

A bi-variant algebraic cobordism via correspondences

SHOJI YOKURA

Abstract: A bi-variant theory $\mathbb{B}(X, Y)$ defined for a pair (X, Y) is a theory satisfying properties similar to those of Fulton–MacPherson’s bivariant theory $\mathbb{B}(X \xrightarrow{f} Y)$ defined for a morphism $f : X \rightarrow Y$. In this paper, using correspondences we construct a bi-variant algebraic cobordism $\Omega^{*,\sharp}(X, Y)$ such that $\Omega^{*,\sharp}(X, pt)$ is isomorphic to Lee–Pandharipande’s algebraic cobordism of vector bundles $\Omega_{-*,\sharp}(X)$. In particular, $\Omega^*(X, pt) = \Omega^{*,0}(X, pt)$ is isomorphic to Levine–Morel’s algebraic cobordism $\Omega_{-*}(X)$. Namely, $\Omega^{*,\sharp}(X, Y)$ is a *bi-variant version* of Lee–Pandharipande’s algebraic cobordism of bundles $\Omega_{*,\sharp}(X)$.

Keywords: (Co)bordism, algebraic cobordism, algebraic cobordism of bundles, correspondence.

1. Introduction

This is a continuation of our previous works [23] and [24] and also partially motivated by a recent book [13] by D. Gaitsgory and N. Rozenblyum.

In [12] W. Fulton and R. MacPherson have introduced bivariant theory $\mathbb{B}(X \xrightarrow{f} Y)$ with an aim to deal with Riemann–Roch type theorems for singular spaces and to unify them. $\mathbb{B}_*(X) := \mathbb{B}^{-*}(X \rightarrow pt)$ becomes a covariant functor and $\mathbb{B}^*(X) := \mathbb{B}^*(X \xrightarrow{\text{id}_X} X)$ a contravariant functor. In this sense $\mathbb{B}(X \xrightarrow{f} Y)$ is called a *bivariant* theory. In [21] (cf. [22] and [25]) the author introduced *an oriented bivariant theory* and *a universal oriented bivariant theory* in order to eventually construct a bivariant algebraic cobordism $\Omega^*(X \xrightarrow{f} Y)$ which is supposed to be a bivariant-theoretic version of Levine–Morel’s algebraic cobordism¹ $\Omega_*(X)$ in such a way that the covariant part $\Omega^{-*}(X \rightarrow pt)$ becomes isomorphic to Levine–Morel’s algebraic cobordism $\Omega_*(X)$, thus $\Omega^*(X \xrightarrow{\text{id}_X} X)$ would become really a new contravariant “cobordism”.

Received June 15, 2022.

2010 Mathematics Subject Classification: Primary 55N35, 55N22; secondary 14C17, 14C40, 14F99, 19E99.

¹It is called “cobordism”, but it is a theory of “bordism”.

Our general universal bivariant theory $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Y)$ defined on a category \mathcal{C} , equipped with a class \mathcal{C} of *confined* morphisms, a class \mathcal{S} of *specialized* morphisms and a class \mathcal{Ind} of independent squares, is defined to be the free abelian group generated by the set of isomorphism classes of confined morphisms $c : W \rightarrow X$ such that *the composite of c and f is a specialized morphism*:

(1.1)

$$\begin{array}{ccc}
 & W & \\
 \mathcal{C} \ni c \swarrow & & \searrow f \circ c \in \mathcal{S} \\
 X & \xrightarrow{f} & Y.
 \end{array}$$

Here we note that confined and specialized morphisms are both *closed under composition and base change* and *the identity morphisms are defined to be both confined and specialized*. In the case of the category of complex algebraic varieties, a proper morphism is confined and a smooth morphism is specialized and in [21] we consider the universal bivariant theory $\mathbb{M}_{sm}^{prop}(X \xrightarrow{f} Y)$ where $prop = \mathcal{C}$ is the class of proper morphisms and $sm = \mathcal{S}$ the class of smooth morphisms, in order to construct the above-mentioned bivariant algebraic cobordism $\Omega^*(X \xrightarrow{f} Y)$. If the coproduct \sqcup is well-defined in a category \mathcal{C} , then we can consider the Grothendieck group $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Y)^+$, which is the quotient group $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Y)$ modulo the relations

$$[W \xrightarrow{h} X] + [V \xrightarrow{k} X] = [W \sqcup V \xrightarrow{h \sqcup k} X].$$

In [2] (also see his Ph.D. Thesis [4]) T. Annala considered the universal bivariant theory $\mathbb{M}_{quism}^{prop}(X \xrightarrow{f} Y)^+ \otimes \mathbb{L}$ where \mathbb{L} is the Lazard ring and $quism = \mathcal{S}$ is the class of quasi-smooth morphisms in the category of derived schemes, instead of considering the above $\mathbb{M}_{sm}^{prop}(X \xrightarrow{f} Y)^+$. A quasi-smooth morphism is closed under composition and base change. Note that a local complete intersection (abbr., *l.c.i.*) morphism is *not necessarily closed under base change* in the classical category of algebraic schemes, but it *is closed under base change in the category of derived schemes* since it is quasi-smooth. Then, by considering some bivariant ideal $\langle R^{LS} \rangle(X \xrightarrow{f} Y)$ of $\mathbb{M}_{quism}^{prop}(X \xrightarrow{f} Y)^+$, he defined the quotient group

(1.2)

$$\frac{\mathbb{M}_{quism}^{prop}(X \xrightarrow{f} Y)^+ \otimes \mathbb{L}}{\langle R^{LS} \rangle((X \xrightarrow{f} Y))} =: \Omega^*(X \xrightarrow{f} Y),$$

which is his *bivariant derived algebraic cobordism* $\Omega^*(X \xrightarrow{f} Y)$. It turns out that $\Omega^*(X \rightarrow pt)$ is isomorphic to the derived algebraic cobordism $d\Omega_*(X)$ of Lowrey–Schürg [20], which is isomorphic to Levine–Morel’s algebraic cobordism $\Omega_*(X)$. Thus T. Annala has succeeded in constructing a bivariant-theoretic version of Levine–Morel’s algebraic cobordism.

A motivation of considering $\mathbb{M}_{sm}^{prop}(X \xrightarrow{f} Y)$, i.e., the form (1.1) above with $prop = \mathcal{C}$ and $sm = \mathcal{S}$, is that cobordism cycles, which are key ingredients of Levine–Morel’s algebraic cobordism $\Omega_*(X)$, are of the form $[W \xrightarrow{p} X; L_1, \dots, L_r]$ where $p : W \rightarrow X$ is a *proper* morphism, W is *smooth* and L_i ’s are line bundles over W . Namely, putting aside line bundles L_i ’s, the data of a *proper morphism* $W \xrightarrow{p} X$ with *smooth* W is the same as the above commutative diagram (1.1) with $Y = pt$ a point, $f = a_X : X \rightarrow pt$ and $a_W = a_X \circ p : W \rightarrow pt$, the morphisms to a point, such that $c : W \rightarrow X$ is *proper* and $a_W : W \rightarrow pt$ is *smooth*. Namely, it is the left commutative diagram below:

$$(1.3) \quad \begin{array}{ccc} & W & \\ p \swarrow & \circlearrowright & \searrow a_W \\ X & \xrightarrow{a_X} & pt. \end{array} \quad \iff \quad \begin{array}{ccc} & W & \\ p \swarrow & & \searrow a_W \\ X & & pt. \end{array}$$

However, it is clear that the left commutative diagram is actually the same as the right diagram (without making it commutative via the map $a_X : X \rightarrow pt$). A general diagram $X \xleftarrow{p} W \xrightarrow{s} Y$ is called a *correspondence*. Then the free abelian group generated by the isomorphism classes of such correspondences $X \xleftarrow{p} W \xrightarrow{s} Y$ with proper p and smooth s is denoted by $\mathbb{M}_{sm}^{prop}(X, Y)$, similarly to $\mathbb{M}_{sm}^{prop}(X \rightarrow Y)$. When $Y = pt$ is a point, we have the canonical isomorphism $\mathbb{M}_{sm}^{prop}(X \rightarrow pt) \cong \mathbb{M}_{sm}^{prop}(X, pt)$. In a general situation we have “forgetting the morphism f ”, $\mathfrak{F} : \mathbb{M}_{sm}^{prop}(X \xrightarrow{f} Y) \hookrightarrow \mathbb{M}_{sm}^{prop}(X, Y)$, which is an embedding or a monomorphism, defined by $\mathfrak{F}([W \xrightarrow{p} X]) := [X \xleftarrow{p} W \xrightarrow{f \circ p} Y]$:

$$(1.4) \quad \begin{array}{ccc} & W & \\ p \swarrow & \circlearrowright & \searrow f \circ p \\ X & \xrightarrow{f} & Y \end{array} \quad \xrightarrow{\text{forget the morphism } f} \quad \begin{array}{ccc} & W & \\ p \swarrow & & \searrow f \circ p \\ X & & Y \end{array}$$

Here we note that if $f : X \rightarrow Y$ is not surjective, then there does not exist a proper morphism $p : W \rightarrow X$ such that the composite $f \circ p : W \rightarrow Y$

is smooth, thus $\mathbb{M}_{sm}^{prop}(X \xrightarrow{f} Y) = 0$, however $\mathbb{M}_{sm}^{prop}(X, Y) \neq 0$. As explained above, in [2] Annala overcomes this kind of drawback $\mathbb{M}_{sm}^{prop}(X \xrightarrow{f} Y) = 0$ by considering *quasi-smooth morphisms in the category of derived schemes*.

In [6] Annala and the author constructed a bivariant theory $\Omega^{*,\sharp}(X \xrightarrow{f} Y)$, which is generated by the isomorphism classes of $[W \xrightarrow{p} X; E]$ where p is proper and the composite $f \circ p$ is quasi-smooth and E is a vector bundle over W . \sharp refers to the rank of a vector bundle E . $\Omega^{*,\sharp}(X \rightarrow pt)$ is isomorphic to Lee–Pandharipande’s algebraic cobordism of vector bundles $\Omega_{*,\sharp}(X)$ [18]. Thus $\Omega^{*,\sharp}(X \xrightarrow{f} Y)$ is a bivariant-theoretic version of $\Omega_{*,\sharp}(X)$.

In §4.3 we show that there is a *bi-variant*² algebraic cobordism $\Omega^{*,\sharp}(X, Y)$, using isomorphism classes of correspondences $[X \xleftarrow{p} W \xrightarrow{s} Y; E]$ with vector bundles E over W , and we have the canonical homomorphism “forgetting the morphism f ”:

$$\mathfrak{F} : \Omega^{*,\sharp}(X \xrightarrow{f} Y) \rightarrow \Omega^{*,\sharp}(X, Y)$$

defined by $\mathfrak{F}([W \xrightarrow{p} X; E]) := [X \xleftarrow{p} W \xrightarrow{f \circ p} Y; E]$. $\Omega^{*,\sharp}(X, pt)$ is isomorphic to Lee–Pandharipande’s algebraic cobordism $\Omega_{*,\sharp}(X)$, hence $\Omega^{*,\sharp}(X, Y)$ is a *bi-variant version* of Lee–Pandharipande’s algebraic cobordism $\Omega_{*,\sharp}(X)$. In the case when we consider $E = \mathbf{0}$ the zero bundles, $\Omega^*(X, Y) = \Omega^{*,0}(X, Y)$ is a *bi-variant version* of Levine–Morel’s algebraic cobordism $\Omega_*(X)$.

For considering the above bi-variant theory $\Omega(X, Y)$, there are a few more motivations other than the above very simple observation (1.3):

1. Speaking of a bivariant theory, there is another bivariant theory, which is called a bivariant K -theory or KK -theory $K(X, Y)$ due to G. Kasparov [15]. This is well-known and has been studied very well in C^* -algebra and operator theories. In [10] (cf. [8, 9]) H. Emerson and R. Meyer described $K(X, Y)$, using correspondences. *Hence it would be reasonable or natural to consider such a KK -theory type “bivariant” theory $\Omega(X, Y)$, instead of $\Omega(X \rightarrow Y)$ in the sense of Fulton–MacPherson.*
2. Since we have the natural homomorphism

$$\mathfrak{F} : \mathbb{M}_{quism}^{prop}(X \xrightarrow{f} Y)^+ \rightarrow \mathbb{M}_{quism}^{prop}(X, Y)^+,$$

it would be quite natural to think that there must be some “bi-variant” version $\Omega^{*,\sharp}(X, Y)$, using $\mathbb{M}_{quism}^{prop}(X, Y)^+$, corresponding to $\Omega^{*,\sharp}(X \xrightarrow{f} Y)$

²In order not to be confused with a *bivariant* theory in the sense of Fulton–MacPherson, we use “bi-variant” in [23].

which is constructed by using $\mathbb{M}_{quism}^{prop}(X \xrightarrow{f} Y)^+$:

$$\begin{array}{ccc}
 \mathbb{M}_{quism}^{prop}(X \xrightarrow{f} Y)^+ & \xrightarrow{\mathfrak{F}} & \mathbb{M}_{quism}^{prop}(X, Y)^+ \\
 \downarrow & & \downarrow \\
 \Omega^{*,\#}(X \xrightarrow{f} Y) & \xrightarrow{\mathfrak{F}} & \Omega^{*,\#}(X, Y) \\
 \downarrow \wr & & \downarrow \wr \\
 \Omega^{*,\#}(X \rightarrow pt) & \xrightarrow[\cong]{\mathfrak{F}} & \Omega^{*,\#}(X, pt).
 \end{array}$$

The bottom isomorphism may be expected from (1.3) above.

3. Levine–Morel’s algebraic cobordism $\Omega_*(X)$ satisfies the following:

- For a proper morphism $f : X \rightarrow Y$ we have the pushforward $f_* : \Omega_*(X) \rightarrow \Omega_*(Y)$, which is covariantly functorial,
- For a smooth morphism $g : X \rightarrow Y$ we have the pullback $g^* : \Omega_*(Y) \rightarrow \Omega_*(X)$, which is contravariantly functorial,
- For a fiber square

$$\begin{array}{ccc}
 X' & \xrightarrow{g'} & X \\
 f' \downarrow & & \downarrow f \\
 Y' & \xrightarrow{g} & Y,
 \end{array}$$

where f is proper and g is smooth, thus so are f' and g' , respectively, we have

$$f'_* \circ (g')^* = g^* \circ f_* : \Omega_*(X) \rightarrow \Omega_*(Y')$$

which is called *base change isomorphism* or *Beck–Chevalley condition*.

Such a functor is called is a “bi-variant” functor, e.g., see [13, Part III Categories of Correspondences, §1.1, p. 271 and §0.1.1., p. 285]. In order to encode such a bi-variant functor, in [13] D. Gaitsgory and N. Rozenblyum consider a category \mathcal{C} (with finite limits) equipped with two classes of morphisms *vert* and *horiz* (both *closed under composition and base change*) and consider correspondences written as fol-

lows:

$$(1.5) \quad \begin{array}{ccc} c_{0,1} & \xrightarrow{g} & c_0 \\ f \downarrow & & \\ & & c_1 \end{array}$$

where f is *vert* and g is *horiz*. For such a correspondence they consider a pushforward $\Phi(f) : \Phi(c_{0,1}) \rightarrow \Phi(c_1)$ and a pullback $\Phi^!(g) : \Phi(c_0) \rightarrow \Phi(c_{0,1})$ for some bi-variant functor Φ . Using these, they consider the composition $\Phi(f) \circ \Phi^!(g) : \Phi(c_0) \rightarrow \Phi(c_1)$, which is considered as a functor defined on the category of correspondences, i.e., a morphism from c_0 to c_1 is this correspondence. They use such correspondences in order to encode a bi-variant functor with values in a ∞ -category and furthermore to encode Grothendieck six-functor formalism. As the above diagram (1.5) indicates, *vert* and *horiz* clearly come from “vertical” and “horizontal”, respectively, however their real roles or properties are exactly the same as those of *confined* and *specialized*, respectively. Hence, it would be *reasonable or natural to consider some theories similar to a bivariant theory in the sense of Fulton–MacPherson and/or our bi-variant theory in Gaitsgory–Rozenblyum’s study of correspondences*. For example, in [1] T. Abe introduces a kind of bivariant theory (in the sense of Fulton–MacPherson, but missing the structure of bivariant product) with values in an ∞ -category, using correspondences.

In §2 we make a quick review of Fulton–MacPherson’s bivariant theory [12] and the author’s universal bivariant theory [21] (cf. [22]). In §3 we recall cobordism bicycles of vector bundles $[X \xleftarrow{p} W \xrightarrow{s} Y; E]$ and their properties, which are key ingredients for our bi-variant theory $\Omega^{*,\#}(X, Y)$. In §4 we define our bi-variant theory

$$\Omega^{*,\#}(X, Y) := \frac{\mathcal{M}^{*,\#}(X, Y)^+}{\langle \mathcal{R}^{LS} \rangle(X, Y)},$$

after defining a bi-variant ideal $\langle \mathcal{R}^{LS} \rangle(X, Y)$, such that $\Omega^{*,\#}(X, pt) \cong \Omega_{-*,\#}(X)$ which is Lee–Pandharipande’s algebraic cobordism of vector bundles [18]. In the final section §5 we discuss a bit some possible generalized versions of Anala’s bivariant algebraic cobordism $\Omega^*(X \rightarrow Y)$ and our bi-variant theory $\Omega^*(X, Y)$.

2. Fulton–MacPherson’s bivariant theory and a universal bivariant theory

We make a quick review of Fulton–MacPherson’s bivariant theory [12] (also see [11]) and the author’s universal bivariant theory [21] (also see [22]).

2.1. Fulton–MacPherson’s bivariant theory

Let \mathcal{C} be a category which has a final object pt and on which the fiber product or fiber square is well-defined. Also we consider a class of morphisms, called “confined morphisms” (e.g., proper morphisms, projective morphisms, in algebraic geometry), which are *closed under composition and base change and contain all the identity morphisms*, and a class of fiber squares, called “independent squares” (or “confined squares”, e.g., “Tor-independent” in algebraic geometry, a fiber square with some extra conditions required on morphisms of the square), which satisfy the following:

(i) if the two inside squares in

$$\begin{array}{ccc}
 X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\
 \downarrow f'' & & \downarrow f' & & \downarrow f \\
 Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ccc}
 X' & \xrightarrow{h''} & X \\
 f' \downarrow & & \downarrow f \\
 Y' & \xrightarrow{h'} & Y \\
 g' \downarrow & & \downarrow g \\
 Z' & \xrightarrow{h} & Z
 \end{array}$$

are independent, then the outside square is also independent,

(ii) any square of the following forms are independent:

$$\begin{array}{ccc}
 X & \xrightarrow{\text{id}_X} & X \\
 f \downarrow & & \downarrow f \\
 Y & \xrightarrow{\text{id}_X} & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \text{id}_X \downarrow & & \downarrow \text{id}_Y \\
 X & \xrightarrow{f} & Y
 \end{array}$$

where $f : X \rightarrow Y$ is *any* morphism.

Remark 2.1. Given an independent square, its transpose is *not necessarily* independent. For example, let us consider the category of topological

spaces and continuous maps. Let *any* map be confined, and we allow a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

to be *independent only if g is proper* (hence g' is also proper). Then its transpose is *not independent unless f is proper*. (Note that the pullback of a proper map by any continuous map is proper, because “proper” is equivalent to “universally closed”, i.e., the pullback by any map is closed.)

A bivariant theory \mathbb{B} on a category \mathcal{C} with values in the category of graded abelian groups is an assignment to each morphism $X \xrightarrow{f} Y$ in the category \mathcal{C} a graded abelian group (in most cases we ignore the grading) $\mathbb{B}(X \xrightarrow{f} Y)$ which is equipped with the following three basic operations. The i -th component of $\mathbb{B}(X \xrightarrow{f} Y)$, $i \in \mathbb{Z}$, is denoted by $\mathbb{B}^i(X \xrightarrow{f} Y)$.

1. **Product:** For morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, the product operation

$$\bullet : \mathbb{B}^i(X \xrightarrow{f} Y) \otimes \mathbb{B}^j(Y \xrightarrow{g} Z) \rightarrow \mathbb{B}^{i+j}(X \xrightarrow{gf} Z)$$

is defined.

2. **Pushforward:** For morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ with f *confined*, the pushforward operation

$$f_* : \mathbb{B}^i(X \xrightarrow{gf} Z) \rightarrow \mathbb{B}^i(Y \xrightarrow{g} Z)$$

is defined.

3. **Pullback:** For an *independent* square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation

$$g^* : \mathbb{B}^i(X \xrightarrow{f} Y) \rightarrow \mathbb{B}^i(X' \xrightarrow{f'} Y')$$

is defined.

An element $\alpha \in \mathbb{B}(X \xrightarrow{f} Y)$ is sometimes expressed as follows:

$$X \xrightarrow[f]{\textcircled{\alpha}} Y$$

These three operations are required to satisfy the following seven compatibility axioms ([12, Part I, §2.2]):

(A₁) **Product is associative:** for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ with $\alpha \in \mathbb{B}(X \xrightarrow{f} Y), \beta \in \mathbb{B}(Y \xrightarrow{g} Z), \gamma \in \mathbb{B}(Z \xrightarrow{h} W)$,

$$(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma).$$

(A₂) **Pushforward is functorial:** for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ with f and g confined and $\alpha \in \mathbb{B}(X \xrightarrow{h \circ g \circ f} W)$

$$(g \circ f)_*(\alpha) = g_*(f_*(\alpha)).$$

(A₃) **Pullback is functorial:** given independent squares

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow f'' & & \downarrow f' & & \downarrow f \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y \end{array}$$

$$(g \circ h)^* = h^* \circ g^*.$$

(A₁₂) **Product and pushforward commute:** for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ with f confined and $\alpha \in \mathbb{B}(X \xrightarrow{g \circ f} Z), \beta \in \mathbb{B}(Z \xrightarrow{h} W)$,

$$f_*(\alpha \bullet \beta) = f_*(\alpha) \bullet \beta.$$

(A₁₃) **Product and pullback commute:** given independent squares

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{h'} & Y \\ g' \downarrow & & \downarrow g \\ Z' & \xrightarrow[h]{} & Z \end{array}$$

with $\alpha \in \mathbb{B}(X \xrightarrow{f} Y), \beta \in \mathbb{B}(Y \xrightarrow{g} Z),$

$$h^*(\alpha \bullet \beta) = h'^*(\alpha) \bullet h^*(\beta).$$

(A₂₃) **Pushforward and pullback commute:** given independent squares

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{h'} & Y \\ g' \downarrow & & \downarrow g \\ Z' & \xrightarrow{h} & Z \end{array}$$

with f confined and $\alpha \in \mathbb{B}(X \xrightarrow{g \circ f} Z),$

$$f'_*(h^*(\alpha)) = h^*(f_*(\alpha)).$$

(A₁₂₃) **Projection formula:** given an independent square with g confined and $\alpha \in \mathbb{B}(X \xrightarrow{f} Y), \beta \in \mathbb{B}(Y' \xrightarrow{h \circ g} Z)$

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X & & \\ f' \downarrow & & \downarrow f & & \\ Y' & \xrightarrow{g} & Y & \xrightarrow{h} & Z \end{array}$$

and $\alpha \in \mathbb{B}(X \xrightarrow{f} Y), \beta \in \mathbb{B}(Y' \xrightarrow{h \circ g} Z),$

$$g'_*(g^*(\alpha) \bullet \beta) = \alpha \bullet g_*(\beta).$$

We also assume that \mathbb{B} has units:

Units: \mathbb{B} has units, i.e., there is an element $1_X \in \mathbb{B}^0(X \xrightarrow{\text{id}_X} X)$ such that $\alpha \bullet 1_X = \alpha$ for all morphisms $W \rightarrow X$ and all $\alpha \in \mathbb{B}(W \rightarrow X)$, such that $1_X \bullet \beta = \beta$ for all morphisms $X \rightarrow Y$ and all $\beta \in \mathbb{B}(X \rightarrow Y)$, and such that $g^*1_X = 1_{X'}$ for all $g : X' \rightarrow X$.

Commutativity: \mathbb{B} is called *commutative* if whenever both

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \xrightarrow{f'} & Y \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{g} & Z \end{array}$$

are independent squares with $\alpha \in \mathbb{B}(X \xrightarrow{f} Z)$ and $\beta \in \mathbb{B}(Y \xrightarrow{g} Z)$,

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha.$$

Let \mathbb{B}, \mathbb{B}' be two bivariant theories on a category \mathcal{C} . A *Grothendieck transformation* from \mathbb{B} to \mathbb{B}' , $\gamma : \mathbb{B} \rightarrow \mathbb{B}'$ is a collection of homomorphisms $\mathbb{B}(X \rightarrow Y) \rightarrow \mathbb{B}'(X \rightarrow Y)$ for a morphism $X \rightarrow Y$ in the category \mathcal{C} , which preserves the above three basic operations:

1. $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
2. $\gamma(f_*\alpha) = f_*\gamma(\alpha)$, and
3. $\gamma(g^*\alpha) = g^*\gamma(\alpha)$.

A bivariant theory unifies both a covariant theory and a contravariant theory in the following sense:

$\mathbb{B}_*(X) := \mathbb{B}^{-*}(X \rightarrow pt)$ becomes a covariant functor for *confined* morphisms and

$\mathbb{B}^*(X) := \mathbb{B}^*(X \xrightarrow{id} X)$ becomes a contravariant functor for *any* morphisms. A Grothendieck transformation $\gamma : \mathbb{B} \rightarrow \mathbb{B}'$ induces natural transformations $\gamma_* : \mathbb{B}_* \rightarrow \mathbb{B}'_*$ and $\gamma^* : \mathbb{B}^* \rightarrow \mathbb{B}'^*$.

Definition 2.1. As to the grading, $\mathbb{B}_i(X) := \mathbb{B}^{-i}(X \rightarrow pt)$ and $\mathbb{B}^j(X) := \mathbb{B}^j(X \xrightarrow{id} X)$.

Definition 2.2 ([12, Part I, §2.6.2 Definition]). Let \mathcal{S}' be a class of maps in \mathcal{V} , closed under composition³ and containing all identity maps. Suppose that to each $f : X \rightarrow Y$ in \mathcal{S}' there is assigned an element $\theta(f) \in \mathbb{B}(X \xrightarrow{f} Y)$ satisfying that

³In the case of confined maps, we require that the pullback of a confined map is confined. For this class \mathcal{S}' we do not require the stability of pullback. For example, in [12] the class of local complete intersection (abbr. *l.c.i.*) morphisms is considered as such a class, and the pullback of an *l.c.i.* morphism is not necessarily a *l.c.i.* morphism.

- (i) $\theta(g \circ f) = \theta(f) \bullet \theta(g)$ for all $f : X \rightarrow Y, g : Y \rightarrow Z \in \mathcal{S}'$ and
- (ii) $\theta(\text{id}_X) = 1_X$ for all X with $1_X \in \mathbb{B}^*(X) := \mathbb{B}^*(X \xrightarrow{\text{id}_X} X)$ the unit element.

Then $\theta(f)$ is called an *canonical orientation* of f .

Gysin homomorphisms Note that such a canonical orientation makes the covariant functor $\mathbb{B}_*(X)$ a contravariant functor for morphisms in \mathcal{S}' , and also makes the contravariant functor \mathbb{B}^* a covariant functor for morphisms in $\mathcal{C} \cap \mathcal{S}'$: Indeed,

1. As to the covariant functor $\mathbb{B}_*(X)$: For a morphism $f : X \rightarrow Y \in \mathcal{S}'$ and a canonical orientation θ on \mathcal{S}' , the following *Gysin (pullback) homomorphism*

$$f^! : \mathbb{B}_*(Y) \rightarrow \mathbb{B}_*(X) \quad \text{defined by} \quad f^!(\alpha) := \theta(f) \bullet \alpha$$

is *contravariantly functorial*.

2. As to contravariant functor \mathbb{B}^* : For a fiber square (which is an independent square by hypothesis)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{id}_X \downarrow & & \downarrow \text{id}_Y \\ X & \xrightarrow{f} & Y, \end{array}$$

where $f \in \mathcal{C} \cap \mathcal{S}'$, the following *Gysin (pushforward) homomorphism*

$$f_! : \mathbb{B}^*(X) \rightarrow \mathbb{B}^*(Y) \quad \text{defined by} \quad f_!(\alpha) := f_*(\alpha \bullet \theta(f))$$

is *covariantly functorial*.

The above Gysin homomorphisms are sometimes denoted by $f_\theta^!$ and $f_!^\theta$ respectively in order to make it clear that the canonical orientation θ is used.

2.2. A universal bivariant theory

- (i) Let \mathcal{S} be another class of morphisms called “specialized morphisms” (e.g., smooth morphisms in algebraic geometry) in \mathcal{C} , which is *closed under composition and base change and contains all identity morphisms*.

(ii) Let \mathcal{S} be as in (i). Furthermore, if the orientation θ on \mathcal{S} satisfies that for an independent square with $f \in \mathcal{S}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

$\theta(f') = g^*\theta(f)$, i.e., the orientation θ preserves the pullback operation, then we call θ a *nice canonical orientation* of \mathbb{B} . We also assume that our category \mathcal{V} satisfies that any fiber square

$$\begin{array}{ccc} P' & \longrightarrow & P \\ f' \downarrow & & \downarrow f \\ Q' & \longrightarrow & Q \end{array}$$

with f being confined, i.e., $f \in \mathcal{C}$, is an independent square. In [21] this condition is called “ \mathcal{C} -independence”.

Theorem 2.1 ([21, Theorem 3.1]). *Let \mathcal{C} be a category with a class \mathcal{C} of confined morphisms, a class $\mathcal{I}nd$ of independent squares and a class \mathcal{S} of specialized morphisms. We define $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Y)$ to be the free abelian group generated by the set of isomorphism classes of confined morphisms $h : W \rightarrow X$ such that the composite of h and f is a specialized morphism: $h \in \mathcal{C}$ and $f \circ h : W \rightarrow Y \in \mathcal{S}$.*

1. The association $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$ is a bivariant theory if the three bivariant operations are defined as follows:

(a) **Product:** For morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, the product

$$\bullet : \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Y) \otimes \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(Y \xrightarrow{g} Z) \rightarrow \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{gf} Z)$$

is defined by $[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{h \circ k'} X]$ and extended linearly, where we consider the following fiber squares

$$\begin{array}{ccccccc} V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W & & \\ k'' \downarrow & & k' \downarrow & & k \downarrow & & \\ V & \xrightarrow{h} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z. \end{array}$$

(b) **Pushforward:** For morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ with f confined, the pushforward

$$f_* : \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{gf} Z) \rightarrow \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(Y \xrightarrow{g} Z)$$

is defined by $f_*([V \xrightarrow{h} X]) := [V \xrightarrow{f \circ h} Y]$ and extended linearly.

(c) **Pullback:** For an independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback

$$g^* : \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Y) \rightarrow \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X' \xrightarrow{f'} Y')$$

is defined by $g^*([V \xrightarrow{h} X]) := [V' \xrightarrow{h'} X']$ and extended linearly, where we consider the following fiber squares:

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ h' \downarrow & & \downarrow h \\ X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y. \end{array}$$

2. For a specialized morphism $f : X \rightarrow Y \in \mathcal{S}$,

$$\theta_{\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}}(f) = [X \xrightarrow{\text{id}_X} X] \in \mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \xrightarrow{f} Y)$$

is a nice canonical orientation of $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$ for \mathcal{S} .

3. (A universality of $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}$) Let \mathbb{B} be a bivariant theory on the same category \mathcal{C} with the same class \mathcal{C} of confined morphisms, the same class $\mathcal{I}nd$ of independent squares and the same class \mathcal{S} of specialized morphisms, and let θ be a nice canonical orientation of \mathbb{B} for \mathcal{S} . Then there exists a unique Grothendieck transformation $\gamma_{\mathbb{B}} : \mathbb{M}_{\mathcal{S}}^{\mathcal{C}} \rightarrow \mathbb{B}$ such that for a

specialized morphism $f : X \rightarrow Y$, $\gamma_{\mathbb{B}} : \mathbb{M}_{\mathbb{S}}^{\mathcal{C}}(X \xrightarrow{f} Y) \rightarrow \mathbb{B}(X \xrightarrow{f} Y)$ satisfies the normalization condition that $\gamma_{\mathbb{B}}(\theta_{\mathbb{M}_{\mathbb{S}}^{\mathcal{C}}}(f)) = \theta_{\mathbb{B}}(f)$.

Proposition 2.1 (Commutativity⁴). *The universal bivariant theory $\mathbb{M}_{\mathbb{S}}^{\mathcal{C}}$ is commutative in the sense that $g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha$ for a fiber square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \textcircled{\alpha} \\ Z' & \xrightarrow{g} & Z \textcircled{\beta} \end{array}$$

Remark 2.2. Here we note that

1. for a *confined* morphism $f : X \rightarrow X'$ we have the pushforward $f_* : (\mathbb{M}_{\mathbb{S}}^{\mathcal{C}})_*(X) \rightarrow (\mathbb{M}_{\mathbb{S}}^{\mathcal{C}})_*(X')$,
2. for a *specialized* morphism $g : Y \rightarrow Y'$ we have the (Gysin) pullback $g^* : (\mathbb{M}_{\mathbb{S}}^{\mathcal{C}})_*(Y') \rightarrow (\mathbb{M}_{\mathbb{S}}^{\mathcal{C}})_*(Y)$, where we use the above canonical orientation $\theta_{\mathbb{M}_{\mathbb{S}}^{\mathcal{C}}}(g)$,
3. for a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

where f is confined and g is specialized, thus so are f' and g' , respectively. Then we have

$$f'_* \circ (g')^* = g^* \circ f_* : (\mathbb{M}_{\mathbb{S}}^{\mathcal{C}})_*(X) \rightarrow (\mathbb{M}_{\mathbb{S}}^{\mathcal{C}})_*(Y')$$

which is called *base change isomorphism* or *Beck–Chevalley condition*.

Hence the functor $(\mathbb{M}_{\mathbb{S}}^{\mathcal{C}})_*$ is a *bi-variant functor* (e.g., see [13, Part III Categories of Correspondences, §1.1, p. 271 and §0.1.1., p. 285]) from the category \mathcal{C} equipped with two classes of morphisms *confined* and *specialized* to the category of abelian groups.

Remark 2.3. Suppose that the coproduct \sqcup (like the disjoint union in the category of sets, the category of topological spaces, etc.) is well-defined⁵ in the

⁴If $g^*(\alpha) \bullet \beta = (-1)^{\deg(\alpha) \deg(\beta)} f^*(\beta) \bullet \alpha$ holds, then it is called *skew-commutative* (see [12, Part I: Bivariant Theories, §2.2]).

⁵Note that there is a category in which the coproduct is not defined, e.g., the category of finite groups.

category \mathcal{C} and $f : X_1 \sqcup X_2 \rightarrow Y$ is defined to be confined (resp., specialized) if and only if the restrictions $f|_{X_i} : X_i \rightarrow Y$ is confined (resp., specialized). Here $f|_{X_i} := f \circ \iota_i : X_i \rightarrow Y$ where $\iota_i : X_i \rightarrow X_1 \sqcup X_2$ is the inclusion. If we consider the following relations on $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \rightarrow Y)$

$$[V_1 \sqcup V_2 \xrightarrow{h} X] = [V_1 \xrightarrow{h|_{V_1}} X] + [V_2 \xrightarrow{h|_{V_2}} X]$$

where we note that $(h_1 \sqcup h_2)|_{V_i} = (h_1 \sqcup h_2) \circ \iota_i = h_i : V_i \rightarrow X$, then we get the Grothendieck group, denoted by $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \rightarrow Y)^+$. Or the set of isomorphism classes of $[V \rightarrow X]$ becomes a commutative monoid by the coproduct

$$[V_1 \xrightarrow{h_1} X] + [V_2 \xrightarrow{h_2} X] := [V_1 \sqcup V_2 \xrightarrow{h_1 \sqcup h_2} X].$$

Then the Grothendieck group $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X \rightarrow Y)^+$ is the group completion of this monoid.

3. Cobordism bicycles of vector bundles

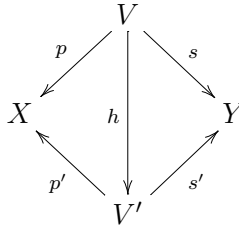
In [23] we considered extending the notion of algebraic cobordism of vector bundles due to Y.-P. Lee and R. Pandharipande [16] (cf. [19]) to “correspondences”. For later use, we give a quick recall of them.

3.1. Cobordism bicycles of vector bundles

Definition 3.1. Let $X \xleftarrow{p} V \xrightarrow{s} Y$ be a correspondence such that p is *proper* and s is *smooth*, and let E be a complex vector bundle over V . Then $(X \xleftarrow{p} V \xrightarrow{s} Y; E)$ is called a *cobordism bicycle of a vector bundle*.

Definition 3.2. Let $(X \xleftarrow{p} V \xrightarrow{s} Y; E)$ and $(X \xleftarrow{p'} V' \xrightarrow{s'} Y; E')$ be two cobordism bicycles of vector bundles of the same rank. If there exists an isomorphism $h : V \cong V'$ such that

1. $(X \xleftarrow{p} V \xrightarrow{s} Y) \cong (X \xleftarrow{p'} V' \xrightarrow{s'} Y)$ as correspondences, i.e., the following diagrams commute:



2. $E \cong h^*E'$,

then they are called isomorphic and denoted by

$$(X \xleftarrow{p} V \xrightarrow{s} Y; E) \cong (X \xleftarrow{p'} V' \xrightarrow{s'} Y; E').$$

The isomorphism class of $(X \xleftarrow{p} V \xrightarrow{s} Y; E)$ is denoted by $[X \xleftarrow{p} V \xrightarrow{s} Y; E]$, which is still called a cobordism bicycle of a vector bundle. For a fixed rank r for vector bundles, the set of isomorphism classes of cobordism bicycles of vector bundles for a pair (X, Y) becomes a commutative monoid by the disjoint sum:

$$\begin{aligned} [X \xleftarrow{p_1} V_1 \xrightarrow{s_1} Y; E_1] + [X \xleftarrow{p_2} V_2 \xrightarrow{s_2} Y; E_2] \\ := [X \xleftarrow{p_1+p_2} V_1 \sqcup V_2 \xrightarrow{s_1+s_2} Y; E_1 \sqcup E_2]. \end{aligned}$$

This monoid is denoted by $\mathcal{M}^r(X, Y)$ and another grading of $[X \xleftarrow{p} V \xrightarrow{s} Y; E]$ is defined by the relative dimension of the smooth morphism s , denoted by $\dim s$, thus by double grading, $[X \xleftarrow{p} V \xrightarrow{s} Y; E] \in \mathcal{M}^{n,r}(X, Y)$, where we set $-n = \dim s$ and $r = \text{rank } E$. The group completion of this monoid, i.e., the Grothendieck group, is denoted by $\mathcal{M}^{n,r}(X, Y)^+$. Using similar notations as in §2.2, $\mathcal{M}^{n,r}(X, Y)^+$ could or should be denoted by $(\mathbb{M}_{sm}^{prop})^{n,r}(X, Y)^+$ where $prop = \mathcal{C}$ is the class of proper morphisms and $sm = \mathcal{S}$ is the class of smooth morphisms. But, in order to avoid some messy notation, we use $\mathcal{M}^{n,r}(X, Y)^+$, mimicking the notation used in [17].

Remark 3.1. The reason for why we set $-n = \dim s$ in the above grading is requiring that $\mathcal{M}_{-n,r}(X, pt) := \mathcal{M}^{n,r}(X, pt)$ in the case when $Y = pt$ and $\mathcal{M}_{-n,r}(X, pt)$ is supposed to be the same as $\mathcal{M}_{-n,r}(X)^+$ considered in Lee–Pandharipande [18], where $-n = \dim s$ is nothing but the dimension of the source variety V . This is just like setting $\mathbb{B}_i(X) := \mathbb{B}^{-i}(X \rightarrow pt)$.

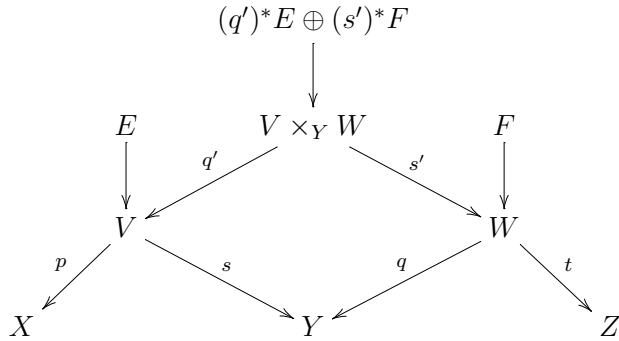
Definition 3.3. For three varieties X, Y, Z , by using Whitney sum \oplus we define the following product⁶

$$\bullet : \mathcal{M}^{m,r}(X, Y)^+ \otimes \mathcal{M}^{n,k}(Y, Z)^+ \rightarrow \mathcal{M}^{m+n,r+k}(X, Z)^+$$

$$\begin{aligned} [X \xleftarrow{p} V \xrightarrow{s} Y; E] \bullet [Y \xleftarrow{q} W \xrightarrow{t} Z; F] \\ := [(X \xleftarrow{p} V \xrightarrow{s} Y) \circ (Y \xleftarrow{q} W \xrightarrow{t} Z); (q')^*E \oplus (s')^*F] \end{aligned}$$

⁶In [23] we consider another product \bullet_{\otimes} using tensor product \otimes . In this paper we do not consider \bullet_{\otimes} .

Here we consider the following diagrams:



The middle diamond square is the fiber product and $(X \xleftarrow{p} V \xrightarrow{s} Y) \circ (Y \xleftarrow{q} W \xrightarrow{s} Z)$ is the composition of correspondences, defined by $(X \xleftarrow{p} V \xrightarrow{s} Y) \circ (Y \xleftarrow{q} W \xrightarrow{s} Z) := (X \xleftarrow{p \circ q'} V \times_Y W \xrightarrow{t \circ s'} Z)$ in the above diagram.

For later use, we define the following special cobordism bicycles:

Definition 3.4.

1. For a smooth morphism $g : Y \rightarrow Y'$, we define

$$\mathbb{1}_g := [Y \xleftarrow{\text{id}_Y} Y \xrightarrow{g} Y'; \mathbf{0}] \in \mathcal{M}^{-\dim g, 0}(Y, Y')^+,$$

2. For a proper morphism $g : Y \rightarrow Y'$, we define

$${}_g\mathbb{1} := [Y' \xleftarrow{g} Y \xrightarrow{\text{id}_Y} Y; \mathbf{0}] \in \mathcal{M}^{0, 0}(Y', Y)^+,$$

3. For any variety Y , $\mathbb{1}_Y := [Y \xleftarrow{\text{id}_Y} Y \xrightarrow{\text{id}_Y} Y; \mathbf{0}] \in \mathcal{M}^{0, 0}(Y, Y)^+$. Here $\mathbf{0}$ is the zero bundle over Y .
4. For a vector bundle E over Y we define

$$[[E]] := [Y \xleftarrow{\text{id}_Y} Y \xrightarrow{\text{id}_Y} Y; E].$$

If we need to emphasize the base Y of the bundle E , then we denote it by $[[E]]_Y$. Otherwise we omit the suffix Y unless there is some possible confusion on which is the base space of the bundle. Hence for the 0-bundle over Y we have $[[\mathbf{0}]] = \mathbb{1}_Y$.

5. For two vector bundles E and F over the same base Y , we have

$$[[E]] \bullet [[F]] = [[E \oplus F]].$$

From now on we identify $[X \xleftarrow{p} V \xrightarrow{s} Y] = [X \xleftarrow{p} V \xrightarrow{s} Y; \mathbf{0}]$. Hence we have

$$\begin{aligned} \mathbb{1}_g &= [Y \xleftarrow{\text{id}_Y} Y \xrightarrow{g} Y'] = [Y \xleftarrow{\text{id}_Y} Y \xrightarrow{g} Y'; \mathbf{0}], \\ g\mathbb{1} &= [Y' \xleftarrow{g} Y \xrightarrow{\text{id}_Y} Y] = [Y' \xleftarrow{g} Y \xrightarrow{\text{id}_Y} Y; \mathbf{0}], \\ \mathbb{1}_Y &= [Y \xleftarrow{\text{id}_Y} Y \xrightarrow{\text{id}_Y} Y] = [Y \xleftarrow{\text{id}_Y} Y \xrightarrow{\text{id}_Y} Y; \mathbf{0}]. \end{aligned}$$

Remark 3.2. We note that by Definitions 3.4 and 3.3 we have

$$(3.1) \quad [X \xleftarrow{p} V \xrightarrow{s} Y] = {}_p\mathbb{1} \bullet \mathbb{1}_s.$$

In general, we can see

$$(3.2) \quad [X \xleftarrow{p} V \xrightarrow{s} Y; E] = {}_p\mathbb{1} \bullet [[E]] \bullet \mathbb{1}_s.$$

So, we have that $[X \xleftarrow{p} V \xrightarrow{s} Y; \mathbf{0}] = {}_p\mathbb{1} \bullet [[\mathbf{0}]] \bullet \mathbb{1}_s = {}_p\mathbb{1} \bullet \mathbb{1}_Y \bullet \mathbb{1}_s = {}_p\mathbb{1} \bullet \mathbb{1}_s$.

Now we define pushforward and pullback of cobordism bicycles of vector bundles:

Definition 3.5.

1. (Pushforward)

- (a) For a *proper* morphism $f : X \rightarrow X'$, the (proper) pushforward *acting on the first factor* $X \xleftarrow{p} V$, denoted by $f_* : \mathcal{M}^{m,r}(X, Y)^+ \rightarrow \mathcal{M}^{m,r}(X', Y)^+$, is defined by

$$f_*([X \xleftarrow{p} V \xrightarrow{s} Y; E]) := [X' \xleftarrow{f \circ p} V \xrightarrow{s} Y; E].$$

Note that $f_*(-) = f\mathbb{1} \bullet (-)$.

- (b) For a *smooth* morphism $g : Y \rightarrow Y'$, the (smooth) pushforward *acting on the second factor* $V \xrightarrow{s} Y$, denoted by $*g : \mathcal{M}^{m,r}(X, Y)^+ \rightarrow \mathcal{M}^{m-\dim g, r}(X, Y')^+$,⁷ is defined by

$$([X \xleftarrow{p} V \xrightarrow{s} Y; E]) *g := [X \xleftarrow{p} V \xrightarrow{g \circ s} Y'; E].$$

Here we emphasize that $*g$ is written on the right side of $([X \xleftarrow{p} V \xrightarrow{s} Y; E])$ *not on the left side*. Note that $(-)*g = (-) \bullet \mathbb{1}_g$.

⁷In [23] we have $\mathcal{M}^{m+\dim g, r}(X, Y')^+$, instead of $\mathcal{M}^{m-\dim g, r}(X, Y')^+$, because in [23] we set $n = \dim s$, not $-n = \dim s$, for the first grading n of $\mathcal{M}^{n,r}(X, Y)^+$ (cf. Remark 3.1 above). This sign change is applied to the other situations below, as long as the relative dimensions of smooth morphisms are involved.

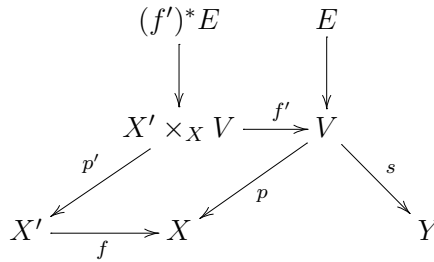
(Note also that $-m = \dim s$ and $\dim(g \circ s) = \dim s + \dim g = -m + \dim g = -(m - \dim g)$.)

2. (Pullback)

- (a) For a *smooth* morphism $f : X' \rightarrow X$, the (smooth) pullback *acting on the first factor* $X \xleftarrow{p} V$, denoted by $f^* : \mathcal{M}^{m,r}(X, Y)^+ \rightarrow \mathcal{M}^{m-\dim f, r}(X', Y)^+$ is defined by

$$f^*([X \xleftarrow{p} V \xrightarrow{s} Y; E]) := [X' \xleftarrow{p'} X' \times_X V \xrightarrow{s \circ f'} Y; (f')^* E].$$

Here we consider the following commutative diagram:



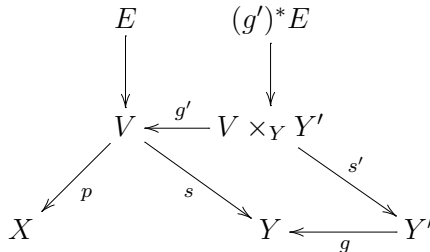
Note that $f^*(-) = \mathbb{1}_f \bullet (-)$.

(Note also that the left diamond is a fiber square, thus $f' : X' \times_X V \rightarrow V$ is smooth and $p' : X' \times_X V \rightarrow X'$ is proper. Note that $\dim f' = \dim f$ and $\dim(s \circ f') = \dim s + \dim f' = -m + \dim f = -(m - \dim f)$.)

- (b) For a *proper* morphism $g : Y' \rightarrow Y$, the (proper) pullback *acting on the second factor* $V \xrightarrow{s} Y$, denoted by $*g : \mathcal{M}^{m,r}(X, Y)^+ \rightarrow \mathcal{M}^{m,r}(X, Y')^+$, is defined by

$$([X \xleftarrow{p} V \xrightarrow{s} Y; E]) * g := [X \xleftarrow{p \circ g'} V \times_Y Y' \xrightarrow{s'} Y'; (g')^* E].$$

Here we consider the following commutative diagram:



Note that $(-)^*g = (-) \bullet_g \mathbb{1}$.

(Note that the right diamond is a fiber square, thus $s' : V \times_Y Y' \rightarrow Y'$ is smooth and $g' : V \times_Y Y' \rightarrow V$ is proper, and $\dim s = \dim s'$.)

Remark 3.3. Since we refer to the formulas noted in the above Definition 3.5 later, we list them here again:

$$(3.3) \quad f_*(-) = f\mathbb{1} \bullet (-),$$

$$(3.4) \quad (-)_*g = (-) \bullet \mathbb{1}_g,$$

$$(3.5) \quad f^*(-) = \mathbb{1}_f \bullet (-),$$

$$(3.6) \quad (-)^*g = (-) \bullet_g \mathbb{1}.$$

Proposition 3.1. *The above three operations of product, pushforward and pullback satisfy the following properties.*

(A₁) **Product is associative:** *For three varieties X, Y, Z, W we have*

$$(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma)$$

where $\alpha \in \mathcal{M}^{m,r}(X, Y)^+, \beta \in \mathcal{M}^{n,k}(Y, Z)^+$ and $\beta \in \mathcal{M}^{\ell,e}(Z, W)^+$.

(A₂) **Pushforward is functorial:**

(a) *For two proper morphisms $f_1 : X \rightarrow X', f_2 : X' \rightarrow X''$, we have*

$$(f_2 \circ f_1)_* = (f_2)_* \circ (f_1)_*$$

where $(f_1)_* : \mathcal{M}^{m,r}(X, Y)^+ \rightarrow \mathcal{M}^{m,r}(X', Y)^+$ and $(f_2)_* : \mathcal{M}^{m,r}(X', Y)^+ \rightarrow \mathcal{M}^{m,r}(X'', Y)^+$.

(b) *For two smooth morphisms $g_1 : Y \rightarrow Y', g_2 : Y' \rightarrow Y''$ we have*

$$*(g_2 \circ g_1) = *(g_1) \circ *(g_2) \quad \text{i.e., } \alpha *(g_2 \circ g_1) = (\alpha *(g_1)) *(g_2)$$

where $*(g_1) : \mathcal{M}^{m,r}(X, Y) \rightarrow \mathcal{M}^{m-\dim g_1, r}(X, Y')$ and $*(g_2) : \mathcal{M}^{m-\dim g_1, r}(X, Y') \rightarrow \mathcal{M}^{m-\dim g_1-\dim g_2, r}(X, Y'')$.

(A₂)' **Proper pushforward and smooth pushforward commute:** *For a proper morphism $f : X \rightarrow X'$ and a smooth morphism $g : Y \rightarrow Y'$ we have*

$$*g \circ f_* = f_* \circ *g, \quad \text{i.e., } (f_*\alpha) *g = f_*(\alpha *g).$$

(A₃) **Pullback is functorial:**

(a) For two smooth morphisms $f_1 : X \rightarrow X'$, $f_2 : X' \rightarrow X''$ we have

$$(f_2 \circ f_1)^* = (f_1)^* \circ (f_2)^*$$

where $(f_2)^* : \mathcal{M}^{m,r}(X'', Y)^+ \rightarrow \mathcal{M}^{m-\dim f_2,r}(X', Y)^+$ and $(f_1)^* : \mathcal{M}^{m-\dim f_2,r}(X', Y)^+ \rightarrow \mathcal{M}^{m-\dim f_2-\dim f_1,r}(X, Y)^+$.

(b) For two proper morphisms $g_1 : Y \rightarrow Y'$, $g_2 : Y' \rightarrow Y''$ we have

$$*(g_2 \circ g_1) = *(g_2) \circ *(g_1) \quad \text{i.e., } \alpha^*(g_2 \circ g_1) = (\alpha^*(g_2))^*(g_1)$$

where $*(g_1) : \mathcal{M}^{m,r}(X, Y')^+ \rightarrow \mathcal{M}^{m,r}(X, Y)^+$ and $*(g_2) : \mathcal{M}^{m,r}(X, Y'')^+ \rightarrow \mathcal{M}^{m,r}(X, Y')^+$.

(A₃)' **Proper pullback and smooth pullback commute:** For a smooth morphism $g : X' \rightarrow X$ and a proper morphism $f : Y' \rightarrow Y$ we have

$$g^* \circ *f = *f \circ g^*, \quad \text{i.e., } g^*(\alpha^*f) = (g^*\alpha)^*f.$$

(A₁₂) **Product and pushforward commute:** Let $\alpha \in \mathcal{M}^{m,r}(X, Y)^+$ and $\beta \in \mathcal{M}^{n,k}(Y, Z)^+$.

(a) For a proper morphism $f : X \rightarrow X'$,

$$f_*(\alpha \bullet \beta) = (f_*\alpha) \bullet \beta \quad (\in \mathcal{M}^{m+n,r+k}(X', Z)^+).$$

(b) For a smooth morphism $g : Z \rightarrow Z'$,

$$(\alpha \bullet \beta)_*g = \alpha \bullet (\beta_*g) \quad (\in \mathcal{M}^{m+n-\dim g,r+k}(X, Z')^+).$$

(A₁₃) **Product and pullback commute:** Let $\alpha \in \mathcal{M}^{m,r}(X, Y)^+$ and $\beta \in \mathcal{M}^{n,k}(Y, Z)^+$.

(a) For a smooth morphism $f : X' \rightarrow X$,

$$f^*(\alpha \bullet \beta) = (f^*\alpha) \bullet \beta \quad (\in \mathcal{M}^{m+n-\dim f,r+k}(X', Z)^+).$$

(b) For a proper morphism $g : Z' \rightarrow Z$,

$$(\alpha \bullet \beta)^*g = \alpha \bullet (\beta^*g) \quad (\in \mathcal{M}^{m+n,r+k}(X, Z')^+).$$

(A₂₃) **Pushforward and pullback commute:** For $\alpha \in \mathcal{M}^{m,r}(X, Y)^+$

- (a) (proper pushforward and proper pullback commute) For proper morphisms $f : X \rightarrow X'$ and $g : Y' \rightarrow Y$ and for $\alpha \in \mathcal{M}^{m,r}(X, Y)^+$

$$(f_*\alpha)^*g = f_*(\alpha^*g) \quad (\in \mathcal{M}^{m,r}(X', Y')^+).$$

- (b) (smooth pushforward and smooth pullback commute) For smooth morphisms $f : X' \rightarrow X$ and $g : Y \rightarrow Y'$ and for $\alpha \in \mathcal{M}^{m,r}(X, Y)^+$

$$f^*(\alpha_*g) = (f^*\alpha)_*g \quad (\in \mathcal{M}^{m-\dim f-\dim g,r}(X', Y')^+).$$

- (c) (proper pushforward and smooth pullback “commute” in the following sense) For the following fiber square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & X'' \\ \tilde{g} \downarrow & & \downarrow g \\ X' & \xrightarrow{f} & X \end{array}$$

with f proper and g smooth, we have

$$g^*f_* = \tilde{f}_*\tilde{g}^*.$$

- (d) (smooth pushforward and proper pullback “commute” in the following sense) For the following fiber square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & Y'' \\ \tilde{g} \downarrow & & \downarrow g \\ Y' & \xrightarrow{f} & Y \end{array}$$

with f proper and g smooth, for $\alpha \in \mathcal{M}^{m,r}(X, Y'')^+$ we have

$$(\alpha_*g)^*f = (\alpha^*\tilde{f})_*\tilde{g}.$$

(A₁₂₃) **“Projection formula”:**

- (a) For a smooth morphism $g : Y \rightarrow Y'$ and $\alpha \in \mathcal{M}^{m,r}(X, Y)^+$ and $\beta \in \mathcal{M}^{n,k}(Y', Z)^+$,

$$(\alpha_*g) \bullet \beta = \alpha \bullet g^*\beta \quad (\in \mathcal{M}^{m+n-\dim g,r+k}(X, Z)^+).$$

(b) For a proper morphism $g : Y' \rightarrow Y$, $\alpha \in \mathcal{M}^{m,r}(X, Y)^+$ and $\beta \in \mathcal{M}^{n,k}(Y', Z)^+$,

$$(\alpha^*g) \bullet \beta = \alpha \bullet g_*\beta \quad (\in \mathcal{M}^{m+n, r+k}(X, Z)^+).$$

(Units) $\mathbb{1}_X = [X \xleftarrow{\text{id}_X} X \xrightarrow{\text{id}_X} X; \mathbf{0}] \in \mathcal{M}^{0,0}(X, X)^+$ satisfies that $\mathbb{1}_X \bullet \alpha = \alpha$ for any element $\alpha \in \mathcal{M}^{m,r}(X, Y)^+$ and $\beta \bullet \mathbb{1}_X = \beta$ for any element $\beta \in \mathcal{M}^{m,r}(Y, X)^+$.

Remark 3.4. We note that by Definition 3.5 (1) we have

$$(3.7) \quad f\mathbb{1} \bullet p\mathbb{1} = f \circ p\mathbb{1} \quad \text{for proper maps } f \text{ and } p,$$

$$(3.8) \quad \mathbb{1}_s \bullet \mathbb{1}_g = \mathbb{1}_{g \circ s} \quad \text{for smooth maps } s \text{ and } g,$$

$$(3.9) \quad \text{id}_X \mathbb{1} = \mathbb{1}_{\text{id}_X} = \mathbb{1}_X \text{ is the unit.}$$

This reflects also the *symmetry* between the classes of proper and smooth morphisms, coming from the *transposition* of correspondences, given by *switching* the two sides of a correspondence. Namely, transposition of $f\mathbb{1} \bullet p\mathbb{1} = f \circ p\mathbb{1}$ is $\mathbb{1}_p \bullet \mathbb{1}_f = \mathbb{1}_{f \circ p}$, thus transposition of (3.7) is (3.8) and vice versa, putting aside properness and smoothness of maps. We also see by (3.8) that $\mathbb{1}_s$ for a smooth map s is exactly a canonical orientation in Definition 2.2. And by Definition 3.5 (2), it follows from both (a) and (b) that for the following fiber square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & X'' \\ \tilde{s} \downarrow & & \downarrow s \\ X' & \xrightarrow{g} & X \end{array}$$

with s (hence \tilde{s} as well) being smooth and g (hence \tilde{g} as well) being proper, we have the following

$$(3.10) \quad \mathbb{1}_s \bullet g\mathbb{1} = \tilde{s}\mathbb{1} \bullet \mathbb{1}_{\tilde{s}}.$$

By Definition 3.5 (2) we have the following:

$$(3.11) \quad \mathbb{1}_f \bullet [[E]] = [[f^*E]] \bullet \mathbb{1}_f,$$

$$(3.12) \quad [[E]] \bullet g\mathbb{1} = g\mathbb{1} \bullet [[g^*E]],$$

which are kind of “projection formula”. Hence, using these properties and the associativity of the product \bullet , for example, we can do the following computation:

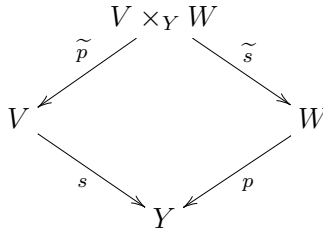
$$\begin{aligned}
 [X \xleftarrow{p} V \xrightarrow{s} Y; E] \bullet [Y \xleftarrow{g} W \xrightarrow{t} Z; F] & \\
 &= (p\mathbb{1} \bullet [[E]] \bullet \mathbb{1}_s) \bullet (g\mathbb{1} \bullet [[F]] \bullet \mathbb{1}_t) \\
 &= p\mathbb{1} \bullet [[E]] \bullet (\mathbb{1}_s \bullet g\mathbb{1}) \bullet [[F]] \bullet \mathbb{1}_t \\
 &= p\mathbb{1} \bullet [[E]] \bullet (\tilde{g}\mathbb{1} \bullet \mathbb{1}_{\tilde{s}}) \bullet [[F]] \bullet \mathbb{1}_t \\
 &= p\mathbb{1} \bullet ([E] \bullet \tilde{g}\mathbb{1}) \bullet (\mathbb{1}_{\tilde{s}} \bullet [[F]]) \bullet \mathbb{1}_t \\
 &= p\mathbb{1} \bullet (\tilde{g}\mathbb{1} \bullet [[(\tilde{g})^*E]]) \bullet ([[(\tilde{s})^*F]]) \bullet \mathbb{1}_{\tilde{s}} \bullet \mathbb{1}_t \\
 &= (p\mathbb{1} \bullet \tilde{g}\mathbb{1}) \bullet ([[(\tilde{g})^*E]]) \bullet ([[(\tilde{s})^*F]]) \bullet (\mathbb{1}_{\tilde{s}} \bullet \mathbb{1}_t) \\
 &= p_{\circ\tilde{g}}\mathbb{1} \bullet [[(\tilde{g})^*E \oplus (\tilde{s})^*F]] \bullet \mathbb{1}_{t\circ\tilde{s}}
 \end{aligned}$$

The above proposition follows directly from the definitions. We also note that all the properties except for (A_1) also follow from the property (A_1) , i.e., the associativity of product \bullet , and the formulas listed in the above Remark 3.3 and Remark 3.4. Indeed, for example,

- (A_2) (a) follows from (3.3), (A_1) and (3.7).
- (A_2) (b) follows from (3.4), (A_1) and (3.8).
- (A_{23}) (c) follows from (3.3), (3.5), (A_1) and (3.10).
- (A_{23}) (d) follows from (3.4), (3.6), (A_1) and (3.10).

The following fact (which also follows from the above (3.10)) is emphasized for a later use.

Proposition 3.2 (Pushforward-Product Property for Units (abbr. PPPU)).
For the following fiber square



with $s : V \rightarrow Y$ smooth and $p : W \rightarrow Y$ proper, we have

$$(\mathbb{1}_V * s) \bullet p_*\mathbb{1}_W = (\tilde{p}_*\mathbb{1}_{V \times_Y W}) * \tilde{s} = \tilde{p}_*(\mathbb{1}_{V \times_Y W} * \tilde{s}) \in \mathcal{M}^{*,0}(V, W)^+.$$

Remark 3.5. We note that for the free abelian group $\mathcal{M}^{*,\#}(X, Y)$ Definitions 3.3 and 3.5 are also well-defined and Propositions 3.1 and 3.2 hold as well.

Here we recall that

$$\mathcal{M}^{i,r}(X \xrightarrow{f} Y)^+$$

is generated by the isomorphism classes of cobordism cycles of the form

$$[V \xrightarrow{p} X; E]$$

such that $p : V \rightarrow X$ is *proper* and the composite $f \circ p : V \rightarrow Y$ is *smooth of relative dimension $-i$* and $r = \text{rank}(E)$. We have the “forgetting the morphism f ”:

$$\mathfrak{F} : \mathcal{M}^{i,r}(X \xrightarrow{f} Y)^+ \rightarrow \mathcal{M}^{i,r}(X, Y)^+.$$

Lemma 3.1. *The “forgetting the morphism f ” $\mathfrak{F} : \mathcal{M}^{i,r}(X \xrightarrow{f} Y)^+ \rightarrow \mathcal{M}^{i,r}(X, Y)^+$ commutes with product \bullet and the pushforward and satisfies the following simple formulas for the canonical orientation and the element $[[E]]$ for a vector bundle:*

1. $\mathfrak{F}(\alpha \bullet \beta) = \mathfrak{F}(\alpha) \bullet \mathfrak{F}(\beta)$.
2. $\mathfrak{F} \circ f_* = f_* \circ \mathfrak{F}$.
3. For a smooth morphism $s : X \rightarrow Y$ we have

(a) $\mathfrak{F}(\theta(s)) = 1_s$, i.e., $\mathfrak{F}([X \xrightarrow{\text{id}_X} X]) = [X \xleftarrow{\text{id}_X} X \xrightarrow{s} Y]$, where $\theta(s) = [X \xrightarrow{\text{id}_X} X]$ is the canonical orientation (see Definition 2.2 and Theorem 2.1 (2)).

(b) $\mathfrak{F} \circ s^! = s^* \circ \mathfrak{F}$, where $s^! = \theta(s) \bullet$ is the Gysin homomorphism.

4. For a vector bundle E over X , we let $[E] := [X \xrightarrow{\text{id}_X} X; E]$. Then for the “forgetting” homomorphism $\mathfrak{F} : \mathcal{M}^{i,r}(X \xrightarrow{\text{id}_X} X)^+ \rightarrow \mathcal{M}^{i,r}(X, X)^+$ we have

$$\mathfrak{F}([E]) = [[E]].$$

Proof. (3) (a) and (4) are obvious. For the sake of convenience, we write down proofs for (1), (2) and (3)(b).

1. For $[V \xrightarrow{h} X; E] \in \mathcal{M}^{*,\#}(X \xrightarrow{f} Y)^+$ and $[W \xrightarrow{k} Y; F] \in \mathcal{M}^{*,\#}(Y \xrightarrow{g} Z)^+$, we have

$$\mathfrak{F}([V \xrightarrow{h} X; E] \bullet [W \xrightarrow{k} Y; F])$$

$$\begin{aligned}
 &= \mathfrak{F}([V' \xrightarrow{h \circ k''} X; k''^* E \oplus (f' \circ h')^* F]) \\
 &= [X \xleftarrow{h \circ k''} V' \xrightarrow{(g \circ f) \circ (h \circ k'')} Z; k''^* E \oplus (f' \circ h')^* F] \\
 &= [X \xleftarrow{h \circ k''} V' \xrightarrow{(g \circ k) \circ (f' \circ f')} Z; k''^* E \oplus (f' \circ h')^* F].
 \end{aligned}$$

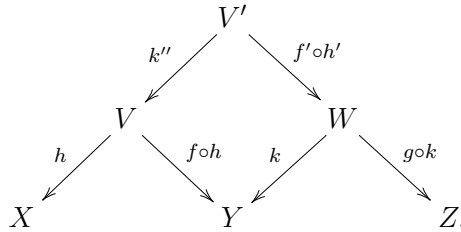
Here we use the following fiber squares

$$\begin{array}{ccccc}
 V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W \\
 k'' \downarrow & & k' \downarrow & & k \downarrow \\
 V & \xrightarrow{h} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z.
 \end{array}$$

On the other hand, we have

$$\begin{aligned}
 &\mathfrak{F}([V \xrightarrow{h} X; E]) \bullet \mathfrak{F}([W \xrightarrow{k} Y; F]) \\
 &= [X \xleftarrow{h} V \xrightarrow{f \circ h} Y; E] \bullet [Y \xleftarrow{k} W \xrightarrow{g \circ k} Z; F] \\
 &= [X \xleftarrow{h \circ k''} V' \xrightarrow{(g \circ k) \circ (f' \circ f')} Z; k''^* E \oplus (f' \circ h')^* F].
 \end{aligned}$$

Here we use the following diagram (reusing the above diagram):



Therefore we do have

$$\mathfrak{F}([V \xrightarrow{h} X; E]) \bullet [W \xrightarrow{k} Y; F] = \mathfrak{F}([V \xrightarrow{h} X; E]) \bullet \mathfrak{F}([W \xrightarrow{k} Y; F]).$$

2. For $[V \xrightarrow{h} X; E] \in \mathcal{M}^{*,\#}(X \xrightarrow{g \circ f} Z)^+$, we have

$$\begin{aligned}
 \mathfrak{F}(f_*[V \xrightarrow{h} X; E]) &= \mathfrak{F}([V \xrightarrow{f \circ h} Y; E]) \\
 &= [Y \xleftarrow{f \circ h} V \xrightarrow{g \circ f \circ h} Z; E] \\
 &= f_*[X \xleftarrow{h} V \xrightarrow{g \circ f \circ h} Z; E] \\
 &= f_*(\mathfrak{F}([V \xrightarrow{h} X; E]))
 \end{aligned}$$

3. (b)

$$\begin{aligned}
 \mathfrak{F}(s^!(\alpha)) &= \mathfrak{F}(\theta(s) \bullet \alpha) \\
 &= \mathfrak{F}(\theta(s)) \bullet \mathfrak{F}(\alpha) \quad (\text{by (1)}) \\
 &= \mathbb{1}_s \bullet \mathfrak{F}(\alpha) \quad (\text{by (3)(a)}) \\
 &= s^*(\mathfrak{F}(\alpha)).
 \end{aligned}$$

Hence we have $\mathfrak{F} \circ s^! = s^* \circ \mathfrak{F}$. □

Remark 3.6. We emphasize that the “forgetting” homomorphism \mathfrak{F} is *not* compatible with pullback. One simple reason is that “pullback” of the bi-variant theory $\mathcal{M}^{*,\sharp}(- \rightarrow -)^+$ and four kinds of “pullback” of the present bi-variant theory $\mathcal{M}^{*,\sharp}(-, -)^+$ are quite different. Thus $\mathfrak{F}(g^*s)$ cannot be expressed as $g^*\mathfrak{F}(s)$. This is a crucial problem and how to circumvent this problem is an issue later.

Remark 3.7. Lemma 3.1 (3) and (4) are obvious, but will play roles as a trick later.

3.2. \mathcal{C} – \mathcal{S} -correspondences

The cobordism bicycle of the zero bundle $[X \xleftarrow{p} V \xrightarrow{s} Y; \mathbf{0}]$ is the same as the isomorphism class $[X \xleftarrow{p} V \xrightarrow{s} Y]$ of a correspondence with a proper morphism $p : V \rightarrow X$ and a smooth morphism $s : V \rightarrow Y$. So this shall be called a *proper-smooth correspondence*. Then the above Grothendieck group $\mathcal{M}^*(X, Y)^+ = \mathcal{M}^{*,0}(X, Y)^+$ with the second degree $r = 0$ is nothing but the Grothendieck group of proper-smooth correspondences.

In general, for the class \mathcal{C} of *confined* morphisms and the class \mathcal{S} of *specialized* morphisms, the isomorphism class $[X \xleftarrow{p} V \xrightarrow{s} Y]$ of a correspondence with $p \in \mathcal{C}$ and $s \in \mathcal{S}$ shall be called a \mathcal{C} – \mathcal{S} -correspondence. Then the free abelian group generated by \mathcal{C} – \mathcal{S} -correspondences is denoted by $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)$. If we assume the conditions as in Remark 2.3, then the Grothendieck group generated by \mathcal{C} – \mathcal{S} -correspondences is denoted by $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)^+$. Then for both $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)$ and $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)^+$ Definitions 3.3 and 3.5 with the data of vector bundles deleted are also well-defined and Propositions 3.1 and 3.2 with the data of vector bundles deleted hold as well.

Remark 3.8. As we know, correspondences make a category, usually called “a category of correspondences”, i.e., a morphism from X to Y is a correspondence $X \xleftarrow{f} V \xrightarrow{g} Y$ and the composition of two morphisms $X \xleftarrow{f} V \xrightarrow{g} Y$ and $Y \xleftarrow{h} W \xrightarrow{k} Z$ is nothing but the composition of these correspondences. In

[24] $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)$ and $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)^+$ are both called *an enriched category of correspondences*, because $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)$ and $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)^+$ are *abelian groups*, whereas in the usual category of correspondences $\text{hom}(X, Y)$ is *a set*. $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)$ and $\mathbb{M}_{\mathcal{S}}^{\mathcal{C}}(X, Y)^+$ can be generalized as follows: Let $\mathfrak{B}(X, Y)$ be abelian groups for pairs (X, Y) which satisfy the following:

1. they are equipped with a product $\bullet' : \mathfrak{B}(X, Y) \otimes \mathfrak{B}(Y, Z) \rightarrow \mathfrak{B}(X, Z)$ which is associative with the unit $1 = 1_X \in \mathfrak{B}(X, X)$,
2. For confined maps $f : Y \rightarrow X$ there exists distinguished elements $f_1 \in \mathfrak{B}(X, Y)$ such that for two confined maps $f : Y \rightarrow X$ and $g : Y \rightarrow Z$ we have $f_1 \bullet'_p 1 = f \circ_p 1$.
3. For specialized maps $s : X \rightarrow Y$ there exist distinguished elements $1_s \in \mathfrak{B}(X, Y)$ such that for two specialized maps $s : X \rightarrow Y$ and $g : Y \rightarrow Z$ we have $1_s \bullet'_g 1_g = 1_{g \circ s}$.
4. $id_X 1 = 1 = 1_{id_X}$ is the unit for the identity map $id_X : X \rightarrow X$,
5. $1_s \bullet'_g 1 = \tilde{1} \bullet'_s 1_{\tilde{s}}$ for the fiber square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & X'' \\ \tilde{s} \downarrow & & \downarrow s \\ X' & \xrightarrow{g} & X \end{array}$$

with s (hence \tilde{s} as well) being specialized and g (hence \tilde{g} as well) being confined.

Then as in Remark 3.3, we can define the following two pushforwards and two pullbacks:

$$\begin{aligned} g_*(-) &:= {}_g 1 \bullet' (-), {}^*g(-) := (-) \bullet'_g 1 \text{ for confined maps } g, \\ s^*(-) &:= 1_s \bullet' (-), {}_*s(-) := (-) \bullet'_s 1_s \text{ for specialized maps } s. \end{aligned}$$

Then we can see that $\mathfrak{B}(X, Y)$ gives rise to a theory satisfying those properties in the above Proposition 3.1. Furthermore, we suppose that

1. there is a well-defined element $[[E]]'$ for each vector bundle over a space and
2. an abelian group $\mathfrak{B}(X, Y)$ is generated by elements of the forms ${}_p 1 \bullet' [[E]]' \bullet'_s 1_s$ for confined maps g and specialized maps s and
3. they satisfy the following properties

$$(3.13) \quad 1_f \bullet' [[E]]' = [[f^*E]] \bullet'_f 1_f.$$

$$(3.14) \quad [[E]]' \bullet'_g 1 = {}_g 1 \bullet'_g [[g^*E]]'.$$

Then ${}_p1 \bullet' 1_s$ and ${}_p1 \bullet' [[E]]' \bullet' 1_s$ shall be respectively called an “abstract” correspondence and an “abstract” cobordism bicycle of vector bundle. Then $\mathfrak{B}(X, Y)$ is respectively a more enriched category of abstract correspondences and a more enriched category of abstract cobordism bicycles of vector bundles, compared with the geometric cobordism bicycle of vector bundle $[X \xleftarrow{p} V \xrightarrow{s} Y; E] = {}_p1 \bullet [[E]] \bullet 1_s$. For such a theory we do have a canonical Grothendieck transformation, e.g.,

$$\gamma : \mathbb{M}_S^c(X, Y) \rightarrow \mathfrak{B}(X, Y)$$

defined by

$$\gamma({}_p1 \bullet [[E]] \bullet 1_s) := {}_p1 \bullet' [[E]]' \bullet' 1_s.$$

Which, in particular, implies that $\gamma({}_p1) = {}_p1$ by considering the special case when $[[E]] = [[\mathbf{0}]] = 1_V$ and $1_s = 1_V$ for the identity map $s = \text{id}_V$, similarly $\gamma([[E]]) = [[E]]'$ and $\gamma(1_s) = 1_s$ by considering the corresponding special cases. Here we emphasize that a theory satisfying those properties in the above Proposition 3.1 does not necessarily come from a certain more enriched category $\mathfrak{B}(X, Y)$ of abstract correspondences or abstract cobordism bicycles of vector bundles.

We also have the following “forgetting the morphism f ” defined by $\mathfrak{F}([V \xrightarrow{p} X]) := [X \xleftarrow{p} V \xrightarrow{f \circ p} Y]$:

$$\begin{aligned} \mathfrak{F} : \mathbb{M}_S^c(X \xrightarrow{f} Y) &\rightarrow \mathbb{M}_S^c(X, Y), \\ \mathfrak{F} : \mathbb{M}_S^c(X \xrightarrow{f} Y)^+ &\rightarrow \mathbb{M}_S^c(X, Y)^+, \end{aligned}$$

which are both embeddings or monomorphisms.

4. A bi-variant algebraic cobordism with bundles $\Omega^{*\#}(X, Y)$

In this section we consider the “correspondence” version of Annala’s bivariant derived algebraic cobordism $\Omega^*(X \rightarrow Y)$ [2] and furthermore that of Annala–Yokura’s bivariant algebraic cobordism with vector bundles $\Omega^{*\#}(X \rightarrow Y)$ [6] (also see [3, 5]).

From now on, as in [2] and [6], we work in derived algebraic geometry, thus we work in the context of quasi-projective derived schemes over a base field of characteristic zero. These are not categories in the classical sense, but ∞ -categories, so that one needs some small modifications. One can work either directly with an ∞ -category with homotopy fiber squares (instead of fiber squares), and equivalence classes of arrows (instead of isomorphism classes)

and a final object unique up to equivalence, e.g., as in Annala’s thesis [4]. Or one works in the underlying homotopy category, which is a usual category so that equivalence classes map to isomorphism classes and final objects map to a usual final object unique up to isomorphism. And instead of fiber products one works with the commutative diagrams induced from the homotopy fiber squares, e.g., as in [2]. So one has to be aware that these are usually not fiber squares in the underlying homotopy category as used before. For simplicity we work in the underlying homotopy category. Then the notion of a “homotopy (or derived) fiber square” in the homotopy category is preserved by isomorphisms of commutative squares, so that all the constructions for universal bivariant theories, or those defined via correspondences, work as before, e.g., any two arrows

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

can be completed to a “homotopy (or derived) fiber square” unique up to isomorphism. Also in this context one can speak of vector bundles E (of constant rank), proper as the confined morphisms and quasi-smooth (of constant (virtual) fiber dimension) as the specialized morphisms (as discussed in more detail in [6]).

4.1. A slight modification of $\mathcal{M}^{*,\sharp}(X, Y)^+$

In order to construct our bi-variant algebraic cobordism $\Omega^{*,\sharp}(X, Y)$, we modify our previous abelian groups $\mathcal{M}^{*,\sharp}(X, Y)^+$ as follows.

Definition 4.1. Let $X \xleftarrow{p} V \xrightarrow{s} Y$ be a correspondence such that $p : V \rightarrow X$ is *proper* and $s : V \rightarrow Y$ is *quasi-smooth*, and let E be a complex vector bundle over V . Then $(X \xleftarrow{p} V \xrightarrow{s} Y; E)$ is still called a *cobordism bicycle of a vector bundle*.

Then we define $\mathcal{M}^{i,r}(X, Y)^+$ (using the same symbol as before) to be an abelian group generated by the isomorphism classes of cobordism bicycles of the form $[X \xleftarrow{p} V \xrightarrow{s} Y; E]$ such that $s : V \rightarrow Y$ is a quasi-smooth morphism of *virtual relative dimension* $-i$ and $r = \text{rank}(E)$.

Here we note that in the case of the bivariant algebraic cobordism with vector bundles $\Omega^{*,\sharp}(X \rightarrow Y)$ we consider

$$\mathcal{M}^{i,r}(X \rightarrow Y)^+ := \mathbb{L} \otimes \mathcal{M}^{i,r}(X \rightarrow Y)^+$$

where $\mathcal{M}^{i,r}(X \xrightarrow{f} Y)^+$ is generated by the isomorphism classes of cobordism cycles of the form

$$[V \xrightarrow{p} X; E]$$

such that $p : V \rightarrow X$ is proper and the composite $f \circ p : V \rightarrow Y$ is quasi-smooth of *virtual relative dimension* $-i$ and $r = \text{rank}(E)$.

Remark 4.1. If we consider the case when E is the zero bundle $\mathbf{0}$, i.e., $[V \xrightarrow{p} X; \mathbf{0}]$, or equivalently we consider $[V \xrightarrow{p} X]$, then we have $\mathcal{M}^{i,0}(X \xrightarrow{f} Y)^+$, which is nothing but $\mathcal{M}^i(X \xrightarrow{f} Y)^+$ used in [2]. If we use the notation in Remark 2.3, $\mathcal{M}^i(X \xrightarrow{f} Y)^+$ is denoted by $(\mathbb{M}_{quasm}^{prop})^i(X \xrightarrow{f} Y)^+$ where $prop = \mathcal{C}$ is the class of proper morphisms and $quasm = \mathcal{S}$ is the class of quasi-smooth morphisms.

Therefore we have the canonical homomorphism, “forgetting the morphism f ”:

$$\mathfrak{F} : \mathcal{M}^{i,r}(X \xrightarrow{f} Y)^+ \rightarrow \mathcal{M}^{i,r}(X, Y)^+$$

defined by

$$\mathfrak{F}([V \xrightarrow{p} X; E]) := [X \xleftarrow{p} V \xrightarrow{f \circ p} Y; E].$$

Then we get the same results for $\mathcal{M}^{i,r}(-, -)^+$ as Proposition 3.1 and Proposition 3.2 as in §3, so omitted for the sake of simplicity.

We recall the following definition of the top Chern class of a vector bundle [6, Definition 5.5]:

Definition 4.2 ([6, Definition 5.5]). Given a vector bundle E of rank n on X , we define its *top Chern class* as

$$c_n(E) = s^* s_!(1_X) \in \mathcal{M}^{n,0}(X \xrightarrow{\text{id}_X} X)^+$$

where $s : X \rightarrow E$ is the zero section.

We note (see [6, Remark 5.6]) that the above $c_n(E)$ can be expressed explicitly as follows:

$$(4.1) \quad c_n(E) = [V(s) \xrightarrow{i_{V(s)}} X; \mathbf{0}]$$

where $V(s)$ is the derived vanishing locus of the zero section and $i_{V(s)}$ is the inclusion. Hence, it is quite natural to define the “correspondence” version of the Chern class, still called the top Chern class of the vector bundle E and denoted by $c_n(E) \in \mathcal{M}^{n,0}(X, X)^+$, as follows:

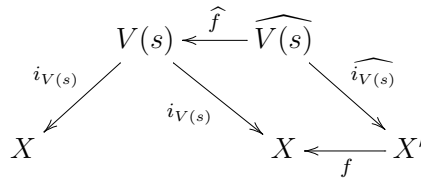
Definition 4.3.

$$c_n(E) := [X \xleftarrow{i_{V(s)}} V(s) \xrightarrow{i_{V(s)}} X; \mathbf{0}] \in CalM^{n,0}(X, X)^+.$$

Lemma 4.1. *Let E be a vector bundle of rank n over X .*

1. *For a proper morphism $f : X' \rightarrow X$, $f_*c_n(f^*E) = c_n(E)^*f$, i.e., $f\mathbb{1} \bullet c_n(f^*E) = c_n(E) \bullet f\mathbb{1}$.*
2. *For a quasi-smooth morphism $f : X' \rightarrow X$, $c_n(f^*E)_*f = f^*c_n(E)$, i.e., $c_n(f^*E) \bullet f\mathbb{1} = f\mathbb{1} \bullet c_n(E)$.*

Proof. We prove only the first one, since that of the second one is similar. By definition we have $c_n(E) = [X \xleftarrow{i_{V(s)}} V(s) \xrightarrow{i_{V(s)}} X]$, where we drop the zero bundle $\mathbf{0}$ for the sake of simplicity. Hence, using the following diagram in which the right diamond is a fiber square:



we have

$$\begin{aligned}
 c_n(E)^*f &= [X \xleftarrow{i_{V(s)} \circ \widehat{f}} \widehat{V(s)} \xrightarrow{\widehat{i_{V(s)}}} X'] \\
 &= [X \xleftarrow{f \circ \widehat{i_{V(s)}}} \widehat{V(s)} \xrightarrow{\widehat{i_{V(s)}}} X'] \\
 &= f_*[X' \xleftarrow{\widehat{i_{V(s)}}} \widehat{V(s)} \xrightarrow{\widehat{i_{V(s)}}} X']
 \end{aligned}$$

Since $\widehat{V(s)}$ is the pullback of the vanishing locus $V(s)$ of the zero section s by the morphism $f : X \rightarrow X'$, $\widehat{V(s)}$ is equal to the vanishing locus of the pullbacked zero section f^*s of the pullbacked vector bundle f^*E , i.e., we have $c_n(f^*E) = [X' \xleftarrow{\widehat{i_{V(s)}}} \widehat{V(s)} \xrightarrow{\widehat{i_{V(s)}}} X']$. Hence we have $f_*c_n(f^*E) = c_n(E)^*f$. □

For a vector bundle L of rank ℓ over X or a vector bundle M of rank m over Y , the homomorphisms

$$c_\ell(L) \bullet : \mathcal{M}^{i,r}(X, Y)^+ \rightarrow \mathcal{M}^{i+\ell,r}(X, Y)^+,$$

$$c_m(M) : \mathcal{M}^{i,r}(X, Y)^+ \rightarrow \mathcal{M}^{i+m,r}(X, Y)^+$$

defined respectively by $c_\ell(L) \bullet \alpha$ and $\beta \bullet c_m(M)$, are called the ‘‘Chern class operators’’.

Proposition 4.1. *The Chern class operators of line bundles satisfy the following properties.*

1. (identity): *If L and L' are line bundles over X and isomorphic and if M and M' are line bundles over Y and isomorphic, then*

$$\begin{aligned} c_1(L) \bullet &= c_1(L') \bullet : \mathcal{M}^{i,r}(X, Y)^+ \rightarrow \mathcal{M}^{i+1,r}(X, Y)^+, \\ \bullet c_1(M) &= \bullet c_1(M') : \mathcal{M}^{i,r}(X, Y)^+ \rightarrow \mathcal{M}^{i+1,r}(X, Y)^+. \end{aligned}$$

2. (commutativity): *If L and L' are line bundles over X and if M and M' are line bundles over Y , then*

$$\begin{aligned} c_1(L) \bullet c_1(L') \bullet &= c_1(L') \bullet c_1(L) \bullet : \mathcal{M}^{i,r}(X, Y)^+ \rightarrow \mathcal{M}^{i+2,r}(X, Y)^+, \\ \bullet c_1(M) \bullet c_1(M') &= \bullet c_1(M') \bullet c_1(M) : \mathcal{M}^{i,r}(X, Y)^+ \rightarrow \mathcal{M}^{i+2,r}(X, Y)^+. \end{aligned}$$

3. (compatibility with product) *Let L be a line bundle over X and N be a line bundle over Z . For $\alpha \in \mathcal{M}^{i,r}(X, Y)^+$ and $\beta \in \mathcal{M}^{j,k}(Y, Z)^+$, then*

$$c_1(L) \bullet (\alpha \bullet \beta) = (c_1(L) \bullet \alpha) \bullet \beta, \quad (\alpha \bullet \beta) \bullet c_1(N) = \alpha \bullet (\beta \bullet c_1(N)).$$

4. (compatibility with pushforward = ‘‘projection formula’’) *For a proper morphism $f : X \rightarrow X'$ and a line bundle L over X' and for a quasi-smooth morphism $g : Y \rightarrow Y'$ and a line bundle M over Y' we have that for $\alpha \in \mathcal{M}^{i,r}(X, Y)^+$*

$$f_*(c_1(f^*L) \bullet \alpha) = c_1(L) \bullet f_*\alpha, \quad (\alpha \bullet c_1(g^*M))_*g = \alpha_*g \bullet c_1(M).$$

5. (compatibility with pullback = ‘‘pullback formula’’) *For a quasi-smooth morphism $f : X' \rightarrow X$ and a line bundle L over X and for a proper morphism $g : Y' \rightarrow Y$ and a line bundle M over Y we have that for $\alpha \in \mathcal{M}^{i,r}(X, Y)^+$*

$$f^*(c_1(L) \bullet \alpha) = c_1(f^*L) \bullet f^*\alpha, \quad (\alpha \bullet c_1(M))^*g = \alpha^*g \bullet c_1(g^*M).$$

6. (Pullback Property for Unit (abbr. PPU)) *For a quasi-smooth morphism $f : X' \rightarrow X$ and a line bundle L over X and for a proper*

morphism $g : Y' \rightarrow Y$ and a line bundle M over Y we have that for $\mathbb{1}_X \in \mathcal{M}^{0,0}(X, X)^+$ and $\mathbb{1}_Y \in \mathcal{M}^{0,0}(Y, Y)^+$

$$\begin{aligned} c_1(f^*L) \bullet f^*\mathbb{1}_X &= f^*\mathbb{1}_X \bullet c_1(L) \in \mathcal{M}^{\dim f+1,0}(X', X)^+, \\ \mathbb{1}_Y *g \bullet c_1(g^*M) &= c_1(M) \bullet \mathbb{1}_Y *g \in \mathcal{M}^{1,0}(Y, Y')^+. \end{aligned}$$

In particular

7. (Commutativity of the unit and Chern class) Let L be a line bundle over X . Then we have

$$c_1(L) \bullet \mathbb{1}_X = \mathbb{1}_X \bullet c_1(L).$$

Proof. It suffices to show (2), since (1), (3) and (7) are clear and (4), (5) and (6) follow from the above Lemma 4.1 and the associativity of product \bullet .

We just show the first one. Let $s : X \rightarrow L$ and $s' : X \rightarrow L'$ be the zero sections. It follows from the fact that the derived fiber product $V(s) \times_X V(s') = V((s, s'))$ where $(s, s') : X \rightarrow L \oplus L'$ is the zero section. Hence by the definition of the product \bullet we have

$$\begin{aligned} c_1(L) \bullet c_1(L') &= [X \xleftarrow{i_{V(s)}} V(s) \xrightarrow{i_{V(s)}} X; \mathbf{0}] \bullet [X \xleftarrow{i_{V(s')}} V(s') \xrightarrow{i_{V(s')}} X; \mathbf{0}] \\ &= [X \xleftarrow{i_{V((s,s'))}} V((s, s')) \xrightarrow{i_{V((s,s'))}} X; \mathbf{0}] \\ &= c_2(L \oplus L') = c_2(L' \oplus L) = c_1(L') \bullet c_1(L). \quad \square \end{aligned}$$

Here we recall the definition of a *bi-variant theory* [23], which is defined as one similar to the definition of Fulton–MacPherson’s bivariant theory. In this sense, it could be called “a bi-variant theory of Fulton–MacPherson-type”.

Definition 4.4 (Bi-variant theory). An association \mathcal{B} assigning to a pair (X, Y) a graded abelian group $\mathcal{B}^*(X, Y)$ is called a *bi-variant theory* provided that

(1) it is equipped with the following three operations

1. (Product) $\bullet : \mathcal{B}^i(X, Y) \times \mathcal{B}^j(Y, Z) \rightarrow \mathcal{B}^{i+j}(X, Z)$
2. (Pushforward)
 - (a) For a *proper* morphism $f : X \rightarrow X'$, $f_* : \mathcal{B}^i(X, Y) \rightarrow \mathcal{B}^i(X', Y)$.
 - (b) For a *smooth* morphism $g : Y \rightarrow Y'$,

$$*g : \mathcal{B}^i(X, Y) \rightarrow \mathcal{B}^{i-\dim g}(X, Y').$$

3. (Pullback)

(a) For a *smooth* morphism $f : X' \rightarrow X$,

$$f^* : \mathcal{B}^i(X, Y) \rightarrow \mathcal{B}^{i-\dim f}(X', Y).$$

(b) For a *proper* morphism $g : Y' \rightarrow Y$, $*g : \mathcal{B}^i(X, Y) \rightarrow \mathcal{B}^i(X, Y')$.

(2) the three operations satisfy the following nine properties as in Proposition 3.1:

- (A₁) Product is associative.
- (A₂) Pushforward is functorial. ((a), (b))
- (A₂)' Proper pushforward and smooth pushforward commute.
- (A₃) Pullback is functorial. ((a), (b))
- (A₃)' Proper pullback and smooth pullback commute.
- (A₁₂) Product and pushforward commute. ((a), (b))
- (A₁₃) Product and pullback commute. ((a), (b))
- (A₂₃) Pushforward and pullback commute. ((a), (b), (c), (d))
- (A₁₂₃) Projection formula. ((a), (b))

(3) \mathcal{B} has units, i.e., there is an element $1_X \in \mathcal{B}^0(X, X)$ such that $1_X \bullet \alpha = \alpha$ for any element $\alpha \in \mathcal{B}(X, Y)$ and $\beta \bullet 1_X = \beta$ for any element $\beta \in \mathcal{B}(Y, X)$.

(4) \mathcal{B} satisfies PPPU (as in Proposition 3.2).

(5) \mathcal{B} is equipped with the Chern class operators satisfying the properties in Proposition 4.1.

Corollary 4.1. $\mathcal{M}^{*,\sharp}(X, Y)^+$ is a bi-variant theory (with respect the grading $*$, or ignoring the grading \sharp).

4.2. Grothendieck transformation and a bi-variant ideal

Definition 4.5. Let $\mathcal{B}, \mathcal{B}'$ be two bi-variant theories on a category \mathcal{V} . A Grothendieck transformation from \mathcal{B} to \mathcal{B}' , $\gamma : \mathcal{B} \rightarrow \mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(X, Y) \rightarrow \mathcal{B}'(X, Y)$ for a pair (X, Y) in the category \mathcal{V} , which preserves the above three basic operations and the Chern class operator:

1. $\gamma(\alpha \bullet_{\mathcal{B}} \beta) = \gamma(\alpha) \bullet_{\mathcal{B}'} \gamma(\beta)$,
2. $\gamma(f_* \alpha) = f_* \gamma(\alpha)$ and $\gamma(\alpha *_g) = \gamma(\alpha) *_g$,
3. $\gamma(g^* \alpha) = g^* \gamma(\alpha)$ and $\gamma(\alpha ^* f) = \gamma(\alpha) ^* f$,
4. $\gamma(c_1(L) \bullet \alpha) = c_1(L) \bullet_{\mathcal{B}} \gamma(\alpha)$ and $\gamma(\alpha \bullet c_1(M)) = \gamma(\alpha) \bullet_{\mathcal{B}} c_1(M)$.

Motivated by the definition of the bivariant ideal considered in [2] and [6], in an analogous manner we can define a “bi-variant” ideal as follows: Let \mathcal{B} be a bi-variant theory. A *bi-variant ideal* $\mathcal{I} \subset \mathcal{B}$ is defined to consist of (graded) subgroups $\mathcal{I}(X, Y) \subset \mathcal{B}(X, Y)$ for each pair (X, Y) such that the following hold: Let $\alpha \in \mathcal{I}(X, Y)$.

1. (pushforward)

- (a) $f_*\alpha \in \mathcal{I}(X', Y)$ for a *proper* morphism $f : X \rightarrow X'$, i.e.,
 $f_* : \mathcal{I}(X, Y) \rightarrow \mathcal{I}(X', Y)$,
- (b) $\alpha_*g \in \mathcal{I}(X, Y')$ for a *quasi-smooth* morphism $g : Y \rightarrow Y'$, i.e.,
 $*g : \mathcal{I}(X, Y) \rightarrow \mathcal{I}(X, Y')$,

2. (pullback)

- (a) $f^*\alpha \in \mathcal{I}(X', Y)$ for a *quasi-smooth* morphism $f : X' \rightarrow X$, i.e.,
 $f^* : \mathcal{I}(X, Y) \rightarrow \mathcal{I}(X', Y)$,
- (b) $\alpha^*g \in \mathcal{I}(X, Y')$ for a *proper* morphism $g : Y \rightarrow Y'$, i.e.,
 $*g : \mathcal{I}(X, Y) \rightarrow \mathcal{I}(X, Y')$,

3. (product)

- (a) $\beta \bullet \alpha \in \mathcal{I}(X', Y)$ for any $\beta \in \mathcal{B}(X', X)$, i.e.,
 $\bullet : \mathcal{B}(X', X) \otimes \mathcal{I}(X, Y) \rightarrow \mathcal{I}(X', Y)$,
- (b) $\alpha \bullet \delta \in \mathcal{I}(X, Y')$ for any $\delta \in \mathcal{B}(Y, Y')$, i.e.,
 $\bullet : \mathcal{I}(X, Y) \otimes \mathcal{B}(Y, Y') \rightarrow \mathcal{I}(X, Y')$,

4. (Chern class operator)

- (a) $c_1(L) \bullet \alpha \in \mathcal{I}(X, Y)$ for any line bundle L over X , i.e.,
 $c_1(L) \bullet : \mathcal{I}(X, Y) \rightarrow \mathcal{I}(X, Y)$,
- (b) $\alpha \bullet c_1(M) \in \mathcal{I}(X, Y)$ for any line bundle M over Y , i.e.,
 $\bullet c_1(M) : \mathcal{I}(X, Y) \rightarrow \mathcal{I}(X, Y)$.

Remark 4.2. We note that in the case of “bi-variant” ideal of $\mathcal{M}^{*,\#}(X, Y)^+$, thanks to the formulas listed in Remark 3.3, the above (1) (pushforward) and (2) (pullback) are special cases of (3) product, and it is also the case for (4) (Chern class operator). Therefore, any “bi-variant ideal” of $\mathcal{M}^{*,\#}(X, Y)^+$ is simply defined just as (3) (product). This simplicity will be used later.

It is easy to see the following:

Proposition 4.2.

1. The (object-wise) kernel of a Grothendieck transformation $\gamma : \mathcal{B} \rightarrow \mathcal{B}'$ is a bi-variant ideal, i.e., the kernel $\gamma^{-1}(X, Y)$ of the homomorphism $\gamma : \mathcal{B}(X, Y) \rightarrow \mathcal{B}'(X, Y)$ gives rise to a bi-variant ideal, denoted by γ^{-1} , of the bi-variant theory \mathcal{B} .
2. Given a bi-variant ideal $\mathcal{I} \subset \mathcal{B}$, the following quotient \mathcal{B}/\mathcal{I} becomes a bi-variant theory by setting

$$(\mathcal{B}/\mathcal{I})(X, Y) := \mathcal{B}(X, Y)/\mathcal{I}(X, Y)$$

and by taking the bi-variant operations to be the ones induced by \mathcal{B} . Namely, they are defined as follows:

(a) (product): the product operation

$$\bullet : (\mathcal{B}/\mathcal{I})^*(X, Y) \otimes (\mathcal{B}/\mathcal{I})^*(Y, Z) \rightarrow (\mathcal{B}/\mathcal{I})^*(X, Z)$$

is defined by $[\alpha] \bullet [\beta] := [\alpha \bullet \beta]$.

(b) (pushforward):

i. for a proper morphism $f : X \rightarrow X'$, the pushforward

$$f_* : (\mathcal{B}/\mathcal{I})(X, Y) \rightarrow (\mathcal{B}/\mathcal{I})(X', Y)$$

is defined by $f_*[\alpha] := [f_*\alpha]$.

ii. for a quasi-smooth morphism $g : Y \rightarrow Y'$, the pushforward

$$*_g : (\mathcal{B}/\mathcal{I})(X, Y) \rightarrow (\mathcal{B}/\mathcal{I})(X, Y')$$

is defined by $[\alpha] *_g := [\alpha *_g]$.

(c) (pullback)

i. for a quasi-smooth morphism $f : X' \rightarrow X$, the pullback

$$f^* : (\mathcal{B}/\mathcal{I})(X, Y) \rightarrow (\mathcal{B}/\mathcal{I})(X', Y)$$

is defined by $f^*[\alpha] := [f^*\alpha]$.

ii. for a proper morphism $g : Y \rightarrow Y'$, the pull

$$*_g : (\mathcal{B}/\mathcal{I})(X, Y) \rightarrow (\mathcal{B}/\mathcal{I})(X, Y')$$

is defined by $[\alpha] *_g := [\alpha *_g]$.

(d) (Chern class operator)

i. for any line bundle L over X the Chern class operator

$$c_1(L) \bullet : (\mathcal{B}/\mathcal{I})(X, Y) \rightarrow (\mathcal{B}/\mathcal{I})(X, Y)$$

is defined by $c_1(L) \bullet [\alpha] := [c_1(L) \bullet \alpha]$.

(To be more precise, $c_1(L) \bullet$ should be written as $[c_1(L)] \bullet$.)

ii. for any line bundle M over Y the Chern class operator

$$\bullet c_1(M) : (\mathcal{B}/\mathcal{I})(X, Y) \rightarrow (\mathcal{B}/\mathcal{I})(X, Y)$$

is defined by $[\alpha] \bullet c_1(M) := [\alpha \bullet c_1(M)]$.

(As above, $\bullet c_1(M)$ should be written as $\bullet [c_1(M)]$.)

Remark 4.3. Proposition 4.2 (2) means, in other words, that the quotient morphism

$$\Theta : \mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}$$

is a Grothendieck transformation of bi-variant theories.

4.3. A bi-variant algebraic cobordism with bundles $\Omega^{*,\#}(X, Y)$

First we recall the definition of the bivariant algebraic cobordism with vector bundles $\Omega^{*,\#}(X \xrightarrow{f} Y)$ [6, Definition 5.7]. First we set

$$\mathcal{M}^{i,r}(X, Y)^+ := \mathbb{L} \otimes \mathcal{M}^{i,r}(X, Y)^+$$

where \mathbb{L} is the Lazard ring.

Definition 4.6 (The bivariant algebraic cobordism with vector bundles).

$$\Omega^{*,\#}(X \xrightarrow{f} Y) := \frac{\mathcal{M}^{*,\#}(X \xrightarrow{f} Y)^+}{\langle \mathcal{R}^{\text{LS}} \rangle(X \xrightarrow{f} Y)}$$

where $\langle \mathcal{R}^{\text{LS}} \rangle(X \xrightarrow{f} Y)$ is the bivariant ideal generated by the bivariant subset $\mathcal{R}^{\text{LS}}(X \xrightarrow{f} Y)$, which is defined as follows:

$$\mathcal{R}^{\text{LS}}(X \xrightarrow{f} Y) := \begin{cases} \mathcal{R}^{\text{LS}}(X \rightarrow pt), & Y = pt, \\ \emptyset, & Y \neq pt. \end{cases}$$

Here $\mathcal{R}^{\text{LS}}(X \rightarrow pt)$ is defined to be the kernel of the (surjective) morphism

$$\mathcal{M}^{-*,0}(X \rightarrow pt)^+ = \mathbb{L} \otimes \mathcal{M}^{-*,0}(X \rightarrow pt)^+ \rightarrow d\Omega_*(X) = \Omega^*(X \rightarrow pt).$$

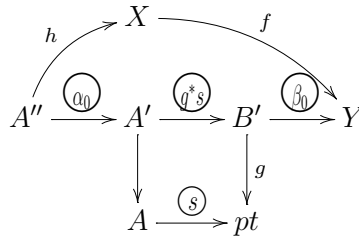
which is the \mathbb{L} -linear map, induced from a group homomorphism $\mathcal{M}^{-*,0}(X \rightarrow pt)^+ \rightarrow d\Omega_*(X)$ together with the ring homomorphism $\mathbb{L} \rightarrow d\Omega_*(pt)$ classifying the *formal group law* for the first Chern class of the tensor product of line bundles.

Here is an explicit description of the bivariant ideal $\langle \mathcal{R}^{LS} \rangle(X \xrightarrow{f} Y)$ [6, Proposition 5.12], which is a “vector bundle” version of [2, Lemma 3.8]:

Proposition 4.3. *The bivariant ideal $\langle \mathcal{R}^{LS} \rangle$ satisfies that $\langle \mathcal{R}^{LS} \rangle(X \xrightarrow{f} Y)$ consists of linear combinations of elements of the form*

$$(4.2) \quad h_*([E] \bullet \alpha_0 \bullet g^*s \bullet \beta_0)$$

where $h : A'' \rightarrow X, g : B' \rightarrow pt, \alpha_0 \in \mathcal{M}^{*,0}(A'' \rightarrow A')^+, \beta_0 \in \mathcal{M}^{*,0}(B' \rightarrow Y)^+$ and $s \in \mathcal{R}^{LS}(A \rightarrow pt)$ are as in the following diagram



E a vector bundle on A'' and $h : A'' \rightarrow X$ is a proper morphism. Here note that $[E] := [A'' \xrightarrow{\text{id}_{A''}} A''; E] \in \mathcal{M}^{*,*}(A'' \xrightarrow{\text{id}_{A''}} A'')$.

Remark 4.4. Here we emphasize that if we let E be the 0 bundle in (4.2) or equivalently if we delete $[E] \bullet$ from (4.2), the above proposition is nothing but [2, Lemma 3.8].

The linear extension of the “forgetting” homomorphism $\mathfrak{F} : \mathcal{M}^{i,r}(X \xrightarrow{f} Y)^+ \rightarrow \mathcal{M}^{i,r}(X, Y)^+$ by the Lazard ring \mathbb{L} shall be denoted by the same symbol $\mathfrak{F} : \mathcal{M}^{i,r}(X \xrightarrow{f} Y)^+ \rightarrow \mathcal{M}^{i,r}(X, Y)^+$, which is \mathbb{L} -linear.

In order to define a “correspondence” version $\langle \mathcal{R}^{LS} \rangle(X, Y)$ of $\langle \mathcal{R}^{LS} \rangle(X \xrightarrow{f} Y)$, one might be tempted to simply define it as the image of $\langle \mathcal{R}^{LS} \rangle(X \xrightarrow{f} Y)$ by $\mathfrak{F} : \mathcal{M}^{i,r}(X \xrightarrow{f} Y)^+ \rightarrow \mathcal{M}^{i,r}(X, Y)^+$:

$$\mathfrak{F} \left(\langle \mathcal{R}^{LS} \rangle(X \xrightarrow{f} Y) \right)$$

which consists of linear combinations of the form $\mathfrak{F}(h_*([E] \bullet \alpha_0 \bullet g^*s \bullet \beta_0))$ by Proposition 4.3. It follows from Lemma 3.1 that we have

$$(4.3) \quad \mathfrak{F}(h_*([E] \bullet \alpha_0 \bullet g^*s \bullet \beta_0)) = h_*\left(\mathfrak{F}([E]) \bullet \mathfrak{F}(\alpha_0) \bullet \mathfrak{F}(g^*s) \bullet \mathfrak{F}(\beta_0)\right).$$

Now we state our theorem:

Theorem 4.1. *There exists a bi-variant ideal $\langle \mathcal{R}^{\text{LS}} \rangle(-, -) \subset \mathcal{M}^{*,\#}(-, -)^+$ which is invariant under the \mathbb{L} -module structure such that the following hold:*

1. $\mathfrak{F}(\langle \mathcal{R}^{\text{LS}} \rangle(X \xrightarrow{f} Y)) \subset \langle \mathcal{R}^{\text{LS}} \rangle(X, Y)$.
2. We set

$$\Omega^{*,\#}(X, Y) := \frac{\mathcal{M}^{*,\#}(X, Y)^+}{\langle \mathcal{R}^{\text{LS}} \rangle(X, Y)}.$$

Then the “forgetting” homomorphism $\mathfrak{F} : \mathcal{M}^{*,\#}(X \xrightarrow{f} Y)^+ \rightarrow \mathcal{M}^{*,\#}(X, Y)^+$ descends to

$$\tilde{\mathfrak{F}} : \Omega^{*,\#}(X \xrightarrow{f} Y) \rightarrow \Omega^{*,\#}(X, Y)$$

defined by $\tilde{\mathfrak{F}}([\alpha]) := [\mathfrak{F}(\alpha)]$, where on the left-hand side

$[\alpha] := \alpha + \langle \mathcal{R}^{\text{LS}} \rangle(X \xrightarrow{f} Y)$ and on the right-hand side $[-]$ is

$[\beta] := \beta + \langle \mathcal{R}^{\text{LS}} \rangle(X, Y)$, i.e., we have the following commutative diagram:

$$(4.4) \quad \begin{array}{ccc} \mathcal{M}^{*,\#}(X \xrightarrow{f} Y)^+ & \xrightarrow{\mathfrak{F}} & \mathcal{M}^{*,\#}(X, Y)^+ \\ \pi \downarrow & & \downarrow \pi \\ \Omega^{*,\#}(X \xrightarrow{f} Y) & \xrightarrow{\tilde{\mathfrak{F}}} & \Omega^{*,\#}(X, Y) \end{array}$$

where both π 's are the quotient maps.

3. When $Y = pt$ is a point, we have the isomorphism:

$$\tilde{\mathfrak{F}} : \Omega^{*,\#}(X \rightarrow pt) \xrightarrow{\cong} \Omega^{*,\#}(X, pt),$$

thus we have $\Omega^{*,\#}(X, pt) \cong \Omega^{*,\#}(X \rightarrow pt) = \Omega_{-*,\#}(X) \cong \omega_{-*,\#}(X)$.

Proof. In order to see how to define a reasonable “correspondence” version $\langle \mathcal{R}^{\text{LS}} \rangle(X, Y)$, we express the right hand side of (4.3) simply as follows:

$$h_*\left(\mathfrak{F}([E]) \bullet \mathfrak{F}(\alpha_0) \bullet \mathfrak{F}(g^*s) \bullet \mathfrak{F}(\beta_0)\right)$$

$$\begin{aligned} &= h_*\mathfrak{F}([E]) \bullet \mathfrak{F}(\alpha_0) \bullet \mathfrak{F}(g^*s) \bullet \mathfrak{F}(\beta_0) \quad (\text{by } A_{12}) \\ &= \left(h_*\mathfrak{F}([E]) \bullet \mathfrak{F}(\alpha_0) \right) \bullet \mathfrak{F}(g^*s) \bullet \mathfrak{F}(\beta_0). \end{aligned}$$

Here clearly we have

$$\begin{aligned} h_*\mathfrak{F}([E]) \bullet \mathfrak{F}(\alpha_0) &\in \mathcal{M}^{*,\sharp}(X, A')^+, \quad \mathfrak{F}(g^*s) \in \mathcal{M}^{*,\sharp}(A', B)^+, \\ \mathfrak{F}(\beta_0) &\in \mathcal{M}^{*,\sharp}(B, Y)^+. \end{aligned}$$

Instead of giving an explicit form for an “correspondence” version $\langle \mathcal{R}^{\text{LS}} \rangle(X, Y)$ just like (4.3), we define the following less explicit one: $\langle \mathcal{R}^{\text{LS}} \rangle(X, Y)$ is defined to be generated (as an abelian group) by elements of the form

$$(4.5) \quad \alpha'_0 \bullet \mathfrak{F}(g^*s) \bullet \beta'_0,$$

where $\alpha'_0 \in \mathcal{M}^{*,\sharp}(X, A')^+, \beta'_0 \in \mathcal{M}^{*,\sharp}(B, Y)^+$, and g and s are as in (4.3). Then clearly it follows from (4.3) that we do have

$$\mathfrak{F} \left(\langle \mathcal{R}^{\text{LS}} \rangle(X \xrightarrow{f} Y) \right) \subset \langle \mathcal{R}^{\text{LS}} \rangle(X, Y).$$

Furthermore it follows from (4.5) that the following product property holds: for $\alpha \in \langle \mathcal{R}^{\text{LS}} \rangle(X, Y)$

1. $\beta \bullet \alpha \in \langle \mathcal{R}^{\text{LS}} \rangle(X', Y)$ for any $\beta \in \mathcal{M}^{*,\sharp}(X', X)^+$,
2. $\alpha \bullet \delta \in \langle \mathcal{R}^{\text{LS}} \rangle(X, Y')$ for any $\delta \in \mathcal{M}^{*,\sharp}(Y, Y')^+$.

Therefore $\langle \mathcal{R}^{\text{LS}} \rangle(-, -)$ is a bi-variant ideal in $\mathcal{M}^{*,\sharp}(-, -)^+$. Moreover $\langle \mathcal{R}^{\text{LS}} \rangle(X, Y)$ is automatically invariant under the \mathbb{L} -module structure, i.e., stable under the \mathbb{L} -action coming from the ring homomorphism $\mathbb{L} \rightarrow \mathcal{M}^{*,\sharp}(Z, Z)^+ : \ell \rightarrow \ell \otimes \mathbb{1}_Z$ for $Z = X, Y$. Clearly we obtain the above commutative diagram (4.4).

Next we observe that the “forgetting” homomorphism

$$\mathfrak{F} : \mathcal{M}^{*,\sharp}(X \rightarrow pt)^+ \rightarrow \mathcal{M}^{*,\sharp}(X, pt)^+$$

is an isomorphism because $\mathfrak{F}([V \xrightarrow{h} X; E]) = [X \xleftarrow{h} V \rightarrow pt; E]$ and the inverse of \mathfrak{F}

$$\mathfrak{F}^{-1} : \mathcal{M}^{*,\sharp}(X, pt)^+ \rightarrow \mathcal{M}^{*,\sharp}(X \rightarrow pt)^+$$

defined by $\mathfrak{F}^{-1}([X \xleftarrow{h} V \rightarrow pt; E]) := [V \xrightarrow{h} X; E]$ is well-defined. As observed above, we have

$$\mathfrak{F}(\langle \mathcal{R}^{\text{LS}} \rangle(X \rightarrow pt)) \subset \langle \mathcal{R}^{\text{LS}} \rangle(X, pt).$$

Now we want to show the following reverse inclusion in the case of $X \rightarrow pt$:

$$(4.6) \quad \mathfrak{F}(\langle \mathcal{R}^{LS} \rangle(X \rightarrow pt)) \supset \langle \mathcal{R}^{LS} \rangle(X, pt).$$

First, we note that, as defined above, $\langle \mathcal{R}^{LS} \rangle(X, pt)$ is an abelian group generated by elements of the form

$$\alpha'_0 \bullet \mathfrak{F}(g^*s) \bullet \beta'_0$$

where $\alpha'_0 \in \mathcal{M}^{*,\#}(X, A')^+, \beta'_0 \in \mathcal{M}^{*,\#}(B, pt)^+$, and g and s are the same as above. Then clearly we do have that $\beta'_0 = \mathfrak{F}(\beta_0)$ for some $\beta_0 \in \mathcal{M}^{*,\#}(B \rightarrow pt)^+$, which is the tautological inverse of β_0 , as observed above. Since \mathfrak{F} is \mathbb{L} -linear, it is enough to assume that $\alpha'_0 = [X \xleftarrow{p} A'' \xrightarrow{s'} A'; E]$ (where p is proper, s' is quasi-smooth and E is a vector bundle over A''), which can be expressed as

$$\begin{aligned} \alpha'_0 &= [X \xleftarrow{p} A'' \xrightarrow{s'} A'; E] \\ &= {}_p\mathbb{1} \bullet [[E]] \bullet \mathbb{1}_{s'} \\ &= {}_p\mathbb{1} \bullet \mathfrak{F}([E]) \bullet \mathfrak{F}(\theta(s')) \\ &\quad \text{(by Lemma 3.1, where “smooth” is replaced by “quasi-smooth”)} \\ &= p_*(\mathfrak{F}([E])) \bullet \mathfrak{F}(\theta(s')) \\ &= \mathfrak{F}(h_*[E]) \bullet \mathfrak{F}(\theta(s')) \quad \text{(by Lemma 3.1 (2))} \\ &= \mathfrak{F}(h_*[E] \bullet \theta(s')) \quad \text{(by Lemma 3.1 (1))} \end{aligned}$$

Therefore we have

$$\begin{aligned} \alpha'_0 \bullet \mathfrak{F}(g^*s) \bullet \beta'_0 &= \mathfrak{F}(h_*[E] \bullet \theta(s')) \bullet \mathfrak{F}(g^*s) \bullet \mathfrak{F}(\beta_0) \\ &= \mathfrak{F}(h_*[E] \bullet \theta(s') \bullet g^*s \bullet \beta_0) \\ &= \mathfrak{F}(h_*([E] \bullet \theta(s') \bullet g^*s \bullet \beta_0)) \\ &\in \mathfrak{F}(\langle \mathcal{R}^{LS} \rangle(X \rightarrow pt)), \end{aligned}$$

which implies the above inclusion (4.6). Hence we have $\mathfrak{F}(\langle \mathcal{R}^{LS} \rangle(X \rightarrow pt)) = \langle \mathcal{R}^{LS} \rangle(X, pt)$. Therefore we see that the above isomorphism $\mathfrak{F} : \mathcal{M}^{*,\#}(X \rightarrow pt)^+ \rightarrow \mathcal{M}^{*,\#}(X, pt)^+$ implies the isomorphism

$$\Omega^{*,\#}(X \rightarrow pt) \cong \Omega^{*,\#}(X, pt). \quad \square$$

Problem 4.1. *Would it be possible to give some conditions on $f : X \rightarrow Y$, X and Y so that we have an isomorphism $\Omega^{*,\sharp}(X \xrightarrow{f} Y) \cong \Omega^{*,\sharp}(X, Y)$?*

Remark 4.5. The “forgetting” homomorphism $\tilde{\mathfrak{F}} : \Omega^{*,\sharp}(X \xrightarrow{f} Y) \rightarrow \Omega^{*,\sharp}(X, Y)$ satisfies the following:

1. $\tilde{\mathfrak{F}}$ is commutative with the product \bullet .
2. $\tilde{\mathfrak{F}}$ is commutative with the pushforward, i.e., $\tilde{\mathfrak{F}} \circ f_* = f_* \circ \tilde{\mathfrak{F}}$.
3. $\tilde{\mathfrak{F}}$ is commutative the Chern class operators, which are special cases of (1).
4. As to the pullback, we cannot expect that the following diagram is commutative:

$$(4.7) \quad \begin{array}{ccc} \Omega^{*,\sharp}(X \xrightarrow{f} Y) & \xrightarrow{g^*} & \Omega^{*,\sharp}(X' \xrightarrow{f'} Y') \\ \tilde{\mathfrak{F}} \downarrow & & \tilde{\mathfrak{F}} \downarrow \\ \Omega^{*,\sharp}(X, Y) & \xrightarrow{(g')^* \circ (*g)} & \Omega^{*,\sharp}(X', Y') \end{array}$$

since the definitions of pullback are different. For the sake of simplicity, we show it. Here we consider the following fiber square

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ h' \downarrow & & \downarrow h \\ X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

for the pullback $g^* : \Omega^{*,\sharp}(X \xrightarrow{f} Y) \rightarrow \Omega^{*,\sharp}(X' \xrightarrow{f'} Y')$.

$$\begin{aligned} (\tilde{\mathfrak{F}} \circ g^*)([V \xrightarrow{h} X; E]) &= \tilde{\mathfrak{F}}([V' \xrightarrow{h'} X'; (g'')^* E]) \\ &= [X' \xleftarrow{h'} V' \xrightarrow{f' \circ h'} Y'; (g'')^* E]. \end{aligned}$$

$$\begin{aligned} (((g')^* \circ (*g)) \circ \tilde{\mathfrak{F}})([V \xrightarrow{h} X; E]) \\ = ((g')^* \circ (*g))([X \xleftarrow{h} V \xrightarrow{f \circ h} Y; E]) \end{aligned}$$

$$\begin{aligned}
 &= (g')^*([X \xleftarrow{h} V \xrightarrow{f \circ h} Y; E]) * g \\
 &= [X' \xleftarrow{h' \circ g''} V' \times_V V' \xrightarrow{f' \circ h' \circ g''} Y'; (\widetilde{g''})^*(g'')^* E].
 \end{aligned}$$

Here we consider the following fiber square:

$$\begin{array}{ccc}
 V' \times_V V' & \xrightarrow{\widetilde{g''}} & V' \\
 \widetilde{g''} \downarrow & & \downarrow g'' \\
 V' & \xrightarrow{g''} & V.
 \end{array}$$

Thus in general we have that $\mathfrak{F} \circ g^* \neq ((g')^* \circ (*g)) \circ \mathfrak{F}$ in the above diagram (4.7).

5. Generalizations

In this section we discuss briefly some possible generalized versions of Annala’s bivariant derived algebraic cobordism and our bi-variant algebraic cobordism treated in the previous section.

5.1. A “generalized” bivariant derived algebraic cobordism $\Omega_{\mathfrak{S}}^*(X \rightarrow Y)$ associated to a system \mathfrak{S} of subgroups $\mathfrak{s}(A)$ of $\mathcal{M}_*(A)$

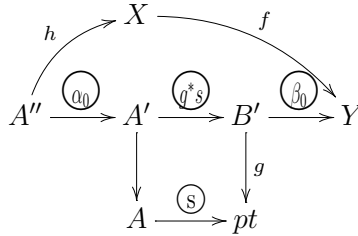
For Annala’s bivariant derived algebraic cobordism

$$\Omega^*(X \rightarrow Y) := \frac{\mathcal{M}^*(X \rightarrow Y)}{\langle \mathcal{R}^{\text{LS}} \rangle(X \rightarrow Y)},$$

the bivariant ideal $\langle \mathcal{R}^{\text{LS}} \rangle(X \rightarrow Y)$ of $\mathcal{M}^*(X \rightarrow Y)$ is constructed by using the kernels $\text{Ker } \eta_A$ of the morphisms $\eta_A : \mathcal{M}^*(A \rightarrow pt) = \mathcal{M}_*(A) \rightarrow d\Omega(A)$. The kernel $\text{Ker } \eta_A$ is a very special subgroup of $\mathcal{M}^*(A \rightarrow pt)$. So, instead of taking this very special subgroup, we can consider an arbitrary subgroup, putting aside the issue of what its geometric or algebraic or topological meaning is. Namely, if we let \mathfrak{S} be a family or system of subgroups $\mathfrak{s}(A) \subset \mathcal{M}^*(A \rightarrow pt)$ where $A \in \text{Obj}(\mathcal{C})$. Then the bivariant ideal $\langle \mathcal{R}^{\mathfrak{S}} \rangle(X \rightarrow Y)$ associated to the system \mathfrak{S} is defined to consist of linear combinations of elements of the following form, as in [2, Lemma 3.8] (also see Proposition 4.3 and Remark 4.4),

$$(5.1) \quad h_*(\alpha_0 \bullet g^* s \bullet \beta_0)$$

where $h : A'' \rightarrow X, g : B' \rightarrow pt, \alpha_0 \in \mathcal{M}^*(A'' \rightarrow A')^+, \beta_0 \in \mathcal{M}^*(B' \rightarrow Y)^+$ and $s \in \mathfrak{s}(A) \subset \mathcal{M}^*(A \rightarrow pt)$ are as in the following diagram



$h : A'' \rightarrow X$ is a proper morphism. Then a “generalized” bivariant derived algebraic cobordism $\Omega_{\mathfrak{S}}^*(X \rightarrow Y)$ associated to a system \mathfrak{S} of subgroups $\mathfrak{s}(A)$ of $\mathcal{M}_*(A)$ is defined to be the following quotient

$$(5.2) \quad \Omega_{\mathfrak{S}}^*(X \rightarrow Y) := \frac{\mathcal{M}^*(X \rightarrow Y)}{\langle \mathcal{R}^{\mathfrak{S}}(X \rightarrow Y) \rangle}.$$

If each $\mathcal{M}^*(A \rightarrow pt)$ contains the kernel $\text{Ker } \eta_A$, then we have the surjection $\Omega^*(X \rightarrow Y) \twoheadrightarrow \Omega_{\mathfrak{S}}^*(X \rightarrow Y)$. If each $\mathcal{M}^*(A \rightarrow pt)$ is contained in the kernel $\text{Ker } \eta_A$, then we have the other-way surjection $\Omega_{\mathfrak{S}}^*(X \rightarrow Y) \twoheadrightarrow \Omega^*(X \rightarrow Y)$.

5.2. A naive “algebraic cobordism” $\Omega_{prop-sm}(X \rightarrow Y)$ using proper and smooth morphisms

Let us replace the class of quasi-smooth morphisms by the class of smooth morphisms in Annala’s bivariant theory $\mathcal{M}^*(X \rightarrow Y)$ and denote the replaced one by $\mathcal{M}_{prop-sm}^*(X \rightarrow Y)$. Of course, if $f : X \rightarrow Y$ is not surjective, then $\mathcal{M}_{prop-sm}^*(X \rightarrow Y) = 0$ the trivial group. Note that this kind of thing *does* happen in bivariant theory as pointed out in [12, Remarks, p. 62]. Then we have the following natural inclusion, since a smooth morphism is a quasi-smooth morphism:

$$\iota : \mathcal{M}_{prop-sm}^*(X \rightarrow Y) \hookrightarrow \mathcal{M}^*(X \rightarrow Y)$$

Then the kernel of the following composite morphism

$$\mathcal{M}_{prop-sm}^*(X \rightarrow pt) \xrightarrow{\iota} \mathcal{M}^*(X \rightarrow pt) = \mathcal{M}_*(X) \rightarrow d\Omega(X)$$

is denoted by $\mathcal{R}_{prop-sm}^{LS}(X \rightarrow pt)$. Then, in the same way as the construction of Annala’s bivariant derived algebraic cobordism $\Omega^*(X \rightarrow Y)$, we define

$\Omega_{prop-sm}^*(X \rightarrow Y)$ as follows: The bivariant ideal $\langle \mathcal{R}_{prop-sm}^{LS} \rangle(X \xrightarrow{f} Y)$ is defined to consist of linear combinations of elements of the above form (5.1) where $s \in \mathcal{R}_{prop-sm}^{LS}(X \rightarrow pt)$. Then we define

$$\Omega_{prop-sm}^*(X \rightarrow Y) := \frac{\mathcal{M}_{prop-sm}^*(X \rightarrow Y)}{\langle \mathcal{R}_{prop-sm}^{LS} \rangle(X \rightarrow Y)}.$$

Here we emphasize that the category considered in this case is the classical category, not the category of derived schemes. Hence the explicit description (4.1) is not well-defined as an element $\Omega_{prop-sm}^*(X \xrightarrow{id_X} X)$ in the classical category, thus the Chern class operator $c_1(L)_{\bullet}$ is not available⁸ in this case. Thus $\Omega_{prop-sm}^*(X \rightarrow Y)$ is a bivariant theory without the data of the Chern class operator of line bundles. It is easy to see that there is an embedding

$$\tilde{\iota} : \Omega_{prop-sm}^*(X \rightarrow Y) \hookrightarrow \Omega^*(X \rightarrow Y)$$

since we have $\iota \left(\langle \mathcal{R}_{prop-sm}^{LS} \rangle(X \rightarrow Y) \right) \subset \langle \mathcal{R}^{LS} \rangle(X \rightarrow Y)$.

Here we note that, as in the above section §5.1, we can consider the following:

$$(5.3) \quad \Omega_{prop-sm, \mathfrak{S}}^*(X \rightarrow Y) := \frac{\mathcal{M}_{prop-sm}^*(X \rightarrow Y)}{\langle \mathcal{R}_{prop-sm}^{\mathfrak{S}} \rangle(X \rightarrow Y)}.$$

5.3. In the general case of *confined* morphisms and *specialized* morphisms or *vert* morphisms and *horiz* morphisms

Let *conf* and *spe* be the classes of confined morphisms and specialized morphisms, respectively and also *vert* and *horiz* be the classes of *vert* morphisms and *horiz* morphisms. Then, in the above (5.3) *prop-sm* can be replaced by *conf-spe* and *vert-horiz*:

$$(5.4) \quad \Omega_{conf-spe, \mathfrak{S}}^*(X \rightarrow Y), \quad \Omega_{vert-horiz, \mathfrak{S}}^*(X \rightarrow Y).$$

Note that in a general case when there is no notion of line bundles or vector bundles available, we cannot consider the Chern class operators $c_1(L)_{\bullet}$ of line bundles. Thus the above are bivariant theories without Chern class operators $c_1(L)_{\bullet}$. However, here we *emphasize* that in the category of derived schemes, if we let *confined* and *specialized* morphisms or *vert* and *horiz* morphisms

⁸If we consider the universal oriented bivariant theory $\mathbb{O}\mathbb{M}_{sm}^{prop}(X \rightarrow Y)$ [21], then the Chern class operator $c_1(L)_{\bullet}$ is available.

be *proper* morphisms and *quasi-smooth* morphisms, respectively, and let \mathfrak{S} be the system consisting of the kernel $\text{Ker } \eta_A$ of the morphism $\eta_A : \mathcal{M}^*(A \rightarrow pt) \rightarrow d\Omega_*(A)$ for $A \in \mathcal{C}$, then $\Omega_{conf-spe, \mathfrak{S}}^*(X \rightarrow Y) = \Omega_{vert-horiz, \mathfrak{S}}^*(X \rightarrow Y)$ is nothing but Annala's bivariant derived algebraic cobordism $\Omega^*(X \rightarrow Y)$, which is equipped with the Chern class operators $c_1(L)^\bullet$ of line bundles.

Acknowledgements

The author would like to thank Jörg Schürmann and the anonymous referee for careful reading of the paper and for their many valuable comments and constructive suggestions. The author is supported by JSPS KAKENHI Grant Numbers JP19K03468 and JP23K03117.

References

- [1] T. ABE, Enhanced bivariant homology theory attached to six functor formalism, *J. Topol.* **15** (2022), no. 4, 1675–1754. [MR4461845](#)
- [2] T. ANNALA, Bivariant derived algebraic cobordism, *J. Algebraic Geom.* **30** (2021), 205–252. [MR4233182](#)
- [3] T. ANNALA, Precobordism and cobordism, *Algebra Number Theory* **15** (2021), 2571–2646. [MR4377859](#)
- [4] T. ANNALA, Derived algebraic cobordism, *Ph.D. Thesis, Univ. British Columbia*, March 2022, [arXiv:2203.12096](#).
- [5] T. ANNALA, Chern classes in precobordism theories, *J. Eur. Math. Soc.*, **25** (2023), no. 4, 1379–1422. [MR4577967](#)
- [6] T. ANNALA AND S. YOKURA, Bivariant algebraic cobordism with bundles, *Algebraic Geom. (Foundation Compositio Mathematica)* **10** (2023), no. 4, 461–488. [MR4606410](#)
- [7] N. CHRISS AND V. GINZBURG, *Representation Theory and Complex Geometry*, Birkhäuser Boston, Inc., Boston, MA, 1997.
- [8] A. CONNES AND G. SKANDALIS, Théorème de l'indice pour les feuilletages, *C. R. Acad. Sci. Paris Sér. I. Math.* **292** (1981), no. 18, 871–876. [MR0623519](#)
- [9] A. CONNES AND G. SKANDALIS, The longitudinal index theorem for foliations, *Publ. Res. Inst. Math. Sci.* **20** (1984), 1139–1183. [MR0775126](#)
- [10] H. EMERSON AND R. MEYER, Bivariant K -theory via correspondences, *Adv. Math.* **225** (2010), 2883–2919. [MR2680187](#)

- [11] W. FULTON, *Intersection Theory*, Springer Verlag, 1981.
- [12] W. FULTON AND R. MACPHERSON, Categorical frameworks for the study of singular spaces, *Mem. Amer. Math. Soc.* **31** (1981), 243. [MR0609831](#)
- [13] D. GAITSGORY AND N. ROZENBLYUM, *A Study in Derived Algebraic Geometry, Volume I: Correspondences and Duality*, Mathematical Surveys and Monographs, volume **221**, Amer. Math. Soc., 2017. [MR3701353](#)
- [14] V. GINZBURG, Geometric methods in the representation theory of Hecke algebras and quantum groups, in: *Representation Theories and Algebraic Geometry* (Montreal, PQ, 1997), volume 514 of *NATO Adv. Sci. Inst. Ser. C, Math. Phys. Sci.*, pages 127–183. Kluwer Acad. Publ., Dordrecht, 1998. [MR1649626](#)
- [15] G. KASPAROV, The operator K -functor and extensions of C^* -algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.* **44** (1980), 571–636. [MR0582160](#)
- [16] Y.-P. LEE AND R. PANDHARIPANDE, Algebraic cobordism of bundles on varieties, *J. Eur. Math. Soc.* **14** (2012), 1081–1101. [MR2928845](#)
- [17] M. LEVINE AND F. MOREL, *Algebraic Cobordism*, Springer Monographs in Mathematics, Springer-Verlag (2007).
- [18] M. LEVINE AND R. PANDHARIPANDE, Algebraic cobordism revisited, *Invent. Math.* **176** (2009), 63–130. [MR2485880](#)
- [19] C.-H. LIU, Y.-J. TZENG AND S.-Y. YAU, Algebraic cobordism of filtered vector bundles on varieties: Notes on a work of Lee and Pandharipande, [arXiv:1104.0280v1](#) [math.AG].
- [20] P. LOWREY AND T. SCHÜRG, Derived algebraic cobordism, *J. Inst. Math. Jussieu* **15** (2016), no. 2, 407–443. [MR3466543](#)
- [21] S. YOKURA, Oriented bivariant theory. I, *Internat. J. Math.* **20** (2009), no. 10, 1305–1334. [MR2574317](#)
- [22] S. YOKURA, Oriented bivariant theory, II – Algebraic cobordism of S -schemes, *Internat. J. Math.* **36** (2019), no. 6, 1950031, 40 pp. [MR3977282](#)
- [23] S. YOKURA, Cobordism bicycles of vector bundles, *New York J. Math.* **26** (2020), 950–1001. [MR4150634](#)
- [24] S. YOKURA, Enriched categories of correspondences and characteristic classes of singular varieties, *Fund. Math.* **253** (2021), 17–60. [MR4178874](#)
- [25] S. YOKURA, Riemann–Roch formulas for universal bivariant theories, *Eur. J. Math.* **9** (2023), 18. doi: [10.1007/s40879-023-00605-4](#). [MR4560561](#)

Shoji Yokura
Graduate School of Science and Engineering
Kagoshima University
21-35 Korimoto 1-chome
Kagoshima 890-0065
Japan
E-mail: yokura@sci.kagoshima-u.ac.jp