

# Actions of finite group schemes on curves

MICHEL BRION

**Abstract:** Every action of a finite group scheme  $G$  on a variety admits a projective equivariant model, but not necessarily a normal one. As a remedy, we introduce and explore the notion of  $G$ -normalization. In particular, every curve equipped with a  $G$ -action has a unique projective  $G$ -normal model, characterized by the invertibility of ideal sheaves of all orbits. Also,  $G$ -normal curves occur naturally in some questions on surfaces in positive characteristics.

**Keywords:** Finite group scheme, curve, surface.

1	Introduction	1065
2	Preliminaries	1067
3	Rational actions	1073
4	$G$ -normality	1079
5	Generically free actions on curves	1087
	Acknowledgements	1093
	References	1093

## 1. Introduction

Much is known about finite groups of automorphisms of algebraic varieties, the case of curves being the most classical and well-understood. By contrast, finite group schemes of automorphisms (in characteristic  $p > 0$ ) seem to have attracted little attention until very recent years, where they have been determined for several natural classes of surfaces; see e.g. [23], [10], [15], [16]. One reason may be that key properties of finite group actions fail for a finite group scheme  $G$ , for example:

---

Received July 17, 2022.

**2010 Mathematics Subject Classification:** Primary 14L30, 14L15; secondary 14E07, 14H37, 14J50.

- If  $G$  acts faithfully on a variety  $X$ , then it may not act freely on a dense open subset.
- The  $G$ -action on  $X$  may not lift to an action on the normalization.

When  $X$  is a normal projective curve, its group of birational automorphisms coincides with the automorphism group, and hence is the group of rational points of an algebraic group. Moreover, every finite group is the full automorphism group scheme of some smooth projective curve over an algebraically closed field (see [14]). These results also do not extend to the schematic setting, already for infinitesimal group schemes of height at most 1 (which are in bijective correspondence with finite-dimensional  $p$ -Lie algebras):

- Every curve admits  $p$ -Lie algebras of rational vector fields of any prescribed dimension.
- But there are strong restrictions on the  $p$ -Lie algebras which can be realized by rational vector fields on a curve, as follows from [20, Thm. 12.1].

In this paper, we propose remedies to some of these failures. As a substitute for “generic freeness”, we show that every action of a finite group scheme  $G$  on a variety  $X$  is “generically transitive” (Corollary 2.5; if  $G$  is infinitesimal of height 1, this is [20, Prop. 5.2]).

As a substitute for the normalization, we introduce and explore the notion of “ $G$ -normalization” in Section 4. In particular, we show that every rational  $G$ -action on  $X$  admits a projective  $G$ -normal model, which is unique if  $X$  is a curve (Corollary 4.4). We also obtain a version of Serre’s criterion for  $G$ -normality (Theorem 4.12), which takes a more specific form in dimension 1: a curve is  $G$ -normal if and only if the ideal sheaf of every  $G$ -orbit is invertible (Corollary 4.14).

Let us emphasize that  $G$ -normal curves are generally singular. On the positive side, they turn out to be geometrically unibranch (Corollary 4.6) and local complete intersections (Corollary 4.18). Moreover, the tangent sheaf of every generically free  $G$ -normal curve is invertible (Proposition 5.4).

Finally,  $G$ -normal curves are related to regular surfaces via the following construction: let  $X$  be a curve equipped with a  $G$ -action, and assume that  $G$  is a subgroup scheme of a smooth connected algebraic group  $G^\#$  of dimension 1. Consider the diagonal action of  $G$  on  $G^\# \times X$ ; then the quotient  $S$  is a surface equipped with a  $G^\#$ -action. Moreover,  $X$  is  $G$ -normal if and only if  $S$  is regular (Proposition 4.17). The case where  $X$  is projective and  $G^\#$  is an elliptic curve is of special interest, since  $S$  is projective; then the above construction goes back to [3]. In this case,  $G$ -normal curves provide the missing ingredient in the recent classification of maximal connected algebraic groups of birational automorphisms of surfaces, see [12] and Proposition 5.6.

This paper is organized as follows. Section 2 begins with preliminary results on finite group scheme actions that we could not find in the literature; we then show that such actions are generically transitive. In Section 3, we investigate rational  $G$ -actions on a variety  $X$ ; in particular, such rational actions correspond bijectively to actions on the generic point. This is then used to construct faithful rational actions of infinitesimal group schemes of height 1. Section 4 makes the first steps in the study of  $G$ -normal varieties, with applications to curves. The final Section 5 is devoted to generically free actions on curves. We obtain in particular a local model for such an action at a fixed point (Proposition 5.5).

We conclude this introduction with two open questions. For a curve equipped with a generically free action of an infinitesimal group scheme  $G$ , it is easy to see that the Lie algebra of  $G$  has dimension 1 (Lemma 5.3). Of course, this holds if  $G$  is a subgroup scheme of a smooth connected algebraic group of dimension 1. Are there any further examples? In particular, are these group schemes commutative?

Also, every curve  $X$  as above may be viewed as a ramified  $G$ -cover of the quotient  $Y = X/G$ ; the latter is normal if  $X$  is  $G$ -normal. Can we then determine  $X$  in terms of ramification data on  $Y$ , in analogy with the known description of abelian covers (see e.g. [18], [1])?

## 2. Preliminaries

We fix a ground field  $k$  of characteristic  $p \geq 0$ , and choose an algebraic closure  $\bar{k}$ . Given a field extension  $K/k$  and a  $k$ -scheme  $X$ , we denote by  $X_K$  the  $K$ -scheme  $X \times_{\text{Spec}(k)} \text{Spec}(K)$ , with projection  $\pi : X_K \rightarrow X$ .

A *variety*  $X$  is a separated, geometrically integral scheme of finite type over  $k$ . A *curve* (resp. a *surface*) is a variety of dimension 1 (resp. 2).

The field of rational functions on a variety  $X$  is denoted by  $k(X)$ . This is a *function field in  $n$  variables* where  $n = \dim(X)$ , i.e., a separable, finitely generated field extension  $K$  of  $k$ , such that  $k$  is algebraically closed in  $K$ . Conversely, every function field  $K$  in  $n$  variables is the field of rational functions on some  $n$ -dimensional variety  $X$ , a *model* of  $K$ .

Throughout this paper, we denote by  $G$  a finite group scheme, and by  $|G|$  its *order*, i.e., the dimension of the  $k$ -vector space  $\mathcal{O}(G) = \Gamma(G, \mathcal{O}_G)$ . If  $p = 0$  then  $G$  is étale by Cartier's theorem (see [8, II.6.1.1]). This fails if  $p > 0$ , where basic examples of non-étale finite group schemes are  $\mu_p$  (the multiplicative group scheme of  $p$ th roots of unity) and  $\alpha_p$  (the kernel of the  $p$ th power map in the additive group).

The connected component of the neutral element  $e \in G(k)$  is denoted by  $G^0$ ; this is an *infinitesimal* group scheme, i.e., a finite group scheme having a unique point. Also,  $G^0$  is a normal subgroup scheme of  $G$ , and  $\pi_0(G) = G/G^0$  is étale. If  $k$  is perfect, then the reduced subscheme  $G_{\text{red}}$  is the largest étale subgroup scheme of  $G$ , and  $G = G^0 \rtimes G_{\text{red}}$ ; in particular,  $G_{\text{red}} \xrightarrow{\sim} \pi_0(G)$  (see [8, II.5.1.1, II.5.2.4]).

Returning to an arbitrary ground field  $k$ , we denote by  $\text{Lie}(G)$  the Lie algebra of  $G$ . Then  $\text{Lie}(G^0) = \text{Lie}(G)$ ; in particular,  $\text{Lie}(G) = 0$  if  $G$  is étale (e.g., when  $p = 0$ ). If  $p > 0$  then  $\text{Lie}(G)$  has the structure of a finite-dimensional  $p$ -Lie algebra, also called a restricted Lie algebra; see [8, II.7.3.4].

Still assuming  $p > 0$ , we denote by

$$F_X : X \longrightarrow X^{(p)}$$

the relative Frobenius morphism of a scheme  $X$ , and by

$$F_X^n : X \longrightarrow X^{(p^n)}$$

its  $n$ th iterate, where  $n$  is a positive integer. Given  $G$  as above, each  $F_G^n$  is a homomorphism of group schemes, with kernel the  *$n$ th Frobenius kernel*  $G_n$ . Moreover,  $G$  is infinitesimal if and only if  $G_n = G$  for  $n \gg 0$ ; then the smallest such  $n$  is the *height*  $\text{ht}(G)$ . The assignment  $G \mapsto \text{Lie}(G)$  yields an equivalence of categories between finite group schemes of height at most 1 and finite-dimensional  $p$ -Lie algebras; moreover, we have  $\text{Lie}(G_1) = \text{Lie}(G)$  (see [8, II.7.3.5, II.7.4.1]).

A  *$G$ -scheme* is a scheme  $X$  equipped with a  $G$ -action

$$\alpha : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x.$$

Note that  $\alpha$  is identified with the projection  $G \times X \rightarrow X$  via the automorphism  $(\text{pr}_1, \alpha)$  of  $G \times X$ . In particular, the morphism  $\alpha$  is finite and locally free. The  $G$ -action is said to be *faithful* if every non-trivial subgroup scheme acts non-trivially.

A morphism of  $G$ -schemes  $f : X \rightarrow Y$  is *equivariant* if  $f(g \cdot x) = g \cdot f(x)$  identically on  $G \times X$ .

Given a  $G$ -scheme  $X$ , the *stabilizer*  $\text{Stab}_G$  is the preimage of the diagonal under the *graph morphism*

$$\gamma : G \times X \longrightarrow X \times X, \quad (g, x) \longmapsto (g \cdot x, x).$$

Via the second projection,  $\text{Stab}_G$  is a closed subgroup scheme of the  $X$ -group scheme  $G \times X$ ; in particular, the projection  $\text{Stab}_G \rightarrow X$  is finite.

We now consider a  $G$ -scheme  $X$  of finite type, and a closed subscheme  $Y \subset X$ . The action  $\alpha$  restricts to a finite morphism  $\alpha_Y : G \times Y \rightarrow X$ , with schematic image denoted by  $G \cdot Y$ . We say that  $Y$  is  $G$ -stable if  $G \cdot Y = Y$ ; equivalently,  $\alpha_Y$  factors through  $Y$ . For an arbitrary closed subscheme  $Y$ , note that  $G \cdot Y$  is the smallest closed  $G$ -stable subscheme of  $X$  containing  $Y$ .

In particular, taking for  $Y$  a closed point  $x \in X$ , we obtain the  $G$ -orbit  $G \cdot x$ . We say that  $x$  is  $G$ -fixed if it is  $G$ -stable and the induced action of  $G$  on  $\text{Spec}(\kappa(x))$  is trivial, where  $\kappa(x)$  denotes the residue field at  $x$ . Equivalently,  $x$  lies in the fixed point subscheme  $X^G$  (the largest  $G$ -stable closed subscheme of  $X$  on which  $G$  acts trivially).

In the opposite direction, the  $G$ -action is said to be *free* at  $x \in X$  if the stabilizer  $\text{Stab}_G(x)$  is trivial. We denote by  $X_{\text{fr}}$  the set of free points of  $X$ ; this is an open  $G$ -stable subset of  $X$ . For a faithful action of an étale group  $G$ , it is easy to see that  $X_{\text{fr}}$  is non-empty. But this does not extend to arbitrary faithful actions, as shown by the example of  $\alpha_p \times \alpha_p$  acting on the affine plane  $\mathbb{A}^2$  via  $(u, v) \cdot (x, y) = (x, y + u + xv)$ .

**Lemma 2.1.** *Let  $X$  be a  $G$ -scheme of finite type such that every  $G$ -orbit is contained in an open affine subset. Then there is a categorical quotient by  $G$ ,*

$$q : X \longrightarrow Y = X/G,$$

where  $Y$  is a scheme of finite type. Moreover,  $q$  is finite and surjective, with fibers at closed points being the  $G$ -orbits (as sets).

If in addition  $G$  acts freely on  $X$ , then  $q$  is faithfully flat and the graph morphism  $\gamma$  induces an isomorphism  $G \times X \xrightarrow{\sim} X \times_Y X$ .

Equivalently,  $q$  is a  $G$ -torsor if the action is free.

Lemma 2.1 is obtained in [17, §12, Thm. 1] under the assumption that  $k$  is algebraically closed; the proof extends unchanged to an arbitrary field (see [8, III.2.6.1] for another proof).

The assumption that every  $G$ -orbit is contained in an open affine subset is satisfied if  $X$  is quasi-projective (then every finite set of points is contained in an open affine subset); in particular, if  $X$  is a curve. This assumption is also satisfied if  $G$  is infinitesimal (then the  $G$ -orbits are just fat points); in that case, the quotient morphism is radicial and bijective, see e.g. [4, Lem. 2.5]. But an example of Hironaka (see [13]) yields an action of the constant group  $\mathbb{Z}/2\mathbb{Z}$  on a smooth proper threefold which admits no categorical quotient.

In the opposite direction, if the categorical quotient  $q : X \rightarrow Y$  exists and is finite, then  $X$  is covered by open affine  $G$ -stable subsets. Moreover, for any open  $G$ -stable subset  $U$  of  $X$ , the image  $V = q(U)$  is open in  $Y$  and the

restriction  $q|_U : U \rightarrow V$  is the categorical quotient. In particular, we have a  $G$ -torsor  $X_{\text{fr}} \rightarrow X_{\text{fr}}/G = Y_{\text{fr}}$ .

Given a closed  $G$ -stable subset  $i : Z \subset X$ , the quotient  $Z \rightarrow Z/G$  also exists and hence comes with a morphism  $i/G : Z/G \rightarrow X/G$ . If  $G$  is linearly reductive, then  $i/G$  is a closed immersion; also, recall that the linearly reductive groups are exactly the extensions of finite étale groups of order prime to  $p$  by groups of multiplicative type (see [8, IV.3.3.6]). For an arbitrary group  $G$ , the morphism  $i/G$  is not necessarily a closed immersion, as shown by the example of  $\alpha_p$  acting on  $\mathbb{A}^2$  via  $u \cdot (x, y) = (x, y + ux)$ ; then the quotient is the morphism  $(x, y) \mapsto (x, y^p)$ . The zero subscheme  $Z$  of  $x$  is a  $G$ -fixed affine line with coordinate  $y$ , and hence has quotient the morphism  $y \mapsto y$ .

Next, recall that the formation of the categorical quotient commutes with flat base change on  $Y$ . As an easy consequence, for any normal subgroup scheme  $N \triangleleft G$ , we obtain an action of  $G/N$  on  $X/N$  such that the quotient morphism  $X \rightarrow X/N$  is equivariant, and the induced morphism  $(X/N)/(G/N) \rightarrow X/G$  is an isomorphism.

**Lemma 2.2.** *Let  $X$  be a  $G$ -scheme of finite type, and  $U \subset X$  an open subset. Then  $U$  contains a dense open affine  $G$ -stable subset.*

*Proof.* The quotient morphism  $X \rightarrow X/G^0$  exists and is finite, radicial and  $G$ -equivariant, where  $G$  acts on  $X/G^0$  via its étale quotient  $G/G^0 = \pi_0(G)$ . Moreover, every open subset of  $X$  is  $G^0$ -stable. Thus, it suffices to prove the assertion for the  $\pi_0(G)$ -scheme  $X/G^0$ .

So we may assume that  $G$  is étale; then  $G_{k'}$  is constant for some finite Galois field extension  $k'/k$ . We may further assume that  $U$  is affine; then  $U_{k'}$  contains  $\bigcap_{g \in G(k')} g \cdot U_{k'}$  as a dense open affine subset, stable by  $G(k')$  and hence by  $G_{k'}$ , and also by the action of the Galois group of  $k'/k$ . The statement follows from this by Galois descent.  $\square$

**Lemma 2.3.** *Let  $X$  be a  $G$ -variety with function field  $K$ . Choose a dense open affine  $G$ -stable subset  $U \subset X$ .*

- (i) *The field of invariants  $L = K^G$  is the fraction field of the ring of invariants  $\mathcal{O}(U)^G$ .*
- (ii) *The scheme  $\text{Spec}(K)$  is the generic fiber of the quotient  $q : U \rightarrow U/G$ .*
- (iii) *There is a unique action of  $G$  on  $\text{Spec}(K)$  such that the morphism  $\text{Spec}(K) \rightarrow X$  is equivariant.*
- (iv) *The extension  $K/L$  is finite. Moreover,  $K/K^{G^0}$  is purely inseparable, and  $K^{G^0}/L$  is separable.*

*Proof.* We may replace  $X$  with  $U$ , and hence assume that  $X = \text{Spec}(R)$  where  $R$  is a finitely generated algebra equipped with a  $G$ -action; moreover,  $R$  is a domain with fraction field  $K$ .

(i) Given  $l \in L$ , the set of those  $r \in R$  such that  $rl \in R$  is a non-zero  $G$ -stable ideal  $I$  of  $R$ . It suffices to show that  $I^G \neq 0$ . For this, we use a norm argument from [17, §12, p. 112]. Observe that  $\mathcal{O}(G \times X) = \mathcal{O}(G) \otimes_k R$  is a finite free  $R$ -module via the co-action  $\alpha^*$ . Denote by  $N : \mathcal{O}(G) \otimes_k R \rightarrow R$  the corresponding norm map. Then  $N(\alpha^*(r)) \in R^G$  for all  $r \in R$ , see loc. cit. If  $r \in I$  then  $\alpha^*(r) \in \mathcal{O}(G) \otimes_k I$  as  $I$  is  $G$ -stable. Using the covariance of the norm (see e.g. [11, II.6.5.4]), it follows that  $N(\alpha^*(r)) \in I^G$ . If in addition  $r \neq 0$ , then  $\alpha^*(r) \neq 0$  and hence  $N(\alpha^*(r)) \neq 0$ . This completes the proof of (i).

Next, we prove (ii), (iii) and (iv) simultaneously. Note that  $R$  is a  $G$ -module, and hence so is the subalgebra  $LR \subset K$  generated by  $L$  and  $R$ . Also,  $R$  is a finite module over  $R^G = \mathcal{O}(X)^G$ , and hence  $LR$  is a finite-dimensional vector space over  $L$ . Since  $LR$  is an integral domain, it follows that it is a field; thus,  $LR = K$  as the latter is the fraction field of  $R$ . This yields a  $G$ -algebra structure on  $K$  extending that on  $R$ . Also,  $K$  is the localization of  $R$  at  $L \setminus \{0\}$ , and hence the natural map  $L \otimes_{R^G} R \rightarrow K$  is an isomorphism; equivalently,  $\text{Spec}(K)$  is the generic fiber of  $q : X \rightarrow X/G$ . It is also the generic fiber of the quotient  $X \rightarrow X/G^0$ . Since the latter is radicial, the extension  $K/K^{G^0}$  is purely inseparable. Finally,  $K^{G^0}/K^G$  is separable as  $X/G^0 \rightarrow X/G$  is the quotient by the finite étale group  $\pi_0(G)$ . □

If  $G$  is a constant group scheme acting faithfully on  $X$ , then  $G = \text{Aut}_L(K)$  is uniquely determined by the invariant subfield  $L \subset K$ . This does not extend to actions of (say) infinitesimal group schemes: for example, the  $p$ th power map of  $\mathbb{A}^1$  is the quotient by the actions of  $\mu_p$  via  $t \cdot x = tx$ , and of  $\alpha_p$  via  $u \cdot x = x + u$ .

**Proposition 2.4.** *Let  $X$  be a  $G$ -variety. Then there exists a dense open affine  $G$ -stable subset  $U \subset X$  such that the graph morphism*

$$G \times U \longrightarrow U \times_{U/G} U, \quad (g, x) \longmapsto (x, g \cdot x)$$

*is faithfully flat.*

*Proof.* This is known in the setting of actions of smooth algebraic groups, as a modern version of a result of Rosenlicht (see [21]). We will deduce the desired statement from this version, after some first reductions.

We may assume that  $X$  is affine; then there exists a quotient morphism  $q : X \rightarrow Y$ . Using generic flatness (see e.g. [11, IV<sub>2</sub>.6.9.1]), we may further assume that  $q$  is faithfully flat.

Next, we may assume that  $G$  is a subgroup scheme of a smooth connected affine algebraic group  $G^\#$  (for example,  $G^\# = \text{GL}_n$  in which  $G$  is embedded via the regular representation). Then  $G^\# \times X$  is an affine variety equipped with a free action of  $G$  via  $g \cdot (g^\#, x) = (g^\# g^{-1}, g \cdot x)$  and the quotient by this action is an affine variety  $X^\# = G^\# \times^G X$  on which  $G^\#$  acts via its action on itself by left multiplication. (The formation of  $X^\#$  is functorial in  $X$  once an embedding  $G \rightarrow G^\#$  is fixed). Note that the open  $G^\#$ -stable subsets of  $X^\#$  are exactly the subsets  $U^\# = G^\# \times^G U$ , where  $U \subset X$  is open and  $G$ -stable; moreover,  $U^\#$  is affine if and only if  $U$  is affine.

The projection  $G^\# \times X \rightarrow G^\#$  induces a morphism  $\psi : X^\# \rightarrow G^\#/G$ , which is  $G^\#$ -equivariant and has fiber  $X$  at the base point of  $G^\#/G$ . Also, the projection  $G^\# \times X \rightarrow X$  induces a morphism  $q^\# : X^\# \rightarrow Y$  which is the categorical quotient by  $G^\#$ ; we have  $\mathcal{O}(Y) = \mathcal{O}(X)^G \xrightarrow{\sim} \mathcal{O}(X^\#)^{G^\#}$  and  $k(Y) = k(X)^G \xrightarrow{\sim} k(X^\#)^{G^\#}$ . As a consequence, the algebra  $\mathcal{O}(X^\#)^{G^\#}$  is finitely generated and its fraction field is  $k(X^\#)^{G^\#}$  (Lemma 2.3). Moreover,  $q^\#$  is faithfully flat, since so are the quotient morphism  $G^\# \times X \rightarrow X^\#$  and the composite morphism  $G^\# \times X \xrightarrow{\text{pr}} X \xrightarrow{q} Y$  (use [11, IV<sub>2</sub>.2.2.11]).

In view of [21, Satz 1.7], it follows that the fiber product  $X^\# \times_Y X^\#$  is a variety, and the graph morphism

$$\varphi^\# : G^\# \times X^\# \longrightarrow X^\# \times_Y X^\#, \quad (g, x) \longmapsto (g \cdot x, x)$$

is dominant. Also,  $\varphi^\#$  is equivariant for the action of  $G^\# \times G^\#$  on  $G^\# \times X^\#$  defined by  $(g_1, g_2) \cdot (g, x) = (g_1 g g_2^{-1}, g_2 \cdot x)$ , and its action on  $X^\# \times_Y X^\#$  via  $(g_1, g_2) \cdot (x_1, x_2) = (g_1 \cdot x_1, g_2 \cdot x_2)$ . So the image of  $\varphi^\#$  contains a dense open subset  $V$  of  $X^\# \times_Y X^\#$ , stable by  $G^\# \times G^\#$ . Thus,  $(\varphi^\#)^{-1}(V)$  is a dense open subset of  $G^\# \times X^\#$ , stable by  $G^\# \times G^\#$  and hence of the form  $G^\# \times U^\#$  where  $U^\# \subset X^\#$  is open and  $G^\#$ -stable. Replacing  $X^\#$  with  $U^\#$ , we may thus assume that  $\varphi^\#$  is surjective. Likewise, using generic flatness and equivariance, we may further assume that  $\varphi^\#$  is flat.

The  $G^\#$ -equivariant morphism  $\psi : X^\# \rightarrow G^\#/G$  yields a  $G^\# \times G^\#$ -equivariant morphism

$$\psi^\# : X^\# \times_Y X^\# \longrightarrow G^\#/G \times G^\#/G,$$

which is faithfully flat by equivariance. The composite morphism

$$\psi^\# \circ \varphi^\# : G^\# \times X^\# \longrightarrow G^\#/G \times G^\#/G$$



is faithfully flat as well, and its fiber at the base point is  $G \times X$ . Moreover, the restriction of the graph morphism  $\varphi^\#$  to this fiber is the analogously defined morphism  $\varphi : G \times X \rightarrow X \times_Y X$ . By the fiberwise criterion for flatness (see [11, IV<sub>3</sub>.11.3.11]), it follows that  $\varphi$  is faithfully flat.  $\square$

As a direct consequence of the above proposition, the  $G$ -action on  $X$  is generically transitive. Indeed, the generic fiber of the structure morphism  $U \times_{U/G} U \rightarrow U/G$  is

$$Z = \text{Spec}(K \otimes_L K) = \text{Spec}(K) \times_{\text{Spec}(L)} \text{Spec}(K)$$

by Lemma 2.3. This is a  $K$ -scheme via the first projection, and has a canonical  $K$ -point  $z$  (the diagonal) corresponding to the multiplication  $K \otimes_L K \rightarrow K$ . Moreover, the  $G$ -action on  $\text{Spec}(K)$  yields a  $G_K$ -action on  $Z$ , and Proposition 2.4 implies that the orbit map  $G_K \rightarrow Z, g \mapsto g \cdot z$  is faithfully flat. Also,  $H = \text{Stab}_{G_K}(z)$  is the generic fiber of the projection  $\text{Stab}_G \rightarrow X$ , i.e., the generic stabilizer. This yields the following:

**Corollary 2.5.** *With the above notation, we have a  $G_K$ -equivariant isomorphism  $Z \simeq G_K/H$ .*

Considering the lengths of the finite  $K$ -schemes  $Z$  and  $G_K/H$ , it follows that  $[K : L] = [G_K : H]$ . This divides  $|G|$ , and equality holds if and only if  $H$  is trivial; equivalently,  $G_K$  acts freely on  $Z$ , i.e.,  $G$  acts freely on  $\text{Spec}(K)$ . We have proved:

**Corollary 2.6.** *With the above notation,  $[K : L]$  divides  $|G|$ . Moreover, equality holds if and only if the  $G$ -action on  $X$  is generically free.*

### 3. Rational actions

**Definition 3.1.** Let  $G$  be a finite group scheme, and  $X$  a variety. A *rational action* of  $G$  on  $X$  is a rational map  $\alpha : G \times X \dashrightarrow X$  which satisfies the following two properties:

- (i) The rational map  $(\text{pr}_1, \alpha) : G \times X \dashrightarrow G \times X$  is birational.
- (ii) The rational maps  $\alpha \circ (\text{id}_G \times \alpha), \alpha \circ (\mu \times \text{id}_X) : G \times G \times X \dashrightarrow X$  are equal, where  $\mu : G \times G \rightarrow G$  denotes the multiplication.

In this definition (adapted from [7, §3]), a *rational map*  $f : Y \dashrightarrow Z$  is an equivalence class of pairs  $(U, \varphi)$ , where  $U$  is a schematically dense open subset of  $Y$ , and  $\varphi : U \rightarrow Z$  a morphism; two pairs  $(U, \varphi), (V, \psi)$  are equivalent if there exists a schematically dense open subset  $W \subset U \cap V$  such that

$\varphi|_W = \psi|_W$ . Every rational map  $f : X \dashrightarrow Y$  has a unique representative  $(U, \varphi)$ , where  $U$  is maximal; then  $U$  is the *domain of definition*  $\text{dom}(f)$ . The formation of the domain of definition commutes with base change by field extensions (see [11, IV<sub>4</sub>.20.3.11]). The rational map  $f$  is *birational* if it admits a representative  $(U, \varphi)$  such that  $\varphi$  is an isomorphism onto a schematically dense open subset of  $Z$ .

Since  $X$  is reduced, every dense open subset  $U \subset X$  is schematically dense. Also, we may replace  $X$  with any dense open subset in Definition 3.1, and hence assume that  $X$  is smooth. Then  $G \times X$  is Cohen-Macaulay (as  $G$  is finite), and hence has no embedded component. As a consequence, we may replace “schematically dense” with “dense” when dealing with the rational maps in (i), (ii).

Let  $\alpha : G \times X \dashrightarrow X$  be a rational map satisfying (i). Then  $\alpha$  is dominant, since it is the composition of the birational map  $(\text{pr}_1, \alpha)$  and the second projection. Thus, the image of  $\alpha$  contains a dense open subset  $W \subset X$ . So the image of  $\text{id}_G \times \alpha$  contains  $G \times W$ . Denote by  $V$  the domain of definition of  $\alpha$ ; then the composite rational map  $\alpha \circ (\text{id}_G \times \alpha)$  is defined on the open subset  $(\text{id}_G \times \alpha)^{-1}(V \cap (G \times W))$ . Moreover, the composite rational map  $\alpha \circ (\mu \times \text{id}_X)$  is defined as  $\mu \times \text{id}_X$  is a morphism (see [11, IV<sub>1</sub>.20.3.1]). So the two compositions of rational maps in (ii) make sense.

For any  $g \in G$ , the intersection  $V_g = \text{pr}_1^{-1}(g) \cap V_{\kappa(g)}$  is identified with a dense open subset of  $X_{\kappa(g)}$ ; moreover,  $g$  induces a birational morphism  $\alpha_g : V_g \rightarrow X_{\kappa(g)}$  (as follows from the condition (i) and the finiteness of  $G$ ). This motivates (i), while (ii) is a rational analogue of the associativity property of actions.

**Proposition 3.2.** *Let  $X$  be a variety equipped with a rational action  $\alpha$  of  $G$ . Then there exists a dense open subset  $U \subset X$  such that  $\alpha$  is defined on  $G \times U$  and induces a  $G$ -action on  $U$ .*

*Proof.* We first consider the case where  $k$  is algebraically closed; then we have  $G = G^0 \rtimes G_{\text{red}}$  and  $G_{\text{red}}$  is the constant group scheme associated with  $G(k)$ . For any  $g \in G(k)$  and  $x \in X$ , we denote  $\alpha(g, x)$  by  $g \cdot x$  whenever it is defined, i.e.,  $x \in V_g$ . By (ii), given  $g, h \in G(k)$  and  $x \in X$ , if  $h \cdot x$  and  $g \cdot (h \cdot x)$  are defined, then  $gh \cdot x$  is defined and equals  $g \cdot (h \cdot x)$ . It follows easily that the birational morphism  $\alpha_e : V_e \rightarrow X$  is just the inclusion. Also,

$$W = \bigcap_{g \in G(k)} V_g$$

is a dense open subset of  $X$ , as well as

$$U = \bigcap_{g \in G(k)} \alpha_g^{-1}(W).$$

We check that  $U$  satisfies our assertions.

Since  $\alpha_e$  is the inclusion, we have  $U \subset W$ . Let  $g \in G(k)$  and  $x \in U$ ; then  $g \cdot x$  is defined and lies in  $W$ . We claim that  $g \cdot x \in U$ . Otherwise, there exists  $h \in G(k)$  such that  $h \cdot (g \cdot x)$  is defined and does not lie in  $W$ . Then  $hg \cdot x$  is defined and not in  $W$ , a contradiction. This proves the claim.

By this claim, we have  $G(k) \times U \subset V$ . Since  $V$  is open in  $G \times X$ , it follows that  $G \times U \subset V$ . Thus,  $\alpha$  yields a morphism  $\alpha_0 : G \times U \rightarrow X$ , which factors through  $U$  by the claim again. This completes the proof when  $k$  is algebraically closed.

Next, we consider an arbitrary field  $k$ . Then  $X_{\bar{k}}$  is equipped with a rational action of  $G_{\bar{k}}$ . The above construction yields a dense open subset  $U' \subset X_{\bar{k}}$  on which  $G_{\bar{k}}$  acts, and which is stable under all automorphisms of  $\bar{k}/k$ . Thus,  $U'$  is the preimage of a dense open subset  $U \subset X$  under the projection  $\pi : X_{\bar{k}} \rightarrow X$ . One may readily check that  $U$  satisfies our assertions, by using the fact that the formation of the domain of definition commutes with field extensions. □

**Corollary 3.3.** *Let  $X$  be a variety equipped with a rational action of  $G$ . Then  $X$  is equivariantly birationally isomorphic to a projective  $G$ -variety.*

*Proof.* Using Proposition 3.2, we may assume that  $X$  is a  $G$ -variety. In view of Lemma 2.2, we may further assume that  $X$  is affine. Then the algebra  $\mathcal{O}(X)$  is generated by a finite-dimensional  $G$ -submodule  $V$ . This yields a closed  $G$ -equivariant immersion of  $X$  in the corresponding affine space  $\mathbb{V}(V) = \text{Spec}(\text{Sym}(V))$ , and hence in its projective completion  $\mathbb{P}(V \oplus k)$  (where  $G$  acts via its linear representation in  $V \oplus k$ ). The schematic image of  $X$  in  $\mathbb{P}(V \oplus k)$  is the desired projective  $G$ -variety. □

**Corollary 3.4.** *Every rational action of  $G$  on  $X$  restricts to a  $G$ -action on the spectrum of the function field  $K = k(X)$ . Conversely, every  $G$ -action on  $\text{Spec}(K)$  extends to a unique rational  $G$ -action on  $X$ .*

*Proof.* The first assertion is a direct consequence of Lemma 2.3 together with Proposition 3.2.

For the converse assertion, consider an action  $\alpha : G \times \text{Spec}(K) \rightarrow \text{Spec}(K)$  and the corresponding co-action

$$\alpha^* : K \longrightarrow \mathcal{O}(G) \otimes K, \quad x_i \longmapsto \sum_j y_{ij} \otimes z_{ij} \quad (i = 1, \dots, m),$$

where  $x_1, \dots, x_m$  generate the field  $K$  over  $k$ . Let  $U$  (resp.  $V$ ) be a dense open affine subset of  $X$  on which the  $x_i$  (resp.  $z_{ij}$ ) are defined; then  $\alpha^*$  yields a homomorphism of algebras  $\mathcal{O}(U) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(V)$ , or equivalently, a morphism  $G \times V \rightarrow U$ . As  $U$  is dense in  $X$ , and  $G \times V$  is (schematically) dense in  $G \times X$  (e.g. by [11, IV<sub>4</sub>.20.3.5]), we get a rational map  $\beta : G \times X \dashrightarrow X$ . It satisfies the properties (i) and (ii) in view of the “local determination of morphisms” (see [11, I.6.5]), since these properties hold for the action of  $G$  on  $\text{Spec}(K)$ . □

**Remark 3.5.** More generally, Corollary 3.3 holds in the setting of rational actions of algebraic groups, by a refinement of Weil’s regularization theorem (see [24], [19, Thm. 1] for the original version, and [6, Thm. 8] for the refinement). But Proposition 3.2 and Corollary 3.4 do not extend to this setting.

We now assume that  $p > 0$ , and use Corollary 3.4 to construct examples of faithful rational actions of infinitesimal group schemes on any variety  $X$ . Recall that the function field  $K = k(X)$  (a separable, finitely generated extension of  $k$  of transcendence degree  $n$ ) admits a *p-basis of length n*, i.e., a sequence  $(x_1, \dots, x_n) \in K^n$  such that the monomials  $x_1^{m_1} \dots x_n^{m_n}$ , where  $0 \leq m_1, \dots, m_n \leq p - 1$ , form a basis of  $K$  over its subfield  $kK^p$  (the composite of  $k$  and  $K^p$  in  $K$ ; this is a function field in  $n$  variables as well). Equivalently, the differentials  $dx_1, \dots, dx_n$  form a basis of the  $K$ -vector space of Kähler differentials  $\Omega_{K/k}^1 = \Omega_{K/kK^p}^1$  (see [11, IV<sub>1</sub>.21.4.2, IV<sub>1</sub>.21.4.5]). We denote the dual basis of the  $K$ -vector space of derivations by  $D_1, \dots, D_n \in \text{Der}_k(K) = \text{Der}_{kK^p}(K)$ . Then the  $D_i$  commute pairwise and satisfy  $D_i^p = 0$  for  $i = 1, \dots, n$ .

Next, recall the equivalence of categories between infinitesimal group schemes  $G$  of height at most 1 and  $p$ -Lie algebras  $\mathfrak{g} = \text{Lie}(G)$  (see [8, II.7.4.1]. Under this equivalence, the  $G$ -actions on  $\text{Spec}(K)$  correspond to the homomorphisms of  $p$ -Lie algebras  $\mathfrak{g} \rightarrow \text{Der}_k(K)$  in view of [8, II.7.3.10]. Also, recall that the  $p$ -Lie algebra  $k$  with trivial  $p$ th power map (resp.  $p$ th power map  $t \mapsto t^p$ ) corresponds to the group scheme  $\alpha_p$  (resp.  $\mu_p$ ).

In particular, for any  $f_1, \dots, f_m \in kK^p$ , the derivations  $f_1 D_1, \dots, f_m D_1$  commute pairwise and satisfy  $(f_i D_1)^p = 0$  for all  $i$ . Choosing  $f_1, \dots, f_m$  linearly independent over  $k$ , this yields:

**Lemma 3.6.** *Every variety of positive dimension admits a faithful rational action of  $\alpha_p^m$  for any  $m \geq 1$ .*

Here  $\alpha_p^m = \alpha_p \times \dots \times \alpha_p$  ( $m$  factors). Likewise, considering the derivations  $x_1 D_1, \dots, x_n D_n$ , we obtain a faithful action of  $\mu_p^n$  on  $\text{Spec}(K)$ , or equivalently,

a faithful  $\mu_p^n$ -action on  $K$  by algebra automorphisms (see e.g. [8, II.2.1.2]). The latter action fixes  $kK^p$  pointwise, and satisfies

$$(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1x_1, \dots, t_nx_n)$$

for all  $(t_1, \dots, t_n) \in \mu_p^n$ . We say that this action is *standard in the  $p$ -basis*  $(x_1, \dots, x_n)$ .

**Lemma 3.7.** *Let  $X$  be a variety of dimension  $n$ . Then  $X$  admits a faithful rational action of  $\mu_p^n$ . Every such action  $\alpha$  is generically free, and standard in some  $p$ -basis  $(x_1, \dots, x_n)$  of  $K = k(X)$ . Moreover, the  $x_i$  are uniquely determined by  $\alpha$ , up to multiplication by non-zero elements of  $kK^p$ . If  $\alpha$  extends to a faithful rational action of an infinitesimal group scheme  $H$  of height 1 normalizing  $\mu_p^n$ , then  $H = \mu_p^n$ .*

*Proof.* Let  $G = \mu_p^n$ . The existence of a faithful rational  $G$ -action on  $X$  follows from Corollary 3.4 together with the preceding construction.

Given such an action  $\alpha$ , the corresponding  $G$ -action on  $K$  yields a grading

$$K = \bigoplus_{\gamma \in \Gamma} K_\gamma,$$

where  $\Gamma = (\mathbb{Z}/p\mathbb{Z})^n$  is the character group of the diagonalizable group  $G$  (see [8, II.2.2.5]). Then  $K_0 = K^G$  is a subfield of  $K$ , and each  $K_\gamma$  is a  $K_0$ -vector space. Moreover, the group  $\Gamma$  is generated by the  $\gamma$  such that  $K_\gamma \neq 0$ , since the  $G$ -action is faithful. But these  $\gamma$  form a subgroup of  $\Gamma$  as  $K$  is a field. It follows that  $K_\gamma \neq 0$  for all  $\gamma \in \Gamma$ . Since  $K_0 \supset kK^p$  and  $[K : kK^p] = p^n = |\Gamma|$ , we must have  $K_0 = kK^p$  and  $\dim_{K_0}(K_\gamma) = 1$  for all such  $\gamma$ . Choosing  $x_i \in K_{\gamma_i}$  where  $\gamma_1, \dots, \gamma_n$  form the standard basis of  $(\mathbb{Z}/p\mathbb{Z})^n$ , we see that the monomials  $x_1^{m_1} \dots x_n^{m_n}$ , where  $0 \leq m_1, \dots, m_n \leq p - 1$ , have pairwise distinct weights, and hence are linearly independent over  $kK^p$ . For dimension reasons, these monomials form a  $p$ -basis. By construction, the action  $\alpha$  is standard in this basis, which is unique up to non-zero elements of  $kK^p$ ; moreover, the generic stabilizer is trivial.

Let  $H$  be an infinitesimal group scheme of height 1 normalizing  $G$ , and equipped with a faithful rational action on  $X$  extending  $\alpha$ . Since the automorphism group scheme of  $G$  is constant (see [8, III.5.3.3]), we see that  $H$  centralizes  $G$ . Then  $\mathfrak{h} = \text{Lie}(H)$  centralizes  $\mathfrak{g} = \text{Lie}(G)$ , and  $\mathfrak{g} \subset \mathfrak{h} \subset \text{Der}_k(K)$ . But we have

$$\text{Der}_k(K) = \bigoplus_{i=1}^n KD_i = \bigoplus kK^p x_1^{m_1} \dots x_n^{m_n} D_i,$$

the latter sum being over  $i = 1, \dots, n$  and  $m_1, \dots, m_n = 0, \dots, p - 1$ . Also,  $x_1^{m_1} \cdots x_n^{m_n} D_i$  is a  $G$ -eigenvector of weight  $(m_1, \dots, m_i - 1, \dots, m_n)$  (viewed in  $(\mathbb{Z}/p\mathbb{Z})^n$ ). It follows that the centralizer of  $\mathfrak{g}$  in  $\text{Der}_k(K)$  is the Lie algebra

$$kK^p \mathfrak{g} = \left\{ \sum_{i=1}^n t_i x_i D_i \mid t_1, \dots, t_n \in kK^p \right\}.$$

So  $\mathfrak{h}$  is a finite-dimensional subspace of the  $k$ -vector space  $kK^p \mathfrak{g}$ , stable under the  $p$ th power map. If  $\sum_i t_i x_i D_i \in \mathfrak{h}$ , then  $\sum_i t_i^p x_i D_i, \sum_i t_i^{p^2} x_i D_i, \dots \in \mathfrak{h}$ . It follows that  $t_i, t_i^p, t_i^{p^2}, \dots$  are linearly dependent over  $k$  for  $i = 1, \dots, n$ . In particular, each  $t_i$  is algebraic over  $k$ . Since  $k$  is algebraically closed in  $K$ , this forces  $t_1, \dots, t_n \in k$  and  $\mathfrak{h} = \mathfrak{g}$ . □

**Remark 3.8.** In particular,  $X$  admits many faithful rational actions of  $\mu_p^n$ , but no faithful rational action of  $\mu_p^{n+1}$ . The latter fact also follows from a classical result in the theory of  $p$ -Lie algebras: choosing a  $p$ -basis  $(x_1, \dots, x_n)$  of  $K/k$  yields an isomorphism of  $kK^p$ -algebras

$$K \simeq kK^p[T_1, \dots, T_n]/(T_1^p - x_1^p, \dots, T_n^p - x_n^p)$$

and hence an isomorphism of  $K$ -algebras

$$K \otimes_{kK^p} K \simeq K[T_1, \dots, T_n]/(T_1^p, \dots, T_n^p).$$

As a consequence, the  $kK^p$ -algebra  $\text{Der}_k(K) = \text{Der}_{kK^p}(K)$  is a form of the  $K$ -algebra  $\text{Der}_K(K[T_1, \dots, T_n]/(T_1^p, \dots, T_n^p))$ . The latter is a  $p$ -Lie algebra over  $K$ , known as the split Jacobson–Witt algebra  $W_n$ . Its maximal tori (i.e., the maximal  $p$ -Lie subalgebras having a basis  $D_1, \dots, D_m$  such that the  $D_i$  commute pairwise and satisfy  $D_i^p = D_i$ ) have been determined in [9]; in particular, they all have dimension  $n$ .

In another direction, the above construction of faithful rational actions of  $\mu_p^n$  via  $p$ -bases can be iterated to yield faithful rational actions of  $\mu_{p^s}^n$  for any  $s \geq 1$ . Indeed, taking  $p$ th powers in the equality

$$K = \bigoplus_{0 \leq m_1, \dots, m_n \leq p-1} kK^p x_1^{m_1} \cdots x_n^{m_n},$$

we obtain

$$kK^p = \bigoplus_{0 \leq m_1, \dots, m_n \leq p-1} kK^{p^2} x_1^{pm_1} \cdots x_n^{pm_n},$$

and hence

$$K = \bigoplus_{0 \leq m_1, \dots, m_n \leq p^2 - 1} kK^{p^2} x_1^{m_1} \cdots x_n^{m_n}.$$

By induction, this yields

$$K = \bigoplus_{0 \leq m_1, \dots, m_n \leq p^s - 1} kK^{p^s} x_1^{m_1} \cdots x_n^{m_n}$$

for any  $s \geq 1$ . We may thus define the desired action of  $\mu_{p^s}^n$  by the same formula as the standard  $\mu_p^n$ -action; its fixed subfield is  $kK^{p^s}$ .

Since these  $\mu_{p^s}^n$ -actions are compatible with the standard embeddings  $\mu_{p^s}^n \rightarrow \mu_{p^{s+1}}^n$ , where  $s \geq 1$ , we get a faithful rational action of the ind-group scheme  $\mu_{p^\infty}^n = \varinjlim_s \mu_{p^s}^n$  on  $X$ . Note that the family  $(\mu_{p^s}^n, s \geq 1)$  is schematically dense in the multiplicative group  $\mathbb{G}_m$ , but  $X$  may admit no faithful rational  $\mathbb{G}_m$ -action (for example if  $X$  is not geometrically uniruled).

Likewise,  $X$  admits a faithful rational action of the ind-group scheme  $\alpha_{p^\infty}^r = \varinjlim_s \alpha_{p^s}^r$  for any  $r \geq 1$ . Also, the  $\alpha_{p^s}^r$  form a schematically dense family in the additive group  $\mathbb{G}_a$ , but  $X$  may admit no faithful rational  $\mathbb{G}_a$ -action.

### 4. $G$ -normality

Recall that  $G$  denotes a finite group scheme.

**Definition 4.1.** A  $G$ -variety  $X$  is  $G$ -normal if every finite birational morphism of  $G$ -varieties  $f : Y \rightarrow X$  is an isomorphism.

**Proposition 4.2.** *Let  $X$  be a  $G$ -variety.*

- (i) *There exists a finite birational morphism of  $G$ -varieties  $\varphi : X' \rightarrow X$ , where  $X'$  is  $G$ -normal.*
- (ii) *For any morphism  $\varphi$  as in (i) and any finite birational morphism of  $G$ -varieties  $f : Z \rightarrow X$ , there exists a unique morphism of  $G$ -varieties  $\psi : X' \rightarrow Z$  such that  $\varphi = f \circ \psi$ .*

*Proof.* (i) Let  $f : Y \rightarrow X$  be a finite birational morphism of  $G$ -varieties. Then the normalization morphism  $\eta = \eta_X : \tilde{X} \rightarrow X$  factors uniquely through the analogous morphism  $\eta_Y : \tilde{X} \rightarrow Y$ . Thus,  $\mathcal{O}_X \subset f_*(\mathcal{O}_Y) \subset (\eta_X)_*(\mathcal{O}_{\tilde{X}})$ . Since  $\eta_X$  is finite, we may choose  $f$  so that the subsheaf  $f_*(\mathcal{O}_Y) \subset (\eta_X)_*(\mathcal{O}_{\tilde{X}})$  is maximal among the direct images of structure sheaves of  $G$ -varieties equipped with a finite birational morphism to  $X$ .

We claim that  $Y$  is  $G$ -normal (and hence  $f : Y \rightarrow X$  is the desired morphism). To check this, consider a finite birational morphism of  $G$ -varieties

$g : Z \rightarrow Y$ . Then again,  $f \circ g : Z \rightarrow X$  factors through  $\eta_Z$ , and hence we have  $f_*(\mathcal{O}_Y) \subset (f \circ g)_*(\mathcal{O}_Z) \subset (\eta_X)_*(\mathcal{O}_{\tilde{X}})$ . By maximality, we obtain  $f_*(\mathcal{O}_Y) = (f \circ g)_*(\mathcal{O}_Z)$  and hence  $\mathcal{O}_Y = g_*(\mathcal{O}_Z)$  as  $f$  is finite surjective. It follows that  $g$  is an isomorphism, proving the claim.

(ii) There exists a dense open  $G$ -stable subset  $U \subset X$  such that the induced morphisms  $f^{-1}(U) \rightarrow U$  and  $\varphi^{-1}(U) \rightarrow U$  are isomorphisms. Thus, we may identify  $U$  with an open subset of the fiber product  $Z \times_X X'$ , stable under the natural  $G$ -action. Let  $Y$  be the schematic closure of  $U$  in  $Z \times_X X'$ ; then  $Y$  is a  $G$ -variety equipped with finite birational  $G$ -morphisms  $\psi : Y \rightarrow Z$ ,  $g : Y \rightarrow X'$  such that the square

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & Z \\ g \downarrow & & \downarrow f \\ X' & \xrightarrow{\varphi} & X \end{array}$$

commutes. Since  $X'$  is  $G$ -normal,  $g$  is an isomorphism; this yields the desired morphism  $X' \rightarrow Z$ . □

With the above notation, we say that  $\varphi : X' \rightarrow X$  is the  $G$ -normalization; it is unique up to unique  $G$ -isomorphism.

**Remark 4.3.** If a  $G$ -variety  $X$  is  $H$ -normal for some subgroup scheme  $H$  of  $G$ , then clearly  $X$  is  $G$ -normal. In particular, every normal  $G$ -variety is  $G$ -normal. The converse holds if  $G$  is étale, since the  $G$ -action lifts uniquely to an action on the normalization (see e.g. [5, Prop. 2.5.1]). So the notion of  $G$ -normalization is only relevant in characteristic  $p > 0$ .

**Corollary 4.4.** *Let  $X$  be a variety equipped with a rational action of  $G$ . Then  $X$  is equivariantly birationally isomorphic to a  $G$ -normal projective variety  $Y$ . If  $X$  is a curve, then  $Y$  is unique.*

*Proof.* The first assertion follows readily from Corollary 3.3 together with the existence of the  $G$ -normalization.

Assume that  $X$  is a  $G$ -curve and consider two projective  $G$ -models  $Y_1, Y_2$ ; then we have a  $G$ -equivariant rational map  $f : Y_1 \dashrightarrow Y_2$ . Using Lemma 2.2, we may find dense open  $G$ -stable subsets  $U_i \subset Y_i$  ( $i = 1, 2$ ) such that  $f$  restricts to an isomorphism  $U_1 \xrightarrow{\sim} U_2$ . By a graph argument as in the proof of Proposition 4.2 (ii), this yields a projective  $G$ -curve  $Y$  equipped with equivariant birational morphisms to  $Y_1, Y_2$ . The second assertion follows from this, as every birational morphism of projective curves is finite. □



**Lemma 4.5.** *Let  $X$  be a  $G$ -variety, and  $N \triangleleft G$  a normal connected subgroup scheme.*

- (i) *If  $X$  is  $G$ -normal, then  $X/N$  is  $G/N$ -normal.*
- (ii) *If  $X/N$  is  $G/N$ -normal and  $N$  acts freely on  $X$ , then  $X$  is  $G$ -normal.*

*Proof.* (i) Let  $f : Y \rightarrow X/N$  be a finite birational morphism of  $G/N$ -varieties. Arguing again as in the proof of Proposition 4.2 (ii), we obtain a  $G$ -variety  $Z$  equipped with finite birational morphisms  $\varphi : Z \rightarrow X$ ,  $\psi : Z \rightarrow Y$  such that the square

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & Y \\ \varphi \downarrow & & \downarrow f \\ X & \xrightarrow{q} & X/N \end{array}$$

commutes. Since  $X$  is  $G$ -normal,  $\varphi$  is an isomorphism. Then the resulting morphism  $\psi \circ \varphi^{-1} : X \rightarrow Y$  is  $N$ -invariant over a dense open subset of  $Y$ , and hence everywhere. Since  $q$  is a categorical quotient, it follows that  $f$  has a section. As  $f$  is a finite morphism of varieties, it is an isomorphism.

(ii) We argue similarly, and consider a finite birational morphism of  $G$ -varieties  $f : Y \rightarrow X$ . Since  $N$  acts freely on  $X$ , it also acts freely on  $Y$ . Thus, the quotient morphisms  $q_X : X \rightarrow X/N$ ,  $q_Y : Y \rightarrow Y/N$  are  $N$ -torsors, and fit in a cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{q_Y} & Y/N \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{q_X} & X/N. \end{array}$$

By fppf descent, it follows that  $g$  is finite. Also,  $g$  is birational as  $f$  restricts to an isomorphism over a dense open  $N$ -stable subset of  $X$ . Thus,  $g$  is an isomorphism, and hence so is  $f$ . □

**Corollary 4.6.** *Let  $X$  be a  $G$ -normal variety. Then  $X/G^0$  is normal. Moreover, the normalization  $\eta : \tilde{X} \rightarrow X$  is radicial and bijective.*

*Proof.* The variety  $X/G^0$  is  $\pi_0(G)$ -normal by Lemma 4.5; this yields the first assertion in view of Remark 4.3.

Recall that the quotient morphism  $q : X \rightarrow X/G^0$  is radicial and bijective. Also,  $q \circ \eta : \tilde{X} \rightarrow X/G^0$  is a finite morphism of normal varieties such that the corresponding extension of function fields is purely inseparable. As a consequence,  $q \circ \eta$  is radicial and bijective as well (see e.g. [11, II.4.3.8]). This implies the second assertion by using [11, I.3.5.6]. □

Next, we relate the  $G$ -normalization of a  $G$ -variety  $X$  with the (usual) normalization of a variety obtained from  $X$  by “induction”, as in the proof of Proposition 2.4. More specifically, embed  $G$  as a closed subgroup scheme of a smooth connected algebraic group  $G^\#$ . Then  $G^\# \times X$  is a variety equipped with a  $G^\# \times G$ -action via  $(a, g) \cdot (b, x) = (abg^{-1}, g \cdot x)$ . Since  $G^0$  is infinitesimal, the quotient variety  $X^\# = G^\# \times^{G^0} X$  exists; it is equipped with an action of  $G^\# \times (G/G^0) = G^\# \times \pi_0(G)$  together with a  $G^\# \times (G/G^0)$ -equivariant morphism

$$\psi : X^\# \longrightarrow G^\# / G^0.$$

Here  $G^\# / G^0$  is identified with the homogeneous space  $(G^\# \times (G/G^0)) / G$ , where  $G$  is embedded diagonally in  $G^\# \times (G/G^0)$ . The fiber of  $\psi$  at the base point of this homogeneous space is  $G$ -equivariantly isomorphic to  $X$ .

Now consider the normalization  $\eta^\# : \tilde{X}^\# \rightarrow X^\#$ . Since  $G^\# \times (G/G^0)$  is smooth, its action on  $X^\#$  lifts uniquely to an action on  $\tilde{X}^\#$  such that  $\eta^\#$  is equivariant (see e.g. [5, Prop. 2.5.1]). Thus,  $\psi \circ \eta^\# : \tilde{X}^\# \rightarrow G^\# / G^0$  is  $G^\# \times (G/G^0)$ -equivariant as well. So its fiber at the base point is a  $G$ -scheme  $Y$  equipped with a  $G$ -equivariant morphism

$$\mu : Y \longrightarrow X$$

Moreover, the morphism  $G^\# \times Y \rightarrow \tilde{X}^\#, (a, y) \mapsto a \cdot y$  factors uniquely through an isomorphism  $G^\# \times^{G^0} Y \xrightarrow{\sim} \tilde{X}^\#$ .

**Lemma 4.7.** *With the above notation,  $\mu$  is the  $G$ -normalization, and the  $G^0$ -normalization as well.*

*Proof.* By construction, we have a cartesian square

$$\begin{CD} G^\# \times Y @>{\text{id} \times \mu}>> G^\# \times X \\ @VVV @VVV \\ \tilde{X}^\# @>{\eta^\#}>> X^\# \end{CD}$$

where the vertical arrows are  $G^0$ -torsors. As  $\eta^\#$  is finite and birational, the same holds for  $\text{id} \times \mu$ , and hence for  $\mu$ ; in particular,  $Y$  is a variety.

Next, the  $G^0$ -normalization  $\varphi : X' \rightarrow X$  yields a  $G^\#$ -equivariant morphism

$$G^\# \times^{G^0} \varphi : G^\# \times^{G^0} X' \longrightarrow G^\# \times^{G^0} X = X^\#$$

which is finite and birational by the above argument. By the universal property of the normalization,  $\eta^\#$  lifts to a unique morphism  $\gamma : \tilde{X}^\# \rightarrow G^\# \times^{G^0} X'$

which is finite and birational, and hence  $G^\#$ -equivariant. Moreover,  $\gamma$  is a morphism of varieties over  $G^\#/G^0$ , and hence restricts to a finite birational  $G^0$ -morphism  $\delta : Y \rightarrow X'$ . Since  $X'$  is  $G^0$ -normal,  $\delta$  is an isomorphism. So  $Y$  is  $G^0$ -normal, and hence  $G$ -normal by Remark 4.15.  $\square$

As a direct consequence of Lemma 4.7, we obtain:

**Corollary 4.8.** *The following conditions are equivalent for a  $G$ -variety  $X$ :*

- (i)  $X$  is  $G$ -normal.
- (ii)  $X$  is  $G^0$ -normal.
- (iii)  $X^\#$  is normal.

**Proposition 4.9.** *Let  $X$  be a  $G$ -variety, and  $k'/k$  a field extension.*

- (i) *If  $X_{k'}$  is  $G_{k'}$ -normal, then  $X$  is  $G$ -normal.*
- (ii) *If  $X$  is  $G$ -normal and  $k'$  is separable over  $k$ , then  $X_{k'}$  is  $G_{k'}$ -normal.*

*Proof.* (i) Consider a finite birational morphism of  $G$ -varieties  $f : Y \rightarrow X$ . Then the base change  $f_{k'} : Y_{k'} \rightarrow X_{k'}$  is a finite birational morphism of  $G_{k'}$ -varieties, and hence an isomorphism. By descent,  $f$  is an isomorphism as well.

(ii) By Corollary 4.8, we may assume that  $G$  is connected. Then the assertion follows from this corollary, since the formation of  $X^\#$  commutes with field extensions, and normality is preserved under separable field extensions (see [11, IV<sub>2</sub>.6.7.4] for the latter assertion).  $\square$

**Remark 4.10.** It is well known that normality may not be preserved under a non-trivial purely inseparable field extension  $k'/k$  (see e.g. [11, IV<sub>2</sub>.6.7.5]). This also holds for  $G$ -normality, as shown by the following example where  $G = \alpha_p$ . Choose  $a \in k$  such that  $a^{1/p} \in k' \setminus k$  and let  $G^\#$  be the affine plane curve with equation  $y^p = x + ax^p$ . Then  $G^\#$  is a smooth connected algebraic group via pointwise addition of coordinates, and its Frobenius kernel (the zero subscheme of  $(x^p, y^p)$ ) is isomorphic to  $G$ . Moreover, one may check that the normal projective completion of  $G^\#$  is the projective plane curve  $X$  with homogeneous equation  $y^p = xz^{p-1} + ax^p$ . The  $G^\#$ -action on itself by translation extends uniquely to an action on  $X$ . In particular,  $X$  is a normal  $G$ -curve, and hence is  $G$ -normal. But  $X_{k'}$  is non-normal, since it has homogeneous equation  $(y - a^{1/p}x)^p = xz^{p-1}$ . As  $G_{k'}^\#$  is smooth, its action on  $X_{k'}$  lifts to an action on the normalization of this curve, and hence to a  $G_{k'}$ -action. It follows that  $X_{k'}$  is not  $G_{k'}$ -normal.

**Remark 4.11.** Consider an *affine*  $G$ -variety  $X$ ; then the normalization  $\tilde{X}$  and the  $G$ -normalization  $X'$  are affine as well, and we have inclusions of rings

$$R = \mathcal{O}(X) \subset R' = \mathcal{O}(X') \subset \tilde{R} = \mathcal{O}(\tilde{X}) \subset K = k(X) = k(X') = k(\tilde{X}).$$

Also, recall that  $K$  is a  $G$ -module, and  $R, R'$  are submodules (but  $\tilde{R}$  is generally not a submodule).

We say that  $f \in K$  is  *$G$ -integral* (over  $R$ ) if the  $G$ -submodule of  $K$  generated by  $f$  is contained in  $\tilde{R}$ . We now claim that  $R'$  is the subset of  $K$  consisting of  $G$ -integral elements.

Indeed, since the direct sum and tensor product of any two  $G$ -modules are  $G$ -modules and the sum and product of integral elements are integral, we see that the  $G$ -integral elements form a subalgebra  $S \subset K$ . Clearly, we have  $R' \subset S \subset \tilde{R}$ , and hence  $S$  is a finite  $R'$ -module. In particular,  $\text{Spec}(S)$  is a  $G$ -variety equipped with a finite birational equivariant morphism to  $X'$ . Thus,  $S = R'$ , proving the claim.

As a direct consequence of this claim, the formation of  $R'$  commutes with localization by  $G$ -invariants of  $R$ .

Next, consider a  $G$ -variety  $X$  admitting a covering by open affine  $G$ -stable subsets. Then the  $G$ -normalizations of these subsets may be glued to a  $G$ -variety, which is readily seen to be the  $G$ -normalization. This provides an algebraic construction of the  $G$ -normalization.

We now obtain an equivariant version of Serre's criterion for normality (see [11, IV<sub>2</sub>.5.8.6]). The latter can be stated as follows in our setting: a variety  $X$  is normal if and only if it satisfies  $(S_2)$  and the ideal sheaf of every closed subvariety is invertible in codimension 1.

**Theorem 4.12.** *Let  $X$  be a  $G$ -variety. Then  $X$  is  $G$ -normal if and only if it satisfies  $(S_2)$  and the ideal sheaf  $\mathcal{I}_Z$  is invertible in codimension 1 for any closed  $G$ -stable subscheme  $Z \subsetneq X$ .*

*Proof.* We use the construction before Lemma 4.7. By Corollary 4.8, it suffices to show the following two equivalences:

- (i)  $X^\#$  satisfies  $(S_2)$  if and only if  $X$  satisfies  $(S_2)$ .
- (ii)  $X^\#$  satisfies  $(R_1)$  if and only if  $\mathcal{I}_Z$  is invertible in codimension 1 for any closed  $G$ -stable subscheme  $Z \subsetneq X$ .

(i) This follows from the fact that  $\psi : X^\# \rightarrow G^\# / G$  is a faithfully flat morphism to a smooth variety, with fibers being obtained from  $X$  via base change by field extensions (use [11, IV<sub>2</sub>.6.6.1]).

For (ii), assume that  $X^\#$  satisfies  $(R_1)$ . Let  $Z \subsetneq X$  be a closed  $G$ -stable subscheme; then  $Z^\# = G^\# \times^{G^0} Z$  is a closed subscheme of  $X^\#$ , and hence  $\mathcal{I}_{Z^\#}$  is invertible in codimension 1. Since the quotient  $G^\# \times X \rightarrow X^\#$  is a  $G^0$ -torsor, it follows that  $\mathcal{I}_{G^\# \times Z} \subset \mathcal{O}_{G^\# \times Z}$  is invertible in codimension 1 as well. This yields the desired assertion.

Conversely, assume that  $\mathcal{I}_Z$  is invertible in codimension 1 for any closed  $G$ -stable subscheme  $Z \subsetneq X$ . Recall that  $X^\#$  is equipped with an action of the smooth algebraic group  $H = G^\# \times \pi_0(G)$ ; thus, the regular locus  $X_{\text{reg}}^\#$  is  $H$ -stable. It follows that the singular locus  $X_{\text{sing}}^\#$  (equipped with its reduced subscheme structure) is  $H$ -stable as well (indeed, the formation of  $X_{\text{sing}}^\#$  commutes with separable field extensions, and hence we may assume  $k$  separably closed. Then  $X_{\text{sing}}^\#$  is stable under  $H(k)$ , and hence under its schematic closure  $H$ ). So  $X_{\text{sing}}^\# = H \times^G Z$  for a unique closed  $G$ -stable subscheme  $Z \subsetneq X$ . By using the  $G$ -torsor  $H \times X \rightarrow X^\#$  as above, it follows that  $\mathcal{I}_{X_{\text{sing}}^\#}$  is invertible in codimension 1. This forces  $\text{codim}_{X^\#}(X_{\text{sing}}^\#) \geq 2$ . □

**Example 4.13.** Assume that  $k$  is algebraically closed. Consider the zero subscheme  $X \subset \mathbb{A}^{n+1}$  of  $y^{p^m} - f(x_1, \dots, x_n)$ , where  $m, n$  are positive integers,  $x_1, \dots, x_n, y$  denote the coordinates on  $\mathbb{A}^{n+1}$ , and  $f \in k[T_1, \dots, T_n]$ . The group scheme  $\alpha_{p^m}$  acts freely on  $\mathbb{A}^{n+1}$  via  $u \cdot (x_1, \dots, x_n, y) = (x_1, \dots, x_n, u+y)$  and this action stabilizes  $X$ . The quotient is the morphism

$$\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{n+1}, \quad (x_1, \dots, x_n, y) \longmapsto (x_1, \dots, x_n, y^{p^m}).$$

Its restriction to  $X$  is identified with the projection  $(x_1, \dots, x_n) : X \rightarrow \mathbb{A}^n$ . So  $X$  is  $\alpha_{p^m}$ -normal by Corollary 4.6. Also,  $X$  is generally singular in codimension 1, e.g., when  $f$  is divisible by the square of a non-constant polynomial; then the singular locus is not stable under  $\alpha_{p^m}$ .

Next, let  $\mu_{p^m}$  act on  $\mathbb{A}^{n+1}$  via  $t \cdot (x_1, \dots, x_n, y) = (x_1, \dots, x_n, ty)$ . Then this action stabilizes  $X$ , and the quotient is as above. One may check by using Theorem 4.12 that  $X$  is not  $\mu_{p^m}$ -normal when  $f$  is divisible by the square of a non-constant polynomial.

Next, we obtain an equivariant version of a classical normality criterion for curves:

**Corollary 4.14.** *The following conditions are equivalent for a  $G$ -curve  $X$ :*

- (i)  $X$  is  $G$ -normal.
- (ii) For any closed point  $x \in X$ , the ideal  $\mathcal{I}_{G \cdot x}$  is invertible.
- (iii) For any closed  $G$ -stable subscheme  $Z \subsetneq X$ , the ideal  $\mathcal{I}_Z$  is invertible.

*Proof.* (i)  $\Rightarrow$  (ii) Let  $x \in X$  be a closed point. Set  $Z = G \cdot x$  and consider the blow-up  $f : \text{Bl}_Z(X) \rightarrow X$ . Then  $\text{Bl}_Z(X)$  is a  $G$ -curve and  $f$  is equivariant. Thus,  $f$  is an isomorphism. By the universal property of the blow-up, this means that  $\mathcal{I}_Z$  is invertible.

(ii)  $\Rightarrow$  (iii) Let  $Z \subsetneq X$  be a closed  $G$ -stable subscheme. We show that  $\mathcal{I}_Z$  is invertible by induction on the length  $\ell(Z) = \dim H^0(X, \mathcal{O}_X/\mathcal{I}_Z)$ . We may assume that  $\ell(Z) \geq 1$ , and hence choose a closed point  $x \in Z$ . Then  $G \cdot x \subset Z$ ; thus,  $\mathcal{I} = \mathcal{I}_Z$  is contained in  $\mathcal{I}_{G \cdot x} = \mathcal{J}$  and the latter is invertible. So  $\mathcal{J}^{-1}\mathcal{I}$  is a  $G$ -stable sheaf of ideals of  $\mathcal{O}_X$ . Denoting by  $W \subset X$  the corresponding closed  $G$ -stable subscheme, we have

$$\mathcal{O}_X/\mathcal{I}_W = \mathcal{O}_X/\mathcal{J}^{-1}\mathcal{I} \simeq \mathcal{J}/(\mathcal{J}^{-1}\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{J}) \simeq \mathcal{J}/\mathcal{I}.$$

As a consequence,  $\ell(W) = \dim H^0(X, \mathcal{J}/\mathcal{I}) < \ell(Z)$ . By the induction assumption,  $\mathcal{I}_W$  is invertible, and hence so is  $\mathcal{I}_Z$ .

(iii)  $\Rightarrow$  (i) This follows readily from Theorem 4.12. □

**Remark 4.15.** Given a smooth closed point  $x$  of a  $G$ -curve  $X$ , the orbit  $G \cdot x$  is contained in the smooth locus of  $X$ , and hence the ideal  $\mathcal{I}_{G \cdot x}$  is invertible. So to check the  $G$ -normality of  $X$ , it suffices to show that  $\mathcal{I}_{G \cdot x}$  is invertible for any non-smooth point  $x$ .

Likewise, if  $k$  is perfect then the  $G$ -normalization of  $X$  is obtained by iterating the process of blowing up the  $G$ -orbits of non-smooth points.

**Corollary 4.16.** *The following conditions are equivalent for a finite group scheme  $G$  of order  $p$  and a  $G$ -curve  $X$ :*

- (i)  $X$  is  $G$ -normal.
- (ii)  $X/G$  is normal and  $X$  is normal at every  $G$ -stable point.

*Proof.* (i)  $\Rightarrow$  (ii) The normality of  $X/G$  follows from Corollaries 4.6 and 4.8. If  $x \in X$  is  $G$ -stable, then the ideal  $\mathcal{I}_x$  is invertible by Corollary 4.14, hence  $X$  is normal at  $x$ .

(ii)  $\Rightarrow$  (i) Let  $x \in X$  be a closed point. If  $x$  is  $G$ -stable, then  $\mathcal{I}_{G \cdot x} = \mathcal{I}_x$  is invertible. Otherwise, we claim that  $\text{Stab}_G(x)$  is trivial. To check this, view  $x$  as a  $\kappa(x)$ -point of the  $G_{\kappa(x)}$ -variety  $X_{\kappa(x)}$ ; then  $x$  is not  $G_{\kappa(x)}$ -stable, since its orbit  $G_{\kappa(x)} \cdot x$  has schematic image  $G \cdot x$  under the projection  $X_{\kappa(x)} \rightarrow X$ . As  $G_{\kappa(x)} \cdot x \simeq G_{\kappa(x)}/\text{Stab}_G(x)$ , it follows that  $\text{Stab}_G(x)$  is strictly contained in  $G_{\kappa(x)}$ . This implies our claim, since  $G$  has order  $p$ .

By the claim, the quotient  $q : X \rightarrow X/G$  is a  $G$ -torsor at  $x$ , and hence  $\mathcal{I}_{G \cdot x} = \mathcal{I}_{q(x)}\mathcal{O}_X$ . Since the curve  $X/G$  is normal,  $\mathcal{I}_{q(x)}$  is invertible; therefore, so is  $\mathcal{I}_{G \cdot x}$ . By Corollary 4.14 again, it follows that  $X$  is  $G$ -normal. □

Next, assume that  $G$  is a subgroup scheme of a smooth connected algebraic group  $G^\#$ . Then the quotient  $G^\# \times^G X$  exists for any  $G$ -curve  $X$ , since  $G^\#$  and  $X$  are quasi-projective.

**Proposition 4.17.** *With the above notation and assumptions,  $X$  is  $G$ -normal if and only if  $G^\# \times^G X$  is regular.*

*Proof.* Note that there is a canonical morphism  $X^\# = G^\# \times^{G^0} X \rightarrow G^\# \times^G X$ , which is a torsor under the finite étale group scheme  $G/G^0 = \pi_0(G)$ .

If  $G^\# \times^G X$  is regular, then so is  $X^\#$  by [11, IV<sub>2</sub>.6.6.1]. So  $X$  is  $G$ -normal in view of Corollary 4.8.

The converse is obtained by arguing as in the end of the proof of Theorem 4.12: the singular locus of  $X^\#$  satisfies  $X^\#_{\text{sing}} = G^\# \times^G Z$  for some closed  $G$ -stable subscheme  $Z \subsetneq X$ . Thus,  $\text{codim}_Z(X) = \text{codim}_{X^\#}(X^\#_{\text{sing}})$ . Since  $X^\#$  is normal (by Corollary 4.8 again), this yields  $\text{codim}_X(Z) \geq 2$  and hence  $Z = \emptyset = X^\#_{\text{sing}}$ . So  $X^\#$  is regular, and hence  $G^\# \times^G X$  is regular as well (see [11, IV<sub>2</sub>.6.6.1] again).  $\square$

**Corollary 4.18.** *Every  $G$ -normal curve is a local complete intersection.*

*Proof.* Given a  $G$ -normal curve  $X$ , consider the natural morphism  $\psi : X^\# = G^\# \times^G X \rightarrow G^\#/G$  with fiber  $X$  at the base point. Note that  $X^\#$  is regular,  $G^\#/G$  is smooth, and  $\psi$  is faithfully flat (e.g., by  $G^\#$ -equivariance). So the assertion follows from [11, IV<sub>4</sub>.19.3.2].  $\square$

**Remark 4.19.** Assume that  $k$  is perfect. Then a  $G$ -curve  $X$  is  $G$ -normal if and only if the quotient stack  $[X/G]$  is smooth. This follows from the above proposition in view of the isomorphism of stacks  $[X/G] \simeq [G^\# \times^G X/G^\#]$ , which in turn is a direct consequence of the definitions of such stacks.

### 5. Generically free actions on curves

Throughout this section, we denote by  $G$  an infinitesimal group scheme of order  $p^n$ , with Lie algebra  $\mathfrak{g}$ .

**Proposition 5.1.** *Let  $X$  be a curve equipped with a rational action of  $G$ ; let  $K = k(X)$  and  $L = K^G$ . Then there exists a unique integer  $m = m(G) \geq 0$  such that  $L = kK^{p^m}$ . Moreover,  $m \leq n$  and equality holds if and only if the rational  $G$ -action on  $X$  is generically free.*

*Proof.* Recall from Lemma 2.3 that the extension  $K/L$  is finite and purely inseparable. Thus,  $[K : L] = p^m$  for some  $m \geq 0$ , and  $kK^{p^m} \subset L$ . Moreover, we have  $[K : kK^{p^m}] = p^m$ , since  $K$  is a function field in one variable (indeed,

using the tower of field extensions  $K = K_0 \supset kK^p = K_1 \supset \dots \supset kK^{p^m} = K_m$  where  $K_{i+1} = kK_i^p$  for  $i = 0, \dots, m-1$ , it suffices to show that  $[K : kK^p] = p$ . But this follows from the fact that every  $x \in K \setminus kK^p$  forms a  $p$ -basis of  $K$  over  $k$ , see e.g. [11, IV<sub>1</sub>.2.1.4]). So  $L = kK^{p^m}$ . The final assertion follows readily from Corollary 2.6.  $\square$

**Remark 5.2.** Let  $X$  be a generically free  $G$ -curve with quotient morphism  $q : X \rightarrow Y = X/G$ . Then the  $n$ th Frobenius morphism  $F_X^n : X \rightarrow X^{(p^n)}$  is finite, surjective and  $G$ -invariant, and  $X^{(p^n)}$  is a curve with function field  $kK^{p^n}$ . Thus,  $F_X^n$  factors uniquely as

$$X \xrightarrow{q} Y \xrightarrow{r} X^{(p^n)},$$

where  $r$  is finite surjective as well; also,  $r$  is birational by Proposition 5.1.

If  $X$  is  $G$ -normal, then  $Y$  is normal (Lemma 4.5) and hence is isomorphic to the normalization of  $X^{(p^n)}$ . As a consequence,  $g(Y_{\bar{k}}) = g(X_{\bar{k}})$  where  $g$  denotes the geometric genus.

**Lemma 5.3.** *The following conditions are equivalent for a curve  $X$  equipped with a faithful action of  $G$ :*

- (i) *The  $G$ -action is generically free.*
- (ii) *We have  $\dim(\mathfrak{g}) = 1$ .*
- (iii) *We have an isomorphism of algebras  $\mathcal{O}(G) \simeq k[T]/(T^{p^n})$ .*

*Under these conditions,  $G$  is either unipotent or a form of  $\mu_{p^n}$ .*

*Proof.* (i)  $\Leftrightarrow$  (ii) We may assume  $k$  algebraically closed. Then (i) is equivalent to the existence of a smooth point  $x \in X(k)$  such that  $\text{Stab}_G(x)$  is trivial. This is in turn equivalent to  $\text{Lie}(\text{Stab}_G(x)) = 0$ , since  $G$  is infinitesimal. But  $\text{Lie}(\text{Stab}_G(x))$  is the kernel of the natural map  $\mathfrak{g} \rightarrow T_x X$  (the differential at  $e$  of the morphism  $G \rightarrow X, g \mapsto g \cdot x$ ), see e.g. [8, III.2.2.6]. So  $\text{Lie}(\text{Stab}_G(x)) = 0$  implies that  $\dim(\mathfrak{g}) = 1$ .

Conversely, assume that  $\dim(\mathfrak{g}) = 1$ . Choose a smooth point  $x \in X(k)$  which is not fixed by  $\mathfrak{g}$  (identified with the Frobenius kernel  $G_1$ ). Then  $\text{Stab}_G(X)_1$  is trivial, and hence so is  $\text{Stab}_G(x)$ .

(ii)  $\Leftrightarrow$  (iii) This follows from the fact that  $\mathcal{O}(G)$  is a local  $k$ -algebra of dimension  $p^n$  and residue field  $k$ .

To show the final assertion, we may again assume  $k$  algebraically closed. If  $G$  contains no copy of  $\alpha_p$ , then  $G$  is diagonalizable by [8, IV.3.3.7]. Thus,  $G \simeq \prod_{i=1}^r \mu_{p^{m_i}}$  in view of the structure of diagonalizable group schemes (see e.g. [8, IV.1.1.2]). Using (ii), it follows that  $r = 1$ .



So we may assume that  $\alpha_p \subset G$ . By (ii) again, we then have  $G_1 = \alpha_p$ . The relative Frobenius  $F_G$  yields an exact sequence

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{f} F(G) \longrightarrow 1,$$

where  $\mathcal{O}(F(G)) = \mathcal{O}(G)^p = k[T]/(T^{p^{n-1}})$  in view of (iii). Arguing by induction on  $n$ , we may assume that  $F(G)$  is either unipotent or isomorphic to  $\mu_{p^{n-1}}$ . In the former case,  $G$  is unipotent. In the latter case,  $G$  is trigonalizable (see [8, IV.2.3.1]) and hence  $G \simeq \alpha_p \rtimes \mu_{p^{n-1}}$  by loc. cit., IV.2.3.5. But then  $\dim(\mathfrak{g}) = 2$ , a contradiction.  $\square$

**Proposition 5.4.** *Let  $X$  a generically free  $G$ -normal curve.*

- (i) *The quotient of  $\Omega_{X/k}^1$  by its torsion subsheaf is invertible.*
- (ii) *The tangent sheaf  $\mathcal{T}_X$  is invertible and effective.*

*Proof.* (i) Choose a non-zero element of  $\mathfrak{g}$ , which yields a morphism of  $\mathcal{O}_X$ -modules  $D : \Omega_{X/k}^1 \rightarrow \mathcal{O}_X$ . Then  $D$  is an isomorphism at the generic point, and hence its kernel is the torsion subsheaf. Also,  $\Omega_{X/k}^1$  is equipped with a  $G$ -linearization, and  $D$  is  $G$ -equivariant (since the adjoint representation of  $G$  in  $\mathfrak{g}$  is trivial by the final assertion of Lemma 5.3). Thus, the image of  $D$  is a  $G$ -stable ideal. As every such ideal is invertible (Corollary 4.14), this yields the assertion.

- (ii) The sheaf  $\mathcal{T}_X$  is invertible by (i), and has non-zero global sections.  $\square$

**Proposition 5.5.** *Let  $X$  a generically free  $G$ -normal curve with quotient  $Y$ . Let  $x \in X$  be a closed  $G$ -fixed point with image  $y \in Y$ . If  $k$  is perfect, then there exists an isomorphism of  $\mathcal{O}_{Y,y}$ -algebras*

$$\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y}[T]/(T^{p^n} - w),$$

where  $w$  generates the maximal ideal of  $\mathcal{O}_{Y,y}$ .

*Proof.* By Corollaries 4.6 and 4.8,  $Y$  is smooth at  $y$ ; also,  $X$  is smooth at  $x$  in view of Corollary 4.14. Since  $G$  is infinitesimal, the fiber  $X_y$  has a unique point  $x$  and the extension  $\kappa(x)/\kappa(y)$  is purely inseparable. But  $\kappa(y)$  is perfect (since so is  $k$ ), and hence  $\kappa(x) = \kappa(y)$ . Thus,  $\mathcal{O}_{X,x}$  and  $\mathcal{O}_{Y,y}$  are discrete valuation rings with the same residue field  $k' = \kappa(x)$ ; moreover,  $\mathcal{O}_{X,x}$  is a finite free module over  $\mathcal{O}_{Y,y}$ . The rank of this module is  $p^n$ , since  $[k(X) : k(Y)] = p^n$  by Proposition 5.1. Thus,  $X_y$  is a finite  $k'$ -subscheme of length  $p^n$  of  $\mathcal{O}_{X,x}$ . As a consequence,  $\mathcal{O}(X_y) = \mathcal{O}_{X,x}/(z^{p^n})$ , where  $z \in \mathcal{O}_{X,x}$  generates the maximal ideal  $\mathfrak{m}_x$ . On the other hand,  $\mathcal{O}(X_y) = \mathcal{O}_{X,x}/(w)$ , where  $w \in \mathcal{O}_{Y,y}$  generates

$\mathfrak{m}_y$ . So  $z^{p^n} = uw$  where  $u$  is a unit in  $\mathcal{O}_{X,x}$ . By Proposition 5.1 and the normality of  $Y$ , we have  $z^{p^n} \in \mathcal{O}_{Y,y}$ . Thus,  $u$  is a unit in  $\mathcal{O}_{Y,y}$ . Replacing  $w$  with  $uw$ , we may thus assume that  $z^{p^n} = w$ . Consider the homomorphism of  $\mathcal{O}_{Y,y}$ -algebras

$$\mathcal{O}_{Y,y}[T]/(T^{p^n} - w) \longrightarrow \mathcal{O}_{X,x}, \quad T \longmapsto z.$$

This is surjective by Nakayama’s lemma, and hence an isomorphism since both sides are free  $\mathcal{O}_{Y,y}$ -modules of rank  $p^n$ . □

Finally, we use the above results and methods to settle an issue in the classification of maximal connected algebraic groups of birational automorphisms of surfaces. This classification (up to conjugation by birational automorphisms) was recently obtained by Fong for smooth projective surfaces over an algebraically closed field, see [12]. One case was left unsettled in positive characteristics, see Proposition 3.25 and Remark 3.26 in loc. cit. This case can be handled as follows:

Assume that  $k$  is algebraically closed and let  $S$  be a smooth projective surface equipped with a faithful action of an elliptic curve  $E$ . By loc. cit., there is an  $E$ -equivariant isomorphism  $S \simeq E \times^G X$ , where  $G \subset E$  is a finite subgroup scheme and  $X \subset S$  is a closed  $G$ -stable subscheme of pure dimension 1 (a priori,  $X$  is not necessarily reduced). This yields an  $E$ -equivariant morphism  $\psi : S \rightarrow E/G$  with fiber  $X$  at the origin of the elliptic curve  $E/G$ . Using the Stein factorization, we may assume that  $\psi_*(\mathcal{O}_S) = \mathcal{O}_{E/G}$ ; then  $X$  is connected.

**Proposition 5.6.** *With the above notation and assumptions,  $X$  is a  $G$ -normal curve.*

*Proof.* We begin with some observations. First,  $X$  is Cohen-Macaulay, as  $\psi$  is flat.

Also, the categorical quotient  $X \rightarrow X/G = Y$  exists, and the natural morphism  $S = E \times^G X \rightarrow Y$  is the categorical quotient by  $E$ . It follows that  $Y$  is a smooth projective curve.

Finally,  $G$  acts faithfully on  $X$ , since  $E$  is a commutative algebraic group and acts faithfully on  $S$ .

We now reduce to the case where  $G$  is infinitesimal. For this, we consider the natural morphism  $f : E \times^{G^0} X \rightarrow E \times^G X = S$ , which is a torsor under the finite étale group scheme  $\pi_0(G)$ . Thus,  $E \times^{G^0} X$  is smooth, projective and of pure dimension 2. It is also connected (since so are  $E$  and  $X$ ) and equipped with a faithful action of  $E$ . This yields the desired reduction.

The scheme  $X$  is irreducible, as the quotient morphism  $E \times X \rightarrow S$  is a homeomorphism. Also, since  $G$  is a subgroup scheme of  $E$ , its Lie algebra

has dimension 1 and hence  $\mathcal{O}(G) \simeq k[T]/(T^{p^n})$  as algebras (Lemma 5.3). As a consequence, the Frobenius kernel  $G_1$  has order  $p$ .

We now show that the  $G$ -action on  $X$  is generically free. Since the  $G_1$ -action is faithful, there exists  $x \in X(k)$  such that  $\text{Stab}_{G_1}(x)$  is trivial (Lemma 5.3 again). Then  $\text{Stab}_G(x)$  has a trivial Frobenius kernel, and hence is trivial as well.

Next, we describe the local structure of  $X$  at  $x \in X_{\text{fr}}(k)$ , by adapting the argument of Proposition 5.5. Let  $y = q(x) \in Y(k)$ . Then the quotient  $q : X \rightarrow Y$  is a  $G$ -torsor at  $y$ , and hence is finite free of rank  $p^n$  at that point. The fiber  $X_y$  satisfies  $\mathcal{O}(X_y) \simeq k[T]/(T^{p^n})$ . Choose a lift  $z \in \mathcal{O}_{X,x}$  of the class of  $T$  in  $\mathcal{O}(X_y)$ . Then  $z^{p^n} \in \mathcal{O}_{X,x}^G$ , since the relative Frobenius morphism  $F_X^n$  is  $G$ -invariant. Thus,  $z^{p^n} = w$  where  $w \in \mathcal{O}_{Y,y}$ . As  $z$  lifts  $T$ , we get  $w(y) = 0$ , i.e.,  $w \in \mathfrak{m}_y$ . So the homomorphism of  $\mathcal{O}_{Y,y}$ -algebras

$$f : \mathcal{O}_{Y,y}[T]/(T^{p^n} - w) \longrightarrow \mathcal{O}_{X,x}, \quad T \longmapsto z$$

induces an isomorphism modulo  $\mathfrak{m}_y$ . By Nakayama’s lemma, it follows that  $f$  is an isomorphism.

Now consider the case where  $w \notin k(Y)^{p^n}$ . Then  $\mathcal{O}_{X,x}$  is reduced. Since  $X$  is Cohen-Macaulay, it is reduced as well. So  $X$  is a  $G$ -curve; it is  $G$ -normal by Proposition 4.17.

It remains to treat the case where  $w \in k(Y)^{p^n}$ . Since  $Y$  is normal, we then have  $w = t^{p^n}$  where  $t \in \mathcal{O}_{Y,y}$ . So  $\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,y}[U]/(U^{p^n})$  as an  $\mathcal{O}_{Y,y}$ -algebra. In geometric terms, there exists a dense open subset  $V \subset Y$  and a section  $\sigma : V \rightarrow X_{\text{fr}}$  of  $q$  above  $V$ . The schematic closure of  $\sigma(V)$  in  $X$  is a projective curve with smooth projective model  $Y$ . Thus,  $\sigma$  extends to a section  $\tau : Y \rightarrow X$  of  $q$ . This yields a morphism

$$\varphi : E \times Y \longrightarrow S, \quad (e, y) \longmapsto e \cdot \tau(y)$$

which is  $E$ -equivariant, where  $E$  acts on  $E \times Y$  by translation on itself. Moreover,  $\varphi$  restricts to an isomorphism  $G \times V \rightarrow q^{-1}(V)$ , since  $q$  induces a  $G$ -torsor  $q^{-1}(V) \rightarrow V$  with section  $\sigma$ . Thus,  $\varphi$  restricts to an isomorphism  $E \times^G (G \times V) \rightarrow E \times^G q^{-1}(V)$ , i.e., an isomorphism of  $E \times V$  onto an open subset of  $S$ . In particular,  $\varphi$  is birational. Also, the triangle

$$\begin{array}{ccc} E \times Y & \xrightarrow{\varphi} & S \\ & \searrow \gamma & \swarrow \psi \\ & & E/G \end{array}$$

commutes, where  $\gamma$  is the composite of the projection  $E \times Y \rightarrow E$  with the quotient map  $E \rightarrow E/G$ . Moreover,  $\gamma$  and  $\psi$  have irreducible fibers. Thus,  $\varphi$  is finite, and hence an isomorphism. Since  $\psi_*(\mathcal{O}_S) = \mathcal{O}_{E/G}$ , it follows that  $G$  is trivial and  $X = Y$ .  $\square$

**Remark 5.7.** Proposition 5.6 can be extended to a perfect ground field  $k$ . But it fails over any imperfect field in view of the existence of non-trivial pseudo-abelian surfaces, see [22, Sec. 6] and [4, Rem. 6.4 (ii)].

**Remark 5.8.** Conversely, given a  $G$ -normal curve  $X$  over an algebraically closed field and an embedding of  $G$  into an elliptic curve  $E$ , we obtain a smooth projective surface  $S = E \times^G X$  by Proposition 4.17. Moreover, the morphism  $f : S \rightarrow X/G = Y$  (the categorical quotient by  $E$ ) is an elliptic fibration with general fiber  $E$ . The multiple fibers of  $f$  are exactly the fibers  $S_y$ , where  $y = f(x)$  and  $x$  has a non-trivial stabilizer; the multiplicity of  $S_y$  is the order of  $\text{Stab}_G(x)$ . Also, one may check that  $R^1 f_*(\mathcal{O}_S) = \mathcal{O}_Y$  when  $G$  is diagonalizable.

If  $X$  is rational (or equivalently,  $Y$  is rational; see Remark 5.2), then the natural morphism  $S \rightarrow E/G$  is the Albanese morphism, since its fibers at closed points are the translates of  $X$ . Moreover, these fibers are exactly the rational curves on  $S$ . If in addition  $X$  is singular, then it follows that  $S$  is minimal. Using the canonical bundle formula for the elliptic fibration  $f$  (see [2]), one may check that  $S$  has Kodaira dimension 1 when  $p > 3$  and  $f$  has at least 3 multiple fibers. But  $S$  may well have Kodaira dimension 0, for example when  $p \leq 3$  and  $S$  is a quasi-hyperelliptic surface (see [3]).

**Example 5.9.** Assume that  $k$  is algebraically closed and consider a projective variant of Example 4.13: let  $X$  be the projective plane curve with equation  $z^{p^n} - f(x, y)$ , where  $f$  is a homogeneous polynomial of degree  $p^n$  and is not a  $p$ th power. Assume in addition that  $p^n \geq 3$ ; then  $X$  is singular.

Consider the action of  $\alpha_{p^n}$  on  $\mathbb{P}^2$  via  $g \cdot (x, y, z) = (x, y, z + g(ax + by))$ , where  $a, b \in k$ . Then  $X$  is stable under this action, and the quotient morphism is the projection  $[x : y] : X \rightarrow \mathbb{P}^1$ . So  $X$  is rational by Remark 5.2 or a direct argument. Using Corollary 4.14 and the subsequent remark, one may check that  $X$  is  $G$ -normal for general  $a, b$ .

Next, let  $\mu_{p^n}$  act on  $\mathbb{P}^2$  via  $t \cdot (x, y, z) = (x, y, tz)$ . Then again,  $X$  is stable under this action, with the same quotient. Moreover,  $X$  is  $\mu_{p^n}$ -normal if and only if  $f$  is square-free.

Every elliptic curve  $E$  contains copies of  $\mu_{p^n}$  (if  $E$  is ordinary) or  $\alpha_{p^n}$  (if  $E$  is supersingular) for all  $n \geq 1$ . Denote by  $G \subset E$  the resulting infinitesimal subgroup scheme, and let  $X$  be a singular  $G$ -normal curve obtained by the

above construction. In view of Remark 5.8, this yields a smooth projective minimal surface  $S = E \times^G X$ , which is uniruled but not ruled. By the canonical bundle formula for the elliptic fibration  $S \rightarrow \mathbb{P}^1$ , the Kodaira dimension of  $S$  is 1 if  $p^n \geq 4$ .

### Acknowledgements

I thank Pascal Fong, Jérémy Guéré, Daniel Litt, Matilde Maccan, Benjamin Morley, Immanuel van Santen, Ronan Terpereau and Susanna Zimmermann for very helpful comments and discussions. Also, many thanks to the anonymous referee for a careful reading and valuable suggestions.

### References

- [1] V. ALEXEEV AND R. PARDINI, *Non-normal abelian covers*, *Compositio Math.* **148** (2012), 1051–1084. [MR2956036](#)
- [2] E. BOMBIERI AND D. MUMFORD, *Enriques' classification of surfaces in char.  $p$ . II*, *Complex analysis and algebraic geometry*, Cambridge Univ. Press, Cambridge, 1977, 23–42. [MR0491719](#)
- [3] E. BOMBIERI AND D. MUMFORD, *Enriques' classification of surfaces in char  $p$ . III*, *Invent. Math.* **35** (1976), 197–232. [MR0491720](#)
- [4] M. BRION, *Algebraic group actions on normal varieties*, *Trans. Moscow Math. Soc.* **78** (2017), 85–107. [MR3738079](#)
- [5] M. BRION, *Some structure results on algebraic groups*, *Proc. Symp. Pure Math.* **94** (2017), 53–125. [MR3645068](#)
- [6] M. BRION, *On models of algebraic group actions*, *Proc. Indian Acad. Sci. (Math. Sci.)* **132** (2022). [MR4504264](#)
- [7] M. DEMAZURE, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, *Ann. Sci. École Norm. Sup. (4)* **3** (1970), 507–588. [MR0284446](#)
- [8] M. DEMAZURE AND P. GABRIEL, *Groupes algébriques*, Masson, Paris, 1970. [MR0302656](#)
- [9] S. DEMUSHKIN, *Cartan subalgebras of simple Lie  $p$ -algebras  $W_n$  and  $S_n$* , *Sib. Math. J.* **11** (1970), 233–245. [MR0262310](#)
- [10] I. DOLGACHEV AND A. DUNCAN, *Automorphisms of cubic surfaces in positive characteristic*, *Izv. Math.* **83** (2019), 424–500. [MR3954305](#)

- [11] A. GROTHENDIECK, *Éléments de géométrie algébrique (rédigés avec la collaboration de J. Dieudonné)*, Pub. Math. I.H.É.S. **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32** (1961–1967). [MR0217083](#)
- [12] P. FONG, *Connected algebraic groups acting on algebraic surfaces*, to appear in Ann. Inst. Fourier (Grenoble).
- [13] H. HIRONAKA, *An example of a non-Kählerian complex-analytic deformation of Kählerian complex structures*, Ann. of Math. (2) **75** (1962), 190–208. [MR0139182](#)
- [14] D. MADDEN AND R. VALENTINI, *The group of automorphisms of algebraic function fields*, J. Reine Angew. Math. **343** (1983), 162–168. [MR0705883](#)
- [15] G. MARTIN, *Infinitesimal automorphisms of algebraic varieties and vector fields on elliptic surfaces*, Algebra & Number Theory **16** (2022), 1655–1704. [MR4496078](#)
- [16] G. MARTIN, *Automorphism group schemes of bielliptic and quasi-bielliptic surfaces*, EpiGA **6** (2022), Article no. 9. [MR4443302](#)
- [17] D. MUMFORD, *Abelian varieties*, Oxford Univ. Press, Oxford, 1970. [MR0282985](#)
- [18] R. PARDINI, *Abelian covers of algebraic varieties*, J. reine angew. Math. **417** (1991), 191–213. [MR1103912](#)
- [19] M. ROSENBLIHT, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443. [MR0082183](#)
- [20] S. SCHRÖER AND N. TZIOLAS, *The structure of Frobenius kernels for automorphism group schemes*, Algebra & Number Theory **17** (2023), 1637–1680. [MR4637512](#)
- [21] T. A. SPRINGER, *Aktionen reductiver Gruppen*, in: Algebraische Transformationsgruppen und Invariantentheorie, DMV Seminar **13**, 3–39, Birkhäuser, Basel, 1989. [MR1044583](#)
- [22] B. TOTARO, *Pseudo-abelian varieties*, Ann. Scient. École Norm. Sup. **46** (2013), 693–721. [MR3185350](#)
- [23] N. TZIOLAS, *Automorphisms of smooth canonically polarized surfaces in positive characteristic*, Adv. Math. **310** (2017), 235–289; corrigendum *ibid.*, 585–593. [MR3620688](#)
- [24] A. WEIL, *On algebraic groups of transformations*, Amer. J. Math. **77** (1955), 355–391. [MR0074083](#)

Michel Brion  
Université Grenoble Alpes  
Institut Fourier  
100 rue des Mathématiques  
38610 Gières  
France  
E-mail: [Michel.Brion@univ-grenoble-alpes.fr](mailto:Michel.Brion@univ-grenoble-alpes.fr)