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Coassociative submanifolds in Joyce's generalised Kummer constructions

Dominik Gutwein

Abstract: This article constructs coassociative submanifolds in G_2-manifolds arising from Joyce's generalised Kummer construction. The novelty compared to previous constructions is that these submanifolds all lie within the critical region of the G_2-manifold in which the metric degenerates. This forces the volume of the coassociatives to shrink to zero when the orbifold-limit is approached.

Keywords: Coassociative submanifolds, G_2-manifolds, Generalised Kummer constructions.

1. Introduction

Associative and coassociative submanifolds are the natural subobjects in 7-dimensional G_2-manifolds. Besides having minimal volume among all submanifolds realising a fixed homology class (and therefore being minimal, cf. [10, Sections 2.4 and 4.1.A-B]), they play a prominent role in the extensively studied gauge theory on G_2-manifolds (see for example [26] and [4]). Moreover, Halverson and Morrison proposed that associative and coassociative submanifolds might play a role in characterising the period domain of a G_2-manifold [9, Section 3] (see also the formulation in [5, Introduction]). More precisely, assume that Y is a simply-connected and compact 7-manifold that admits torsion-free G_2-structures. In analogy to the Kähler cone of a Calabi–Yau 3-fold, Halverson and Morrison [9, Section 3] proposed that (the G_2-period domain)

$$\mathcal{Q}(Y) \coloneqq \{([\phi], [*_\phi \phi]) \in \mathrm{H}^3(Y) \oplus \mathrm{H}^4(Y) \mid \phi \in \Omega^3(Y) \text{ is a torsion-free } \mathrm{G}_2\text{-structure}\}$$

might be fully characterised by the following inequalities:[1]

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[1]Where we ignore for the moment the issue that the notions of G_2-instantons, associative-, and coassociative submanifolds themselves depend on ϕ. Furthermore,

1. A topological condition: $\int_Y \alpha \wedge \alpha \wedge \phi < 0$ for every nonzero $[\alpha] \in \mathrm{H}^2(Y)$.
2. A characteristic class condition: $\int_Y p_1(E) \wedge \phi < 0$ for any vector bundle E over Y admitting a non-flat G_2-instanton.
3. Two calibrated submanifold conditions:
   - $\int_P \phi > 0$ for any associative submanifold P.
   - $\int_M *\phi > 0$ for any coassociative submanifold M.

If Halverson and Morrison's proposal is indeed true, then certain degenerations of G_2-structures would be detectable by the vanishing of one of the above integrals. As a step towards this proposal, Dwivedi, Platt, and Walpuski constructed therefore in [5] associative submanifolds in families of G_2-manifolds arising from Joyce's generalised Kummer construction [11, 12]. These associative submanifolds have the property that their volume shrinks to zero as the G_2-manifold approaches its (singular) orbifold-limit. (This is equivalent to $\int_P \phi_t \to 0$ where P denotes the mentioned associative and ϕ_t corresponds to the degenerating path of G_2-structures.) The purpose of the article at hand is to augment their work by the analogous construction of coassociative submanifolds. We hereby proceed as follows:

In Section 2 we review the necessary background on the generalised Kummer construction and asymptotically locally Euclidean (ALE) hyperkähler 4-manifolds. Section 3 is devoted to the analysis of our construction. In Theorem 3.7 we prove a perturbation result for coassociative submanifolds whose spirit is well-known from gluing constructions in gauge theory. It roughly states that whenever two closed G_2-structures ϕ and ϕ_0 on a 7-manifold Y are 'close' (in a quantified sense) to one another, then a ϕ_0-coassociative submanifold can be perturbed to a ϕ-coassociative. In Proposition 4.2 we prove that this theorem is applicable to a certain class of submanifolds that occur very frequently in generalised Kummer constructions. These submanifolds are modelled on (or covered by) the product of a 2-torus and a holomorphic sphere where the latter lies in the exceptional divisor of the glued-in ALE hyperkähler 4-manifold appearing in the Kummer construction (cf. Example 3.2). Subsequently, we find in Section 4 numerous examples of coassociative submanifolds in various resolutions of G_2-orbifolds constructed in [12] and [24]. Our construction leaves the freedom of choosing a 'basepoint' of the 2-torus and we mention in Remark 4.3 that moving this basepoint produces the full deformation family of the coassociatives constructed in Proposition 4.2. In our examples, this family is either homeomorphic to S^1 or a closed interval.

note that the integration is carried out with respect to the orientation determined by $\frac{1}{7}[\phi] \cup [*_\phi \phi] \in \mathrm{H}^7(Y)$.

The coassocative submanifolds in the latter case are embedded for the inner values of the interval and factor at the endpoints through a double-cover over an (embedded) rigid coassociative submanifold. Ultimately, we give in Appendix B all choices of ALE hyperkähler 4-manifold that can be used to resolve the G_2-orbifolds of [24] that were treated in Section 4.

There already exists a vast literature on the construction of coassociative submanifolds (see [15, Chapter 12] and [21, Section 6] for an overview). Here we only mention that Joyce [12, Section 4.2] constructed coassociative submanifolds inside his generalised Kummer constructions as fixed-point sets of anti G_2-involutions. At least one part of their support lies outside the critical region of the ambient manifold in which the orbifold singularities develop. In contrast, the coassociatives in the article at hand are all constructed to lie completely within this region. This is ultimately the reason why their volume shrinks to zero.

2. Background

2.1. Joyce's generalised Kummer construction

The generalised Kummer construction, as developed (and extended) by Joyce in [11, 12, 14], produces compact manifolds with holonomy contained in G_2 as desingularisations of certain G_2-orbifolds. This section follows the presentation in [5] very closely. The following class of examples serve as models for the singularities considered in this article:

Example 2.1. Let $(X, \underline{\omega})$ be a hyperkähler 4-orbifold with hyperkähler structure $\underline{\omega} \in \Omega^2(X, \operatorname{Im}\mathbb{H}^*)$. Denote by $\mathrm{Vol} \in \Omega^3(\operatorname{Im}\mathbb{H})$ and $\underline{\sigma} \in \Omega^1(\operatorname{Im}\mathbb{H}, \operatorname{Im}\mathbb{H})$ the volume form and the canonical isomorphism $T\operatorname{Im}\mathbb{H} \to \operatorname{Im}\mathbb{H} \times \operatorname{Im}\mathbb{H}$, respectively. In the following we denote by $\langle \underline{\sigma} \wedge \underline{\omega} \rangle$ the 3-form on $\operatorname{Im}\mathbb{H} \times X$ obtained by wedging and pairing $\operatorname{Im}\mathbb{H} \otimes \operatorname{Im}\mathbb{H}^* \to \mathbb{R}$.

1. The product $\operatorname{Im}\mathbb{H} \times X$ carries a torsion-free G_2-structure defined by

$$\phi := \mathrm{Vol} - \langle \underline{\sigma} \wedge \underline{\omega} \rangle \in \Omega^3(\operatorname{Im}\mathbb{H} \times X). \tag{1}$$

2. Assume there is a group action $\rho\colon G \to \mathrm{Isom}(X)$ by $G < \mathrm{SO}(\operatorname{Im}\mathbb{H}) \ltimes \operatorname{Im}\mathbb{H}$ that preserves the hyperkähler structure in the sense that for any $(R, v) \in G$

$$(\rho(R, v)^* \otimes R^*)\underline{\omega} = \underline{\omega}. \tag{2}$$

The 3-form ϕ is invariant under the product action on $\operatorname{Im}\mathbb{H} \times X$ and descends to a torsion-free G_2-structure on the quotient $Y \coloneqq (\operatorname{Im}\mathbb{H} \times X)/G$. We denote the corresponding 3-form on Y by ϕ as well. Note that whenever G is Bieberbach (i.e. discrete, cocompact, and torsion-free) then the action is free and taking the quotient does not introduce additional singularities in Y.

Let now (Y_0, ϕ_0) be a compact and flat G_2-orbifold such that its singularities are locally modelled on $\mathbb{R}^3 \times \mathbb{H}/\Gamma$ for a finite group $\Gamma < \mathrm{Sp}(1)$. More precisely, we demand:

Assumption 2.2. Denote by $\mathcal{S}$ the set of connected components of the singular set of Y_0. We assume that for every $S \in \mathcal{S}$ there exist

1. A finite subgroup $\Gamma_S < \mathrm{Sp}(1)$, a Bieberbach group $G_S < \mathrm{SO}(\operatorname{Im}\mathbb{H}) \ltimes \operatorname{Im}\mathbb{H}$, and a group action $\rho\colon G_S \to N_{\mathrm{SO}(\mathbb{H})}(\Gamma_S) \subset \mathrm{Isom}(\mathbb{H}/\Gamma_S)$. Denote by
$$(Y_S \coloneqq (\operatorname{Im}\mathbb{H} \times \mathbb{H}/\Gamma_S)/G_S, \phi_S)$$
the corresponding G_2-orbifold from Example 2.1.
2. An open set
$$U_S \coloneqq (\operatorname{Im}\mathbb{H} \times B_{2R_S}(0)/\Gamma_S)/G_S \subset Y_S$$
for $R_S > 0$ and an open embedding $\mathtt{J}_S\colon U_S \to Y_0$ with $S \subset \mathtt{J}_S(U_S)$ and $\mathtt{J}_S^*\phi_0 = \phi_S$. The R_S are chosen such that $\mathtt{J}_{S_1}(U_{S_1}) \cap \mathtt{J}_{S_2}(U_{S_2}) = \emptyset$ for any two $S_1 \neq S_2 \in \mathcal{S}$.

Remark 2.3. All (non-trivial) finite subgroups $\Gamma < \mathrm{Sp}(1)$ were classified by Klein [17]. These are isomorphic to:

(A_k) The cyclic group C_{k+1} for $k \geq 1$
(D_k) The dicyclic group Dic_{k-2} for $k \geq 3$
(E_6) The binary tetrahedral group $2T$
(E_7) The binary octahedral group $2O$
(E_8) The binary icosahedral group $2I$

(See also [24, Section 2] for a description on how these groups lie inside $\mathrm{Sp}(1)$.)

Definition 2.4. Let (Y_0, ϕ_0) be a flat G_2-orbifold satisfying Assumption 2.2. A set of resolution data consists for every $S \in \mathcal{S}$ of the following:

1. An asymptotically locally Euclidean (ALE) hyperkähler manifold which is asymptotic to $\mathbb{H}/\Gamma_S$. That is, a hyperkähler 4-manifold $(\hat{X}_S, \underline{\hat{\omega}}_S)$ together with a diffeomorphism $\tau_S\colon\ \hat{X}_S \setminus \hat{K}_S \to (\mathbb{H} \setminus B_{R_S}(0))/\Gamma_S$ outside a compact set $\hat{K}_S \subset \hat{X}_S$ that satisfies
$$|\nabla^k(\tau_{S*}\underline{\hat{\omega}}_S - \underline{\omega})| = \mathcal{O}(r^{-4-k}).$$
The norm and covariant derivatives are hereby taken with respect to the flat metric on $(\mathbb{H} \setminus \{0\})/\Gamma_S$.
2. A group action $\rho_S\colon\ G_S \to \operatorname{Isom}(\hat{X}_S)$ which leaves $\hat{K}_S$ and $\underline{\hat{\omega}}_S$ invariant (in the sense of (2)) and makes τ_S equivariant.

For a given orbifold Y_0, a set of resolution data, and a positive parameter $t > 0$ we define the following sets:

$$\begin{aligned}
V &\coloneqq \bigsqcup_{S\in\mathcal{S}} V_S \quad \text{for } V_S \coloneqq (\operatorname{Im}\mathbb{H} \times B_{R_S}(0)/\Gamma_S)/G_S \subset (\operatorname{Im}\mathbb{H} \times \mathbb{H}/\Gamma_S)/G_S\\
U &\coloneqq \bigsqcup_{S\in\mathcal{S}} U_S \quad \text{for } U_S \coloneqq (\operatorname{Im}\mathbb{H} \times B_{2R_S}(0)/\Gamma_S)/G_S \subset (\operatorname{Im}\mathbb{H} \times \mathbb{H}/\Gamma_S)/G_S\\
\hat{V} &\coloneqq \bigsqcup_{S\in\mathcal{S}} \hat{V}_S \quad \text{for } \hat{V}_S \coloneqq (\operatorname{Im}\mathbb{H} \times \hat{K}_S)/G_S \subset (\operatorname{Im}\mathbb{H} \times \hat{X}_S)/G_S\\
\hat{U}_t &\coloneqq \bigsqcup_{S\in\mathcal{S}} \hat{U}_S^t \quad \text{for } \hat{U}_S^t \coloneqq (\operatorname{Im}\mathbb{H} \times (t\tau_S)^{-1}(B_{2R_S}(0)/\Gamma_S))/G_S \subset (\operatorname{Im}\mathbb{H} \times \hat{X}_S)/G_S
\end{aligned}$$

Denote by $\mathsf{J}\colon\ U \to Y_0$ and $t\tau\colon\ \hat{U}_t \to U$ the maps induced by all $\{\mathsf{J}_S\}_{S\in\mathcal{S}}$ and $\{t\tau_S\}_{S\in\mathcal{S}}$, respectively.

Definition 2.5 ([12, proof of Theorem 2.2.1])**.** Given a flat G_2-orbifold (Y_0, ϕ_0) and a set of resolution data, Joyce defines a 1-parameter family of smooth manifolds by
$$\hat{Y}_t \coloneqq (Y_0 \setminus \mathsf{J}(V)) \cup (\hat{U}_t \cup \hat{V})/\sim$$
where $\hat{U}_t \ni x \sim \mathsf{J}(t\tau(x)) \in \mathsf{J}(U \setminus V)$.

Furthermore, Joyce equips each $\hat{Y}_t$ with a closed G_2-structure $\tilde{\phi}_t$ that has the following property: On any $\hat{V}_S \subset \hat{Y}_t$, $\tilde{\phi}_t$ agrees with the Model Structure (1) associated to the (rescaled) hyperkähler structure $t^2\underline{\hat{\omega}}_S$ on $\hat{K}_S$.

Remark 2.6. Instead of working with $\tilde{\phi}_t$ we follow [5] and work with the rescaled G_2-structure $t^{-3}\tilde{\phi}_t$.

The following existence theorem was first proven by Joyce in [11] and later reproven with improved estimates by Platt in [23]. The following formulation is taken from [5, Theorem 2.19].

Theorem 2.7 ([23, Theorem 4.58]). *Let (Y_0, ϕ_0) be a compact and flat* G_2*-orbifold satisfying Assumption 2.2 and let $\mathcal{R}$ be a set of resolution data. Furthermore, let $\alpha \in (0, 1/16)$ be a chosen Hölder exponent. Then there are $T_0 = T_0(\mathcal{R})$ and $c = c(\mathcal{R}, \alpha) > 0$ such that for any $t \in (0, T_0)$ there exists a torsion-free* G_2*-structure ϕ_t on $\hat{Y}_t$ with $[\phi_t] = [\tilde{\phi}_t] \in \mathrm{H}^3(\hat{Y}_t)$ and*

$$\left\| t^{-3}(\phi_t - \tilde{\phi}_t) \right\|_{C^{1,\alpha}} < ct^{5/2}.$$

The $C^{1,\alpha}$-norm above is taken with respect to the metric $t^{-2}\tilde{g}_t$ (induced by $t^{-3}\tilde{\phi}_t$).

Remark 2.8. Note that the formulation of Theorem 2.7 in [23, Theorem 4.58] bounds the $C^{1,\alpha}$-norm of $\phi_t - \tilde{\phi}_t$ only by $t^{3/2-\alpha}$. However, if one uses the $C^{1,\alpha}$-norm with respect to $t^{-2}\tilde{g}_t$ (instead of $\tilde{g}_t$) one obtains the estimate in Theorem 2.7 as a direct consequence of the estimate with respect to the weighted norm given in [23, Theorem 4.58].

Remark 2.9. The formulation of Theorem 2.7 in [23] only considers G_2-orbifolds whose singularities are resolved via Eguchi–Hanson spaces. Its proof relies on the property that the set

$$\left\{\omega \in \Omega^2(X_{\mathrm{EH}}) \mid \Delta\omega = 0 \text{ and } |\nabla^\ell \omega| = \mathcal{O}(r^{\beta-\ell}) \text{ for each } \ell \in \mathbb{N}_0\right\}$$

is independent of $\beta \in [-4, 0)$. It was explained to us by Thomas Walpuski that this property holds for every ALE 4-manifold. One way to prove this is as [28, Proposition 5.10] using the improved Kato inequality for harmonic 2-forms in [25, Theorem 1]. The proof of Theorem 2.7 given in [23] adapts therefore to resolutions of G_2-orbifolds by arbitrary ALE hyperkähler 4-manifolds.

2.2. Asymptotically locally Euclidean hyperkähler 4-manifolds

Recall from Definition 2.4 that a resolution of a flat G_2-orbifold requires the choice of an ALE hyperkähler 4-manifold together with a lift of the action of the Bieberbach group. In the following section we review how these can be constructed. All these spaces contain holomorphic spheres which are the main ingredient in our construction of coassociative submanifolds later in this article.

Note that Section 2.2.2 is only relevant for Example 4.9 and may be skipped at the reader's preference.

2.2.1. The Gibbons–Hawking Ansatz For $N \in \mathbb{N}$ let $C_N < \mathrm{Sp}(1)$ be the cyclic subgroup generated by right-multiplication with $e^{2\pi i/N}$. Concrete models of ALE spaces asymptotic to $\mathbb{H}/C_N$ were first constructed for $N = 2$ by Eguchi and Hanson [6] and then by Gibbons and Hawking [7] for general N. A detailed treatment of the following material can be found in [27, Section 59] (see also [5, Remark 2.12] and [8, Section 3.5]):

1. For any

$$\zeta \in \Delta \coloneqq \{[\zeta_1, \dots, \zeta_N] \in (\operatorname{Im}\mathbb{H})^N / S_N \mid \zeta_1 + \cdots + \zeta_N = 0\}$$

define $Z_\zeta \coloneqq \{\zeta_1, \dots, \zeta_N\} \subset \operatorname{Im}\mathbb{H}$, $B_\zeta \coloneqq \operatorname{Im}\mathbb{H} \setminus Z_\zeta$, and $f_\zeta \in C^\infty(B_\zeta)$ by

$$f_\zeta(q) \coloneqq \sum_{a=1}^{N} \frac{1}{2|q - \zeta_a|}.$$

The function f_ζ is a sum of harmonic functions and one can furthermore check that the cohomology class $[*_3 \,\mathrm{d} f_\zeta]$ lies inside the image of the canonical inclusion $H^2(B_\zeta, 2\pi\mathbb{Z}) \hookrightarrow H^2(B_\zeta, \mathbb{R})$. This implies that there exists a (up to isomorphism) unique principal U(1)-bundle $\pi_\zeta \colon X_\zeta^\circ \to B_\zeta$ together with a connection 1-form $i\theta \in \Omega^1(X_\zeta^\circ, i\mathbb{R})$ that satisfies $\mathrm{d}\theta = \pi_\zeta^*(*_3 \,\mathrm{d} f_\zeta)$.
For $\zeta_i \in \zeta$ denote by N_{ζ_i} the number of entries of ζ equal to ζ_i. Around any sphere $S^2 \subset B_\zeta$ whose inner ball only contains $\zeta_i \in Z_\zeta$, the restriction $(X_\zeta^\circ)_{|S^2}$ is isomorphic to the quotient of the Hopf-fibration by $C_{N_{\zeta_i}}$.
2. The Gibbons–Hawking Ansatz defines a hyperkähler structure on the total space X_ζ° as follows: The connection induces a horizontal distribution $X_\zeta^\circ \times \operatorname{Im}\mathbb{H} \subset TX_\zeta^\circ$. Furthermore, we identify the vertical tangent bundle $X_\zeta^\circ \times i\mathbb{R}$ with $X_\zeta^\circ \times \operatorname{Re}\mathbb{H}$ via $(x, it) \mapsto (x, t/f_\zeta(x))$. This induces a canonical hypercomplex structure $\underline{I}_\zeta$ on $TX_\zeta^\circ \cong X_\zeta^\circ \times \mathbb{H}$ which is compatible with the metric defined by

$$g_\zeta^\circ \coloneqq f_\zeta^{-1} \cdot \theta \otimes \theta + f_\zeta \cdot \pi_\zeta^*(g_{\operatorname{Im}\mathbb{H}}).$$

The corresponding hyperhermitian form $\underline{\omega}_\zeta$ is closed and therefore hyperkähler.
3. It turns out that $(X_\zeta^\circ, \underline{\omega}_\zeta)$ can be extended to a complete hyperkähler orbifold $(X_\zeta, \underline{\omega}_\zeta)$ by adding $\#Z_\zeta$ points, one over each element in Z_ζ. In fact, whenever

$$\zeta \in \Delta^\circ \coloneqq \{[\zeta_1, \dots, \zeta_N] \in \Delta \mid \zeta_i \neq \zeta_j \text{ for } i \neq j\},$$

then X_ζ is a manifold.
Outside a ball $B_{R^2}(0)$ containing all of Z_ζ, the bundle $(X_\zeta)_{|\operatorname{Im}\mathbb{H}\setminus B_{R^2}(0)}$ has Chern class $-N \in \mathbb{Z} \cong H^2(\operatorname{Im}\mathbb{H}\setminus B_{R^2}(0))$. It is therefore isomorphic to the principal U(1)-bundle

$$\begin{aligned}\pi_0\colon\ (\mathbb{H}\setminus B_R(0))/C_N &\to \operatorname{Im}\mathbb{H}\setminus B_{R^2}(0)\\ [q] &\mapsto qi\bar{q}.\end{aligned}$$

With the right choice of such an isomorphism τ_ζ (e.g. using parallel transport in radial direction and 'matching' the connections θ_ζ and θ_0 at the sphere at infinity) one can show that $\underline{\omega}_\zeta$ approaches the standard hyperkähler structure on $\mathbb{H}/C_N$ as in Definition 2.4, Point 1. The Gibbons–Hawking spaces are therefore ALE asymptotic to $\mathbb{H}/C_N$.

4. Let $R \in N_{\mathrm{SO}(\mathbb{H})}(C_N)$. Identify[2] the space of self-dual 2-vectors $\Lambda^2_+\mathbb{H}$ with $\operatorname{Im}\mathbb{H}$ and denote by $\Lambda^2_+ R \in \mathrm{SO}(\operatorname{Im}\mathbb{H})$ the induced map. Furthermore, define

$$\alpha_R := \begin{cases} 1, & \text{if } R \in Z_{\mathrm{SO}(\mathbb{H})}(C_N)\\ -1, & \text{else}\end{cases}$$

where $Z_{\mathrm{SO}(\mathbb{H})}(C_N)$ denotes the centralizer of C_N in $\mathrm{SO}(\mathbb{H})$. If $\zeta \in \Delta$ satisfies $\Lambda^2_+ R\zeta = \alpha_R\zeta$, then there exists an $\hat{R} \in \operatorname{Isom}(X_\zeta)$ satisfying

$$(\hat{R}^* \otimes \Lambda^2_+ R^*)\underline{\omega}_\zeta = \underline{\omega}_\zeta \quad \text{and} \quad \tau_\zeta \circ \hat{R} = R \circ \tau_\zeta$$

where R acts on $\mathbb{H}/C_N$ in the obvious way. This is explained in more detail in Section 2.2.2, Point 3. However, note that whenever $R \in N_{\mathrm{SO}(\mathbb{H})}(C_N)$ for $N \geq 3$ (which holds in all examples of Section 4), then $\hat{R}$ acts as a bundle (anti-) isomorphism. In this case it can be uniquely characterised by the lift of R along τ_ζ and demanding that the connection $i\theta_\zeta$ gets mapped onto itself.

5. Let $\zeta_0 \neq \zeta_1 \in Z_\zeta$ and assume that the line segment

$$\ell := \{t\zeta_1 + (1-t)\zeta_0 \mid t \in [0,1]\} \subset \operatorname{Im}\mathbb{H}$$

intersect Z_ζ only in its endpoints. The preimage $\Sigma_\ell := \pi_\zeta^{-1}(\ell) \subset X_\zeta$ is a smoothly embedded sphere, which is holomorphic with respect to the complex structure $I_{\zeta,\hat{\xi}} := \langle \underline{I}_\zeta, \hat{\xi}\rangle$ for $\hat{\xi} := \frac{\zeta_1-\zeta_0}{|\zeta_1-\zeta_0|} \in \operatorname{Im}\mathbb{H}$.

[2] Via $\operatorname{Im}\mathbb{H} \ni \xi \mapsto \langle \underline{\omega}, \xi\rangle \in \Omega^2(\mathbb{H})$ where $\underline{\omega} = \mathrm{d}q \wedge \mathrm{d}\bar{q} \in \Omega^2(\mathbb{H}, (\operatorname{Im}\mathbb{H})^*)$ is the standard hyperkähler structure on $\mathbb{H}$.

Let now $R \in N_{\mathrm{SO}(\mathbb{H})}(C_N)$ satisfy $\Lambda^2_+ R\zeta = \alpha_R \zeta$ and denote by $\hat{R}$ its lift to X_ζ (as described in Point 4 above). Then $\hat{R}(\Sigma_\ell) = \Sigma_{\alpha_R \Lambda^2_+ R(\ell)}$, where $\alpha_R \Lambda^2_+ R(\ell)$ denotes the line segment coming from applying $\alpha_R \Lambda^2_+ R \in \mathrm{O}(\mathrm{Im}\,\mathbb{H})$ to ℓ.

2.2.2. Kronheimer's construction of ALE spaces All ALE hyperkähler 4-manifolds asymptotic to $\mathbb{H}/\Gamma$ for any finite subgroup $\Gamma < \mathrm{Sp}(1)$ were constructed and classified by Kronheimer in [18] and [19]. The following summary follows the one given in [5, Remark 2.15]. Note also that for $\Gamma = C_N$ this treatment is equivalent to Section 2.2.1.

1. Let $\mathbb{C}[\Gamma] := \mathrm{Maps}(\Gamma, \mathbb{C})$ denote the regular representation equipped with its standard hermitian inner product. Furthermore, define
$$S := (\mathbb{H} \otimes_{\mathbb{R}} \mathfrak{u}(\mathbb{C}[\Gamma]))^\Gamma \quad \text{and} \quad G := \mathbb{P}\mathrm{U}(\mathbb{C}[\Gamma])^\Gamma$$
and equip S with the canonical flat hyperkähler structure. The adjoint action of G on S has a distinguished hyperkähler moment map
$$\mu\colon\ S \to (\mathrm{Im}\,\mathbb{H})^* \otimes \mathfrak{g}^*.$$
Let $\mathfrak{z}^* \subset \mathfrak{g}^*$ be the annihilator of $[\mathfrak{g}, \mathfrak{g}]$, i.e. all elements in $\mathfrak{g}^*$ fixed by the coadjoint action of G. For any value $\zeta \in (\mathrm{Im}\,\mathbb{H})^* \otimes \mathfrak{z}^*$, the hyperkähler quotient $X_\zeta := \mu^{-1}(\zeta)/G$ is a hyperkähler orbifold asymptotic to $\mathbb{H}/\Gamma$ [18, Lemma 3.3 and Proposition 3.14].
2. Remark 2.3 associates a root system Φ to Γ. Kronheimer [18, Proposition 4.1] defines an isomorphism between $\mathfrak{z}^*$ and the associated Cartan algebra $\mathfrak{h} := (\mathbb{R}\Phi)^*$. For any root $\theta \in \Phi$ let $D_\theta := \ker\theta \subset \mathfrak{h}$ be the associated wall of the Weyl chambers. If
$$\zeta \in \tilde{\Delta}^\circ := ((\mathrm{Im}\,\mathbb{H})^* \otimes \mathfrak{h}) \setminus \cup_{\theta\in\Phi}((\mathrm{Im}\,\mathbb{H})^* \otimes D_\theta),$$
then X_ζ is a manifold [18, Proposition 2.8].
3. Any $R \in N_{\mathrm{SO}(\mathbb{H})}(\Gamma)$ acts on Γ by conjugation. We extend this to a complex linear map $C_R \in \mathrm{U}(\mathbb{C}[\Gamma])$. The standard representation of R on $\mathbb{H}$ tensored with the Adjoint action of C_R on $\mathfrak{u}(\mathbb{C}[\Gamma])$ induces an action on S. The hyperkähler moment map satisfies
$$\mu \circ (R \otimes \mathrm{Ad}_{C_R}) = (\Lambda^2_+ R \otimes \mathrm{Ad}^*_{C_R}) \circ \mu$$
where $\Lambda^2_+ R$ is as in Section 2.2.1, Point 4 and $\mathrm{Ad}^*_{C_R}$ denotes the coadjoint representation of C_R on $\mathfrak{h} \cong \mathfrak{z}^* \subset \mathfrak{g}^*$. (See also [13, Section 3] where

Joyce interprets $\mathrm{Ad}^*_{C_R}\colon \mathfrak{h} \to \mathfrak{h}$ as being induced by an automorphism of the underlying Dynkin diagram.) Thus, if

$$(\Lambda^2_+ R \otimes \mathrm{Ad}^*_{C_R})\zeta = \zeta,$$

we obtain an induced isometry $\hat{R} \in \mathrm{Isom}(X_\zeta)$ satisfying

$$(\hat{R}^* \otimes \Lambda^2_+ R^*)\underline{\omega}_\zeta = \underline{\omega}_\zeta \quad \text{and} \quad \tau_\zeta \circ \hat{R} = R \circ \tau_\zeta.$$

4. Let $\zeta \in \tilde{\Delta}^\circ$ be fixed and $\theta \in \Phi$ be a root. Define $\xi \in \mathrm{Im}\,\mathbb{H}$ by

$$\langle \xi, \cdot \rangle = \zeta(\theta) \in (\mathrm{Im}\,\mathbb{H})^*$$

and let $\hat{\xi} \coloneqq \xi/|\xi|$. Inside X_ζ lies a nodal Riemann surface Σ_θ which is holomorphic with respect to the complex structure $I_{\zeta,\hat{\xi}} \coloneqq \langle \underline{I}_\zeta, \hat{\xi} \rangle$.
If $\theta_1, \theta_2 \in \Phi$ are two roots such that $\theta = \theta_1 + \theta_2$ and $|\zeta(\theta)| = |\zeta(\theta_1)| + |\zeta(\theta_2)|$, then Σ_θ is the union of the (nodal) $I_{\hat{\xi}}$-holomorphic curves Σ_{θ_1} and Σ_{θ_2} attached along one new pair of nodes. If no decomposition with this property exists, then Σ_θ is itself an embedded 2-sphere.
Let now $R \in N_{\mathrm{SO}(\mathbb{H})}(\Gamma)$ satisfy $(\Lambda^2_+ R \otimes \mathrm{Ad}^*_{C_R})\zeta = \zeta$ and denote by $\hat{R}$ its lift as described in Point 3. This isometry maps Σ_θ to the surface $\hat{R}(\Sigma_\theta) = \Sigma_{\mathrm{Ad}^*_{C_R}(\theta)}$.
5. Denote by W the Weyl group of Φ. If two elements $\zeta_1, \zeta_2 \in \Delta^\circ$ are related by an element in W, then X_{ζ_1} and X_{ζ_2} are isomorphic as hyperkähler ALE spaces (cf. [19, Section 3] and [1, Section 3]). This isomorphism identifies the holomorphic spheres $\Sigma_\theta \subset X_{\zeta_1}$ and $\Sigma_{w\theta} \subset X_{\zeta_2}$ where $w \in W$ satisfies $\zeta_2 = w\zeta_1$. Furthermore, one can arrange for this isomorphism to intertwine the asymptotic coordinates τ_{ζ_1} and τ_{ζ_2}. We can therefore replace $\zeta \in \tilde{\Delta}^\circ$ in the previous discussion by

$$[\zeta] \in \Delta^\circ \coloneqq \tilde{\Delta}^\circ / W.$$

3. Perturbing coassociative submanifolds

Throughout this section, (Y, ϕ) denotes a 7-manifold equipped with a closed G_2-structure.

Definition 3.1 ([10, Corollary IV.1.20])**.** A 4-dimensional immersed submanifold $\iota\colon M \to Y$ is called coassociative if $\iota^*\phi = 0$. If we want to emphasize the underlying G_2-structure we will write ϕ-coassociative.

Example 3.2. Let $(X, \underline{\omega})$ be a hyperkähler 4-manifold together with an action $\rho\colon G \to \mathrm{Isom}(X)$ by a Bieberbach group G. We denote the corresponding G_2-manifold from Example 2.1 by (Y, ϕ). Furthermore, note that the normal subgroup $\Lambda := G \cap \mathrm{Im}\,\mathbb{H} < \mathrm{Im}\,\mathbb{H}$ is a lattice. An immersed coassociative submanifold inside of Y can now be constructed from the following data:

1. An embedded Riemann surface $\iota_\Sigma\colon \Sigma \to X$ which is holomorphic with respect to $I_{\hat{\xi}_1} = \langle \underline{I}, \hat{\xi}_1 \rangle$ for $\hat{\xi}_1 \in S^2 \subset \mathrm{Im}\,\mathbb{H}$.
2. Two linearly independent $\xi_2, \xi_3 \in \{\hat{\xi}_1\}^\perp \cap \Lambda \subset \mathrm{Im}\,\mathbb{H}$ such that $\rho(\xi_2)(\Sigma) = \Sigma = \rho(\xi_3)(\Sigma)$.

Furthermore, we require the choice of basepoint $q \in \mathrm{Im}\,\mathbb{H}$. We then define

$$M := ((\mathbb{R}\xi_2 + \mathbb{R}\xi_3) \times \Sigma)/\langle \xi_2, \xi_3 \rangle_{\mathbb{Z}}$$

and $\iota_q\colon M \to Y$ as $\iota_q([y, z]) := [q + y, \iota_\Sigma(z)]$. It immediately follows from (1) that ι_q is a coassociative immersion. Next, we discuss conditions under which ι_q is an embedding or factors through a covering map over an embedding. For this we assume that

3. $\xi_2, \xi_3 \in \Lambda$ are primitive and $\rho(\Lambda) = \{1\}$.

We can then regard $T_q^2 := [q] + (\mathbb{R}\xi_2 + \mathbb{R}\xi_3)/\langle \xi_2, \xi_3 \rangle_{\mathbb{Z}} \subset (\mathrm{Im}\,\mathbb{H})/\Lambda$ as an embedded submanifold. Assume further that

4. $G/\Lambda \cong H_1 \times H_2$ where H_1, H_2 are (possibly trivial) groups that satisfy the following:
   (a) The only element $h \in H_1$ with $\rho(h)(\Sigma) \cap \Sigma \neq \emptyset$ and $h \cdot T_q \cap T_q \neq \emptyset$ is $h = 1$. Note here and below that G/Λ canonically acts on X as $\rho(\Lambda) = \{1\}$.
   (b) Every $h \in H_2$ satisfies $\rho(h)(\Sigma) = \Sigma$ and $h \cdot T_q = T_q$.

In this case, the (free) H_2 action lifts to M and ι_q descends to an embedding $\overline{\iota_q}\colon M/H_2 \to Y$. Note that the conditions in Point 4 (and the groups H_1, H_2) depend on the choice of q.

Remark 3.3. By varying the chosen basepoint $q \in \mathrm{Im}\,\mathbb{H}$ in the construction of ι_q in the previous example as well as the $I_{\hat{\xi}_1}$-holomorphic embedding $\iota_\Sigma\colon \Sigma \to X$, one produces a (up to reparametrisation) $(1 + b_1(\Sigma))$-dimensional family of coassociative immersions (one dimension comes from varying q and $b_1(\Sigma)$ from varying ι_Σ). This is of course in accordance with [22, Theorem 4.5] which implies that the moduli space of coassociative immersions $\iota\colon M \to Y$ is itself an orbifold of dimension $b^2_+(M) = 1 + b_1(\Sigma)$.

It is well-known (cf. [22, Proposition 4.2]) that for a coassociative immersion $\iota\colon M \to Y$, the mapping

$$\iota^* TY \ni v \mapsto \iota^*(i_v \phi) \in \Lambda^2 T^* M$$

descends to an isomorphism between the normal bundle and $\Lambda^2_+ T^* M$ (the bundle of self-dual 2-forms).

Definition 3.4. A tubular neighbourhood of the coassociative immersion $\iota\colon M \to Y$ is a convex open neighbourhood $U \subset \Lambda^2_+ T^* M$ of the zero section together with an open immersion $\mathtt{J}\colon U \to Y$ which restricts to ι at the zero section. Additionally, we demand that for any $u \in U$ the image of $\partial_t(\mathtt{J}(tu))_{|t=0} \in \iota^* TY$ in $\Lambda^2_+ T^* M$ under the isomorphism described above is again u.

Remark 3.5. Subsequently, we may simply use the tubular neighbourhood induced by the Levi–Civita connection of the ambient manifold Y.

Let now $\mathtt{J}\colon U \to Y$ be a tubular neighbourhood. For any $\omega \in \Gamma(U)$ we denote by $\mathtt{J}_\omega\colon M \to Y$ the immersion $x \mapsto \mathtt{J}(\omega_x)$. Furthermore, we define $F_{\mathtt{J}}\colon \Gamma(U) \to \Omega^3(M)$ by $F_{\mathtt{J}}(\omega) = \mathtt{J}_\omega^* \phi$. By definition, the immersed submanifold $\mathtt{J}_\omega\colon M \to Y$ for $\omega \in \Gamma(U)$ is coassociative if and only if $F_{\mathtt{J}}(\omega) = 0$.

Proposition 3.6 ([22, Theorem 4.5])**.** *Let $\mathtt{J}\colon U \subset \Lambda^2_+ M \to Y$ be a tubular neighbourhood of a coassociative immersion $\iota\colon M \to Y$. Then the map $F_{\mathtt{J}}$ has image contained in $\mathrm{d}\Omega^2(M)$. Furthermore, there exists a smooth map $\mathcal{N}_{\mathtt{J}} \in C^\infty(\Gamma(U), \mathrm{d}\Omega^2(M))$ that satisfies*

$$F_{\mathtt{J}}(\omega) = \mathrm{d}\omega + \mathcal{N}_{\mathtt{J}}(\omega)$$

and such that for each $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$ there is a constant $c = c(\mathtt{J}, k, \alpha) > 0$ with

$$\|\mathcal{N}_{\mathtt{J}}(\omega) - \mathcal{N}_{\mathtt{J}}(\eta)\|_{C^{k,\alpha}} \leq c(\|\omega\|_{C^{k+1,\alpha}} + \|\eta\|_{C^{k+1,\alpha}})\|\omega - \eta\|_{C^{k+1,\alpha}}$$

for any $\omega, \eta \in \Gamma(U)$.

The proof of this proposition except the estimate on $\mathcal{N}_{\mathtt{J}}$ can be found in [22, proof of Theorem 4.5]. As the arguments are short, we have included them here for the reader's convenience.

Proof. Since $F_{\mathtt{J}}(0) = 0$ and the cohomology class doesn't change under homotopies, we have that $[F_{\mathtt{J}}(t\omega)] = 0 \in \mathrm{H}^3(M)$ for every $t \in [0,1]$. This proves the first point.

For the second point we observe that the Fundamental Theorem of Calculus and $F_{\mathtt{J}}(0) = 0$ imply

$$F_{\mathtt{J}}(\omega) = D_0 F_{\mathtt{J}}(\omega) + \int_0^1 \partial_t F_{\mathtt{J}}(t\omega) - D_0 F_{\mathtt{J}}(\omega)\,\mathrm{d}t$$

where $D_0 F_{\mathtt{J}}$ denotes the linearisation of $F_{\mathtt{J}}$ at the zero section. It therefore remains to check that $D_0 F_{\mathtt{J}}(\omega)$ equals $\mathrm{d}\omega$ and

$$\mathcal{N}_{\mathtt{J}}(\omega) \coloneqq \int_0^1 \partial_t F_{\mathtt{J}}(t\omega) - D_0 F_{\mathtt{J}}(\omega)\,\mathrm{d}t$$

satisfies the quadratic estimate.

For every point $x \in M$ there exists a vector field $v \in \Gamma(TY)$ such that in an open neighbourhood around x we have $\omega = \iota^*(i_v \phi)$ and $\varphi_t^v \circ \iota = \mathtt{J}_{t\omega}$ for $t \in (-\varepsilon, \varepsilon)$ where φ^v denotes the flow of v. Since ϕ is closed, we obtain around x:

$$D_0 F_{\mathtt{J}}(\omega) = \partial_t (\iota^* (\varphi_t^v)^* \phi)_{|t=0} = \iota^*\,\mathrm{d}(i_v \phi) = \mathrm{d}\omega.$$

The estimate for $\mathcal{N}_{\mathtt{J}}$ is standard but rather lengthy and can be found in Appendix A. ☐

Theorem 3.7. *Let $\alpha \in (0,1)$ be a fixed Hölder-exponent and $\beta, \gamma, c, R > 0$ be constants with $\beta > 2\gamma$. Then there are $T, c_v > 0$ depending only on β, γ, c, R with the following significance: Let ϕ, ϕ_0 be two closed G_2-structures on Y and $\iota\colon M \to Y$ be an immersed ϕ_0-coassociative submanifold with tubular neighbourhood $\mathtt{J}\colon U \subset \Lambda^2_+ M \to Y$ that satisfy*

1. $B_R(0) \subset U$
2. $\iota^*[\phi] = 0 \in \mathrm{H}^3(M)$
3. $\|\mathtt{J}^*(\phi - \phi_0)\|_{C^{1,\alpha}} \leq ct^\beta$
4. $\mathrm{d}\colon (\mathcal{H}^2_+)^\perp \subset \Omega^2_+(M) \to \Omega^3(M)$ *satisfies* $\|\omega\|_{C^{2,\alpha}} \leq ct^{-\gamma}\|\mathrm{d}\omega\|_{C^{1,\alpha}}$, *where $(\mathcal{H}^2_+)^\perp$ denotes the L^2-orthogonal complement of the space of harmonic self-dual 2-forms*
5. $\|\mathcal{N}_{\mathtt{J}}(\omega) - \mathcal{N}_{\mathtt{J}}(\eta)\|_{C^{1,\alpha}} \leq c(\|\omega\|_{C^{2,\alpha}} + \|\eta\|_{C^{2,\alpha}})\|\omega - \eta\|_{C^{2,\alpha}}$

for some $t \in (0, T)$. Then there is a unique section $\omega \in \Gamma(U) \cap (\mathcal{H}^2_+)^\perp$ with $\|\omega\|_{C^{2,\alpha}} \leq c_v t^{\beta-\gamma}$ (where $c_v > 0$ is determined in the proof) such that $\mathtt{J}_\omega$ is ϕ-coassociative.

The analogue statement for associative submanifolds can be found in [5, Proposition 3.19] and its proof carries over with only minor adaptations. We have included it here for the convenience of the reader.

Proof. To ease notation we drop the subscript $\mathtt{J}$ and instead write $F_{(0)}(\omega) := \mathtt{J}_\omega^*\phi_{(0)}$. Since $\mathrm{d}_{|\Omega^2_+(M)}\colon \Omega^2_+(M) \to \mathrm{d}\Omega^2(M)$ is surjective and $\mathrm{image}(F_{(0)}) \subset \mathrm{d}\Omega^2(M)$ by Proposition 3.6 and the second assumption, we can define

$$E(\omega) := \mathrm{d}^{-1}_{|(\mathcal{H}^2_+)^\perp \cap \Omega^2_+(M)}(F_0(\omega) - F(\omega) - \mathcal{N}_0(\omega)).$$

By our assumptions there is a positive constant $c_E = c_E(c, R)$ such that for every $r \in (0, R)$ and $\omega, \eta \in \overline{B_r(0)} \subset C^{2,\alpha}\Gamma(U)$ the following two inequalities hold:

$$\begin{aligned} \|E(0)\|_{C^{2,\alpha}} &\leq c_E t^{\beta-\gamma} \\ \|E(\omega) - E(\eta)\|_{C^{2,\alpha}} &\leq c_E(r + t^\beta)t^{-\gamma}\|\omega - \eta\|_{C^{2,\alpha}}. \end{aligned}$$

Therefore, E restricts to a contraction on $\overline{B_r(0)}$ provided that

$$c_E(r + t^\beta)t^{-\gamma} < 1 \qquad \text{and} \qquad c_E t^{\beta-\gamma} + c_E(r + t^\beta)t^{-\gamma} r \leq r.$$

This holds if we choose $T = T(\beta, \gamma, c, R)$ sufficiently small and for $t \in (0, T)$ the radius $r := 2c_E t^{\beta-\gamma}$.

Let now $\omega \in \overline{B_r(0)}$ be the unique fixpoint of E. By definition, this satisfies

$$0 = \mathrm{d}\omega + \mathcal{N}_0(\omega) - F_0(\omega) + F(\omega) = F(\omega)$$

and gives therefore rise to a ϕ-coassociative submanifold (of regularity $C^{2,\alpha}$). For sufficiently small T this section and the corresponding submanifold are smooth by elliptic regularity (cf. [20, Proposition 7.16]). □

Remark 3.8. If M in the previous theorem is compact and $\iota\colon M \to Y$ an embedding, then $\mathtt{J}_\omega$ will also be an embedding once t is sufficiently small.

3.1. The linear estimate for surface fibrations over tori

The following subsection establishes Point 4 of Theorem 3.7 in the case of Example 3.2. We quickly review the relevant set-up: Let Σ be a closed Riemann surface equipped with a Riemannian metric g_Σ and $\xi_2, \xi_3 \in \mathbb{R}^2$ be linearly independent. Furthermore, assume that there is a group action $\rho\colon \langle\xi_2, \xi_3\rangle_\mathbb{Z} \to \mathrm{Isom}(\Sigma)$. Our coassociative submanifold in Example 3.2 was then defined as

$$M = (\mathbb{R}^2 \times \Sigma)/\langle\xi_2, \xi_3\rangle_\mathbb{Z}$$

equipped with the induced metric coming from g_Σ and $g_{\mathbb{R}^2}$. We will also need the following rescaled version:

$$M_t := (\mathbb{R}^2 \times \Sigma)/\langle t^{-1}\xi_2, t^{-1}\xi_3\rangle_{\mathbb{Z}}$$

for $t > 0$ where $\rho_t\colon \langle t^{-1}\xi_2, t^{-1}\xi_3\rangle_{\mathbb{Z}} \to \mathrm{Isom}(\Sigma)$ is given by $\rho(t\cdot)$. The induced metric on M_t is denoted by g_t.

Observe that the natural projection $p_t\colon M_t \to T_t^2 := \mathbb{R}^2/\langle t^{-1}\xi_2, t^{-1}\xi_3\rangle_{\mathbb{Z}}$ gives rise to a fiber bundle. The orthogonal complement $H_t := V_t^\perp$ of its vertical tangent bundle $V_t = \ker(Dp_t)$ defines a flat Ehresmann connection. This induces a decomposition $\Omega^\ell(M_t) = \oplus_{p+q=\ell}\Omega^{p,q}$ with $\Omega^{p,q} := \Gamma(\Lambda^p H^* \otimes \Lambda^q V^*)$. Furthermore, the operator $\mathrm{d} + \mathrm{d}^*\colon \Omega^k(M_t) \to \Omega^{k+1}(M_t) \oplus \Omega^{k-1}(M_t)$ splits into

$$\mathrm{d}_H + \mathrm{d}_H^*\colon \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p-1,q} \quad \text{and} \quad \mathrm{d}_V + \mathrm{d}_V^*\colon \Omega^{p,q} \to \Omega^{p,q+1} \oplus \Omega^{p,q-1}.$$

Definition 3.9. We define the following operators acting on $\Omega^\ell(M_t)$:

1. Denote by $\Pi_t \in \mathrm{End}(\Omega^\ell(M_t))$ the L^2-projection onto $\ker(\mathrm{d} + \mathrm{d}^*)$.
2. For any $y \in T_t^2$, let $\mathrm{res}_y\colon \Omega^{p,q}(M_t) \to \Lambda^p T_y^* T_t^2 \otimes \Omega^q(p_t^{-1}(y))$ be the composition
$$\Omega^{p,q}(M_t) \to \Gamma(p_t^{-1}(y), \Lambda^p H^* \otimes \Lambda^q V^*) \cong \Lambda^p T_y^* T_t^2 \otimes \Omega^q(p_t^{-1}(y)).$$
3. The operator $\mathrm{d}_V + \mathrm{d}_V^*$ restricts for every $y \in T_t^2$ to an elliptic operator on $\Lambda^p T_y^* T_t^2 \otimes \Omega^q(p_t^{-1}(y))$. Denote by π_y the L^2-orthogonal projection onto its kernel.
4. Finally, denote by $\hat{\pi} \in \mathrm{End}(\Omega^\ell(M_t))$ the operator which maps $\omega \in \Omega^\ell(M_t)$ to the unique $\hat{\pi}(\omega) \in \Omega^\ell(M_t)$ with $\mathrm{res}_y \hat{\pi}(\omega) := \pi_y(\mathrm{res}_y \omega)$ for every $y \in T_t^2$.

Remark 3.10. In all examples of Section 4 the fiber bundle $M_t = T_t^2 \times \Sigma$ is trivial. In this case $\Omega^{p,q} \cong \Omega^p(T_t^2, \Omega^q(\Sigma))$ and $\mathrm{d}_V + \mathrm{d}_V^*$ becomes $\mathrm{d}_\Sigma + \mathrm{d}_\Sigma^*$ acting upon $\Omega^q(\Sigma)$. The operator $\hat{\pi}\colon \Omega^{p,q} \to \Omega^{p,q}$ is then simply the L^2-projection onto $\ker(\mathrm{d}_\Sigma + \mathrm{d}_\Sigma^*)$. Furthermore, $\mathrm{d}_H + \mathrm{d}_H^* = \mathrm{d}_{T_t^2} + \mathrm{d}_{T_t^2}^*$.

The main result of this section is the following Fredholm estimate:

Proposition 3.11. *For every $\alpha \in (0,1), k \geq 1$ there is a constant $c = c(k, \alpha, M_1, g_1)$ such that for every $t \in \mathbb{R}^+$ and $\omega \in \Omega^\ell(M_t)$,*

$$\|\omega\|_{C^{k,\alpha}} \leq c((1 + t^{-1})\| \mathrm{d}\omega + \mathrm{d}^*\omega\|_{C^{k-1,\alpha}} + \|\Pi_t\omega\|_{C^{k,\alpha}}).$$

For this we use the following results on harmonic forms on $\mathbb{R}^2 \times \Sigma$ and M_t which are an immediate consequence of [28, Lemma A.1].

Lemma 3.12 ([23, Corollary 4.13])**.** *Every harmonic $\omega \in \Omega^\ell(\mathbb{R}^2 \times \Sigma)$ with $\|\omega\|_{C^0} < \infty$ is a sum of terms of the form $\eta_1 \otimes \eta_2$, where $\eta_1 \in \Omega^p(\mathbb{R}^2)$ is constant and $\eta_2 \in \Omega^q(\Sigma)$ is harmonic. Identifying the space of constant forms on $\mathbb{R}^2$ with $\Lambda^*\mathbb{R}^2$ we therefore have*

$$\mathcal{H}^\ell(\mathbb{R}^2 \times \Sigma) \cap C^0\Omega^\ell(\mathbb{R}^2 \times \Sigma) = \bigoplus_{p+q} \Lambda^p\mathbb{R}^2 \otimes \mathcal{H}^q(\Sigma).$$

Corollary 3.13. *The pull-back of the quotient map $q_t\colon \mathbb{R}^2 \times \Sigma \to M_t$ induces an isomorphism*

$$\mathcal{H}^\ell(M_t) \cong \bigoplus_{p+q=\ell} \Lambda^p\mathbb{R}^2 \otimes \mathcal{H}^q(\Sigma)^{\langle \xi_2, \xi_3 \rangle_\mathbb{Z}},$$

where $\mathcal{H}^*(\Sigma)^{\langle \xi_2, \xi_3 \rangle_\mathbb{Z}}$ denotes the space of harmonic forms on Σ invariant under the action of $\langle \xi_2, \xi_3 \rangle_\mathbb{Z}$ by the pull-back of ρ.

The next two lemmas prove Proposition 3.11 for elements which respectively lie inside and orthogonal to the kernel of $\mathrm{d}_V + \mathrm{d}_V^*$.

Lemma 3.14. *For every $\omega \in \Omega^\ell(M_t)$*

$$\|(1 - \hat{\pi})\omega\|_{C^{k,\alpha}} \leq c_2 \|(\mathrm{d} + \mathrm{d}^*)(1 - \hat{\pi})\omega\|_{C^{k-1,\alpha}}$$

holds independently of t. It even holds on $\mathbb{R}^2 \times \Sigma$.

Proof. We prove the estimate on $\mathbb{R}^2 \times \Sigma$. Since the quotient maps are isometries, this implies the lemma.

Suppose the estimate does not hold on $\mathbb{R}^2 \times \Sigma$ to produce a contradiction. Then we find a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \Omega^\ell(\mathbb{R}^2 \times \Sigma)$ with

$$\|(1 - \hat{\pi})\omega_n\|_{C^{k,\alpha}} = 1 \quad \text{and} \quad \|(\mathrm{d} + \mathrm{d}^*)(1 - \hat{\pi})\omega_n\|_{C^{k-1,\alpha}} \to 0.$$

Since both expressions are invariant under translations, we can assume that

$$\|(1 - \hat{\pi})\omega_n\|_{C^{k,\alpha}(B_1(0)\times\Sigma)} \geq \frac{1}{4(k+1)} \qquad \text{for every } n \in \mathbb{N}. \tag{3}$$

By the Arzelà-Ascoli Theorem we find a subsequence (again denoted by $(\omega_n)_{n\in\mathbb{N}}$) such that $((1-\hat{\pi})\omega_n)_{n\in\mathbb{N}}$ converges in C^{k-1}_{loc} to $\omega_\infty \in C^{k-1}\Omega^\ell(\mathbb{R}^2\times\Sigma)$. This limit satisfies $\mathrm{d}\omega_\infty + \mathrm{d}^*\omega_\infty = 0$ (for $k = 1$ in the distributional sense) and is therefore smooth by elliptic regularity. As $\|\omega_\infty\|_{C^{k-1}} \leq 1$, Lemma 3.12 implies that ω_∞ is a sum of terms of the form $\eta_1 \otimes \eta_2$ where $\eta_1 \in \Omega^p(\mathbb{R}^2)$ is parallel and $\eta_2 \in \Omega^q(\Sigma)$ is harmonic. Therefore, $\omega_\infty = \hat{\pi}\omega_\infty$ and since $\hat{\pi}(1-\hat{\pi})\omega_n = 0$ for every $n \in \mathbb{N}$, we obtain further $\omega_\infty = \hat{\pi}\omega_\infty = 0$. However, bootstrapping improves the convergence inside $B_1(0) \times \Sigma$ to $C^{k,\alpha}$ and therefore $\|\omega_\infty\|_{C^{k,\alpha}(B_1(0)\times\Sigma)} \geq 1/(4(k+1))$ holds by (3). This gives the sought contradiction. $\square$

Lemma 3.15. *For every $t \in \mathbb{R}^+$ and $\omega \in \Omega^\ell(M_t)$ we have*

$$\|\hat{\pi}\omega\|_{C^0} \leq c_3 t^{-1}\|(\mathrm{d}+\mathrm{d}^*)\hat{\pi}\omega\|_{C^0} + \|\Pi\omega\|_{C^0},$$

where c_3 is independent of t.

Proof. We first prove the estimate for $t = 1$ and then for arbitrary t by scaling.

The estimate for $t = 1$ follows from Morrey's inequality and Fredholm theory.

The estimate for general $t \in \mathbb{R}^+$**:** Denote by $\Phi_t \colon M_1 \to M_t$ the map $[(y,z)] \mapsto [(t^{-1}y, z)]$. One can check that for any $\omega \in \Omega^\ell(M_t)$ we have

$$\begin{aligned} |\Phi_t^*\omega|_{g_1} &= \sum_{p+q=\ell} t^{-p}\big(|\omega^{p,q}|_{g_t} \circ \Phi_t\big), \\ (\mathrm{d}+\mathrm{d}_1^*)\Phi_t^*\omega &= \Phi_t^*\big(\mathrm{d}\omega + t^{-2}\,\mathrm{d}_{H_t}^*\omega + \mathrm{d}_{V_t}^*\omega\big), \\ \Pi_1\Phi_t^* &= \Phi_t^*\Pi_t, \end{aligned}$$

where $\omega^{p,q}$ denotes the projection onto $\Lambda^p H^* \otimes \Lambda^q V^*$ and where the last equality uses Corollary 3.13. The following estimate uses $\|\cdot\|_{C_t^0}$ and $\|\cdot\|_{C_1^0}$ to denote the C^0-norms with respect to the metrics g_t and g_1. Since the decomposition $\Lambda^\ell T^* M_t = \oplus_{p+q=\ell}\Lambda^p H^* \otimes \Lambda^q V^*$ is orthogonal, the previous step implies that for any $\omega \in \operatorname{im}\hat{\pi}$:

$$\begin{aligned} \|\omega\|_{C_t^0} &= \Big\|\sum_{p+q=\ell}\omega^{p,q}\Big\|_{C_t^0} = \Big\|\sum_{p+q=\ell} t^p\Phi_t^*\omega^{p,q}\Big\|_{C_1^0} \\ &\leq c_3\Bigg(\Big\|(\mathrm{d}+\mathrm{d}_1^*)\sum_{p+q=\ell} t^p\Phi_t^*\omega^{p,q}\Big\|_{C_1^0} + \Big\|\sum_{p+q=\ell} t^p\Pi_1\Phi_t^*\omega^{p,q}\Big\|_{C_1^0}\Bigg) \\ &= c_3\Bigg(\Big\|\sum_{p+q=\ell} t^p\Phi_t^*\,\mathrm{d}_{H_t}\omega^{p,q}\Big\|_{C_1^0} + \Big\|\sum_{p+q=\ell} t^{p-2}\Phi_t^*\,\mathrm{d}_{H_t}^*\omega^{p,q}\Big\|_{C_1^0} + \Big\|\Pi_t\omega\Big\|_{C_t^0}\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= c_3\Big(\Big\|\sum_{p+q=\ell} t^{-1}\,\mathrm{d}_{H_t}\omega^{p,q}\Big\|_{C^0_t} + \Big\|\sum_{p+q=\ell} t^{-1}\,\mathrm{d}^*_{H_t}\omega^{p,q}\Big\|_{C^0_t} + \Big\|\Pi_t\omega\Big\|_{C^0_t}\Big)\\
&= c_3\big(t^{-1}\|(\mathrm{d}+\mathrm{d}^*)\omega\|_{C^0_t} + \|\Pi_t\omega\|_{C^0_t}\big).
\end{aligned}
$$
□

Proof of Proposition 3.11. The Schauder estimate

$$\|\omega\|_{C^{k,\alpha}} \leq c_4\big(\|(\mathrm{d}+\mathrm{d}^*)\omega\|_{C^{k-1,\alpha}} + \|\omega\|_{C^0}\big)$$

and Lemmas 3.14 and 3.15 imply

$$\|\omega\|_{C^{k,\alpha}} \leq c_5(1+t^{-1})\big(\|(\mathrm{d}+\mathrm{d}^*)\omega\|_{C^{k-1,\alpha}} + \|(\mathrm{d}+\mathrm{d}^*)\hat{\pi}\omega\|_{C^{k-1,\alpha}} + \|\Pi\omega\|_{C^0}\big).$$

The observation $(\mathrm{d}+\mathrm{d}^*)\hat{\pi} = \hat{\pi}(\mathrm{d}+\mathrm{d}^*)$ finishes the proof. □

4. Examples

Let (Y_0, ϕ_0) be a flat G_2-orbifold together with a chosen set of resolution data. Denote by $\tilde{\phi}_t$ the closed G_2-structure from Definition 2.5 on the resolution $\hat{Y}_t$ and by ϕ_t the torsion-free G_2-structure of Theorem 2.7.

Assumption 4.1. Assume that for some element $((\hat{X}_S, \hat{\omega}_S, \tau_S), (\rho_S\colon G_S \to \mathrm{Isom}(\hat{X}_S)))$ of the resolution data we have

1. An embedded closed surface $\iota_\Sigma\colon \Sigma \to \hat{X}_S$ which is holomorphic with respect to $I_{\hat{\xi}_1} = \langle \underline{I}, \hat{\xi}_1\rangle$ for $\hat{\xi}_1 \in S^2 \subset \operatorname{Im}\mathbb{H}$.
2. Two linearly independent $\xi_2, \xi_3 \in \{\hat{\xi}_1\}^\perp \cap \Lambda_S \subset \operatorname{Im}\mathbb{H}$ such that $\rho(\xi_2)(\Sigma) = \Sigma = \rho(\xi_3)(\Sigma)$. Here Λ_S is the lattice $G_S \cap \operatorname{Im}\mathbb{H} < \operatorname{Im}\mathbb{H}$.

Proposition 4.2. *For every triple (Σ, ξ_2, ξ_3) as in Assumption 4.1 and for every choice of basepoint $q \in \operatorname{Im}\mathbb{H}$ there exists a $T > 0$ (independent of q) such that there is an immersed ϕ_t-coassociative submanifold $\iota_t\colon M \to \hat{Y}_t$ for $t \in (0, T)$. As t approaches 0, the induced volume on M shrinks to 0. Furthermore, if (G_S, ρ_S), (Σ, ξ_2, ξ_3), and q satisfy Points 3 and 4 listed at the end of Example 3.2, then there exists a free group action of $H_2 < G_S/\Lambda_S$ (as specified in Example 3.2) such that ι_t descends to an embedding of M/H_2 once t is sufficiently small.*

Proof. Throughout the proof we work with the rescaled G_2-structures $t^{-3}\tilde{\phi}_t$ and $t^{-3}\phi_t$. Example 3.2 gives rise to an immersed $(t^{-3}\tilde{\phi}_t)$-coassociative submanifold $\tilde{\iota}_t\colon M \to \hat{Y}_t$. Let $\mathsf{J}\colon U \to \hat{Y}_t$ be its tubular neighbourhood induced by the Levi–Civita connection associated to $t^{-2}\tilde{g}_t$. Without loss of generality we may assume that $\mathsf{J}(U) \subset \hat{V}$ where $\hat{V}$ is as in Definition 2.5.

The compactness of M, Theorem 2.7, and Proposition 3.11 imply that there exist t- and q-independent constants $c > 0$, $R > 0$, $\beta := 5/2$, and $\gamma := 1$ such that the first three points in Theorem 3.7 are satisfied. (All $C^{k,\alpha}$-norms are hereby taken with respect to $t^{-2}\tilde{g}_t$ and we tacitly assume $\alpha < 1/16$.) Furthermore, one can check that in our set-up the estimates in Lemma A.2 are independent of t. Thus, by enlarging c if necessary we may assume that the fourth point is also satisfied. We therefore, obtain a $(t^{-3}\phi_t)$-coassociative submanifold (or analogously, a ϕ_t-coassociative submanifold) $\iota_t\colon M \to \hat{Y}_t$ contained in $\hat{V}$ that satisfies

$$\|\iota_t - \widetilde{\iota}_t\|_{C^{2,\alpha}_{t^{-2}\tilde{g}_t}} \leq c_v t^{3/2} \qquad \text{and} \qquad \|\iota_t - \widetilde{\iota}_t\|_{C^{2,\alpha}_{\tilde{g}_t}} \leq c_v t^{1/2-\alpha}.$$

Direct inspection reveals that with respect to the family of metrics $\tilde{g}_t$ (and therefore also with respect to g_t) the fibers of M collapse to points as t tends to 0.

If (G_S, ρ_S), (Σ, ξ_2, ξ_3), and q satisfy the conditions given in Points 3 and 4, then there exists a free H_2-action on M under which $\widetilde{\iota}_t$ and the tubular neighbourhood chosen above are invariant (cf. Example 3.2). By the uniqueness of the section in Theorem 3.7, ι_t is also invariant under H_2 and descends therefore to $\overline{\iota_t}\colon M/H_2 \to \hat{Y}_t$. This is just a perturbation of the embedding $\overline{\widetilde{\iota}_t}\colon M/H_2 \to \hat{Y}_t$ (cf. Example 3.2) and therefore also an embedding once t is sufficiently small. □

Remark 4.3. The previous proposition produces coassociative submanifolds by perturbing the model-immersion from Example 3.2. This requires the choice of a basepoint $q \in \operatorname{Im}\mathbb{H}$. By varying this basepoint Proposition 4.2 produces a (up to reparametrization) 1-dimensional family of coassociative immersions.[3] Since $b^2_+(M) = 1$ (as $b_1(\Sigma) = 0$ for all immersed (holomorphic) Riemann surfaces in ALE hyperkähler 4-manifolds), all coassociative deformations of $\iota_t\colon M \to \hat{Y}_t$ are obtained this way (cf. Remark 3.3).

Example 4.4. Joyce [12, Examples 7–14] constructs seven examples of flat G_2-orbifolds whose respective singular strata are all given by tori: $S = T^3$. More precisely, neighbourhoods of the singularities in all these orbifolds are described by Example 2.1 with $X = \mathbb{H}/\Gamma_S$ for $\Gamma_S = C_2$. The corresponding Bieberbach groups are given by lattices $\mathrm{G}_S = \Lambda_S \cong \mathbb{Z}^3$ (whose exact form are irrelevant for our purpose) and the group actions $\rho_S\colon G_S \to \operatorname{Isom}(\mathbb{H}/\Gamma_S)$ are trivial.

[3] To make this statement rigorous one could use a family version of Theorem 3.7 as in [5, Proposition 3.19].

All these singularities can be resolved via Gibbons–Hawking spaces. This requires a choice of $\zeta \in \Delta^\circ$ (cf. Section 2.2.1). To simply find *some* resolution data, any such choice suffices.

In order to construct coassociative submanifolds, we pick for every singular stratum S two primitive elements $\xi_2, \xi_3 \in \Lambda_S$. Furthermore, we choose $\zeta \coloneqq [-\zeta_1, \zeta_1] \in \Delta^\circ$ with $0 \neq \zeta_1 \in \{\xi_2, \xi_3\}^\perp$. The corresponding Gibbons–Hawking space X_ζ contains a holomorphic sphere Σ such that (Σ, ξ_2, ξ_3) and any choice of $q \in \operatorname{Im}\mathbb{H}$ satisfy the conditions of Proposition 4.2. Furthermore, Points 3 and 4 of Example 3.2 are satisfied with $H_2 = \{1\}$ (as $G/\Lambda = \{1\}$). We therefore obtain embedded coassociative submanifolds in all the critical loci.

Remark 4.5. Joyce [12, Examples 3, 4, 5, 6, 15, 16] constructs further examples of flat G_2-orbifolds whose transverse singularities are modelled upon $\mathbb{H}/\Gamma_S$ for $\Gamma_S \in \{C_2, C_3\}$. [5, Examples 4.3, 4.4, 4.9] describes possible choices for the resolution data and points out holomorphic spheres inside the corresponding Gibbons–Hawking spaces. It is not difficult to check that every singular stratum of these orbifolds admits at least one choice of resolution data such that Proposition 4.2 gives rise to an embedded coassociative submanifold in the resolution.

The following examples all treat G_2-orbifolds constructed in [24, Section 5.4.3]. A neighbourhood of the singular strata in all these orbifolds can be described by Example 2.1 together with the data in Table 1. For this we list:

- The diffeomorphism type of the singular strata S.
- The orbifold group Γ_S such that the transverse singularity is modelled upon $X_S \coloneqq \mathbb{H}/\Gamma_S$.
- The generators of the Bieberbach group G_S as follows: Every G_S is generated by the lattice $\Lambda_S = \langle i, j, k\rangle \subset \operatorname{Im}\mathbb{H}$. Furthermore, we indicate whether the following two additional generators appear (✓ = appears, ✗ = does not appear):

$$\left(R_+, \frac{i+k}{2}\right) \qquad \text{and} \qquad \left(R_-, \frac{j}{2}\right)$$

where $R_\pm \in \mathrm{GL}(\Lambda_S) \cong \mathrm{GL}_3(\mathbb{Z})$ are given by

$$R_\pm \coloneqq \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4}$$

Table 1: Description of those singular strata appearing in [24, Section 5.3.4] which were not treated in [5, Section 4]. For each stratum the Bieberbach group G_S is generated by $\Lambda_S = \langle i, j, k \rangle \subset \operatorname{Im}\mathbb{H}$. Whether the two additional generators $(R_+, \frac{i+k}{2})$ and $(R_-, \frac{j}{2})$ (for $R_\pm$ as in (4)) appear is indicated (✓ = appears, ✗ = does not appear). The homomorphism $\rho_S \colon G_S \to \operatorname{Isom}(\mathbb{H}/\Gamma_S)$ maps Λ_S to Id and the other generators as indicated. A neighbourhood of any singular stratum is then described by Example 2.1 together with the respective data of this table

			G_S and $\rho_S\colon G_S \to \operatorname{Isom}(\mathbb{H}/\Gamma_S)$	
#	S	Γ_S	$(R_+, \frac{i+k}{2})$ with $\rho_S(R_+, \frac{i+k}{2})[q] = [iqi]$	$(R_-, \frac{j}{2})$ with $\rho_S(R_-, \frac{j}{2})[q] = [jqj]$
1.	T^3/C_2^2	C_2	✓	✓
2.	T^3/C_2	C_3	✗	✓
3.	T^3/C_2^2	C_3	✓	✓
4.	T^3/C_2^2	C_4	✓	✓
5.	T^3/C_2^2	C_6	✓	✓
6.	T^3/C_2^2	Dic_3	✓	✓

- The action $\rho_S \colon G_S \to \operatorname{Isom}(\mathbb{H}/\Gamma_S)$ of the generators of G_S as follows: The lattice Λ_S acts trivially in all examples. Furthermore, $(R_+, \frac{i+k}{2})$ and $(R_-, \frac{j}{2})$ act (whenever they appear as generators) via $\rho_S(R_+, \frac{i+k}{2})[q] = [iqi]$ and $\rho_S(R_-, \frac{j}{2})[q] = [jqj]$ for $[q] \in \mathbb{H}/\Gamma_S$, respectively.

Example 4.6. Reidegeld [24, Section 5.3.4] constructs an example of a flat G_2-orbifold whose singular strata split into two types. Both types can be described via Example 2.1 together with the data of rows 2 and 3 in Table 1, respectively.

These singularities can be resolved by Gibbons–Hawking spaces. This requires a choice of parameter $\zeta \in \Delta^\circ$ (cf. Section 2.2). All parameters such that the G_S-action lifts to the Gibbons–Hawking space X_ζ can be found in Appendix B.

The parameter $\zeta \coloneqq [-i, 0, i]$ works for both types of singularities. The corresponding Gibbons–Hawking space contains two I_i-holomorphic spheres which together with $\xi_2 \coloneqq j$, $\xi_3 \coloneqq k$, and any choice of $q \in \operatorname{Im}\mathbb{H}$ satisfy the conditions of Proposition 4.2. Thus, the resolution admits coassociative submanifolds in all the critical regions.

The conditions stated in Points 3 and 4 of Example 3.2 are satisfied by the above choices. However, the group H_2 in Point 4 depends on the value of the basepoint q which we set as $q := si$ with $s \in \mathbb{R}$ in the following. For the resolution of the singularities described by row 2 we then have $H_2 = C_2$ if and only if $s \in \frac{1}{2}\mathbb{Z}$ and for all other values $H_2 = \{1\}$. Note further that the coassociatives for $q = si$, $q = (s+1)i$, and $q = -si$ all agree (up to reparametrisation) in $\hat{Y}_t$. The family of coassociative submanifolds that we obtain by varying q (cf. Remark 4.3) is therefore parametrised by the interval $[0, 1/2]$ (more precisely, by $S^1/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts via reflection). For all inner points $s \in (0, 1/2)$ the corresponding submanifolds are embedded and for $s \in \{0, 1/2\}$ they factor through double cover over embedded rigid coassociatives.

Similarly, the coassociatives inside the resolution of the singularities described by row 3 are embedded for $q := si$ with $s \notin \frac{1}{4}\mathbb{Z}$ and factor through a double cover over an embedded submanifold for these critical values. Furthermore, coassociatives for $q = si, q = (s + 1/2)i$, and $q = -si$ are (up to reparametrisation) identified in $\hat{Y}_t$. As before, we therefore obtain that the deformation family is given by the interval $[0, 1/4]$ with embedded coassociatives for $s \in (0, 1/4)$ and double covers for $s \in \{0, 1/4\}$.

Example 4.7. Reidegeld [24, Section 5.3.4] constructs an example of a flat G_2-orbifold whose singular strata split into two types. Both types can be described via Example 2.1 together with the data of rows 1 and 4 in Table 1, respectively.

We choose a set of resolution data by a collection of certain Gibbons–Hawking spaces. This requires choices of the parameter $\zeta \in \Delta^\circ$ (cf. Section 2.2). All parameters such that the G_S-action lifts to the corresponding Gibbons–Hawking space X_ζ can be found in Appendix B.

As an example, we pick the following:

1. $\zeta = [-i, i]$ for strata of type described by row 1. The corresponding Gibbons–Hawking space contains one I_i-holomorphic sphere which together with $\xi_2 := j$, $\xi_3 := k$, and any basepoint $q \in \operatorname{Im}\mathbb{H}$ satisfies the conditions of Proposition 4.2. As in Example 4.6, these are embedded for generic choices of $q \in \operatorname{Im}\mathbb{H}$ and otherwise factor through a double-cover over an embedded rigid coassociative.
4. $\zeta = [-2i, -i, i, 2i]$ for strata of type described by row 4. The associated Gibbons–Hawking space contains 3 I_i-holomorphic spheres which together with $\xi_2 := j$, $\xi_3 := k$, and any $q \in \operatorname{Im}\mathbb{H}$ satisfy the conditions of Proposition 4.2. The resulting submanifolds are again embedded for generic q and factor otherwise through double-cover over embedded coassociatives.

Example 4.8. Reidegeld [24, Section 5.3.4] constructs an example of a flat G_2-orbifold whose singular strata split into four types. All types can be described via Example 2.1 together with the data of rows 1, 2, 3, and 5 in Table 1, respectively.

All singularities can be resolved by certain Gibbons–Hawking spaces. This requires choices of the parameter $\zeta \in \Delta^\circ$ (cf. Section 2.2). All parameters such that the G_S-action lifts to the Gibbons–Hawking space X_ζ can be found in Appendix B.

The singular strata described by rows 1-3 have been treated in the previous examples. For the strata of type 5 we may exemplary pick the element $\zeta := [-3i, -2i, -i, i, 2i, 3i]$. The corresponding Gibbons–Hawking space contains five I_i-holomorphic spheres. Each one of these together with $\xi_2 := j, \xi_3 := k$, and any $q \in \operatorname{Im}\mathbb{H}$ satisfies the conditions of Proposition 4.2. As in Example 4.6, these are generically embedded and factor otherwise through double-cover over embedded coassociatives.

Example 4.9. Reidegeld [24, Section 5.3.4] constructs an example of a flat G_2-orbifold whose singular strata split into four types. All types can be described via Example 2.1 together with the data of rows 1, 3, 4, and 6 in Table 1, respectively.

Singularities of types described by rows 1, 3, and 4 have been treated in the previous examples. We therefore focus on the strata determined by row 6. A resolving ALE space can be constructed via Kronheimer's method and requires a choice of parameter $\zeta \in \Delta^\circ$ (cf. Section 2.2.2). All parameters such that the G_S-action lifts to the ALE space X_ζ can be found in Appendix B.

For example, $\zeta := [0, i, 2i, 3i, 4i]$ works and the corresponding ALE space contains 5 I_i-holomorphic spheres (cf. Section 2.2.2). Each one together with $\xi_2 := j$, $\xi_3 := k$, and any choice of $q \in \operatorname{Im}\mathbb{H}$ satisfies the conditions of Proposition 4.2 and gives rise to a coassociative submanifold. The lift of the action of $\rho(R_-, \frac{j}{2})$ to X_ζ fixes four of these spheres and maps the fifth sphere to one that does not intersect the original sphere. One of the resulting coassociative submanifolds is therefore embedded for every $q \in \operatorname{Im}\mathbb{H}$ and the other four behave as in Example 4.6.

Remark 4.10. Reidegeld [24, Section 5.3.4] constructs two further examples of orbifolds whose transverse singularities are modelled on $\mathbb{H}/\Gamma_S$ for $\Gamma_S \in \{C_2, C_4, \mathrm{Dic}_2\}$. These are treated in [5, Examples 4.5 and 4.6] and we remark that Proposition 4.2 produces coassociative submanifolds in all critical loci.

Remark 4.11. In [16] Joyce and Karigiannis extended the generalised Kummer construction to certain non-flat G_2-orbifolds. If similar estimates as in Theorem 2.7 continue to hold, then it seems plausible that the construction method for coassociative submanifolds presented in the current article can be extended to these new manifolds.

Remark 4.12. Assume for simplicity the following situation: Let (Y_0, ϕ_0) be a flat G_2-orbifold whose singularities are all modelled upon $T^3 \times \mathbb{H}/C_2$ where $T^3 = \operatorname{Im}\mathbb{H}/\mathbb{Z}^3$ (this is for example the case in [12, Example 3]). These singularities can be resolved by Gibbons–Hawking spaces X_ζ for any choice of parameter $\zeta \coloneqq [-x, x] \in (\operatorname{Im}\mathbb{H} \setminus \{0\})^2/\{\pm 1\}$. However, in order to apply Proposition 4.2, we need the line $\ell \coloneqq \mathbb{R}x$ to intersect $\mathbb{Z}^3 \subset \operatorname{Im}\mathbb{H}$ (this is precisely the second condition of Assumption 4.1). The following regards the situation where this condition fails:

Assume that $x \in \operatorname{Im}\mathbb{H} \setminus \{0\}$ is such that the line $\mathbb{R}x \subset \operatorname{Im}\mathbb{H}$ is 'irrational' (i.e. does not intersect $\mathbb{Z}^3$). Then one could approximate x by a sequence $(x_n)_{n\in\mathbb{N}} \subset \operatorname{Im}\mathbb{H}\setminus\{0\}$ such that all corresponding lines $\mathbb{R}x_n$ are rational (i.e. do intersect $\mathbb{Z}^3$). For each resolution by $T^3 \times X_{\zeta_n}$ with $\zeta_n \coloneqq [-x_n, x_n]$ we obtain a ϕ_t-coassociative submanifold $M_n \subset \hat{Y}_t$ for $t < T_n$ by Proposition 4.2. However, as $n \to \infty$ we have that $T_n \to 0$. Thus (after rescaling) these coassociatives only converge to a (non-compact) coassociative inside the limiting $\mathbb{R}^3 \times X_\zeta$ for $\zeta = [-x, x]$. One might however hope that once $x_n \to x$ converges sufficiently faster than $T_n \to 0$,[4] then *some instance* of this limiting coassociative is already visible inside the resolution by the irrational $T^3 \times X_\zeta$ shortly before the orbifold limit is reached.

Unfortunately, Theorem 3.7 seems to be of little help when addressing this question. This is because the two G_2-structures $\phi_t(\zeta_n)$ and $\phi_t(\zeta)$ on $\hat{Y}_t$ constructed by resolving respectively with a rational ζ_n and the irrational ζ lie in different cohomology classes. The $\phi_t(\zeta_n)$-coassociatives constructed in Proposition 4.2 can then not be perturbed further to $\phi_t(\zeta)$-coassociatives because the second condition of Theorem 3.7 is violated.

Appendix A. The quadratic estimate

This section establishes the quadratic estimate for the map $\mathcal{N}_{\mathrm{J}}$ in Proposition 3.6.

Lemma A.1. *Let $v, w \in \Gamma(TM)$ be vector fields and $\eta \in \Omega^\ell(M)$ be an ℓ-form. Then the following identities hold for any torsion free connection ∇:*

$$\mathcal{L}_w \eta = \nabla_w \eta + \langle \nabla w \wedge \eta \rangle$$

[4] See [2] for an overview on the measure-theoretic properties of irrational numbers that are approximable by rationals with a given rate.

$$\pounds_v \pounds_w \eta = \nabla_v \nabla_w \eta + \langle \nabla w \wedge \nabla_v \eta \rangle + \langle \nabla^2_{v,\cdot} w \wedge \eta \rangle \\ + \langle \nabla v \wedge \nabla_w \eta \rangle + \langle \nabla v \wedge \langle \nabla w \wedge \eta \rangle \rangle$$

where $\langle \cdot \wedge \cdot \rangle \colon T^*M \otimes TM \otimes \Lambda^k T^*M \to \Lambda^k T^*M$ *contracts the second and third* $TM \otimes T^*M \cong \underline{\mathbb{R}}$ *component and takes the the wedge product afterwards. Furthermore,* $\nabla^2_{v,w} = \nabla_v \nabla_w - \nabla_{\nabla_v w}$ *denotes the second covariant derivative.*

Proof. Since ∇ is torsion-free, the equality

$$(\pounds_w \eta)(u_1, \ldots, u_k) = (\nabla_w \eta)(u_1, \ldots, u_k) \\ + \sum_i (-1)^{i+1} \eta(\nabla_{u_i} w, u_1, \ldots \hat{u}_i, \ldots, u_k)$$

holds. This is the first identity and the second is proven similarly. $\square$

Recall from Section 3 that $\iota \colon M \to Y$ is a coassociative immersion equipped with a tubular neighbourhood $\mathtt{J} \colon U \to Y$. Furthermore, let $F_{\mathtt{J}}$ and $\mathcal{N}_{\mathtt{J}}$ be defined as in Proposition 3.6.

Lemma A.2. *Let* $u, v, w \in \Gamma(U)$. *The second derivative of* $F_{\mathtt{J}}$ *can be estimated by*

$$\|(D_u DF_{\mathtt{J}})(v)(w)\|_{C^{k,\alpha}} \leq c(1 + \|u\|_{C^{k+1,\alpha}})\|v\|_{C^{k+1,\alpha}}\|w\|_{C^{k+1,\alpha}}$$

where the differential is a map $DF_{\mathtt{J}} \colon \Gamma(U) \to \mathrm{Hom}(\Omega^2_+(M), \Omega^3(M))$ *and accordingly,* $(D_u DF_{\mathtt{J}})(v) \in \mathrm{Hom}(\Omega^2_+(M), \Omega^3(M))$.

Proof. Lift the sections $v, w \in \Gamma_M(U)$ to vector fields $\hat{v}, \hat{w} \in \Gamma_U(TU)$ via $\hat{v}(u_m) := \frac{\mathrm{d}}{\mathrm{d}t} u_m + tv(m)_{|t=0}$ where $m \in M$ denotes the basepoint of u_m (and analogously for $\hat{w}$). Denote their respective flows by $\varphi^{\hat{v}}$, and $\varphi^{\hat{w}}$. Then

$$(D_u DF_{\mathtt{J}})(v)(w) = \partial_t \partial_s F_{\mathtt{J}}(u + tv + sw) = \partial_t \partial_s u^* (\varphi^{\hat{v}}_t)^* (\varphi^{\hat{w}}_s)^* (\mathtt{J}^* \phi) \\ = u^* \pounds_{\hat{v}} \pounds_{\hat{w}} (\mathtt{J}^* \phi).$$

Thus, $\|D_u DF_{\mathtt{J}}(v)(w)\|_{C^{k,\alpha}} \leq c_1 \|Du\|_{C^{k,\alpha}} \|\pounds_{\hat{w}} \pounds_{\hat{v}} (\mathtt{J}^* \phi)\|_{C^{k,\alpha}}$.

The connection on $\Lambda^2_+ T^*M$ induces a decomposition of the tangent bundle TU into vertical V and horizontal component H. The vertical part of the differential $Du \in \Gamma(\mathrm{Hom}(TM, u^*TU))$ is given (up to the identification of u^*V with $\Lambda^2_+ T^*M$) by ∇u and the horizontal component is up to the identification $u^*H \cong TM$ given by the identity map. Therefore, $\|D_u DF_{\mathtt{J}}(v)(w)\|_{C^{k,\alpha}} \leq c_2(1 + \|\nabla u\|_{C^{k,\alpha}}) \|\pounds_{\hat{w}} \pounds_{\hat{v}} (\mathtt{J}^* \phi)\|_{C^{k,\alpha}}$.

To estimate the Lie derivative, we invoke the previous lemma. The only two terms that might require an explanation are $\nabla_{\hat{v}}\nabla_{\hat{w}}(\mathtt{J}^*\phi)$ and $\langle\nabla^2_{\hat{v},\cdot}\hat{w}\wedge(\mathtt{J}^*\phi)\rangle$. The first can be estimated by

$$\begin{aligned}\|\nabla_{\hat{v}}\nabla_{\hat{w}}(\mathtt{J}^*\phi)\|_{C^{k,\alpha}} &\leq \|i_{\hat{w}}(\nabla_{\hat{v}}\nabla(\mathtt{J}^*\phi))\|_{C^{k,\alpha}} + \|\nabla_{\nabla_{\hat{v}}\hat{w}}(\mathtt{J}^*\phi))\|_{C^{k,\alpha}}\\ &\leq c_3\|w\|_{C^{k,\alpha}}\|v\|_{C^{k,\alpha}}(\|\nabla\nabla(\mathtt{J}^*\phi)\|_{C^{k,\alpha}} + \|\nabla(\mathtt{J}^*\phi)\|_{C^{k,\alpha}}).\end{aligned}$$

Note that in the second line there is no additional derivative of $\hat{w}$ coming from $\nabla_{\hat{v}}\hat{w}$. This is because $(\nabla_{\hat{v}}\hat{w})(u_m)$ only depends on $v(m)$ and $w(m)$. (In fact, one can define a map $\Phi\colon U\times_M U\to TU$ by $(u_1,u_2)\mapsto\nabla_{\hat{u}_1}\hat{u}_2$.)

Similarly,

$$\|\langle\nabla^2_{v,\cdot}w\wedge(\mathtt{J}^*\phi)\rangle\|_{C^{k,\alpha}} \leq c_4(\|v\|_{C^{k,\alpha}}\|w\|_{C^{k,\alpha}} + \|v\|_{C^{k+1,\alpha}}\|w\|_{C^{k+1,\alpha}})\|\mathtt{J}^*\phi\|_{C^{k,\alpha}}$$

which together with with the observation that $\|\mathtt{J}^*\phi\|_{C^{k+2,\alpha}}$ is bounded finishes the proof. □

Proposition A.3. *The quadratic estimate*

$$\begin{aligned}\|\mathcal{N}_{\mathtt{J}}(v)-\mathcal{N}_{\mathtt{J}}(w)\|_{C^{k,\alpha}} \leq c(1&+\|v\|_{C^{k+1,\alpha}}+\|w\|_{C^{k+1,\alpha}}+\|v-w\|_{C^{k+1,\alpha}})\\ &\times\|v-w\|_{C^{k+1,\alpha}}(\|v\|_{C^{k+1,\alpha}}+\|w\|_{C^{k+1,\alpha}})\end{aligned}$$

holds.

Proof. This follows immediately from the previous lemma and

$$\begin{aligned}\mathcal{N}_{\mathtt{J}}(v)-\mathcal{N}_{\mathtt{J}}(w) &= \int_0^1 D_{tv}F_{\mathtt{J}}(v-w) - D_0F_{\mathtt{J}}(v-w)\\ &\quad + (D_{tv}F_{\mathtt{J}} - D_{tw}F_{\mathtt{J}})(w)\,\mathrm{d}t\\ &= \int_0^1\int_0^t (D_{sv}DF_{\mathtt{J}})(v)(v-w)\\ &\quad + (D_{tw+s(v-w)}DF_{\mathtt{J}})(v-w)(w)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$ □

Appendix B. Resolution data for Reidegeld's orbifolds

In this section we describe how to construct resolution data for the G_2-orbifolds of [24, Section 5.3.4] that were used in Section 4. A neighbourhood of the singular strata in all these orbifolds can be described by Example 2.1 using the data from Table 1 (cf. Section 4).

The resolution data for singular strata which are described in rows 1.-5. of Table 1 can be constructed via the Gibbons–Hawking Ansatz (cf. Section 2.2.1) or, equivalently, via Kronheimer's construction (cf. Section 2.2.2).

Recall that in order to obtain a smooth manifold we need to choose for either method a parameter ζ from

$$\Delta^\circ \coloneqq \{[\zeta_1, \dots, \zeta_N] \in (\operatorname{Im}\mathbb{H})^N/S_N \mid \zeta_1 + \dots + \zeta_N = 0 \text{ and } \zeta_i \neq \zeta_j \text{ for } i \neq j\}.$$

To lift the action of G_S we need to restrict further to the following sets:

$$\begin{aligned}
&1.\ (\Delta^\circ)^{R_+,(-R_-)} = && \{[\zeta_1, R_+\zeta_1] \in \Delta^\circ \mid \zeta_1 \in \big((\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\big) \setminus \mathbb{R}i\} \cup \\
& && \{[-\zeta_1, \zeta_1] \in \Delta^\circ \mid \zeta_1 \in \mathbb{R}i\} \\
&2.\ (\Delta^\circ)^{(-R_-)} = && \{[\zeta_1, \zeta_2, -R_-\zeta_2] \in \Delta^\circ \mid \zeta_1 \in (\mathbb{R}j)^\perp, \zeta_2 \notin (\mathbb{R}j)^\perp\} \cup \\
& && \{[\zeta_1, \zeta_2, \zeta_3] \in \Delta^\circ \mid \zeta_1, \zeta_2, \zeta_3 \in (\mathbb{R}j)^\perp\} \\
&3.\ (\Delta^\circ)^{R_+,(-R_-)} = && \{[\zeta_1, \zeta_2, R_+\zeta_2] \in \Delta^\circ \mid \zeta_1 \in \mathbb{R}i, \\
& && \qquad \zeta_2 \in \big((\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\big) \setminus \mathbb{R}i\} \cup \\
& && \{[\zeta_1, \zeta_2, \zeta_3] \in \Delta^\circ \mid \zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}i\} \\
&4.\ (\Delta^\circ)^{R_+,(-R_-)} = && \{[\zeta_1, R_+\zeta_1, -R_-\zeta_1, -R_+R_-\zeta_1] \in \Delta^\circ \mid \\
& && \qquad \zeta_1 \notin (\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\} \cup \\
& && \{[\zeta_1, R_+\zeta_1, \zeta_2, R_+\zeta_2] \in \Delta^\circ \mid \\
& && \qquad \zeta_1, \zeta_2 \in \big((\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\big) \setminus \mathbb{R}i\} \cup \\
& && \{[\zeta_1, \zeta_2, \zeta_3, R_+\zeta_3] \in \Delta^\circ \mid \\
& && \qquad \zeta_1, \zeta_2 \in \mathbb{R}i, \zeta_3 \in \big((\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\big) \setminus \mathbb{R}i\} \cup \\
& && \{[\zeta_1, \zeta_2, \zeta_3, \zeta_4] \in \Delta^\circ \mid \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{R}i\} \\
&5.\ (\Delta^\circ)^{R_+,(-R_-)} = && \{[\zeta_1, R_+\zeta_1, \zeta_2, R_+\zeta_2, -R_-\zeta_2, -R_+R_-\zeta_2] \in \Delta^\circ \mid \\
& && \qquad \zeta_1 \in \big((\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\big) \setminus \mathbb{R}i, \\
& && \qquad \zeta_2 \notin (\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\} \cup \\
& && \{[\zeta_1, \zeta_2, \zeta_3, R_+\zeta_3, -R_-\zeta_3, -R_+R_-\zeta_3] \in \Delta^\circ \mid \\
& && \qquad \zeta_1, \zeta_2 \in \mathbb{R}i, \zeta_2 \notin (\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\} \cup \\
& && \{[\zeta_1, R_+\zeta_1, \zeta_2, R_+\zeta_2, \zeta_3, R_+\zeta_3] \in \Delta^\circ \mid \\
& && \qquad \zeta_1, \zeta_2, \zeta_3 \in \big((\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\big) \setminus \mathbb{R}i\} \cup \\
& && \{[\zeta_1, \zeta_2, \zeta_3, R_+\zeta_3, \zeta_4, R_+\zeta_4] \in \Delta^\circ \mid \\
& && \qquad \zeta_1, \zeta_2 \in \mathbb{R}i, \zeta_3, \zeta_4 \in \big((\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\big) \setminus \mathbb{R}i\} \cup \\
& && \{[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, R_+\zeta_5] \in \Delta^\circ \mid \\
& && \qquad \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{R}i, \\
& && \qquad \zeta_5 \in \big((\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp\big) \setminus \mathbb{R}i\} \cup \\
& && \{[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6] \in \Delta^\circ \mid \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6 \in \mathbb{R}i\}.
\end{aligned}$$

To resolve singular strata described in row 6. of Table 1 we need Kronheimer's construction as reviewed in Section 2.2.2.

The root system of D_5 is given by (cf. [3, Chapter VI.4.8])

$$\Phi = \{\pm e_i \pm e_j \in \mathbb{R}^5 \mid i \neq j \in \{1, \ldots, 5\}\}$$

and one possible choice of simple roots consists of

$$\alpha_i \coloneqq e_i - e_{i+1} \text{ for } i = 1, \ldots, 4 \quad \text{and} \quad \alpha_5 \coloneqq e_5 + e_4.$$

Its Weyl group $W = C_2^4 \rtimes S_5$ acts on $\mathbb{R}^5$ by permuting and changing the signs of an even number of coordinates. Thus, in order to obtain a smooth manifold, we must choose the value of the moment map from

$$\Delta^\circ = \{[\zeta_1, \ldots, \zeta_5] \in ((\operatorname{Im} \mathbb{H})^* \otimes \mathbb{R}^5)/W \mid \zeta_i \neq \pm\zeta_j \text{ for } i \neq j\}.$$

In order to lift the action of G_S, we need to restrict further to a value which is invariant under the $\Lambda^2_+ \rho_S(g) \otimes \mathrm{Ad}^*_{C_{\rho_S(g)}}$ for any $g \in G_S$. Here is how one can understand the action $\mathrm{Ad}^*_{C_{\rho_S(g)}} \colon \mathfrak{h} \to \mathfrak{h}$ (under the identification $\mathfrak{h} \cong \mathbb{R}\Phi$ via the inner product): Let $(R_1, \rho_1), \ldots, (R_5, \rho_5)$ be irreducible (non-trivial, complex) representations of Dic_3 which are pairwise non-isomorphic. Identify the set $\{(R_1, \rho_1), \ldots, (R_5, \rho_5)\}$ with $\{\alpha_1, \ldots, \alpha_5\}$ as in [18, Section 2]. Then $\mathrm{Ad}^*_{C_{\rho_S(g)}}$ acts on $\{\alpha_1, \ldots, \alpha_5\} \cong \{(R_1, \rho_1), \ldots, (R_5, \rho_5)\}$ by mapping (R_i, ρ_i) to the irreducible representation $R_j \cong (R_i, \rho_i \circ C_{\rho_S(g)})$ (where we precompose the representation with conjugation by $\rho_S(g) \in N_{\mathrm{SO}(\mathbb{H})}(\mathrm{Dic}_3)$). The map $\mathrm{Ad}^*_{C_{\rho_S(g)}} \colon \mathfrak{h} \to \mathfrak{h}$ is the linear extension of this action.[5]

One can then check that $\mathrm{Ad}^*_{C_{\rho_S(R_-, \frac{j}{2})}} = 1$ and $\mathrm{Ad}^*_{C_{\rho_S(R_+, \frac{i+k}{2})}} = \sigma_5$, where $\sigma_5 \colon \mathbb{R}^5 \to \mathbb{R}^5$ is the reflection $(x_1, \ldots, x_5) \mapsto (x_1, \ldots, x_4, -x_5)$. In order to lift the action of G_S, we therefore need to choose a parameter from the following set:

$$\begin{aligned} 6.\quad (\Delta^\circ)^{(R_+\sigma_5), R_-} = &\{[\zeta_1, R_+\zeta_1, R_-\zeta_1, R_+R_-\zeta_1, \zeta_2] \in \Delta^\circ \mid \\ &\qquad \zeta_1 \notin (\mathbb{R}i)^\perp \cup (\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp \text{ and } \zeta_2 \in \mathbb{R}j\} \cup \\ &\{[\zeta_1, R_a\zeta_1, \zeta_2, R_b\zeta_2, \zeta_3] \in \Delta^\circ \mid \\ &\qquad \zeta_1, \zeta_2 \in ((\mathbb{R}i)^\perp \cup (\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp) \\ &\qquad\qquad \setminus (\mathbb{R}i \cup \mathbb{R}j \cup \mathbb{R}k), \end{aligned}$$

[5] This can be seen when using the isomorphism $\tau \colon \mathfrak{z}^* \to \mathfrak{h}$ in [18, Equation (2.7)]. Note further, that [18, Proposition 4.1] implies that τ and the isomorphism in [18, Section 4] only differ by a conformal transformation.

$$
\begin{aligned}
&\qquad R_a, R_b \in \{R_+, R_-, R_+R_-\},\ \text{and}\ \zeta_3 \in \mathbb{R}j\} \cup \\
&\{[\zeta_1, R_a\zeta_1, \zeta_2, t\zeta_2, \zeta_3] \in \Delta^\circ \mid \\
&\qquad \zeta_1 \in ((\mathbb{R}i)^\perp \cup (\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp) \\
&\qquad\qquad \setminus (\mathbb{R}i \cup \mathbb{R}j \cup \mathbb{R}k), \\
&\qquad R_a \in \{R_+, R_-, R_+R_-\}, \\
&\qquad \zeta_2 \in \mathbb{R}i \cup \mathbb{R}j \cup \mathbb{R}k, t \in \mathbb{R} \setminus \{-1\}, \\
&\qquad \text{and}\ \zeta_3 \in \mathbb{R}j\} \cup \\
&\{[\zeta_1, R_a\zeta_1, \zeta_2, \zeta_3, 0] \in \Delta^\circ \mid \\
&\qquad \zeta_1 \in ((\mathbb{R}i)^\perp \cup (\mathbb{R}j)^\perp \cup (\mathbb{R}k)^\perp) \\
&\qquad\qquad \setminus (\mathbb{R}i \cup \mathbb{R}j \cup \mathbb{R}k), \\
&\qquad R_a \in \{R_+, R_-, R_+R_-\}, \zeta_2 \in \mathbb{R}i, \\
&\qquad \text{and}\ \zeta_3 \in \mathbb{R}k\} \cup \\
&\{[\zeta_1, t_1\zeta_1, \zeta_2, t_2\zeta_2, \zeta_3] \in \Delta^\circ \mid \zeta_1, \zeta_2 \in \mathbb{R}i \cup \mathbb{R}j \cup \mathbb{R}k, \\
&\qquad t_1, t_2 \in \mathbb{R} \setminus \{-1\},\ \text{and}\ \zeta_3 \in \mathbb{R}j\} \cup \\
&\{[\zeta_1, t\zeta_1, \zeta_2, \zeta_3, 0] \in \Delta^\circ \mid \zeta_1 \in \mathbb{R}i \cup \mathbb{R}j \cup \mathbb{R}k, \\
&\qquad t \in \mathbb{R} \setminus \{-1\}, \zeta_2 \in \mathbb{R}i,\ \text{and}\ \zeta_3 \in \mathbb{R}k\}.
\end{aligned}
$$

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Dominik Gutwein
Humboldt-Universität zu Berlin
Berlin
Germany
E-mail: gutweind@mathematik.hu-berlin.de