we obtain also the *augmented* chain complex $\tilde{\Omega} = \{\Omega_p\}_{p \geq -1}$ of ∂ -invariant paths, where Ω_p with $p \geq 0$ is as before and $\Omega_{-1} = \mathcal{R}_{-1}$, as well as the *reduced* homology groups $\{\tilde{H}_p\}_{p \geq -1}$ where $\tilde{H}_{-1} = \{0\}$, $\tilde{H}_p = H_p$ for $p \geq 1$ and $H_0 \cong \tilde{H}_0 \oplus \mathbb{R}$. The reduced Euler characteristic is

$$(5.1) \quad \tilde{\chi}(P) = \sum_{p \geq -1} (-1)^p \dim \Omega_p = \sum_{p \geq -1} (-1)^p \dim \tilde{H}_p = \chi(P) - 1.$$

Denote by $\langle \cdot, \cdot \rangle$ the standard inner product in \mathcal{R}_p defined by (3.1) in the case $p \geq 0$ and by $\langle e, e \rangle = 1$ for $p = -1$. If $u \in \mathcal{R}_p$ and $v \in \mathcal{R}_q$ with $p \neq q$ then set $\langle u, v \rangle = 0$. As before, we denote by ι the standard inner product structure in \mathcal{R} .

5.2. Join of path complexes and digraphs

Let V be a finite set.

Definition 5.1. For any paths $u \in \mathcal{R}_p(V)$ and $v \in \mathcal{R}_q(V)$ with $p, q \geq -1$ define their *join* $u \cdot v \in \mathcal{R}_{p+q+1}(V)$ as follows: first set

$$e_{i_0 \dots i_p} \cdot e_{j_0 \dots j_q} = e_{i_0 \dots i_p j_0 \dots j_q}$$

and then extend this definition by linearity.

In particular, we have $e_{i_0 \dots i_p} \cdot e = e_{i_0 \dots i_p}$. The following product formula holds for the chain complex $\tilde{\mathcal{R}}(V)$:

$$(5.2) \quad \partial(u \cdot v) = (\partial u) \cdot v + (-1)^{p+1} u \cdot (\partial v),$$

(see [11, Lemma 2.2] and [10, Lemma 2.4]).

Let X, Y be two finite disjoint sets, set $Z = X \sqcup Y$. Then all paths on X and Y can be considered as paths on Z . It follows easily from definition of $u \cdot v$ that

$$u \in \mathcal{R}_p(X) \text{ and } v \in \mathcal{R}_q(Y) \Rightarrow u \cdot v \in \mathcal{R}_{p+q+1}(Z).$$

Also, for the standard inner product \langle, \rangle given by (3.1), we have

$$(5.3) \quad \langle u \cdot v, \varphi \cdot \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle,$$

for all $u \in \mathcal{R}_p(X)$, $v \in \mathcal{R}_q(Y)$, $\varphi \in \mathcal{R}_{p'}(X)$ and $\psi \in \mathcal{R}_{q'}(Y)$ (see also [10, Lemma 3.10]).

Let us extend the property (2.4) of the definition of a path complex P also to n -paths with $n = 0$, that is, we allow in (2.4) also $n = 0$. Then necessarily the empty path e belongs to P .

Definition 5.2. Let $P(X)$ and $P(Y)$ be two path complexes over finite disjoint sets X and Y , respectively. Define the chain complex $P(Z)$ over the set $Z = X \sqcup Y$ as follows: $P(Z)$ consists of all the paths of the form $u \cdot v$ where $u \in P(X)$ and $v \in P(Y)$. The path complex $P(Z)$ is called the *join* of $P(X), P(Y)$ and is denoted by $P(Z) = P(X) * P(Y)$.

Clearly, $P(X)$ and $P(Y)$ are subsets of $P(Z)$.

Definition 5.3. If X and Y are digraphs then define their *join* as a digraph $Z = X * Y$ where the set of vertices is $X \sqcup Y$ and the set of arrows consists of all arrows of X , all arrows of Y as well as of all arrows of the form $x \rightarrow y$ where $x \in X$ and $y \in Y$.

It is easy to see that

$$P(X * Y) = P(X) * P(Y)$$

so that the operation of join of digraphs is compatible with join of path complexes (cf. [11]).

It is clear from the definition of $P(Z)$ that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \Rightarrow u \cdot v \in \mathcal{A}_{p+q+1}(Z).$$

It follows from (5.2) that, for all $p, q \geq -1$,

$$u \in \Omega_p(X) \text{ and } v \in \Omega_q(Y) \Rightarrow u \cdot v \in \Omega_{p+q+1}(Z)$$

(see [11, Prop 5.4]). Furthermore, the following version of the Künneth formula is true for join: for any $r \geq -1$

$$(5.4) \quad \Omega_r(Z) = \bigoplus_{\{p,q \geq -1:p+q+1=r\}} \Omega_p(X) \otimes \Omega_q(Y),$$

where $u \otimes v$ for $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ is identified with the element $u \cdot v$ of $\Omega_r(Z)$ (see [11, Thm. 5.5]).

As a consequence of (5.4) we obtain that

$$(5.5) \quad \tilde{\chi}(Z) = -\tilde{\chi}(X)\tilde{\chi}(Y),$$

where the minus comes from the additional 1 in $r = p + q + 1$.

5.3. Operators ∂^* and Δ on joins

We always assume in what follows that all the spaces \mathcal{R}_p under consideration are endowed with the standard inner product (3.1), in particular, (5.3) is satisfied.

Lemma 5.4. *Let $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$. Then*

$$(5.6) \quad \partial^*(u \cdot v) = (\partial^*u) \cdot v + (-1)^{p+1}u \cdot (\partial^*v).$$

Proof. By definition, we have, for any $w \in \Omega_{p+q+2}(Z)$

$$\langle \partial^*(u \cdot v), w \rangle = \langle u \cdot v, \partial w \rangle.$$

Any $w \in \Omega_*(Z)$ admits a representation

$$w = \sum_k \varphi_k \cdot \psi_k,$$

where the sum is finite and

$$\varphi_k \in \Omega_{p_k}(X) \quad \text{and} \quad \psi_k \in \Omega_{q_k}(Y)$$

with $p_k + q_k + 1 = p + q + 2$ (see [10, Thm. 5.1] and [11, Thm. 5.15]). Then we have using (5.3)

$$\begin{aligned} \langle \partial^*(u \cdot v), w \rangle &= \left\langle u \cdot v, \sum \partial(\varphi_k \cdot \psi_k) \right\rangle \\ &= \left\langle u \cdot v, \sum (\partial\varphi_k \cdot \psi_k + (-1)^{p_k+1}\varphi_k \cdot \partial\psi_k) \right\rangle \end{aligned}$$

$$\begin{aligned}
 &= \sum \langle u \cdot v, \partial \varphi_k \cdot \psi_k \rangle + (-1)^{p_k+1} \langle u \cdot v, \varphi_k \cdot \partial \psi_k \rangle \\
 &= \sum \langle u, \partial \varphi_k \rangle \langle v, \psi_k \rangle + (-1)^{p_k+1} \langle u, \varphi_k \rangle \langle v, \partial \psi_k \rangle \\
 &= \sum \langle \partial^* u, \varphi_k \rangle \langle v, \psi_k \rangle + (-1)^{p_k+1} \langle u, \varphi_k \rangle \langle \partial^* v, \psi_k \rangle.
 \end{aligned}$$

Note that if $p_k \neq p$ then $\langle u, \varphi_k \rangle = 0$. Hence, we can replace p_k everywhere by p and obtain

$$\begin{aligned}
 \langle \partial^*(u \cdot v), w \rangle &= \sum \langle \partial^* u, \varphi_k \rangle \langle v, \psi_k \rangle + (-1)^{p+1} \langle u, \varphi_k \rangle \langle \partial^* v, \psi_k \rangle \\
 &= \sum \langle \partial^* u \cdot v, \varphi_k \cdot \psi_k \rangle + \langle (-1)^{p+1} u \cdot \partial^* v, \varphi_k \cdot \psi_k \rangle \\
 &= \left\langle \partial^* u \cdot v + (-1)^{p+1} u \cdot \partial^* v, \sum \varphi_k \cdot \psi_k \right\rangle \\
 &= \langle \partial^* u \cdot v + (-1)^{p+1} u \cdot \partial^* v, w \rangle,
 \end{aligned}$$

whence (5.6) follows. □

For the *Hodge Laplacian*

$$\Delta u = \partial \partial^* u + \partial^* \partial u$$

we have then the following identity.

Lemma 5.5. *For all $u \in \Omega_p(X)$ and $v \in \Omega_q(X)$ we have*

$$(5.7) \quad \Delta(u \cdot v) = (\Delta u) \cdot v + u \cdot \Delta v.$$

Proof. Indeed, by (5.6) we have

$$\begin{aligned}
 \partial \partial^*(u \cdot v) &= \partial(\partial^* u \cdot v + (-1)^{p+1} u \cdot \partial^* v) \\
 &= \partial(\partial^* u \cdot v) + (-1)^{p+1} \partial(u \cdot \partial^* v) \\
 &= \partial \partial^* u \cdot v + (-1)^{p+2} \partial^* u \cdot \partial v \\
 &\quad + (-1)^{p+1} (\partial u \cdot \partial^* v + (-1)^{p+1} u \cdot \partial \partial^* v) \\
 &= \partial \partial^* u \cdot v + (-1)^p \partial^* u \cdot \partial v + (-1)^{p+1} \partial u \cdot \partial^* v + u \cdot \partial \partial^* v
 \end{aligned}$$

and by (5.2)

$$\begin{aligned}
 \partial^* \partial(u \cdot v) &= \partial^*(\partial u \cdot v + (-1)^{p+1} u \cdot \partial v) \\
 &= \partial^*(\partial u \cdot v) + (-1)^{p+1} \partial^*(u \cdot \partial v) \\
 &= \partial^* \partial u \cdot v + (-1)^p \partial u \cdot \partial^* v \\
 &\quad + (-1)^{p+1} (\partial^* u \cdot \partial v + (-1)^{p+1} u \cdot \partial^* \partial v)
 \end{aligned}$$

$$= \partial^* \partial u \cdot v + (-1)^p \partial u \cdot \partial^* v + (-1)^{p+1} \partial^* u \cdot \partial v + u \cdot \partial^* \partial v.$$

Adding up the two identities, we see that the terms $\partial^* u \cdot \partial v$ and $\partial u \cdot \partial^* v$ cancel out, and we obtain (5.7). □

5.4. Torsion of joins

Let P be a path complex over a set V with the standard inner product structure ι given by (3.1). By means of the augmented chain complex $\tilde{\Omega}(P)$, let us define the *reduced* analytic torsion $\tilde{T}(P)$ by

$$\log \tilde{T}(P) = \log T(\tilde{\Omega}(P), \iota) = \frac{1}{2} \sum_{p=-1}^N (-1)^p p \zeta'_p(0).$$

In the previous sections we used the standard analytic torsion $T(P)$ given by

$$\log T(P) = \log T(\Omega(P), \iota) = \frac{1}{2} \sum_{p=0}^N (-1)^p p \zeta'_p(0).$$

The relation between $T(P)$ and $\tilde{T}(P)$ is given by the following formula.

Lemma 5.6. *We have*

$$(5.8) \quad T(P) = \sqrt{|V|} \tilde{T}(P).$$

Proof. The zeta function $\zeta_p(s)$ is determined by the operator Δ_p that is the same for the chain complexes Ω and $\tilde{\Omega}$ for all $p \geq 1$. For $p = 0$ the operators Δ_p are different for these two complexes, but the value $p = 0$ does not give any contribution to the analytic torsions. Hence, the difference is determined by $p = -1$, that is,

$$(5.9) \quad \log \tilde{T}(P) = \log T(P) + \frac{1}{2} \zeta'_{-1}(0).$$

For $e \in \Omega_{-1}$ we have $\partial e = 0$ and

$$\partial^* e = \sum_{i \in V} e_i,$$

because for any i

$$\langle \partial^* e, e_i \rangle = \langle e, \partial e_i \rangle = \langle e, e \rangle = 1.$$

Hence,

$$\Delta e = \partial \partial^* e + \partial^* \partial e = \partial \sum_{i \in V} e_i = |V|e.$$

Therefore, $\zeta_{-1}(s) = |V|^{-s}$ and $\zeta'_{-1}(0) = -\log |V|$. Substituting into (5.9) we obtain

$$(5.10) \quad \log \tilde{T}(P) = \log T(P) - \frac{1}{2} \log |V|,$$

which is equivalent to (5.8). □

The next theorem is our main result about torsion on joins.

Theorem 5.7. *For the join path complex $P(Z) = P(X) * P(Y)$ we have*

$$(5.11) \quad \log \tilde{T}(Z) = -\tilde{\chi}(Y) \log \tilde{T}(X) - \tilde{\chi}(X) \log \tilde{T}(Y),$$

where $\tilde{\chi}$ is the reduced Euler characteristic.

Proof of Theorem 5.7. The proof is similar to that of Theorem 4.8. Let $E_p(\lambda)$ be the eigenspace of Δ_p with the eigenvalue λ , and set

$$n_p(\lambda) = \dim E_p(\lambda).$$

Lemmas 4.9 and 4.10 go unchanged also for the augmented chain complex $\tilde{\Omega} = \{\Omega_p\}_{p \geq -1}$. The same argument as in the proof of Lemma 4.11 gives for any $\lambda > 0$ that

$$(5.12) \quad \sum_{p=-1}^N (-1)^p n_p(\lambda) = 0,$$

because $n''_{-1}(\lambda) = 0$: indeed, for any $\varphi \in E''_{-1}(\lambda)$ we have $\partial \varphi = 0$ and, hence, $\varphi = \frac{1}{\lambda} \partial^* \partial \varphi = 0$.

Arguing as in the proof of Theorem 4.8 and using the Künneth formula (5.4) we obtain in place of (4.11) the following identity:

$$\zeta_{r,Z}(s) = \sum_{\lambda + \mu > 0} \sum_{p+q+1=r} (\lambda + \mu)^{-s} n_p(\lambda, X) n_q(\mu, Y)$$

for any $r \geq -1$. It follows that

$$\sum_{r \geq -1} (-1)^r r \zeta_{r,Z}(s)$$

$$\begin{aligned}
 &= \sum_{\lambda+\mu>0} (\lambda + \mu)^{-s} \sum_{p \geq -1} \sum_{q \geq -1} (-1)^{p+q+1} (p + q + 1) n_p(\lambda, X) n_q(\mu, Y) \\
 (5.13) \quad &= - \sum_{\lambda+\mu>0} (\lambda + \mu)^{-s} \sum_{p \geq -1} (-1)^p p n_p(\lambda, X) \sum_{q \geq -1} (-1)^q n_q(\mu, Y) \\
 (5.14) \quad &- \sum_{\lambda+\mu>0} (\lambda + \mu)^{-s} \sum_{p \geq -1} (-1)^p n_p(\lambda, X) \sum_{q \geq -1} (-1)^q q n_q(\mu, Y) \\
 (5.15) \quad &- \sum_{\lambda+\mu>0} (\lambda + \mu)^{-s} \sum_{p \geq -1} \sum_{q \geq -1} (-1)^{p+q} n_p(\lambda, X) n_q(\mu, Y).
 \end{aligned}$$

By (5.12), if $\lambda > 0$ then

$$\sum_{p \geq -1} (-1)^p n_p(\lambda, X) = 0$$

and if $\mu > 0$ then

$$\sum_{q \geq -1} (-1)^q n_q(\mu, Y) = 0.$$

Hence, the double sum in (5.15) is equal to zero, while in (5.13)–(5.14) all the terms with $\lambda > 0$ and $\mu > 0$ vanish. We obtain

$$\begin{aligned}
 \sum_{r \geq -1} (-1)^r r \zeta_{r,Z}(s) &= - \sum_{\lambda>0, \mu=0} \lambda^{-s} \sum_{p \geq -1} (-1)^p p n_p(\lambda, X) \sum_{q \geq -1} (-1)^q n_q(0, Y) \\
 &- \sum_{\mu>0, \lambda=0} \mu^{-s} \sum_{p \geq -1} (-1)^p n_p(0, X) \sum_{q \geq -1} (-1)^q q n_q(\mu, Y) \\
 &= -\tilde{\chi}(Y) \sum_{p \geq -1} (-1)^p p \zeta_{p,X}(s) - \tilde{\chi}(X) \sum_{q \geq -1} (-1)^q q \zeta_{q,Y}(s).
 \end{aligned}$$

Taking derivative of the both sides at $s = 0$ and using the definition of analytic torsion, we obtain (5.11). □

For the standard analytic torsion T we obtain the following.

Corollary 5.8. *We have*

$$\begin{aligned}
 \log T(Z) &= -\tilde{\chi}(Y) \log T(X) - \tilde{\chi}(X) \log T(Y) \\
 (5.16) \quad &+ \frac{1}{2} \log |Z| + \frac{\tilde{\chi}(Y)}{2} \log |X| + \frac{\tilde{\chi}(X)}{2} \log |Y|.
 \end{aligned}$$

Proof. Using (5.8) and (5.1), we obtain

$$\log T(Z) = \frac{1}{2} \log |Z| + \log \tilde{T}(Z)$$

$$\begin{aligned}
 &= \frac{1}{2} \log |Z| - (\tilde{\chi}(Y) \log \tilde{T}(X) + \tilde{\chi}(X) \log \tilde{T}(Y)) \\
 &= \frac{1}{2} \log |Z| - \tilde{\chi}(Y) \left(\log T(X) - \frac{1}{2} \log |X| \right) \\
 &\quad - \tilde{\chi}(X) \left(\log T(Y) - \frac{1}{2} \log |Y| \right) \\
 &= \frac{1}{2} \log |Z| + \tilde{\chi}(Y) \frac{1}{2} \log |X| + \tilde{\chi}(X) \frac{1}{2} \log |Y| \\
 &\quad - \tilde{\chi}(Y) \log T(X) - \tilde{\chi}(X) \log T(Y). \quad \square
 \end{aligned}$$

For any $n \geq 2$ define on the set $X^{*n} = \underbrace{X \sqcup \dots \sqcup X}_{n \text{ times}}$ the following path complex

$$P(X^{*n}) = \underbrace{P(X) * \dots * P(X)}_{n \text{ times}} = P(X)^{*n}.$$

Corollary 5.9. *For $Y = X^{*n}$ we have*

$$(5.17) \quad \log \tilde{T}(Y) = n(-\tilde{\chi}(X))^{n-1} \log \tilde{T}(X)$$

and

$$(5.18) \quad \log T(Y) = n(1 - \chi(X))^{n-1} \log T(X) - \frac{1}{2} n(1 - \chi(X))^{n-1} \log |X| + \frac{1}{2} \log(n|X|).$$

Proof. Denote $\log \tilde{T}(X^{*n}) = x_n$ and $-\tilde{\chi}(X) = a$. Then $-\tilde{\chi}(X^{*n}) = a^n$, and we have by (5.11)

$$x_{n+1} = ax_n + a^n x_1.$$

By induction we obtain $x_n = na^{n-1}x_1$, which proves (5.17).

Using (5.8) (or (5.10)) and (5.1), we obtain from (5.17)

$$\begin{aligned}
 \log T(X^{*n}) &= \log \tilde{T}(X^{*n}) + \frac{1}{2} \log |X^{*n}| \\
 &= n(1 - \chi(X))^{n-1} \left(\log T(X) - \frac{1}{2} \log |X| \right) + \frac{1}{2} \log(n|X|),
 \end{aligned}$$

whence (5.18) follows. □

For example, if $\chi(X) = 0$ then (5.18) yields

$$(5.19) \quad \log T(X^{*n}) = n \log T(X) - \frac{n-1}{2} \log |X| + \frac{1}{2} \log n,$$

if $\chi(X) = 1$ then (5.18) yields

$$(5.20) \quad T(X^{*n}) = \sqrt{n|X|},$$

and if $\chi(X) = 2$ then

$$(5.21) \quad \log T(X^{*n}) = n(-1)^{n-1} \log T(X) + \frac{1 + n(-1)^n}{2} \log |X| + \frac{1}{2} \log n.$$

Example 5.10. Let $O = \{\bullet\}$ be a trivial digraph of one vertex. It is easy to see that $\chi(O) = 1$. The join $O * O = O^{*2}$ is the interval I from Example 3.9, the join $O * I = O^{*3}$ is the triangle from Example 3.10. More generally, O^{*n} can be regarded as an $(n - 1)$ -dimensional digraph *simplex* (see Fig. 8).

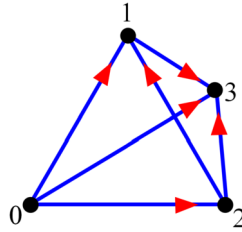


Figure 8: Simplex $O^{*4} = I^{*2}$.

From (5.20) we obtain that

$$T(O^{*n}) = \sqrt{n}.$$

For example, $T(O^{*3}) = \sqrt{3}$ as we have seen in Example 3.10.

Example 5.11. Consider the digraph $D = \{\bullet, \bullet\}$ consisting of two disjoint vertices. The join D^{*2} is a quadrilateral, and D^{*3} is an octahedron (see Fig. 9). The digraph D^{*n} can be regarded as a digraph analogue of an $(n - 1)$ -dimensional sphere.

We have $\chi(D) = 2$ and $\tau(D) = T(D) = 1$. By (5.21) we obtain that

$$\log T((D^{*n})) = \frac{1 + n(-1)^n}{2} \log 2 + \frac{1}{2} \log n,$$

that is

$$T(D^{*n}) = \sqrt{n} 2^{\frac{1+n(-1)^n}{2}}.$$

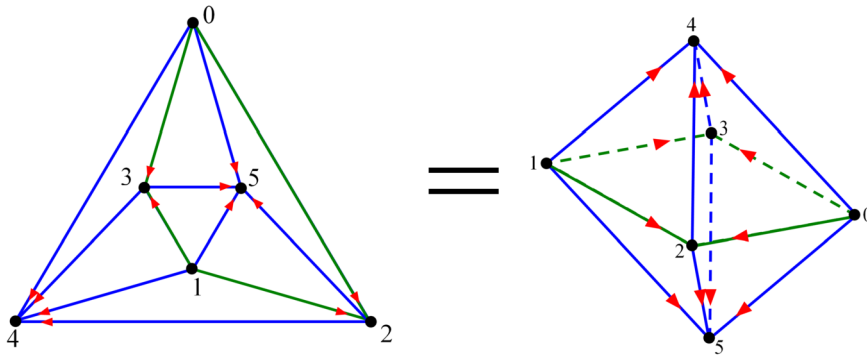


Figure 9: The octahedron D^{*3} is shown in two ways. The green subgraph is D^{*2} .

For example,

$$T(D^{*2}) = \sqrt{2}2^{\frac{3}{2}} = 4$$

and

$$T(D^{*3}) = \sqrt{3}2^{-1} = \frac{\sqrt{3}}{2}.$$

Example 5.12. Let G be a cyclic digraph from Example 3.12 with m vertices. Since $\chi(G) = 0$ and $T(G) = m$, we obtain by (5.19)

$$\log T(G^{*n}) = n \log m - \frac{n-1}{2} \log m + \frac{1}{2} \log n$$

and

$$T(G^{*n}) = \sqrt{nm}^{\frac{n+1}{2}}.$$

Acknowledgements

We would like to thank Wang Chong for the computation of torsions of digraphs in some examples.

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