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Mirror symmetry for open r-spin invariants

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Abstract: We show that a generating function for open r-spin enumerative invariants produces a universal unfolding of the polynomial x^r. Further, the coordinates parametrizing this universal unfolding are flat coordinates on the Frobenius manifold associated to the Landau-Ginzburg model $(\mathbb{C}, x^r)$ via Saito-Givental theory. This result provides evidence for the same phenomenon to occur in higher dimension, proven in the sequel [GKT22].

1. Introduction

Buryak, Clader, and Tessler recently constructed an open r-spin enumerative theory [BCT21, BCT18], following the case of descendent invariants on the moduli space of holomorphic disks developed by Pandharipande, Solomon and Tessler in [PST14]. Roughly stated, they construct a moduli space of r-stable orbidisks with r-spin structures that have prescribed twists at both internal and boundary marked points. In turn, this gives a closed form expression for the open r-spin invariants

$$\left\langle \prod_{i=1}^{l} \tau_0^{a_i} \sigma^{k+1} \right\rangle^{\frac{1}{r}, o}$$

corresponding to genus 0 orbidisks with $k+1$ boundary marked points with twist $r-2$ and l internal marked points with twists $a_1, \ldots, a_l$.

These invariants are analogous to the closed A-model enumerative theory for Landau-Ginzburg models constructed in a sequence of papers [JKV01, FJR07, FJR08, FJR13], using ideas of Witten [Wit93]. In these papers, the authors build an enumerative theory in the case of gauged Landau-Ginzburg models $(\mathbb{C}^n, W, G)$ where W is an invertible polynomial and G is a subgroup of the diagonal automorphism group of W. Berglund and Hübsch predicted mirror pairs between such Landau-Ginzburg models. They propose that the mirror to a gauged LG model $(\mathbb{C}^n, W, G)$ should be a mirror gauged LG model

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$(\mathbb{C}^n, W^T, G^T)$, where W^T is the so-called transposed polynomial and G^T is the dual group [BH92]. One such example is the mirror pair

$$(\mathbb{C}^n, x_1^{r_1} + \cdots + x_n^{r_n}, \mu_{r_1} \times \cdots \times \mu_{r_n}) \leftrightarrow (\mathbb{C}^n, x_1^{r_1} + \cdots + x_n^{r_n}, 1). \tag{1.1}$$

The open r-spin invariants stated above correspond to the open A-model invariants for the Landau-Ginzburg model $(\mathbb{C}, x^r, \mu_r)$, following the analogue of the closed case established in [FJR11]. Thus from (1.1) when $n = 1$, the open r-spin invariants should correspond to an open B-model enumerative theory $(\mathbb{C}^1, x^r, 1)$.

The (closed) B-model side of the story for the right-hand side of the correspondence (1.1) was developed in [Sai83a, Sai83b, Giv96, Dub96], and was more recently used in Landau-Ginzburg mirror symmetry in [LLSS17, HLSW22]. The B-model side, put as simply as possible, is a Saito-Givental theory, which involves calculating oscillatory integrals of the form

$$\int_\Gamma e^{W_{\mathbf{t}}/\hbar} f(x_1, \ldots, x_n, \mathbf{t}) dx_1 \wedge \cdots \wedge dx_n. \tag{1.2}$$

Here $x_1, \ldots, x_n$ are coordinates on $\mathbb{C}^n$, $\hbar$ is a coordinate on an auxiliary $\mathbb{C}^*$, $\mathbf{t} = \{t_{ij} \,|\, 1 \leq i \leq n, 0 \leq j \leq r_i - 2\}$ is a set of coordinates on the parameter space for a universal unfolding $W_{\mathbf{t}}$ of W, and $f \in \mathbb{C}[x_1, \ldots, x_n][[\mathbf{t}]]$. Finally Γ runs over some suitable non-compact cyles in $\mathbb{C}^n$. The requirement on f is that the form $f dx_1 \wedge \cdots \wedge dx_n$ is a so-called primitive form in the sense of Saito-Givental theory.

While Saito-Givental theory in general gives a Frobenius manifold structure to the universal unfolding of W, determining this structure can be quite difficult. However, experience with mirror symmetry for toric Fano varieties [Gro10, FOOO10] suggests that mirror symmetry becomes much more transparent when a specific perturbation $W_{\mathbf{t}}$ of the original potential is used. Previous work on the Fano/LG mirror correspondence (see e.g., [CO06, FOOO10, Gro10]) found that there is a "correct" universal unfolding $W_{\mathbf{t}}$ which is a generating function for counting Maslov index two disks with boundary on a Lagrangian torus. The advantage of this "correct" universal unfolding is that there is a canonical choice of primitive form (which in our case will just be $\Omega = dx$) and that flat coordinates coincide with a natural choice of coordinates. Using the same philosophy here, we show that the "correct" universal unfolding is a generating function for the open r-spin invariants.

Theorem 1.1. *The deformed potential*

$$W_{\mathbf{t}} = \sum_{k\geq 0, l\geq 0} \sum_{\{a_i\}\in\mathscr{A}_l} (-1)^{l-1} \frac{\left\langle \prod_{i=1}^{l} \tau_0^{a_i} \sigma^{k+1} \right\rangle^{\frac{1}{r},o}}{k!|\mathrm{Aut}(\{a_i\})|} \left(\prod_{i=1}^{l} t_{a_i}\right) x^k, \tag{1.3}$$

has the primitive form $\Omega = \mathrm{d}x$ and $t_0, \dots, t_{r-2}$ are flat coordinates for the Frobenius manifold constructed via Saito-Givental theory for the LG model $W = x^r$. Here, $\mathscr{A}_l$ denotes the set of multi-sets $\{a_1, \dots, a_l\}$ with $0 \leq a_i \leq r-2$ for each i, and for $A \in \mathscr{A}_l$, $\mathrm{Aut}(A)$ denotes the group of permutations σ of $\{1, \dots, l\}$ with $a_i = a_{\sigma(i)}$.

Remark 1.2. Recall that a Frobenius manifold is a manifold with a flat connection, a compatible metric g, and an associative product rule on its tangent spaces arising from a potential. Once one knows the explicit description of the primitive form and flat coordinates, it is not difficult to derive this other structure. In particular, the metric g in this case takes the form $g(\partial_{t_i}, \partial_{t_j}) = \delta_{i,r-2-j}$ (see, e.g., [HLSW22, Definition 2.11]). On the other hand, the potential is easily derived from the Givental J-function, which is obtained directly from the oscillatory integrals (1.2), with the integrand being the primitive form. See Chapter 2 of [Gro11] for an exposition of this.

In this paper, we use the closed form for open r-spin invariants in [BCT18] as a black box to prove the mirror Theorem 1.1. The flat structure of the Frobenius manifold for the Landau-Ginzburg model $(\mathbb{C}^1, x^r, 1)$ has been studied in the past in its own right in integrable systems. In the 1980s, Noumi and Yamada found algorithms for finding the flat coordinates [Nou84, NY98]. These techniques were transcendental in nature, whereas we prove Theorem 1.1 through purely algebraic and combinatorial means once given the open r-spin invariants. We aim for this paper to serve as a link from the integrable systems literature for flat structures to the Landau-Ginzburg mirror symmetry analogue of the Fano/LG correspondence developed in the $n = 2$ case in our sequel paper [GKT22].

In [Bur20], Buryak describes the flat structure Frobenius manifold via the extended r-spin invariants introduced in [BCT19]. There, he shows that the change of coordinates from the versal deformation to the flat coordinates can be derived from particular differentials of a generating function built from extended invariants (Theorem 3.1 in [Bur20]). The extended invariants are closely related to the open r-spin invariants [BCT18, Theorem 1.3]. The proof presented here is different from that in [Bur20] as the author uses transcendental techniques from integrable systems. Here, the generating function using open invariants is direct and combinatorial.

While it is true that, in dimension one, one can derive the flat structure using either extended or open r-spin invariants, when one considers dimension greater than one the relation with the extended theory breaks. In the $n = 2$ case constructed in [GKT22], we use open FJRW invariants to construct flat coordinates for the singularity $W = x_1^{r_1} + x_2^{r_2}$. These in turn can be used to compute the closed extended invariants (with descendents). In this case, the open invariants exhibit the exact same wall crossing phenomenon as the flat coordinates, whereas the extended theory is not subject to any wall crossing. We aim for this paper to help solidify the import of the perspective taken from the Fano/LG mirror correspondence to Landau-Ginzburg mirror symmetry.

In Section 2, we outline the relevant Saito-Givental theory for the Landau-Ginzburg models $(\mathbb{C}^n, x_1^{r_1} + \cdots + x_n^{r_n}, 1)$. In Section 3, we define primitive forms and flat coordinates in the context of the LG models in (1.1) and give an example. In Section 4, we prove Theorem 1.1.

2. The *B*-model state space for Fermat polynomials

In this section, we describe the enumerative theory associated to a Landau-Ginzburg B-model, due to Saito and Givental and described in the case of the B-model of FJRW theory by He, Li, Li, Saito, Shen and Webb in [LLSS17, HLSW22]. Let (X, W) be a Landau-Ginzburg model, i.e., X a variety and $W : X \to \mathbb{C}$ a regular function. We will not allow for a group of symmetries. We shall quickly review Saito-Givental theory in this context. For a much more in-depth exposition in our framework and notation, see Chapter 2 of [Gro11] and references therein. A principal object of study is the twisted de Rham complex

$$(\Omega_X^\bullet, d + \hbar^{-1} dW \wedge -),$$

where Ω_X^i is the sheaf of algebraic i-forms on X and $\hbar \in \mathbb{C}^*$ is an auxiliary parameter. We restrict our attention to the Landau-Ginzburg model

$$(X, W) = \Big(\mathbb{C}^n, W = \sum_i x_i^{r_i}\Big). \tag{2.1}$$

Proposition 2.1. *Consider the Landau-Ginzburg model (X, W) in (2.1). Then the hypercohomology group $\mathbb{H}^n(X, (\Omega_X^\bullet, d + \hbar^{-1} dW \wedge -))$ has dimension $\prod_{i=1}^n (r_i - 1)$ and is generated by the basis*

$$M = \left\{ \prod_i x_i^{a_i} \Omega \;\middle|\; 0 \le a_i \le r_i - 2 \right\},$$

where $\Omega = dx_1 \wedge \cdots \wedge dx_n$.

Proof. First, since $\mathbb{C}^n$ is affine, the cohomology of the sheaves $\Omega^i_{\mathbb{C}^n}$ vanishes in degree at least one. Thus by the hypercohomology spectral sequence, it is enough to compute the cohomology of the complex

$$0 \to \Omega^0_{\mathbb{C}^n} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^{n-1}_{\mathbb{C}^n} \xrightarrow{\delta} \Omega^n_{\mathbb{C}^n} \to 0,$$

where $\delta = d + \hbar^{-1} dW \wedge -$. Thus the hypercohomology we are interested in can be written as

$$\mathbb{H}^n(X, (\Omega^\bullet_X, d + \hbar^{-1} dW \wedge -)) = \Omega^n_{\mathbb{C}^n} / \delta(\Omega^{n-1}_{\mathbb{C}^n}). \tag{2.2}$$

First note that

$$\begin{aligned} &\delta(x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n) \\ &\qquad = (-1)^{i-1} \hbar^{-1} r_i x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_i^{r_i - 1} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n} \Omega. \end{aligned} \tag{2.3}$$

Next, when $a_1 \dots a_n \in \mathbb{Z}_{\geq 0}$ and $a_i > 0$ we have that

(2.4)
$$\begin{aligned} &\delta(x_1^{a_1} \cdots x_n^{a_n} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n) \\ &\quad = (-1)^{i-1} a_i x_1^{a_i} \cdots x_i^{a_i - 1} \cdots x_n^{a_n} \Omega + (-1)^{i-1} \hbar^{-1} r_i x_1^{a_1} \cdots x_i^{r_i + a_i - 1} \cdots x_n^{a_n} \Omega. \end{aligned}$$

Thus we have in the quotient $\Omega^n_{\mathbb{C}^n} / \delta(\Omega^{n-1}_{\mathbb{C}^n})$ the relations

$$\begin{aligned} x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_i^{r_i - 1} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n} \Omega &= 0 \\ x_i^{r_i} x_1^{a_1} \cdots x_n^{a_n} \Omega &= -\hbar \tfrac{a_i + 1}{r_i} x_1^{a_1} \cdots x_n^{a_n} \Omega. \end{aligned} \tag{2.5}$$

Thus, any section of $\Omega^n_{\mathbb{C}^n}$ of the form

$$x_1^{a_1} \cdots x_n^{a_n} \Omega$$

with $a_i \geq r_i$ can have its exponent reduced by r_i and any section with $a_i = r_i - 1$ vanishes, yielding the result. $\square$

Remark 2.2. Note that the right-hand side of (2.2) is the exact expression for the formally completed version of the Brieskorn lattice $\mathcal{H}^{(0)}_W$ associated to W stated in §3.1 of [LLSS17].

Using Proposition 2.1, there is a vector bundle $\mathcal{R}^\vee$ on the $\hbar$-plane $\mathbb{C}$ whose fibre over $\hbar \in \mathbb{C}^*$ is the cohomology group $\mathbb{H}^n(X, (\Omega^\bullet_X, d + \hbar^{-1} dW \wedge -))$, and M yields a frame for $\mathcal{R}^\vee$ which extends across the origin; see [Gro11], §2.2.2 and in particular Definition 2.37.

There is a homology group dual to the hypercohomology group $\mathbb{H}^n(X, (\Omega^\bullet_X, d + \hbar^{-1} dW \wedge -))$

$$H_n(X, \operatorname{Re} W/\hbar \ll 0; \mathbb{C}),$$

which roughly is given by possibly unbounded cycles for which $\operatorname{Re} W/\hbar$ tends to $-\infty$ in the unbounded directions. More precisely, these groups can be defined as rapid decay homology as in [Hie07], [KKP08]. There is then a natural perfect pairing

(2.6)

$$\begin{aligned} H_n(X, \operatorname{Re} W/\hbar \ll 0; \mathbb{C}) \times \mathbb{H}^n(X, (\Omega^\bullet_X, d + \hbar^{-1} dW \wedge -)) &\longrightarrow \mathbb{C}, \\ (\Xi, \omega) &\longmapsto \int_\Xi e^{W/\hbar} \omega. \end{aligned}$$

Thus there must be a dual basis for $H_n(X, \operatorname{Re} W/\hbar \ll 0; \mathbb{C})$ for any basis of the hypercohomology group.

Example 2.3. Consider the Landau-Ginzburg model $(\mathbb{C}, x^r)$. Given a fixed value of $\hbar \in \mathbb{C}^*$, set $\Psi_j := \{ t e^{((\pi + \arg \hbar)\sqrt{-1} + 2\pi j \sqrt{-1})/r} \mid t \in \mathbb{R}_{\geq 0} \}$. Note this requires a choice of $\arg \hbar$: if $\arg \hbar$ is replaced with $\arg \hbar + 2\pi$, Ψ_j becomes Ψ_{j+1}. This will yield a multi-valuedness for the cycles Ξ^r_j constructed below. In the formulas below, we make a specific choice of branch of $\arg \hbar$, which will then give a well-defined choice of $\hbar^{1/r}$.

We now show there exist cycles $\Xi^r_j \in H_1(X, \operatorname{Re}(x^r/\hbar) \ll 0; \mathbb{C})$ for $0 \leq j \leq r-2$ so that

$$\int_{\Xi^r_j} x^k e^{x^r/\hbar} dx = \delta_{jk} \tag{2.7}$$

for all $0 \leq j \leq r-2$, where δ_{jk} is the Kronecker delta function.

First note that $W/\hbar$ is real and negative on Ψ_j, going to $-\infty$ in the unbounded direction of Ψ_j. Thus $\Psi_{j+1} - \Psi_j \in H_1(X, \operatorname{Re} W/\hbar \ll 0; \mathbb{C})$. By a direct computation of exponential integrals, we find that

$$A_{jk} := \int_{\Psi_{j+1} - \Psi_j} x^k e^{x^r/\hbar} dx = \frac{1}{r} C_k \zeta^{j(k+1)},$$

where $\zeta := e^{2\pi\sqrt{-1}/r}$ and

$$C_k = e^{\pi i \frac{k+1}{r}} \hbar^{(k+1)/r} \Gamma\left(\frac{k+1}{r}\right) \left(e^{2\pi i (k+1)/r} - 1\right).$$

We find the inverse of the matrix $A = (A_{jk}) \in \mathrm{Mat}_{(r-1)\times(r-1)}(\mathbb{C}(\hbar^{1/r}))$ to explicitly compute the cycles Ξ_j as a linear combination of the $(\Psi_{j+1} - \Psi_j)$. Let $C = \mathrm{diag}(C_k)_{k=0}^{r-2}$. If we postmultiply by C^{-1}, we obtain the matrix

$$B := AC^{-1} = \left(\frac{1}{r}\zeta^{j(k+1)}\right)_{0\leq j,k\leq r-2}.$$

The inverse matrix here can be computed to be $(B^{-1})_{jk} = (\zeta^{-k(j+1)} - \zeta^{j+1})$. So then

$$(A^{-1})_{jk} = (C^{-1}B^{-1})_{jk} = \frac{1}{C_j}(\zeta^{-k(j+1)} - \zeta^{j+1}).$$

We then have an explicit description of Ξ_j^r:

(2.8)

$$\Xi_j^r := \frac{1}{e^{\pi i \frac{j+1}{r}}\hbar^{(j+1)/r}\Gamma(\frac{j+1}{r})(\zeta^{j+1}-1)}\sum_{k=0}^{r-2}(\zeta^{-k(j+1)} - \zeta^{j+1})(\Psi_{k+1} - \Psi_k).$$

The following more general integrals will also be important for us in proving Theorem 1.1:

Lemma 2.4. *For all $r, n \in \mathbb{N}$:*

$$\begin{aligned}\int_{\Xi_d^r} x^{nr+k}e^{x^r/\hbar}dx &= (-1)^n\hbar^n\left(\prod_{i=1}^{n}\left(i-1+\frac{k+1}{r}\right)\right)\int_{\Xi_d^r} x^k e^{x^r/\hbar}dx \\ &= (-1)^n\hbar^n\frac{\Gamma(n+\frac{k+1}{r})}{\Gamma(\frac{k+1}{r})}\int_{\Xi_d^r}x^k e^{x^r/\hbar}dx.\end{aligned} \tag{2.9}$$

Proof. This is proven by iterating integration by parts or the relation given in (2.5). □

Using this example, we can describe a similar set of cycles for the Fermat Landau-Ginzburg model

$$(\mathbb{C}^n, W_0), \qquad W_0 := x_1^{r_1} + \cdots + x_n^{r_n}.$$

Define the set $D := \{\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n \mid 0 \leq \mu_i \leq r_i - 2\}$. For any $\mu \in D$, we then can define the cycles

$$\Xi_\mu := \Xi_{\mu_1}^{r_1} \times \cdots \times \Xi_{\mu_n}^{r_n} \tag{2.10}$$

as products of those defined in the previous example. Then we have that

$$\int_{\Xi_\mu} x^{\mu'} e^{(x_1^{r_1}+\cdots+x_n^{r_n})/\hbar}\Omega = \delta_{\mu\mu'} \tag{2.11}$$

with $\Omega = dx_1 \wedge \cdots \wedge dx_n$.

When $n = 1$ and $W = x^r$, we will suppress notation and take $\Xi_j := \Xi_j^r$.

3. Flat coordinates for Fermat polynomials

We restrict to the case where $(X, W_0) = (\mathbb{C}^n, x_1^{r_1} + \cdots + x_n^{r_n})$. Note that the elements x^μ for $\mu \in D$ are a basis for the Jacobian ring

$$\mathbb{C}[x_1, \ldots, x_n]/(\partial W_0/\partial x_1, \ldots, \partial W_0/\partial x_n).$$

Thus we may write a universal unfolding of W_0 parameterized by a germ $\mathcal{M}$ of the origin in the Jacobian ring, viewed as a vector space. We use coordinates y_μ on $\mathcal{M}$, parameterized by $\mu \in D$. Here $\{y_\mu\}$ is the dual basis to $\{x^\mu\}$. The versal deformation W for W_0 on $\mathcal{M} \times X$ is then given by

$$W = x_1^{r_1} + \cdots + x_n^{r_n} + \sum_{\mu \in D} y_\mu x^\mu.$$

For a polynomial function $W : \mathbb{C}^n \to \mathbb{C}$ and a function $f \in \mathbb{C}[x_1, \ldots, x_n][[y_\mu]]$, we now consider the following collection of oscillatory integrals, each of which we can view as a formal power series in the variables y_μ:

$$\int_{\Xi_\mu} e^{W/\hbar} f\Omega = \sum_{j=-\infty}^{\infty} \varphi_{\mu,j}(\mathbf{y})\hbar^{-j}$$

where $\varphi_{\mu,j}(\mathbf{y}) \in \mathbb{C}[[y_\mu]]$. Note that this expansion should be viewed as formal. Such an expansion can be be obtained by expanding the exponential in a Taylor series as is done in (4.5) below, and then evaluating the integrals over the dual basis as in (2.7) using formulas similar to those of Lemma 2.4.

The following is an oversimplification of Saito's theory of primitive forms, but is sufficient for our purposes:

Definition 3.1. If $\varphi_{\mu,j} \equiv 0$ for all $j < 0$ and $\varphi_{\mu,0} = \delta_{\mu\mathbf{0}}$, then we say that $f\Omega$ is a *primitive form*. Further, in this case, $\varphi_{\mu,1}$ form a set of coordinates on the universal unfolding $\mathcal{M}$ called *flat coordinates*.

Notation 3.2. We typically will use the variables $t_\mu := \varphi_{\mu,1}$ for the flat coordinates.

Example 3.3. Consider the Landau-Ginzburg model $(\mathbb{C}, x^4)$. In this case, we have $M = \{1, x, x^2\}$ and we consider the versal deformation

$$W = x^4 + y_2x^2 + y_1x + y_0.$$

Using Lemma 2.4, we obtain the following expansions showing the lowest degree terms in $\hbar^{-1}$:

$$\begin{aligned} \int_{\Xi_0} e^{W/\hbar}\Omega &= 1 + (y_0 - \tfrac{1}{8}y_2^2)\hbar^{-1} + O(\hbar^{-2}) \\ \int_{\Xi_1} e^{W/\hbar}\Omega &= y_1\hbar^{-1} + O(\hbar^{-2}) \\ \int_{\Xi_2} e^{W/\hbar}\Omega &= y_2\hbar^{-1} + O(\hbar^{-2}) \end{aligned} \tag{3.1}$$

Thus in this case Ω is already a primitive form, and flat coordinates are given by $t_0 = y_0 - y_2^2/8$, $t_1 = y_1$, $t_2 = y_2$. However, we may then rewrite the universal unfolding using this change of variables, obtaining

$$W_{\mathbf{t}} = x^4 + t_2x^2 + t_1x + t_0 + \tfrac{1}{8}t_2^2.$$

Then

$$\int_{\Xi_d} e^{W_{\mathbf{t}}/\hbar}\Omega = \delta_{0d} + t_d\hbar^{-1} + \cdots$$

for all $d \in \{0, 1, 2\}$.

Remark 3.4. He, Li, Shen and Webb in [HLSW22] use Saito's general framework for constructing Frobenius manifold structures on the universal unfoldings of potentials (see [Sai83a, Sai83b] or Section III.8 of [Man99]) to construct the B-model Frobenius manifold for LG models $(\mathbb{C}^n, W)$, where W is an invertible polynomial. Our definition of flat coordinates coincides with the definition given in Equation (13) in §2.2.2 of [HLSW22], with different notation.

4. Open Landau-Ginzburg mirror symmetry in dimension 1

In the case of the mirror pair of Landau-Ginzburg models

$$(\mathbb{C}, x^r, \mu_r) \longleftrightarrow (\mathbb{C}, x^r),$$

the relevant open invariants for the Landau-Ginzburg model $(\mathbb{C}, x^r, \mu_r)$, i.e., open r-spin invariants, have been already constructed in [BCT21, BCT18]. Recall the following theorem:

Theorem 4.1 (Theorem 1.2 of [BCT18]). *Take $k, l \geq 0$ and $0 \leq a_1, \ldots, a_l \leq r-1$. Suppose we consider the open r-spin invariant associated to $k+1$ boundary marked points with twist $r-2$ and l internal marked points with twists $a_1, \ldots a_l$. Then we have*

$$\left\langle \prod_{i=1}^{l} \tau_0^{a_i} \sigma^{k+1} \right\rangle^{\frac{1}{r}, o} = \begin{cases} \frac{(k+l-1)!}{(-r)^{l-1}}, & \text{if } \frac{(r-2)k+2\sum_i a_i}{r} = 2l+k-2; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.2. We note that an open r-spin invariant with no boundary marked points must vanish. This is a corollary to Lemma 6.5(3) of [BCT18].

Proposition 4.3. *If the open r-spin invariant $\langle \prod_{i=1}^{l} \tau_0^{a_i} \sigma^{k+1} \rangle^{\frac{1}{r}, o}$ is nonzero, then $k \leq r$ and $l \leq r$. In particular, the number of nonzero primary open r-spin invariants is finite.*

Proof. The constraint in Theorem 4.1

$$\frac{(r-2)k + 2\sum_i a_i}{r} = 2l + k - 2 \tag{4.1}$$

requires that $2r = 2(lr - \sum_i a_i) + 2k$. Since $lr - \sum_i a_i \geq l$, there are at most $r+1$ boundary marked points and at most r internal marked points. $\square$

We also have the following lemma:

Lemma 4.4. *Suppose $I = \{a_1, \ldots, a_l\}$ is a nonempty multiset of internal marking twists with $0 \leq a_i \leq r-2$. Let $r(I) \in \{0, \ldots, r-1\}$ be such that $\sum_i a_i \equiv r(I) \pmod r$. Then the open r-spin invariant $\langle \prod_{i=1}^{l} \tau_0^{a_i} \sigma^{k+1} \rangle^{\frac{1}{r}, o}$ is nonzero if and only if $k = r(I)$ and $\sum_i a_i = r(I) + (l-1)r$.*

Proof. By Observation 2.1 of [BCT21], in order for the moduli space corresponding to the open r-spin invariant $\langle \prod_{i=1}^{l} \tau_0^{a_i} \sigma^{k+1} \rangle^{\frac{1}{r}, o}$ to be nonempty, the following integrality condition must hold:

$$e := \frac{2\sum_i a_i + k(r-2)}{r} = \frac{2(\sum_i a_i - k)}{r} + k \in \mathbb{Z}.$$

Moreover, by Observation 2.10 of [BCT21], we must also have that $e \equiv k \pmod 2$. This implies that $\frac{2(\sum_i a_i - k)}{r}$ is an even integer, hence $\sum_i a_i - k \in r\mathbb{Z}$, so $k \equiv r(I) \pmod r$.

By Proposition 4.3, we have $k \le r$, so $k = r(I)$ unless $k = r$. If $k = r$, then, by (4.1), we must have $\sum_i a_i = rl$ hence $l = 0$. However, we are assuming the set of internal markings is non-empty, so this does not occur. Thus $k = r(I)$.

Now, suppose that $\sum_i a_i = r(I) + pr$ for some p. Then we have that

$$\frac{(r-2)k + 2(r(I)+pr)}{r} = \frac{(r-2)r(I) + 2r(I) + 2pr}{r} = r(I) + 2p. \tag{4.2}$$

Note by Theorem 4.1 we must have $r(I) + 2p = 2l + k - 2$ in order to have a nonzero open r-spin invariant, but since $k = r(I)$ this implies that $2p = 2l-2$, or $p = l-1$ as desired.

The converse is a straightforward computation from Theorem 4.1. $\square$

Corollary 4.5. *Consider a multiset I as in Lemma 4.4 so that $\sum_i a_i = r(I) + (|I|-1)r$.*

(a) For any $I' \subset I$, we have $\sum_{i\in I'} a_i = r(I') + (|I'|-1)r$. Moreover, there exists a unique $k_{I'} \in \{0,\dots,r-1\}$ so that $\langle \prod_{i\in I'} \tau_0^{a_i}\sigma^{k_{I'}+1}\rangle^{\frac{1}{r},o}$ is nonzero.

(b) For a multiset partition $I = I_1 \cup \dots \cup I_h$, we have $r(I_1) + \dots + r(I_h) = r(I) + (h-1)r$.

Proof. We first prove item (a). Note the condition that $\sum_{i\in I'} a_i = r(I') + (|I'|-1)r$ for any $I' \subseteq I$ is equivalent to $\sum_{i\in I'}(r-a_i) = r - r(I')$. As $r(I') \in \{0,\dots,r-1\}$, this is equivalent to $0 < \sum_{i\in I'}(r-a_i) \le r$. The left-hand inequality is automatic as $a_i < r$ for all i. We have $\sum_{i\in I}(r-a_i) \le r$ by assumption, and hence $\sum_{i\in I'}(r-a_i) \le r$ for any subset $I' \subseteq I$. Thus $\sum_{i\in I'} a_i = r(I') + (|I'|-1)r$.

For (b), note that

$$\begin{aligned} r(I) + (|I|-1)r &= \sum_{i\in I} a_i = \sum_j \sum_{i\in I_j} a_i \\ &= \sum_j (r(I_j) + (|I_j|-1)r) = (|I|-h)r + \sum_j r(I_j), \end{aligned}$$

from which the result follows. $\square$

For any $l \ge 0$, set $[l] := \{1,2,\dots,l\}$ and let $\mathscr{A}_l$ be the set of multisets $\{a_1,\dots,a_l\}$ of integers where $0 \le a_i \le r-2$. If $A \in \mathscr{A}_l$, we define $\mathrm{Aut}(A)$ to be the group of permutations $\sigma : [l] \to [l]$ such that $a_i = a_{\sigma(i)}$ for all $i \in [l]$. For example, the multiset $A = \{1,2,2,3,3,3\}$ has $|\mathrm{Aut}(A)| = 1!\cdot 2!\cdot 3! = 12$.

We are now ready to prove Theorem 1.1 which we repeat here:

Theorem 4.6. *With the potential*

(4.3)

$$W_{\mathbf{t}} = \sum_{k\geq 0, l\geq 0} \sum_{\{a_i\}\in\mathscr{A}_l} (-1)^{l-1} \frac{\langle \prod_{i=1}^{l} \tau_0^{a_i}\sigma^{k+1}\rangle^{\frac{1}{r},o}}{k!|\mathrm{Aut}(\{a_i\})|} \left(\prod_{i=1}^{l} t_{a_i}\right) x^k \in \mathbb{C}[t_0,\dots,t_{r-2},x],$$

$\Omega = \mathrm{d}x$ *is a primitive form and* $t_0,\dots,t_{r-2}$ *are flat coordinates. Further, if* $\mathcal{I}_d$ *denotes the ideal in* $\mathbb{C}[t_0,\dots,t_{r-2}]$ *generated by all degree* d *monomials, then*

$$W_{\mathbf{t}} = x^r + t_{r-2}x^{r-2} + \cdots + t_0 \mod \mathcal{I}_2. \tag{4.4}$$

Proof. Note that by Lemma 4.4, all terms of $W_{\mathbf{t}}$ arising from invariants with internal markings are of degree at most $r-1$ in the variable x. On the other hand, by Theorem 4.1, the only non-zero invariant $\langle\sigma^{k+1}\rangle^{\frac{1}{r},o}$ with no internal markings has $k=r$ and the invariant is $-r!$. Thus x^r appears in $W_{\mathbf{t}}$ and is the only monomial in $W_{\mathbf{t}}$ with degree greater than or equal to r. That is,

$$W_{\mathbf{t}} \equiv x^r \mod \mathcal{I}_1.$$

A similar argument from Theorem 4.1 for invariants with one internal marked point then gives (4.4).

We now expand

$$\int_{\Xi_d} e^{W_{\mathbf{t}}/\hbar}\Omega = \int_{\Xi_d} \left(\sum_{l\geq 0} \frac{(W_{\mathbf{t}}-x^r)^l}{l!\hbar^l}\right) e^{x^r/\hbar}\Omega. \tag{4.5}$$

Any monomial summand in the formal power series $\sum_{l\geq 1}\frac{(W_{\mathbf{t}}-x^r)^l}{l!\hbar^l}$ will be of the form $cx^{nr+k}\hbar^{-l}$ for some $c\in\mathbb{C}[t_0,\dots,t_{r-2}]$, $n\in\mathbb{Z}_{\geq 0}$ and $k\in\{0,\dots,r-1\}$. By Lemma 2.4, it thus follows that

$$\int_{\Xi_d} cx^{nr+k}\hbar^{-l}e^{x^r/\hbar}\Omega = \tilde{c}\hbar^{n-l}\int_{\Xi_d} x^k e^{x^r/\hbar}\Omega = \tilde{c}\hbar^{n-l}\delta_{dk}$$

for some $\tilde{c}\in\mathbb{C}[t_0,\dots,t_{r-2}]$. Since $nr+k\leq(r-1)l$, we have that $n<l$. Thus, the highest power of $\hbar$ appearing in $\int_{\Xi_d}(\sum_{l\geq 1}\frac{(W_{\mathbf{t}}-x^r)^l}{l!\hbar^l})e^{x^r/\hbar}\Omega$ is -1. Hence

$$\int_{\Xi_d} e^{W_{\mathbf{t}}/\hbar}\Omega = \delta_{0d} + \varphi_d\hbar^{-1} + \cdots \tag{4.6}$$

where $\varphi_d\in\mathbb{C}[[t_0,\dots,t_{r-2}]]$, for all d. Note that the δ_{0d} term comes from (2.7) and the $l=0$ term of the Taylor expansion in (4.5).

We will prove that

$$\varphi_d \equiv t_d \bmod \mathcal{I}_l \tag{4.7}$$

for all l, and hence $\varphi_d = t_d$. First, by (4.4), it is clear that $\varphi_d = t_d \bmod \mathcal{I}_2$.

Consider a multiset $I = \{a_1, \ldots, a_l\}$. Suppose that there is a multiset partition $I = \bigcup_{j=1}^h I_j$ of I such that there exists a non-negative integer k_{I_j} with $\langle \prod_{i \in I_j} \tau^{a_i} \sigma^{k_{I_j}+1} \rangle^{\frac{1}{r},o} \neq 0$ for all j. Then we obtain in the expansion of $(W - x^r)^h$ a term of the form

$$cx^{\sum_{j=1}^h k_{I_j}} \hbar^{-h} \prod_{j=1}^h \left\langle \prod_{i \in I_j} \tau^{a_i} \sigma^{k_{I_j}+1} \right\rangle^{\frac{1}{r},o}$$

where $c \in \mathbb{C}[t_0, \ldots, t_{r-2}]$. Note that $k_{I_j} = r(I_j)$ by Lemma 4.4, so the above term, after taking the integral, will contribute a term of the form $\hbar^{-1}$ if and only if $\sum_{j=1}^h r(I_j) = r(I) + (h-1)r$. So let us assume this is the case, as we are not interested in contributions of the form $\hbar^{-d}$, $d \geq 2$. Then, again by Lemma 4.4 and this assumption,

$$\sum_{i \in I} a_i = \sum_{j=1}^h \big[r(I_j) + (|I_j| - 1)r\big] = r(I) + \big(|I| - 1\big)r$$

and the hypotheses of Corollary 4.5 hold.

On the way to proving that Equation (4.7) holds for all l, we will introduce the following notation. Let $\mathrm{Part}_h(I)$ denote the unordered partitions of I into h multisets. For example, with $h = 2$,

$$\begin{aligned}\mathrm{Part}_h(\{1,1,2,2\}) = \Big\{ &\{\{1\},\{1,2,2\}\}, \{\{2\},\{1,1,2\}\}, \\ &\{\{1,1\},\{2,2\}\}, \{\{1,2\},\{1,2\}\} \Big\}.\end{aligned}$$

On the other hand, we write $\mathrm{Part}_h([l])$ for partitions of $[l]$ into h disjoint sets. There is a map

$$\mathbf{a} : \mathrm{Part}_h([l]) \to \mathrm{Part}_h(I)$$

given by $\mathbf{a}(\{Q_1, \ldots, Q_h\}) = \{I_1, \ldots, I_h\}$, where $I_j = \{a_i \,|\, i \in Q_j\}$. This map is surjective but not injective: e.g., in the above example,

$$\mathbf{a}\big(\{\{1,3\},\{2,4\}\}\big) = \mathbf{a}\big(\{\{1,4\},\{2,3\}\}\big) = \{\{1,2\},\{1,2\}\}.$$

For $\{I_1, \ldots, I_h\} \in \mathrm{Part}_h(I)$, we write $\mathrm{Aut}(\{I_1, \ldots, I_h\})$ for the set of permutations $\sigma : [h] \to [h]$ with $I_{\sigma(i)} = I_i$.

With this notation, note that

$$|\mathrm{Aut}(I)| = |\mathbf{a}^{-1}(\{I_1,\dots,I_h\})| \cdot |\,\mathrm{Aut}(\{I_1,\dots,I_h\})| \cdot \prod_{j=1}^{h} |\,\mathrm{Aut}(I_j)|. \tag{4.8}$$

Using Equations (4.3) and (4.5), we can see that the summand in the integral that corresponds to the coefficient of the monomial $t_I := \prod_{a_i\in I} t_{a_i}$ is:

$$t_I \sum_{h=1}^{|I|} \frac{1}{h!} \sum_{\{I_1,\dots,I_h\}\in \mathrm{Part}_h(I)} \Bigg(\frac{h!}{\mathrm{Aut}(\{I_1,\dots,I_h\})} x^{\sum_j r(I_j)} \hbar^{-h} \cdot \prod_{j=1}^{h} (-1)^{|I_j|-1} \frac{\langle \prod_{i\in I_j} \tau^{a_i} \sigma^{k_{I_j}+1}\rangle^{\frac{1}{r},o}}{k_{I_j}!|\mathrm{Aut}(I_j)|} \Bigg),$$

where $k_{I_j} = r(I_j)$ as above.

Using Corollary 4.5(b) combined with Lemma 2.4, we can see that the only integral that will have a nonzero contribution is that corresponding to the cycle $\Xi_{r(I)}$ and that the $\hbar^{-1}$ term will have a summand of the form $\Lambda_I t_I$ where

$$\Lambda_I := \sum_{h=1}^{|I|} \frac{1}{h!} \sum_{\{I_1,\dots,I_h\}\in \mathrm{Part}_h(I)} \Bigg(\frac{h!}{\mathrm{Aut}(\{I_1,\dots,I_h\})} (-1)^{h-1} \frac{\Gamma(\frac{1+\sum_j k_{I_j}}{r})}{\Gamma(\frac{1+k_I}{r})} \cdot \prod_{j=1}^{h} (-1)^{|I_j|-1} \frac{\langle \prod_{i\in I_j} \tau^{a_i} \sigma^{k_{I_j}+1}\rangle^{\frac{1}{r},o}}{k_{I_j}!|\mathrm{Aut}(I_j)|} \Bigg). \tag{4.9}$$

We remind the reader that $k_I = r(I)$ by Lemma 4.4. By using the closed form for the open r-spin invariants given Theorem 4.1 and cancelling signs, we then have that

$$\Lambda_I = \sum_{h=1}^{|I|} \frac{1}{h!} \sum_{\{I_1,\dots,I_h\}\in \mathrm{Part}_h(I)} \Bigg(\frac{h!}{\mathrm{Aut}(\{I_1,\dots,I_h\})} (-1)^{h-1} \frac{\Gamma(\frac{1+\sum_j k_{I_j}}{r})}{\Gamma(\frac{1+k_I}{r})} \cdot \prod_{j=1}^{h} \frac{(k_{I_j}+|I_j|-1)!}{k_{I_j}!|\mathrm{Aut}(I_j)|r^{|I_j|-1}} \Bigg). \tag{4.10}$$

By (4.8),

$$\frac{1}{|\mathrm{Aut}(\{I_1,\dots,I_h\})| \cdot \prod_{j=1}^{|I|} |\mathrm{Aut}(I_j)|} = \frac{|\mathbf{a}^{-1}(\{I_1,\dots,I_h\})|}{|\mathrm{Aut}(I)|} \tag{4.11}$$

and hence by replacing the sum over elements of $\mathrm{Part}_h(I)$ with the larger sum over elements of $\mathrm{Part}_h([l])$ and multiplying by $|\mathrm{Aut}(I)|$, we obtain the following simplification:

(4.12)

$$|\mathrm{Aut}(I)|\Lambda_I = \sum_{h=1}^{|I|} \sum_{\{Q_1,\ldots,Q_h\}\in\mathrm{Part}_h([l])} (-1)^{h-1} \frac{\Gamma(\frac{1+\sum_j k_{I_j}}{r})}{\Gamma(\frac{1+k_I}{r})} \prod_{j=1}^{h} \frac{(k_{I_j}+|I_j|-1)!}{k_{I_j}!r^{|I_j|-1}}.$$

We claim that $|\mathrm{Aut}(I)|\Lambda_I$ vanishes. To prove this, we will instead first view the twists a_i as formal variables and then change coordinates to new formal variables $b_i := r - a_i$. We set the notation $b_A := \sum_{i\in A} b_i$. Note that, since $\sum_{i\in A} a_i = r(A) + (|A|-1)r$ for all subsets $A \subset I$, $b_A = r - r(A)$ and the number b_A is in the set $\{0, \ldots, r\}$.

Note now that we can rewrite $|\mathrm{Aut}(I)|\Lambda_I$ in the following way:

(4.13)

$$|\mathrm{Aut}(I)|\Lambda_I = \sum_{h=1}^{|I|} \sum_{\{Q_1,\ldots,Q_h\}\in\mathrm{Part}_h([l])} (-1)^{h-1} \frac{\Gamma(h+\frac{1-b_I}{r})}{\Gamma(\frac{1+r-b_I}{r})} \prod_{j=1}^{h} \frac{(r-1+|I_j|-b_{I_j})!}{(r-b_{I_j})!r^{|I_j|-1}}.$$

This is further rearranged as:

(4.14)

$$\begin{aligned}
|\mathrm{Aut}(I)|\Lambda_I &= \sum_{h=1}^{|I|} \frac{(-1)^{h-1}}{r^{l-1}} \sum_{\{Q_1,\ldots,Q_h\}\in\mathrm{Part}_h([l])} \left(\prod_{j=1}^{h-1}(jr+1-b_I)\right) \\
&\qquad \cdot \left(\prod_{j=1}^{h} \frac{(r-1+|I_j|-b_{I_j})!}{(r-b_{I_j})!}\right) \\
&= \frac{1}{r^{l-1}} \sum_{h=1}^{|I|}(-1)^{h-1} \sum_{\{Q_1,\ldots,Q_h\}\in\mathrm{Part}_h([l])} \left(\prod_{j=1}^{h-1}(jr+1-b_I)\right) \\
&\qquad \cdot \left(\prod_{j=1}^{h}\prod_{i=1}^{|I_j|-1}(r-b_{I_j}+i)\right).
\end{aligned}$$

From (4.14), it is transparent that $|\mathrm{Aut}(I)|\Lambda_I$ is a polynomial in the variables b_i. If we show that $|\mathrm{Aut}(I)|\Lambda_I$ is the zero polynomial in the variables b_i, then this implies that Λ_I vanishes. Therefore, the theorem reduces to showing that $|\mathrm{Aut}(I)|\Lambda_I$ is the zero polynomial for any I with cardinality at least two. We proceed by induction.

Let $l = 2$ and consider the multiset $I = \{a_1, a_2\}$. Then $\mathrm{Part}_1([l]) = \{I\}$ and $\mathrm{Part}_2([l])$ only contains the one (unordered) partition $\{\{a_1\}, \{a_2\}\}$. In this case, (4.14) reduces to:

$$|\mathrm{Aut}(I)|\Lambda_I = \frac{1}{r}(r - b_I + 1) - \frac{1}{r}(r + 1 - b_I) = 0.$$

Moving to the case where I has cardinality greater than two, by the induction hypothesis, we know for any subset $I' \subset I$ of cardinality above 1 that $\Lambda_{I'} = 0$ viewed as a polynomial in the variables b_i.

Note that $|\mathrm{Aut}(I)|\Lambda_I$ is a symmetric polynomial of degree at most $l-1$ in l variables. Recall that for an arbitrary polynomial P_l of degree at most $l-1$ in l variables b_i that if $P_l|_{\{b_i=0\}} = 0$ for all i then the polynomial P_l is the zero polynomial. Since Λ_I is symmetric, it suffices to show that $\Lambda_I|_{b_l=0} = 0$. Note that $\Lambda_I|_{b_l=0}$ is a polynomial in $l-1$ variables, and we will now work to relate it to the polynomial $\Lambda_{I\setminus\{a_l\}}$.

Take $\mathrm{Part}([l]) = \bigcup_{h=1}^{l} \mathrm{Part}_h([l])$ and let $\mathrm{Part}([l-1]) = \bigcup_{h=1}^{l-1} \mathrm{Part}_h([l-1])$. We then can define a surjective function $f : \mathrm{Part}([l]) \to \mathrm{Part}([l-1])$ by:

(4.15)
$$f(\{Q_1, \ldots, Q_h\}) = \begin{cases} \{Q_1, \ldots, Q_j \setminus \{l\}, \ldots, Q_h\} & \text{if } l \in Q_j \text{ and } |Q_j| > 1; \\ \{Q_1, \ldots, \widehat{Q_j}, \ldots, Q_h\} & \text{if } Q_j = \{l\}. \end{cases}$$

We will denote by $\mathrm{Cont}(\{Q_1, \ldots, Q_h\})$ the polynomial

$$\mathrm{Cont}(\{Q_1, \ldots, Q_h\}) := (-1)^{h-1} \left(\prod_{j=1}^{h-1} (jr + 1 - b_I) \right) \left(\prod_{j=1}^{h} \prod_{i=1}^{|I_j|-1} r - b_{I_j} + i \right),$$

which is the contribution to Λ_I given by the partition $\{Q_1, \ldots, Q_h\}$, where

$$\{I_1, \ldots, I_h\} = \mathbf{a}(\{Q_1, \ldots, Q_h\}).$$

We compare the contribution $\mathrm{Cont}(Q)$ for $Q \in \mathrm{Part}([l-1])$ with the sum $\sum_{Q' \in f^{-1}(Q)} \mathrm{Cont}(Q')$. We claim:

$$\sum_{Q' \in f^{-1}(Q)} \mathrm{Cont}(Q')|_{b_l=0} = (l-2)\,\mathrm{Cont}(Q).$$

Indeed, take $Q = \{Q_1, \ldots, Q_h\} \in \mathrm{Part}_h([l-1])$, with $\mathbf{a}(Q) = \{I_1, \ldots, I_h\}$. Its inverse image under f consists of exactly $h+1$ partitions, namely $Q^0 =$

$\{Q_1, \ldots, Q_h, \{a_l\}\}$ and $Q^j = \{Q_1, \ldots, Q_j \cup \{l\}, \ldots, Q_h\}$ for $1 \leq j \leq h$. Note that

$$\begin{aligned} \operatorname{Cont}(Q^0)|_{b_l=0} &= -(hr + 1 - b_{I\setminus\{a_l\}}) \operatorname{Cont}(Q), \\ \operatorname{Cont}(Q^j)|_{b_l=0} &= (r - b_{I_j} + |Q_j|) \operatorname{Cont}(Q) \quad \text{for } 1 \leq j \leq h. \end{aligned} \tag{4.16}$$

Hence, by using that $\sum_{j=1}^h |Q_j| = l - 1$, we see that

$$\sum_{Q' \in f^{-1}(Q)} \operatorname{Cont}(Q')|_{b_l=0} = \sum_{j=0}^{h} \operatorname{Cont}(Q^j)|_{b_l=0} = (l-2) \operatorname{Cont}(Q).$$

We now can use this recursion with Equation (4.14), the surjectivity of f, and the induction hypothesis to see that

$$\begin{aligned} (|\operatorname{Aut}(I)|\Lambda_I)|_{b_l=0} &= \frac{1}{r^{l-1}} \sum_{Q \in \operatorname{Part}([l])} \operatorname{Cont}(Q)|_{b_l=0} \\ &= \frac{1}{r^{l-1}} \sum_{Q \in \operatorname{Part}([l-1])} (l-2) \operatorname{Cont}(Q) \\ &= \frac{1}{r^{l-1}} \left(r^{l-2}(l-2)|\operatorname{Aut}(I \setminus \{a_l\})|\Lambda_{I\setminus\{a_l\}} \right) \\ &= 0. \end{aligned} \tag{4.17}$$

This consequently implies that $\Lambda_I = 0$ for all $|I| = l \geq 2$. Therefore, Ω is a primitive form and the coordinates $t_0, \ldots, t_{r-2}$ are flat coordinates. $\square$

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