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Deformations of Fano manifolds with weighted solitons

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Abstract: We consider weighted solitons on Fano manifolds which include Kähler-Ricci solitons, Mabuchi solitons and base metrics inducing Calabi-Yau cone metrics outside the zero sections of the canonical line bundles (Sasaki-Einstein metrics on the associated $U(1)$-bundles). In this paper, we give a condition for a weighted soliton on a Fano manifold M_0 to extend to weighted solitons on small deformations M_t of the Fano manifold M_0. More precisely, we show that all the members M_t of the Kuranishi family of a Fano manifold M_0 with a weighted soliton have weighted solitons if and only if the dimensions of T-equivariant automorphism groups of M_t are equal to that of M_0, and also if and only if the T-equivariant automorphism groups of M_t are all isomorphic to that of M_0, where the weight functions are defined on the moment polytope of the Hamiltonian T-action. This generalizes a result of Cao-Sun-Yau-Zhang for Kähler-Einstein metrics.

Keywords: Deformations of complex structures, Kähler manifolds.

1. Introduction

Let M be a Fano manifold, i.e. a compact complex manifold with positive first Chern class, of complex dimension m. We regard $2\pi c_1(M)$ as a Kähler class. The Kähler form ω is expressed as

$$\omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}$$

and the Kähler metric $g_{i\bar{j}}$ is often identified with the Kähler form ω. Let T be a real compact torus in the automorphism group $\mathrm{Aut}(M)$, and assume that ω is T-invariant. Since M is Fano and simply connected the T-action is Hamiltonian with respect to ω. Since the T-action naturally lifts to the anti-canonical line bundle K_M^{-1} we have a canonically normalized moment map $\mu_\omega : M \to \mathfrak{t}^*$ where $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{t}^*$ its dual space, cf. Appendix in [19]. Let $\Delta := \mu_\omega(M)$ be the moment polytope. Then Δ

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is independent of $\omega \in 2\pi c_1(M)$. Let v be a positive smooth function on Δ. Regarding μ as coordinates on Δ using the action angle coordinates, we may sometimes write $v(\mu)$ instead of v. The pull-back $\mu_\omega^* v$ is a smooth function on M, and for this we write $v(\mu_\omega) = \mu_\omega^* v = v \circ \mu_\omega$.

We say that a Kähler metric ω in $2\pi c_1(M)$ a *weighted v-soliton* or simply *v-soliton* if

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\overline{\partial} \log v(\mu_\omega)$$

where $\mathrm{Ric}(\omega) = -i\partial\overline{\partial}\log\omega^m$ is the Ricci form. We also call ω simply a *weighted soliton* when it is v-soliton for some v, or when v is obvious from the context. Examples of weighted solitons are a Kähler-Ricci soliton when $v(\mu) = e^{\langle \mu, \xi\rangle}$ for some $\xi \in \mathfrak{t}$, a Mabuchi solitons when $v(\mu) = \langle \mu, \xi\rangle + a$ for some positive constant a, and a basic metric which which induces Calabi-Yau cone metrics outside the zero sections of the canonical line bundle and hence a Sasaki-Einstein metrics on the $U(1)$-bundle of K_M^{-1} when $v(\mu) = (\langle \mu, \xi\rangle + a)^{-m-2}$, see [23, 30, 1, 2, 22, 34].

In this paper we consider the Kuranishi family $\varpi : \mathfrak{M} \to B$ of deformations of a Fano manifold M which is a complex analytic family of Fano manifolds where B is an open set in $\mathbf{C}^n$ containing the origin 0 and we write $M_t := \varpi^{-1}(t)$ and require $M_0 = M$, cf. [25, 26, 27, 38, 41, 18]. Note that there is no obstruction for Fano manifolds since

$$H^2(M_0, \Theta) \cong H^{m-2}(M_0, \Omega^1(K_{M_0})) = 0$$

by Serre duality and Kodaira-Nakano vanishing. For a given Kähler form $\omega \in 2\pi c_1(M)$ let $f \in C^\infty(M)$ satisfy

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\overline{\partial} f.$$

The Kuranishi family we consider in this paper is described by a family of vector valued 1-forms parametrized by $t \in B$

$$\varphi(t) = \sum_{i=1}^k t^i \varphi_i + \sum_{|I|\geq 2} t^I \varphi_I \ \in \ A^{0,1}(T'M)$$

such that

(1)
$$\begin{cases} \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]; \\ \overline{\partial}_f^*\varphi(t) = 0; \\ \varphi_1, \dots, \varphi_k \\ \quad \text{form a basis of the space of all } T'M\text{-valued } \Delta_f\text{-harmonic } (0,1)\text{-forms} \end{cases}$$

where $\Delta_f = \overline{\partial}_f^*\overline{\partial} + \overline{\partial}\,\overline{\partial}_f^*$ is the weighted Hodge Laplacian with $\overline{\partial}_f^*$ the formal adjoint of $\overline{\partial}$ with respect to the weighted L^2-inner product $\int_M (\cdot,\cdot)e^f\omega^m$. See [18] for more detail about this Kuranishi family. We showed in [18] that the Kähler form ω on $M_0 = M$ remains to be a Kähler form on M_t. The main result of this paper is stated as follows.

Theorem 1.1. *Suppose that M_0 has a weighted v-soliton. Consider the Kuranishi family (1) with $f = \log v(\mu_\omega)$. Then, shrinking B if necessary, the following statements are equivalent.*

(1) M_t has a weighted v-soliton for all $t \in B$.
(2) T is included in $\mathrm{Aut}(M_t)$, and for the centralizer $\mathrm{Aut}^T(M_t)$ of T in $\mathrm{Aut}(M_t)$, $\dim \mathrm{Aut}^T(M_t) = \dim \mathrm{Aut}^T(M_0)$ for all $t \in B$.
(3) T is included in $\mathrm{Aut}(M_t)$, and the identity component $\mathrm{Aut}_0^T(M_t)$ of $\mathrm{Aut}^T(M_t)$ is isomorphic to $\mathrm{Aut}_0^T(M_0)$ for all $t \in B$.

Although there are extensive studies on the existence of Kähler-Einstein metrics on Fano manifolds, e.g. [4], currently, there are not many existence results on weighted solitons on Fano manifolds. Because of this lack of examples of weighted solitons, it is not easy to find non-trivial applications of Theorem 1.1.

As for the deformations of complex structures of polarized manifolds with weighted cscK metrics, there is a result by Hallam [20] which states that a small deformation of a polarized manifold with a weighted cscK metric has a weighted cscK metric if and only if it is weighted K-polystable with respect to smooth T-equivariant test configurations. The "only if" part follows from a result of Apostolov-Jubert-Lahdili [2]. The result of Hallam extends the results of Brönnle [7] and Szekelyhidi [40] for cscK metrics.

The outline of the proof of Theorem 1.1 is as follows. The proof of Theorem 1.1 above is largely parallel to that of Theorem 1.1 of Cao-Sun-Yau-Zhang [9]. Just as the notion of Kähler-Einstein metrics are generalized to constant scalar curvature Kähler (cscK for short) metrics and further to extremal Kähler metrics [8], the notion of weighted solitons are generalized to weighted cscK metrics and further to weighted extremal metrics [30, 23, 24]. In Section 2, we extend a result of Rollin-Simanca-Tipler [39] for extremal Kähler metrics to show that a weighted extremal metric on M_0 can be extended to weighted extremal metrics on small deformations M_t if the maximal torus in the reduced automorphism group acts on B trivially. In Section 3 we review how weighted solitons are regarded as weighted cscK metrics. In Section 4 we finish the proof of Theorem 1.1. We first review results in [18] about the Kähler forms and their Ricci potentials for the Kuranishi family.

Next, we show a lemma which implies that the action of the maximal torus on B is trivial so that we can apply the result obtained in Section 2. We then show using the formula of the Ricci potential obtained in [18] that the weighted extremal metrics obtained in Section 2 are in fact weighted cscK metrics which are in this case weighted solitons, proving (2) implies (1). That (1) implies (3) is proved using the K-polystability characterization obtained by the works of Han-Li [22], Li [34], Blum-Liu-Xu-Zhuang [6] and closely following the arguments of [9]. That (3) implies (2) is trivial.

2. Weighted scalar curvature

In this section we review the weighted scalar curvature, which is also called the (v, w)-scalar curvature, introduced by Lahdili [30], see also Inoue [23, 24] for a similar idea.

Let M be a compact Kähler manifold and Ω its Kähler class. Recall that the Lie algebra of $\mathrm{Aut}(M)$ is the Lie algebra $\mathfrak{h}(M)$ of all holomorphic vector fields. We denote by $\mathrm{Aut}_r(M) \subset \mathrm{Aut}(M)$ the reduced automorphism group, i.e. the Lie algebra $\mathfrak{h}_r(M)$ of $\mathrm{Aut}_r(M)$ consists of holomorphic vector fields with non-empty zeros. They are obtained in the form $\mathrm{grad}' u$, i.e. the $(1, 0)$-part of the gradient vector field, of some complex valued smooth functions u, see e.g. [32].

Let T be a compact real torus in $\mathrm{Aut}_r(M)$. As in the Introduction $\mathrm{Aut}_r^T(M)$ denotes the centralizer of T in $\mathrm{Aut}_r(M)$, i.e. the subgroup consisting of T-equivariant automorphisms. In the Fano case $\mathrm{Aut}_r^T(M) = \mathrm{Aut}^T(M)$. Let $\omega \in \Omega$ be a T-invariant Kähler form. Then T acts on (M, ω) in the Hamiltonian way. Let $\mu_\omega : M \to \mathfrak{t}^*$ be the moment map where $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{t}^*$ its dual space. Then $\Delta := \mu_\omega(M)$ is a compact convex polytope. This is independent of $\omega \in \Omega$ up to translation, but the ambiguity of translation is fixed by giving a normalization of μ_ω which specifies the average by the integration. Let v be a positive smooth function on Δ.

As in Section 1, we also write $v = v(\mu)$ as a function on Δ by considering μ to constitute the action-angle coordinates, and also write $v(\mu_\omega) := \mu_\omega^* v$ as a positive smooth function on M. We define *v-scalar curvature* $S_v(\omega)$ of a T-invariant Kähler form ω by

$$S_v(\omega) := v(\mu_\omega)S(\omega) + 2\Delta_\omega v(\mu_\omega) + \langle g_\omega, \mu_\omega^* Hess(v)\rangle$$

where $S(\omega)$ denotes the Kähler geometers' scalar curvature

$$S(\omega) = -g^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial z^{\bar{j}}} \log\det(g_{l\bar{\ell}})$$

of ω, $\Delta_\omega = \overline{\partial}^*\overline{\partial}$ the Hodge $\overline{\partial}$-Laplacian on functions, and

$$\langle g_\omega, \mu_\omega^* Hess(v)\rangle = g^{i\overline{j}} v_{\alpha\beta}\mu_i^\alpha \mu_{\overline{j}}^\beta$$

is the trace of the pull-back by μ_ω of the Hessian $Hess(v)$ of v on $\mathfrak{t}^*$ in which we express the moment map $\mu_\omega : M \to \mathfrak{t}^*$ as $\mu_\omega(p) = (\mu^1(p), \dots, \mu^\ell(p))$ with $d\mu^\alpha = i(X^\alpha)\omega$ for a basis $X^1, \dots, X^\ell$ of $\mathfrak{t}$. Thus, our S_v is half of that in [30].

Let w be another positive smooth function on Δ. We define (v, w)-*scalar curvature* $S_{v,w}$ by

$$S_{v,w} = \frac{S_v}{w(\mu_\omega)}.$$

The notion of $S_{v,w}$-scalar curvature was originally introduced as a generalization of conformally Kähler, Einstein-Maxwell metrics after extensive studies such as [31, 3, 16, 17, 28, 29]. Later it turned out that the (v, w)-cscK metrics include much more unexpected examples as mentioned in Section 1 (hence if g is a (v, w)-extremal metric then $\mathrm{grad}' S_{v,w} \subset \widetilde{\mathfrak{t}}$). But we do not assume this for the moment; see also Section 3.

We call g a *weighted extremal metric* or (v, w)-*extremal metric* if

$$\mathrm{grad}' S_{v,w} = g^{i\overline{j}} \frac{\partial S_{v,w}}{\partial z^{\overline{j}}} \frac{\partial}{\partial z^i}$$

is a holomorphic vector field.

Remark 2.1. *In Section 3.2 of [30], Lahdili also defined (v, w)-extremal Kähler metrics. But his definition is slightly different from ours in that the extremal vector field belongs to $\mathfrak{t}$ in his case but does not in our case.*

Define $L_v\varphi$ for complex valued smooth functions φ by

$$L_v\varphi = \nabla^i\nabla^j(v(\mu_\omega)\nabla_i\nabla_j\varphi),$$

and call L_v the v-twisted Lichnerowicz operator. Obviously, L_v is self-adjoint elliptic operator;

$$\int_M (L_v\varphi)\,\overline{\psi}\,\omega^m = \int_M \varphi\,\overline{L_v\psi}\,\omega^m$$

where $m = \dim M$. $L := L_1$ is the standard Lichnerowicz operator. The kernel of L_v consists of complex valued smooth functions u such that $\mathrm{grad}' u$ is a holomorphic vector fields, an thus $\mathrm{Ker}\, L_v = \mathfrak{h}_r(M) = Lie(\mathrm{Aut}_r(M))$. We also define $L_{v,w}$ by

$$L_{v,w} = \frac{1}{w(\mu_\omega)}\, L_v.$$

Consider the one parameter family of metrics $g_{ti\bar{j}} = g_{i\bar{j}} + t\varphi_{i\bar{j}}$. By straightforward computations one can show

$$\frac{d}{dt}\Big|_{t=0} S_v(g_t) = -L_v\varphi + S_v^i\,\varphi_i, \tag{2}$$

$$\frac{d}{dt}\Big|_{t=0} S_{v,w}(g_t) = -L_{v,w}\varphi + S_{v,w}^i\,\varphi_i. \tag{3}$$

It is also straightforward using (3) to show the following Proposition 2.2.

Proposition 2.2. *A critical point of the weighted Calabi functional*

$$g \mapsto \int_M S_{v,w}^2(g)\, w(\mu_\omega)\,\omega^m$$

is a weighted extremal metric.

As in [13] we can define the following invariants.

Proposition 2.3. *Let* $h_X \in \operatorname{Ker} L_v$ *be the real Killing potential of* $X \in \mathfrak{t}$*, i.e.* $i\,\mathrm{grad}'\,h_X = X'$*. Then* Fut_v *and* $\mathrm{Fut}_{v,w}$ *defined by*

$$\mathrm{Fut}_v(X) = \int_M (S_v - c_v)\,h_X\,\omega^m, \tag{4}$$

and

$$\mathrm{Fut}_{v,w}(X) = \int_M (S_{v,w} - c_{v,w})\,h_X\,w(\mu_\omega)\,\omega^m \tag{5}$$

are independent of choice of $\omega \in \Omega$ *where* $c_{v,w} = \int_M S_v\,\omega^m / \int_M w(\mu_\omega)\,\omega^m$ *and* $c_v = c_{v,1}$ *which are independent of* $\omega \in \Omega$*.*

Proof. If $h_X(\omega)$ is the real Killing potential as in the statement of the proposition for $\omega \in \Omega$ then, under the normalization $\int_M h_X\,\omega^m = 0$, we have $h_X(\omega_t) = h_X + th_X^i\varphi_i$ where ω_t is the Kähler form of g_t which was defined two lines above the equation (2). Hence

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \int_M S_v(\omega_t)\,h_X(\omega_t)\,\omega_t^m \\
&\quad = \int_M \left(\left(-L_v\varphi + S_v^i\varphi_i\right)h_X + S_v h_X^i\varphi_i + S_v h_X\Delta\varphi\right)\omega^m \\
&\quad = -\int_M \varphi L_v h_X \omega^m = 0,
\end{aligned}$$

and

$$\frac{d}{dt}\Big|_{t=0} \int_M h_X(\omega_t)\,\omega_t^m = 0.$$

Thus Fut_v is independent of $\omega \in \Omega$. Note however that the expression of (4) does not depend on the normalization of h_X. By a similar computation one can show that $\mathrm{Fut}_{v,w}$ is independent of $\omega \in \Omega$. □

Remark 2.4. *In [30] Lahdili shows for smooth test configurations, the slope of the weighted Mabuchi functional is the weighted Donaldson-Futaki invariant. Applying this to the case of product test configurations also yields Proposition 2.2.*

Remark 2.5. *If g is a (v, w)-extremal metric with non-constant $S_{v,w}$ then*

$$\mathrm{Fut}_{v,w}(J\,\mathrm{grad}\,S_{v,w}) = \int_M (S_{v,w} - c_{v,w})^2\, w(\mu_\omega)\,\omega^m > 0.$$

Remark 2.6. *A decomposition theorem similar to that proved by Calabi [8] for extremal Kähler metrics holds for weighted extremal Kähler metrics, see Theorem B.1 in [30], also [17, 28]. A consequence of this is that, if g is a weighted extremal metric, then the centralizer of $\mathrm{grad}'\,S_{v,w}$ in $\mathfrak{h}_r^T(M)$ is the complexification of the real Lie algebra of all T-equivariant Killing vector fields with non-empty zeros. In particular, if g has constant (v, w)-scalar curvature, then the identity component of $\mathrm{Aut}_r^T(M)$ is the complexification of the identity component of $\mathrm{Isom}_r^T(M)$ consisting of isometries with non-empty fixed point set.*

Let (M, g) be a compact Kähler manifold of complex dimension m. Let $\widetilde{T}$ be the maximal torus in $\mathrm{Aut}_r(M)$ including T. Let $L_k^2(M)$ be the k-th Sobolev space with respect to the metric g with weight $w(\mu_\omega)$. Here the weight $w(\mu_\omega)$ means that L^2-inner product is given by

$$(\phi, \psi) = \int_M \phi\psi\, w(\mu_\omega)\,\omega^m.$$

We take k sufficiently large so that $L_k^2(M)$-functions form an algebra. Let $L_{k,\widetilde{T}}^2(M)$ be the subspace of $L_k^2(M)$ consisting of $\widetilde{T}$-invariant functions.

Let $\mathcal{H}_g$ be the space of Killing potentials corresponding to $\widetilde{\mathfrak{t}}$. Then the functions in $\mathcal{H}_g$ are purely imaginary. As in Proposition 2.3 we call a function in $i\mathcal{H}_g$ a real Killing potential. Let $W_{k,g}$ be the orthogonal complement of $i\mathcal{H}_g$ in $L^2_{k,\widetilde{T}}$:

$$L^2_{k,\widetilde{T}} = i\mathcal{H}_g \oplus W_{k,g}$$

with L^2-orthogonal projections

$$\pi_g^H : L^2_{k,\widetilde{T}} \to i\mathcal{H}_g, \qquad \pi_g^W : L^2_{k,\widetilde{T}} \to W_{k,g}.$$

Then $\pi_g^H(S_{v,w}(g))$ is a smooth function independent of the choice of k, and the gradient vector field $\mathrm{grad}'\,\pi_g^H(S_{v,w}(g))$ is independent of the choice of g in the same Kähler class, see Theorem 3.3.3 in [14], also [15], and also Theorem 1.5 in [35] for conformally Kähler Einstein-Maxwell metrics. The proof in the weighted (v,w)-case is identical to [35]. This vector field is called the *extremal Kähler vector field*. If g is a weighted extremal metric then

$$\pi_g^H(S_{v,w}(g)) = S_{v,w}(g).$$

Hence g is a weighted extremal metric if and only if

$$\pi_g^W(S_{v,w}(g)) = 0.$$

Definition 2.7. *We call $S^{\mathrm{red}}_{v,w}(g) := \pi_g^W(S_{v,w}(g))$ the reduced (v,w)-scalar curvature, or simply reduced scalar curvature of g. Thus, g is a weighted (v,w)-extremal metric if and only if $S^{\mathrm{red}}_{v,w}(g) = 0$.*

We can then modify (3) as

$$\frac{d}{dt}\bigg|_{t=0} S^{\mathrm{red}}_{v,w}(g_t) = -L_{v,w}\varphi + S^{\mathrm{red}\,i}_{v,w}\,\varphi_i. \tag{6}$$

Let $\varpi : \mathcal{M} \to B$ a complex analytic family of complex deformations with $M_0 = M$ where B is an open set in $\mathbf{C}^k$ containing 0 and we put $M_t := \varpi^{-1}(t)$. We assume $(M_0, g_0) = (M, g)$ is a compact (v,w)-extremal Kähler manifold. By the rigidity theorem of Kodaira-Spencer, M_t is Kähler for all small t, see e.g. [38]. Note that the dimension of the Dolbeault cohomology is only upper semi-continuous on compact *complex* manifolds. But on a compact *Kähler* manifold M with continuously varying integral complex structure J_t, we have the Hodge decomposition

$$\oplus_{p+q=r} H^{p,q}(M, J_t) = H^r(M, \mathbf{C}).$$

Since the dimension of $H^r(M, \mathbf{C})$ is a topological invariant and independent of t then the upper semi-continuity of the dimension of each component of the left hand side implies that the dimension of $H^{p,q}(M, J_t)$ is independent of t.

Let Ω_t be a smooth family of Kähler classes of M_t, i.e. Ω_t gives a smooth section of the vector bundle $\{H^2(M_t)\}_{t\in B}$. Suppose that $\widetilde{T}$ acts holomorphically on $\mathcal{M} \to B$ and trivially on B. Thus $\widetilde{T}$ acts on M_t holomorphically for each $t \in B$. Taking the average over the $\widetilde{T}$-action we have a smooth family g_t of $\widetilde{T}$-invariant Kähler metrics such that the associated Kähler forms ω_t represent Ω_t.

We denote by $L^2_{k,\widetilde{T}}(M)$ the space of $\widetilde{T}$-invariant real valued L^2_k-functions with respect to $g = g_0$ with weight $w(\mu_\omega)$. We shall write the L^2-inner product with weight $w(\mu_\omega)$ by $L^2(w)$. We put

$$H_t(M) = H^{1,1}(M_t, g_t) \cap H^2(M, \mathbf{R}).$$

For $\phi \in L^2_{k+4,\widetilde{T}}(M)$ and $\alpha \in H_t(M)$ we put

$$\omega_{t,\alpha,\phi} = \omega_t + \alpha + i\partial\overline{\partial}\phi$$

which is a $\widetilde{T}$-invariant real closed $(1,1)$-form on (M, J_t) and

$$[\omega_{t,\alpha,\phi}] = \Omega_t + [\alpha].$$

Shrinking B if necessary $H(M) = \cup_{t\in B} H_t(M)$ forms a trivial vector bundle over B. Let $h : B \times H_0(M) \to H(M)$, $(t, \alpha) \mapsto h_t(\alpha)$, be an isomorphism of vector bundles. Note that the Sobolev spaces $L^2_{k,\widetilde{T}}(M)$ is independent of g_t for all small $t \in B$. Thus we consider $L^2_{k,\widetilde{T}}$ as possessing varying norm corresponding to g_t. Let

$$L^2_{k,\widetilde{T}}(M) = i\mathcal{H}_{t,\alpha,\phi} \oplus W_{k,t,\alpha,\phi}$$

be the splitting of $L^2_{k,\widetilde{T}}(M)$ into the space $i\mathcal{H}_{t,\alpha,\phi}$ of real Killing potentials and its orthogonal complement $W_{k,t,\alpha,\phi}$ with respect to $\omega_{t,\alpha,\phi}$. Let $P : L^2_{k,\widetilde{T}}(M) \to W_{k,0}$ be the projection with respect to g_0. Thus $P = \pi^W_{g_0}$ in the previous notation. Consider the map $\Phi : U \to B \times W_{k,0}$ defined on a small open neighborhood U of $(0,0,0)$ in $B \times H_0(M) \times W_{k+4,0}$ to $B \times W_{k,0}$ by

$$\Phi(t, \alpha, \phi) = (t, P(S^{\mathrm{red}}_{v,w}(g_{t,h_t(\alpha),\phi}))).$$

Here $g_{t,\beta,\phi}$ is the Kähler metric corresponding to the Kähler form $\omega_{t,\beta,\phi}$, and $S^{\mathrm{red}}_{v,w}(g_{t,\beta,\phi})$ is the reduced (v,w)-scalar curvature, cf. Definition 2.7.

Proposition 2.8 (cf. [39])**.** *Let g be a (v,w)-extremal metric where v and w are defined on the image of the moment map $\mu_\omega : M \to \mathfrak{t}^*$. Let $\widetilde{T}$ be a maximal torus in* $\mathrm{Isom}(M,g)$ *containing T. Suppose that $\widetilde{T}$ acts holomorphically on $\mathcal{M} \to B$ and trivially on B. Then, by shrinking B to a sufficiently small neighborhood of the origin if necessary, for arbitrary small perturbations Ω_t of the Kähler class $\Omega = \Omega_0$, there are weighted extremal metrics g_t in Ω_t.*

Proof. We consider the map Φ above with $g_{0,0,0}$ a (v,w)-extremal metric, and thus $S^{\mathrm{red}}_{v,w}(g_{0,0,0}) = 0$. Using (6) one can show

$$d\Phi_{(0,0,0)}(1, \dot\alpha, \dot\phi) = \begin{pmatrix} 1 & 0 \\ * & -L_{v,w}\dot\phi + P(dS^{\mathrm{red}}_{v,w}(\dot\alpha)) \end{pmatrix}.$$

If $\psi \in W_{k,0}$ is in the cokernel of $d\Phi_{(0,0,0)}$ then

$$L_{v,w}\psi = 0 \quad \text{and} \quad (P(dS^{\mathrm{red}}_{v,w}(\dot\alpha)), \psi)_{L^2(w)} = 0.$$

But $L_{v,w}\psi = 0$ implies that ψ is a $\widetilde{T}$-invariant real Killing potential. Since $\widetilde{T}$ is a maximal torus we have $\psi \in i\mathcal{H}_0$ and the second condition above is automatically satisfied. Thus $\psi \in i\mathcal{H}_0 \cap W_{k,0} = \{0\}$. By the implicit function theorem the proposition follows. □

Instead of the maximal torus $\widetilde{T}$, one could use a smaller torus T' such that $T \subset T' \subset \widetilde{T}$, and argue as in [39]. Then a non-degeneracy condition considered in [32] is required as in Theorem 1 in [39]. In fact, if we use a smaller torus T' such that $T \subset T' \subset \widetilde{T}$, then we need to take $\mathcal{H}_g$ to be the space of Killing potentials corresponding to $\mathfrak{t}'$, the Lie algebra of T'. Then $L_{v,w}\psi = 0$ implies that ψ is a Killing potential but it does not imply that it belongs to $\mathcal{H}_g$, so ψ needs not be zero. Hence, in order to be able to use the implicit function theorem we need the following condition: "If $(P(dS^{\mathrm{red}}_{v,w}(\dot\alpha)), \psi)_{L^2(w)} = 0$ for any $\dot\alpha \in H_0(M)$ then $\psi = 0$." This is the non-degeneracy condition in [39] and [32] where, in the case of [32], $T = T' = \{1\}$ and P is the identity. See also Lemma 6 in [39].

3. Weighted solitons on Fano manifolds

In this section we consider weighted solitons on Fano manifolds which form a subclass of weighted cscK metrics. Let M be a Fano manifold, and $\omega \in 2\pi c_1(M)$ be a Kähler form.

Definition 3.1. *Let v be a positive smooth function on the image of the moment map of a Hamiltonian T-action. We say that ω is a weighted v-soliton (or simply weighted soliton, also v-soliton) if*

$$\mathrm{Ric}(\omega) - \omega = i\partial\overline{\partial}\log v(\mu_\omega).$$

Examples of v-solitons are

(i) Kähler-Ricci soliton for $v(\mu) = \exp(\langle \mu, \xi\rangle)$ for $\xi \in \mathfrak{t}$ where the linkage with $S_{v,w}$-cscK metrics was first found by Inoue [23, 24],
(ii) Mabuchi soliton for $v(\mu) = \langle \mu, \xi\rangle + a$ a positive affine-linear function [37], and
(iii) base metric which induce Calabi-Yau cone metrics outside the zero sections of the canonical line bundles (Sasaki-Einstein metrics on the associated $U(1)$-bundles) for $v(\mu) = (\ell(\xi))^{-(m+2)}$ where $\ell(\xi) = \langle \mu, \xi\rangle + a$ is a positive affine-linear function (see Proposition 2 in [2]).

A T-invariant Kähler form $\omega \in 2\pi c_1(M)$ is a v-soliton if and only if ω is $S_{v,w} = 1$ metric with

$$w(\mu) = (m + \langle d\log v, \mu\rangle)v(\mu). \tag{7}$$

This can be seen from the formula

$$S_v - w(\mu_\omega) = v(\mu_\omega)\Delta_v(\log v(\mu_\omega) - f) \tag{8}$$

where $f \in C^\infty(M)$ is the Ricci potential of ω, i.e. $S - m = \Delta f$, and $\Delta_v = v^{-1}\circ \overline{\partial}^* \circ v \circ \overline{\partial}$ in which v and v^{-1} denote the multiplications by $v(\mu_\omega)$ and $v(\mu_\omega)^{-1}$. By (8) we have

$$\int_M (S_v - w(\mu_\omega))\omega^m = 0,$$

and thus $c_{v,w} = 1$ and

$$\mathrm{Fut}_{v,w}(X) = \int_M (S_v - w(\mu_\omega))\, h_X\, \omega^m.$$

Using (8) this can be rewritten as

$$\mathrm{Fut}_{v,w}(X) = \int_M (JX)(\log v(\mu_\omega) - f)\, v(\mu_\omega)\, \omega^m. \tag{9}$$

A characterization of the existence of weighted solitons by Ding-poly-stability and K-polystability was described by Li [34], Theorem 1.17 and Theorem 1.21. The story to this result may be summarized as follows. After the resolution of Yau-Tian-Donaldson conjecture by [10, 42, 12, 11] where the Gromov-Hausdorff convergence was used, a variational proof without using Gromov-Hausdorff convergence was given in [5] under the condition of uniform K-stability. Further in [33], the existence was shown under the condition of G-uniform stability. The work of [36] shows that when G contains the maximal torus G-uniform stability is equivalent to K-polystability. Generalizing the result of [33] for Kähler-Einstein metrics, Han-Li [22] proved the existence of weighted solitons under the condition of G-uniform stability for weighted case. In [6] and [34], the equivalence of G-uniform stability when G contains the maximal torus and K-polystability for weighted case was shown.

4. Geometry of Kuranishi family

Let $\varpi : \mathfrak{M} \to B$ be the Kuranishi family of a Fano manifold M satisfying (1) as described in Section 1. Then the v-soliton ω on $M_0 = M$ remains to be Kähler forms on M_t, $t \in B$, by Theorem 1.4 in [18]. Further, it was shown in Theorem 1.5 in [18] that the Ricci form $\mathrm{Ric}(M_t, \omega)$ of (M_t, ω) is given by

$$\mathrm{Ric}(M_t, \omega) = \omega + \partial_t\overline{\partial}_t(f_0 + \log\det(I - \varphi(t)\overline{\varphi(t)})). \tag{10}$$

But since we assume (M_0, ω_0) with $\omega_0 = \omega$ is a v-soliton we have

$$f_0 = \log v(\mu_{\omega_0}). \tag{11}$$

Recall that $\varphi(t)$ in (1) can be considered as

$$\begin{aligned} &\varphi(t) \in A^{0,1}(T'M_0) \cong \Gamma(\mathrm{Hom}(T'^*M_0, T''^*M_0)) = \Gamma(T'M_0 \otimes T''^*M_0), \\ &\varphi(t) = \varphi^i{}_{\overline{j}}(t)\frac{\partial}{\partial z^i} \otimes dz^{\overline{j}}. \end{aligned} \tag{12}$$

Here $z^i = z^i_0$ are local holomorphic coordinates of M_0, and we keep this notation below. Then, T'^*M_t is spanned by

$$e^i := dz^i + \varphi^i{}_{\overline{j}}(t)dz^{\overline{j}}, \qquad i = 1, \dots, m, \tag{13}$$

or equivalently, $T''M_t$ is spanned by

$$T_{\overline{j}} := \frac{\partial}{\partial z^{\overline{j}}} - \varphi^i{}_{\overline{j}}(t)\frac{\partial}{\partial z^i}, \qquad j = 1, \dots, m. \tag{14}$$

Lemma 4.1. *Suppose that (2) of Theorem 1.1 is satisfied. Then the identity component* $\mathrm{Aut}_0^T(M)$ *of* $\mathrm{Aut}^T(M)$ *acts on* $H^1(M_0, T'M_0) \cong T_0'B$ *trivially, and hence on* B *trivially.*

Proof. We closely follow the arguments in [9, p. 823]. But some missing computations in [9] are supplemented, which are (15)–(19) below, for the reader's convenience. Since the Kuranishi family is a complex analytic family (Proposition 2.6, 2.7 in Chapter 4 of [38], or Theorem 6.5, [25]), by the assumption there are T-invariant holomorphic vector fields $v_1(t), \dots, v_\ell(t)$ which form a basis of $H^0(M_t, T'M_t)^T$ and holomorphic in t. We regard these are vector fields on $M_0 \cong M$ since all M_t are diffeomorphic to M_0. Since $(I - \varphi\overline{\varphi})^{-1} - \overline{\varphi}(I - \varphi\overline{\varphi})^{-1}$ is invertible for small t we may put

$$\widetilde{v}_p := ((I - \varphi\overline{\varphi})^{-1} - \overline{\varphi}(I - \varphi\overline{\varphi})^{-1})^{-1} v_p.$$

Let $z^1, \dots, z^m$ and $w^1, \dots, w^m$ be local holomorphic coordinates for M_0 and M_t respectively defined on a common open set U of M. Note that (4.4) and (4.5) in [18] imply

$$((I - \varphi\overline{\varphi})^{-1})^i{}_j = \frac{\partial z^i}{\partial w^\alpha}\frac{\partial w^\alpha}{\partial z^j} \tag{15}$$

and

$$-\overline{\varphi^i{}_{\overline{j}}}((I - \varphi\overline{\varphi})^{-1})^j{}_\ell = \frac{\partial z^{\overline{i}}}{\partial w^\alpha}\frac{\partial w^\alpha}{\partial z^j}. \tag{16}$$

Then we can see using (15) and (16) that

$$v_p = \widetilde{v}_p^j \frac{\partial w^\alpha}{\partial z^j}\frac{\partial}{\partial w^\alpha}. \tag{17}$$

Since v is holomorphic on M_t, we have $T_{\overline{j}} v^\alpha = 0$, that is,

$$\left(\frac{\partial}{\partial z^{\overline{j}}} - \varphi^i{}_{\overline{j}}(t)\frac{\partial}{\partial z^i}\right)\left(\frac{\partial w^\alpha}{\partial z^k}\widetilde{v}_p^k\right) = 0. \tag{18}$$

On the other hand

$$\left(T_{\overline{j}}\left(\frac{\partial w^\alpha}{\partial z^k}\right)\right)\widetilde{v}_p^k = (\widetilde{v}_p\varphi)^i{}_{\overline{j}}\frac{\partial w^\alpha}{\partial z^i}. \tag{19}$$

From (18) and (19) we get

$$\overline{\partial}_0 \widetilde{v}_p = -[\widetilde{v}_p, \varphi]. \tag{20}$$

Since $\varphi(0) = 0$ we obtain

$$\overline{\partial}_0 \left(\frac{\partial}{\partial t_k}\Big|_{t=0} \widetilde{v}_p(t) \right) = -[\widetilde{v}_p, \varphi_k]. \tag{21}$$

This implies the infinitesimal generators of $\mathrm{Aut}_0^T(M)$ acts on $H^1(M_0, T'M_0)$ trivially. This completes the proof. □

Proof of Theorem 1.1. We first prove that (2) implies (1). Let $G := \mathrm{Isom}_0^T(M_0, \omega)$ be the identity component of the T-equivariant isometries of (M_0, ω) so that G preserves both ω and J_0. Then since $\omega = \omega_0$ is a weighted v-soliton it is a (v, w)-cscK metric with $w(\mu) = (m + \langle d\log v, \mu\rangle)v(\mu)$ and $G^{\mathbf{C}} = \mathrm{Aut}_0^T(M)$. By Lemma 4.1, G acts on $H^1(M_0, T'M_0)$ trivially, which implies that G preserves $\varphi(t)$ since $\varphi(t)$ is uniquely determined by $\sum_{i=1}^k t^i \varphi_i$ in Kuranishi's equation (1). Hence G also preserves J_t, and thus $G \subset \mathrm{Isom}_0^T(M_t, \omega)$. But since

$$\dim G^{\mathbf{C}} = \dim \mathrm{Aut}^T(M) = \dim \mathrm{Aut}^T(M_t) \geq \dim_{\mathbf{R}} \mathrm{Isom}_0^T(M_t, \omega)$$

we have $G = \mathrm{Isom}_0^T(M_t, \omega)$. This implies that the Hamiltonian vector fields for (M_0, ω) remain to be Hamiltonian vector fields of (M_t, ω), and the moment map μ_{ω_t} is unchanged as t varies. Thus

$$v(\mu_{\omega_t}) = v(\mu_\omega) \tag{22}$$

for all $t \in B$.

Let $\widetilde{T}$ be the maximal torus in G containing T. Then since $\widetilde{T} \subset \mathrm{Aut}_0^T(M)$ Lemma 4.1 implies that $\widetilde{T}$ acts on B trivially. By Proposition 2.8, shrinking B if necessary, M_t admits a (v, w)-extremal metric for any $t \in B$. We wish to show this (v, w)-extremal metric is a (v, w)-cscK metric so that it is a v-soliton. To see this, by Remark 2.5, it is sufficient to show the invariant $\mathrm{Fut}_{v,w}(t)$ in (9) for M_t vanishes. By (10) and (11), we need to take f in (9), to be

$$f_t := \log v(\mu_{\omega_0}) + \log\det(I - \varphi(t)\overline{\varphi(t)}).$$

Hence using (22) we have

$$\mathrm{Fut}_{v,w}(t)(X) = -\int_M (JX)(\log\det(I - \varphi(t)\overline{\varphi(t)}))\, v(\mu_\omega)\, \omega^m. \tag{23}$$

But since any automorphism of M_t preserves $\varphi(t)$ the derivative by JX on the right hand side of (23) vanishes. Thus $\mathrm{Fut}_{v,w}(t)$ vanishes, and by Remark 2.5

the extremal (v, w)-extremal metric must be a v-soliton. This proves that (2) implies (1).

Next we prove that (1) implies (3). We first show the action of $G := \mathrm{Isom}_0^T(M_0, \omega)$ and $G^{\mathbf{C}}$ on B is trivial.

For this purpose we show that if this is not the case then there is a non-product $T^{\mathbf{C}}$-equivariant test configuration $\{(M_t, K_{M_t}^{-k})\}$ using arguments similar to [9, pp. 822–823]. Because of the construction of the Kuranishi family, the nontrivial action of $G^{\mathbf{C}}$ on B induces a one parameter subgroup $\lambda : \mathbf{C}^* \to G^{\mathbf{C}}$ whose $T^{\mathbf{C}}$-equivariant action on $T_0'B \cong H^1(M_0, T'M_0)$ is nontrivial. We can choose a basis $e_1, \dots, e_\ell$ of $H^1(M_0, T'M_0)$ such that $\lambda(s)e_i = s^{\kappa_i} e_i$ with $\kappa_i \in \mathbf{Z}$. Since this action is nontrivial some κ_i is non-zero, and we choose and fix one of such i's, and we may assume $\kappa_i > 0$ by replacing λ by λ^{-1}. Consider the one-dimensional subfamily $\{M_t \mid t = (0, \dots, 0, t_i, 0, \dots, 0), |t_i| < \epsilon\}$ of $\mathcal{M} \to B$ for small $\epsilon > 0$. Then we have an action of $\{s \mid |s| < 1\}$ corresponding to the λ-action expressed by $M_t \to M_{st}$. All M_t with $t \neq 0$ are biholomorphic because of the action of $\{s \mid 0 < |s| < 1\}$. Then the Kodaira-Spencer map $T'B_t \to H^1(M_t, T'M_t)$ is only surjective (see e.g. Theorem 2.1, (3) in [9]) but not isomorphic for $t \neq 0$, while $T'B_0 \to H^1(M_0, T'M_0)$ is isomorphic. It follows that, for $t \neq 0$, M_t is not biholomorphic to M_0. Hence after a suitable base change we obtain a non-product $T^{\mathbf{C}}$-equivariant test configuration $\{(M_t, K_{M_t}^{-k})\}$.

But this is impossible since M_t has a v-soliton and K-polystable with respect to $T^{\mathbf{C}}$-equivariant test configurations, the central fiber M_0 also has a v-soliton and the Donaldson-Futaki invariant is zero (see Theorem 1.17 and 1.21 in [34], or [2], or Theorem 1.0.7 in [20], or [21]). Thus the action of G on B is trivial.

Then as we argued at the beginning of this proof, G preserves both ω and $\varphi(t)$, and thus we have an inclusion $G \subset \mathrm{Isom}_0(M_t, \omega)$. In particular $T \subset \mathrm{Isom}_0(M_t, \omega) \subset \mathrm{Aut}_0(M_t)$ and $G \subset \mathrm{Isom}_0^T(M_t, \omega)$, $G^{\mathbf{C}} = \mathrm{Aut}_0^T(M_0) \subset \mathrm{Aut}_0^T(M_t)$. But since $\dim H^0(M_t, T'M_t)^T$ is upper semi-continuous we obtain $G^{\mathbf{C}} = \mathrm{Aut}^T(M_t)$ for all $t \in B$. This proves that (1) implies (3). That (3) implies (2) is trivial. This completes the proof of Theorem 1.1 □

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