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Fibonacci polynomials

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Abstract: The Fibonacci polynomials $\{F_n(x)\}_{n\geq 0}$ have been studied in multiple ways, [1, 6, 7, 9]. In this paper we study them by means of the theory of heaps of Viennot [11, 12]. In this setting our polynomials form a basis $\{P_n(x)\}_{n\geq 0}$ with $P_n(x)$ monic of degree n. This given, we are forced to set $P_n(x) = F_{n+1}(x)$. The heaps setting extends the Flajolet view [4] of the classical theory of orthogonal polynomials given by a three term recursion [3, 10]. Thus with heaps most of the identities for the polynomials $P_n(x)$'s can be derived by combinatorial arguments. Using the present setting we derive a variety of new identities. We must mention that the theory of heaps is presented here without restrictions. This is much more than needed to deal with the Fibonacci polynomials. We do this to convey a flavor of the power of heaps. In [5] there is a chapter dedicated to heaps where most of its contents are dedicated to applications of the theory. In this paper we improve upon the developments in [5] by adding details that were omitted there.

Introduction

The sequence $\{P_n(x)\}_{n\geq 0}$ is defined by the recursion

$$P_{n+1}(x) = x\,P_n(x) + P_{n-1}(x) \tag{I.1}$$

and initial conditions

$$1)\quad P_{-1}(x) = 0, \qquad\qquad 2)\quad P_0(x) = 1. \tag{I.2}$$

Calling $F(x;t) = \sum_{n\geq 0} t^n\, P_n(x)$ the generating function of these polynomials, we see that (I.1) and (I.2) are equivalent to

$$\sum_{n\geq 0} t^{n+1} P_{n+1}(x) = xt\Big(\sum_{n\geq 0} t^n P_n(x)\Big) \;+\; t^2 \sum_{n\geq 1} t^{n-1} P_{n-1}(x) \tag{I.3}$$

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$$\to \quad F(x;t) = \frac{1}{1 \,-\, x\,t \,\,-\, t^2}$$

To give a flavor of the identities we will prove by means of this setting, we need to define the following sequence of "*moments*" (see [8])

(I.4)

$$\nu_n \;=\; \begin{cases} \frac{(-1)^m}{m+1}\binom{2m}{m} = (-1)^m\, 4^{m+1} \frac{\int_0^1 x^m \sqrt{\frac{1-x}{x}}\, dx}{2\pi} & \text{if n=2m,} \\ 0 & \text{otherwise,} \quad \text{(for all } n \geq 0). \end{cases}$$

This given, we will prove that

(I.5)

$$P_n(x) = \frac{1}{d_{n-1}} det \begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_n \\ \nu_1 & \nu_2 & \cdots & \nu_{n+1} \\ \cdots & \cdots\cdots & \cdots & \\ \nu_{n-1} & \nu_n & \cdots & \nu_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix},$$

$$d_n = det \begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_n \\ \nu_1 & \nu_2 & \cdots & \nu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \nu_{n-1} & \nu_n & \cdots & \nu_{2n-1} \\ \nu_n & \nu_{n+1} & \cdots & \nu_{2n} \end{pmatrix} = (-1)^{\lceil n/2 \rceil}.$$

We will also prove the identity

(I.6)
$$J(x,0,-1) \;=\; 1 + \cfrac{1}{1 + \cfrac{x^2}{1 + \cfrac{x^2}{\cdots}}} = \sum_{m \geq 0} \frac{(-1)^m}{m+1}\binom{2m}{m} x^{2m}$$

In the classical theory of polynomial bases $\{Q_n(x)\}$ satisfying a three term recursion the moments are non negative numbers (see for instance Favard [3]). This is obviously not always true in (I.4). So to prove these identities we need to apply the theory to a suitable substitute. We will eventually need to do so, but first we must give a detailed presentation of the setting of Viennot's "*monomer-dimer*" view of continued fractions and derive the tools that we need to prove the validity of our construction of the Fibonacci polynomials.

We develop here the general classical theory of orthogonal polynomials generated by a three term recursion in the Heap setting of Viennot [11, 12].

This is much more than we need to achieve our goal. However, it would be a disservice to the Algebraic Combinatorial audience not to expose them to the general theory of heaps. A collection of lecture notes (see [5]) has recently been published by Springer, this includes a chapter on heaps that contains some of the most beautiful applications of the theory. What we show here is that the classical theory can be extended to the case where the scalar product is still definite but not necessarily positive definite.

The contents are divided into five sections.

In section **1. The "moment" scalar product**,
we define the scalar product of two polynomials with real coefficients using "moments" and present some results of the classical theory of orthogonal polynomials. The characterization of these polynomials and their "three term recursion" is the high point of this section.

In section **2. Heaps of monomers and dimers**,
we introduce the Viennot theory of heaps of "monomers" and "dimers" and terminate with a theorem characterizing Motzkin and Dyck paths as pyramids of heaps of monomers and dimers.

In section **3. Moments and Motzkin paths**,
the most important section of the paper, we show how the theory of heaps is related to the Flajolet "continued fraction" setting of the classical theory of orthogonal polynomials defined by a three term recursion. In Theorems 3.1, 3.2, 3.3 and 3.4 we prove mostly by combinatorial arguments the basic identities of the classical theory that we need to validate our construction of the Fibonacci polynomials. The section ends with two theorems of the classical theory that are of general interest. Their proofs have been omitted since their results are not used in the sequel.

In section **4. Heap identities for the Catalan polynomials**,
we use here a basis $\{Q_n(x)\}_{n\geq 0}$ defined by the two term recursion $Q_{n+1}(x) = x\,Q_n(x) - Q_{n-1}(x)$, we use them as the closest substitute to the Fibonacci polynomials that can be found in the classical theory. This section applies some of the results of section 3 to obtain identities for the basis $\{Q_n(x)\}_{n\geq 0}$.

In section **5. Proofs of Fibonacci polynomials identities**,
we use here a basis $\{Q_n(x;\lambda)\}_{n\geq 0}$ as a substitute for the Fibonacci polynomials. These polynomials contain the extra parameter sequence $\{\lambda_i\}_{i\geq 1}$ and they are generated by the two terms recursion $Q_{n+1}(x;\lambda) = xQ_n(x;\lambda) - \lambda_n\,Q_{n-1}(x;\lambda)$. Our first goal here is to apply the results of

section 3 to obtain the identities satisfied by the polynomials $Q_n(x;\lambda)$ and all their closely related facts. This done, we transfer by means of the specialization $\lambda_i \to -1$ for all $i \geq 1$ to obtain identities satisfied by the Fibonacci polynomials.

An important fact that needs to be mentioned here is that, in the classical theory, the moments are given numerically. In particular the parameters λ_i and c_i occurring in the three term recursion are real numbers and the λ_i are actually positive (see Proposition 5.2 for the reason why this is true). In the Flajolet setting the parameters $\lambda_1, \lambda_2, \lambda_3, \ldots$ and $c_0, c_1, c_2, \ldots$ can be commutative indeterminates. To introduce the initial Viennot setting we are forced to view $\lambda_1, \lambda_2, \lambda_3, \ldots$ and $c_0, c_1, c_2, \ldots$ as non-commutative indeterminates. When Viennot passes from his original setting to the Flajolet continued fraction setting the coefficients λ_i and c_i occurring in the recursion must be allowed to commute.

In this writing we found that it is more convenient to permit this flexibility of point of view with the proviso to make clear the point of view that is being adopted at the very least from the context. Basically, if we start with the *moments* then each parameter λ_n and c_n has a formula in terms of the scalar product of elements of the orthogonal basis. The flexibility we adopted permits stating and proving that the moments and the coefficients of the orthogonal basis are polynomials in the variables $\lambda_1, \lambda_2, \lambda_3, \ldots$ and $c_0, c_1, c_2, \ldots$. This flexibility allows even the Rogers-Ramanujan continued fraction to be an application of the theory of heaps. This is one of the examples treated in the chapter on heaps in [5].

1. The "moment" scalar product

In the classical theory (see [3, 9]) the scalar product of two polynomials with real coefficients $A(x) = \sum_{r=0}^{d_a} a_r x^r$ and $B(x) = \sum_{s=0}^{d_b} b_s x^s$ is defined by setting

$$\text{(1.1)} \qquad \langle A, B\rangle_\alpha = \int_0^{+\infty} A(x)B(x)\, d\alpha = \sum_{r=0}^{d_a}\sum_{s=0}^{d_b} a_r\, b_s\, \mu^\alpha_{r+s}\,,$$

where $\alpha(x)$ is a weekly increasing function increasing from 0 to 1 in a finite interval. The scalar product in (1.1) is well defined since the matrix $A_n = \|\mu_{i+j}\|_{i,j=1}^n$ has positive eigenvalue for every $n \geq 0$ (see section 5 for a proof). The definition in (1.1) also shows that all we need are the "*moments*"

$$\text{(1.2)} \qquad \mu^\alpha_n = \int_0^{+\infty} x^n\, d\alpha.$$

The existence of a measure giving $d\alpha$ and μ_n^α by integration is classically referred to as the *moment problem* and it may be of considerable analytical difficulty depending on what properties the measure is required to satisfy. Our interest here lies on the nature of the relations between the sequence of orthogonal polynomials $\{Q_n\}_{n\geq 0}$, with respect to the scalar product in 1.1, the sequence of moments $\{\mu_n^\alpha\}_{n\geq 0}$ and two additional sequences $\{c_n\}_{n\geq 0}$, $\{\lambda_n\}_{n\geq 1}$. It develops that these relations may be beautifully expressed by means of the theory of continued fractions. What is remarkable about Flajolet's contribution to this subject is to have noticed that many identities of the classical theory can be established by combinatorial methods. The corresponding identities are the contents of the following sequence of theorems which combine classical results of Jacobi, Rogers, Stieltjes and others. We start with a result that shows that each μ_n^α may actually be expressed as a polynomial in the $c\,'s$ and the $\lambda\,'s$.

We develop here the classical theory by showing that a basis of monic polynomials $\{Q_n(x)\}_{n\geq 0}$ is orthogonal with respect to the scalar product in (1.1), that is

$$(1.3) \qquad \begin{aligned} &1) \quad Q_n(x) \,=\, x^n + \sum_{k=0}^{n-1} a_{n,k}\, x^k, \\ &2) \quad \langle\, Q_n \,,\, Q_m \,\rangle_\alpha \,=\, 0 \quad \textit{when} \quad n \neq m. \end{aligned}$$

if and only if it satisfies the following "*three-term recursion*" (see for instance Favard [3])

$$(1.4) \qquad Q_{n+1} = (x - c_n)Q_n - \lambda_n\, Q_{n-1},$$

with initial conditions

$$(1.5) \qquad 1) \quad Q_{-1}(x) \,=\, 0\,, \qquad\qquad 2) \quad Q_0(x) \,=\, 1.$$

It is easy to show that (1.3) and (1.4) imply that $\langle x^m, Q_n\rangle = 0$ for all $0 \leq m \leq n$, moreover we have

$$(1.6) \qquad \lambda_n \,=\, \frac{\big\langle Q_{n-1}, xQ_n\big\rangle_\alpha}{\big\langle Q_{n-1}, Q_{n-1}\big\rangle_\alpha}\,, \qquad\qquad c_n \,=\, \frac{\big\langle Q_n, xQ_n\big\rangle_\alpha}{\big\langle Q_n, Q_n\big\rangle_\alpha}\,.$$

The reason for this is very simple, the orthogonality of the basis $\{Q_n(x)\}_{n\geq 0}$

implies that every polynomial $P(x)$ of degree d has an expansion of the form

$$P(x) \;=\; \sum_{i=0}^{d} \frac{\langle\, P \,,\, Q_i \,\rangle_\alpha}{\langle\, Q_i \,,\, Q_i \,\rangle_\alpha}\, Q_i(x). \tag{1.7}$$

Since, by the previous observation

$$\big\langle xQ_n, Q_i \big\rangle_\alpha \;=\; \big\langle Q_n, xQ_i \big\rangle_\alpha \;=\; 0 \qquad\qquad \text{(for all } i \leq n-2\text{)}, \tag{1.8}$$

it follows that

$$\begin{aligned} xQ_n(x) = {} & \frac{\langle\, xQ_n \,,\, Q_{n-1} \,\rangle_\alpha}{\langle\, Q_{n-1} \,,\, Q_{n-1} \,\rangle_\alpha}\, Q_{n-1}(x) \;+\; \frac{\langle\, xQ_n \,,\, Q_n \,\rangle_\alpha}{\langle\, Q_n \,,\, Q_n \,\rangle_\alpha}\, Q_n(x) \\ & +\; \frac{\langle\, xQ_n \,,\, Q_{n+1} \,\rangle_\alpha}{\langle\, Q_{n+1} \,,\, Q_{n+1} \,\rangle_\alpha}\, Q_{n+1}(x). \end{aligned} \tag{1.9}$$

Since from (1.3) we derive that $\big\langle xQ_n, Q_{n+1} \big\rangle_\alpha = \big\langle x^{n+1}, Q_{n+1} \big\rangle_\alpha$ we can write

$$xQ_n(x) \;=\; \lambda_n Q_{n-1}(x) \;+\; c_n Q_n(x) \;+\; Q_{n+1}(x).$$

or better

$$Q_{n+1}(x) = (x - c_n)Q_n(x) - \lambda_n\, Q_{n-1}(x).$$

Since from the definition in (1.1) of the scalar product $\big\langle\,,\,\big\rangle_\alpha$ it follows from (1.6) that each λ_n is a positive real number, we can clearly see by comparing (I.1) to (1.4) that our Fibonacci polynomials cannot be automatically absorbed into the theory of heaps. So to prove identities for Fibonacci polynomials we need first to see what comes out of the classical theory for polynomials constructed from the recursion (1.4) when all $c_n = 0$ and all $\lambda_n = 1$. That is

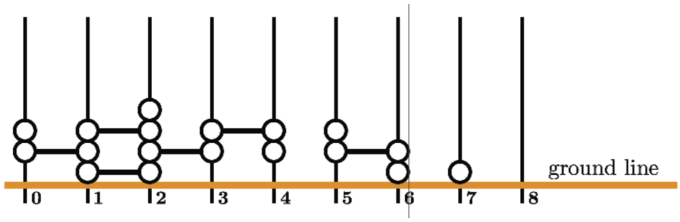
$$Q_{n+1} = xQ_n \;-\; Q_{n-1}. \tag{1.10}$$

In the rest of this paper we will omit the dependence on α and simply use μ_n to represent the moments.

2. Heaps of monomers and dimers

The monomer-dimer setting of Viennot makes the classical theory even more combinatorial. But before making definitions it is best to start with an example. In the figure below we have an instance of a *monomer-dimer heap*

(2.1)



Let us imagine that the vertical lines represent needles and that there are two basic pieces:

1. A *monomer*, which is a billiard ball pierced along a diameter for threading by the needles.
2. A *dimer* which consists of two billiard balls joined by a metal bar.

To put together a heap we simply pick a bunch of monomers and dimers and stack them on top of each other, threading them by the needles, as indicated in Figure (2.1). As it is depicted there the needles are perpendicular to the ground line at the points of coordinates $0, 1, 2, 3 \ldots$. Heaps of monomers and dimers will be represented by words in the alphabet

$$\mathcal{A} = \{ m_0, m_1, m_2, \ldots ; d_1, d_2, d_3, \ldots \},$$

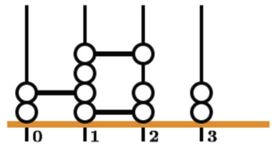
replacing each monomer of ground coordinate i by the letter m_i and each dimer projecting onto the interval $[i-1, i]$ by the letter d_i. The corresponding word is obtained by processing in this manner the successive pieces of the heap from left to right within a row, starting from the bottom row and proceeding upwards. For instance, this procedure applied to the heap of Figure (2.1) yields the word

$$w = d_2 m_6 m_7 d_1 d_3 m_4 d_6 m_0 d_2 d_4 m_5 m_2 \ .$$

Conversely, given any word $w \in \mathcal{A}^*$ we can construct a heap by reversing the process above. That is we read the letters of w from left to right and replace each m_i by a monomer of ground coordinate i and each d_i by a dimer spanning $[i-1, i]$. Of course we must also thread the corresponding monomers and dimers down the needles in the precise succession they are encountered as we read w. The final configuration is obtained by letting the pieces settle as far down as they can. This procedure applied to the word

$w_1 = m_0 d_2 m_2 d_1 m_1 d_2 m_3 m_3$ produces the heap

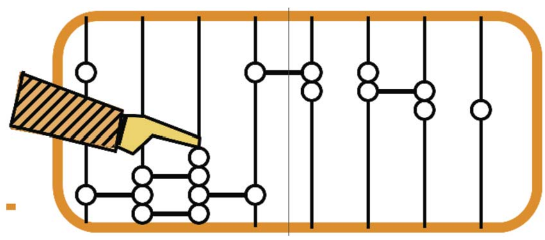
(2.2)



Now it is easy to see that the word $w_2 = m_3 d_2 m_0 d_1 m_3 m_1 m_2 d_2$ produces the same heap. At the same time the word which corresponds to this heap by the construction given above should be $w = m_0 d_2 m_3 d_1 m_2 m_3 m_1 d_2$. Mathematically speaking a "heap" should represent an equivalence class of $\mathcal{A}$-words. Two words being equivalent if and only if they yield the same heap. Thus our procedure of constructing the word corresponding to a heap is just one of the ways of selecting a representative from each equivalence class of words.

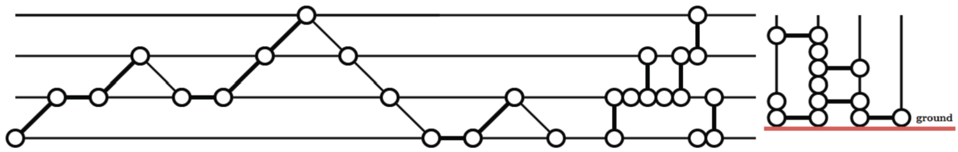
Imagine now that the needles of Figure (2.1) are set into a top and bottom bar as in an abacus and we push down the monomer m_2. This will result in the configuration

(2.3)



We can see that there are heaps that are brought down by pushing on a single piece. Such is also the case for the heap obtained by removing the monomers m_0 and m_7 from the heap in Figure (2.3) and adding on top the dimer d_5. Such a heap will be called a *pyramid* and the top piece the *summit* of the pyramid.

There is a very simple way of transforming Motzkin paths into heaps [5] which, as we shall see, has remarkable mathematical consequences. This transformation is obtained by replacing each *East* step in the path by a monomer and each *North-East* step by a dimer. We only need one example here to get across what we have in mind. For instance, carrying out these replacements (from left to right) on the path on the left yields the configuration in the middle. The latter is then rotated 90^o clockwise so that the pieces settle down to the ground. This results in the heap given on the right of our display.



The transformation is even simpler at the word level. In fact, the a, b, c-word corresponding to the Motzkin path is $w = a_0c_1a_1b_2c_1a_1a_2b_3b_2b_1c_0a_0b_1$ (here we label the a, b, c letters by the height of the starting point). The sequence of monomers and dimers depicted in the middle of the display is $d_1, m_1, d_2, m_1, d_2, d_3, m_0, d_1$. The word of the final heap is $w' = d_1d_3m_0d_2m_1d_2m_1d_1$. We see that to go from the word w of the path to the word w' of the corresponding heap, we simply read the letters of w from right to left, replace each a by a d, each c by an m and remove all $b\,'s$. From our example we can easily extract the following

Theorem 2.1. *Our construction yields a bijection between Motzkin paths and heaps of monomers and dimers with the following properties.*

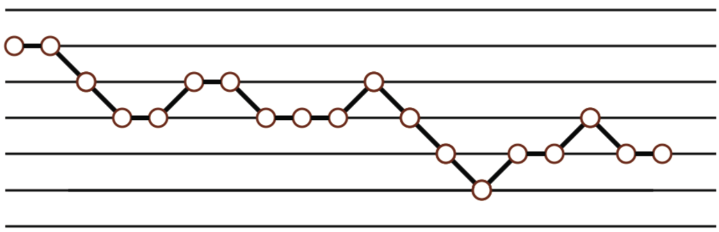
1. *The image heap is always a pyramid with summit a monomer m_0 or a dimer d_1.*
2. *Paths whose maximum height does not exceed n correspond to pyramids whose projection is in the interval $[0, n]$.*
3. *Dyck paths (no East steps) are sent into pyramids of dimers with summit d_1.*
4. *If the image pyramid has d dimers and m monomers then the corresponding path has $2d + m$ steps.*

3. Moments and Motzkin paths

Our goal in this section is to derive from the theory of heaps all the identities of the classical theory that are needed to prove the validity of our construction of the Fibonacci polynomials. We will see that using heaps, the needed classical identities will become visually self evident.

To this end we will deal here with a more general class of Motzkin paths. These are lattice paths that proceed by *North-East, East* and *South-East* steps and remain throughout weakly above the x-axis without restrictions on the heights of the starting or ending points. For instance we give below a

Motzkin path that starts at level 5 and ends at level 2

(3.1) 

Here and after the collection of Motzkin paths that start at level r and end at level s will be denoted by $\Pi_{r,s}$. To such a path π we shall associate a word $w(\pi)$ by replacing (from left to right) each *North-East* edge by an a_i, each *East* edge by a c_i and each *South-East* edge by a b_i, the subscript i giving the starting height of the edge. For instance, carrying this out on the path π in (3.1) yields the word

$$w(\pi) \;=\; c_5 b_5 b_4 c_3 a_3 c_4 b_4 c_3 c_3 a_3 b_4 b_3 b_2 a_1 c_2 a_2 b_3 c_2.$$

Clearly, we can recover a path from its word. In this context it is important to regard a_i, b_i and c_i as sequences of non commuting variables, sometimes using "word" should be the clue. Nevertheless, in other contexts it simplifies our notation to allow these letters to commute. We will do so in these contexts as long as there is no loss. We will adhere to the convention, established by the definition of $n!$, that the word of an empty path is 1.

Proposition 3.1. *Let*

$$\tilde{h}_{n,k} \;=\; \sum_{\pi\in\Pi^{=n}_{0,k}} w(\pi)\Big|_{\substack{a_{i-1}\to\lambda_i,\\ c_i\to c_i, b_i\to 1}}\,, \tag{3.2}$$

where $\Pi^{=n}_{0,k}$ denotes the collection of Motzkin paths that go from height 0 to height k in n steps. Then the $\tilde{h}_{n,k}$ satisfy the following recursion and initial conditions

$$\begin{aligned} &a)\quad \tilde{h}_{n,k} \;=\; \lambda_k \tilde{h}_{n-1,k-1} \;+\; c_k\tilde{h}_{n-1,k} \;+\; \tilde{h}_{n-1,k+1} \text{ for } 1\le k\le n,\\ &b)\quad \tilde{h}_{0,0}=1 \text{ with } \tilde{h}_{n,k}=0 \text{ for } n<k \end{aligned}$$

Proof. Note that every non empty path in $\Pi^{=n}_{0,k}$ must come either from height $k-1$ by a *North-East* step or from height k by an *East* step or from height $k+1$ by a *South-East* step. This observation yields the recursion

$$\tilde{h}_{n,k} \;=\; \lambda_k\,\tilde{h}_{n-1,k-1} \;+\; c_k\,\tilde{h}_{n-1,k} \;+\; \tilde{h}_{n-1,k+1}$$

this proves a) of (3.3).

Note further that every path that starts at level 0 and has n steps cannot reach level $k > n$. Thus $\tilde{h}_{n,k} = 0$ when $k > n$. When $n = k = 0$, the word of the path is 1, thus the sum contains 1, and we will set $\tilde{h}_{0,0} = 1$. This proves b) of (3.3) and completes our proof.

Let now L be the formal power series given by the following summation

$$L \;=\; \sum_{\pi\in\Pi_{0,0}} x^{n(\pi)}\, w(\pi), \tag{3.4}$$

where π is a Motzkin path and $n(\pi)$ denotes the number of edges of π.

This given, a moment's reflection should reveal that this formal power series must satisfy the following identity

$$L \;=\; 1 \;+\; c_0\, x\, L \;+\; x^2 a_0\, (SL)\, b_1\, L\,, \tag{3.5}$$

where S denotes the "shift" operator that replaces a letter indexed by i by the same letter indexed by $i+1$.

Passing to commutative variables we get

$$L \;=\; 1 \;+\; L\,(\,c_0 x \;+\; x^2 a_0\, b_1 (SL)\,), \tag{3.6}$$

equivalently

$$L \;-\; L(\,c_0 x \;+\; x^2 a_0 b_1 SL) \;=\; 1.$$

This gives

$$L\big(1 \;-\; c_0 x \;-\; x^2 a_0 b_1 SL\big) \;=\; 1$$

or better

$$L \;=\; \frac{1}{1 \;-\; c_0 x \;-\; x^2 a_0 b_1 SL} \tag{3.7}$$

Successive iterations of (3.7) then yield that L must be given by the continued fraction

$$L = L(x;a,b,c) \;=\; \cfrac{1}{1 - c_0 x - \cfrac{a_0 b_1 x^2}{1 - c_1 x - \cfrac{a_1 b_2 x^2}{1 - c_2 x - \cfrac{a_2 b_3 x^2}{1 - c_3 x - \cdots}}}} \tag{3.8}$$

From (3.4) it follows that we have

$$L = \sum_{n\geq 0} x^n \sum_{\pi\in\Pi_{0,0}^{=n}} w(\pi), \tag{3.9}$$

provided it is understood that "$w(\pi)$" means that it is only a rearrangement of the non commutative letters of the word of π. Here $\Pi_{0,0}(n)$ denotes the collection of paths with n edges from levels 0 to 0.

We can easily see that any specialization of the sequences $\{a_i\}$, $\{b_i\}$ that makes $a_{i-1}b_i = \lambda_i$ reduces $L(x;a,b,c)$ to the continued fraction

$$J(x;c,\lambda) = \cfrac{1}{1-c_0x-\cfrac{\lambda_1x^2}{1-c_1x-\cfrac{\lambda_2x^2}{1-c_2x-\cfrac{\lambda_3x^2}{1-c_3x-\cdots}}}} \tag{3.10}$$

Here and after we shall assume that the equality $L(x;a,b,c) = J(x;c,\lambda)$ holds true.

This given, with the same understanding about the meaning of "$w(\pi)$", we also have

$$J(x;c,\lambda) = \sum_{n\geq 0} x^n \sum_{\pi\in\Pi_{0,0}(n)} w(\pi). \tag{3.11}$$

Finally we are able to prove the following basic facts.

Theorem 3.1. *Let $h_{n,k} = \langle x^n, Q_k\rangle$ then*

$$h_{n,k} = \lambda_k\, h_{n-1,k-1} + c_k\, h_{n-1,k} + h_{n-1,k+1}, \tag{3.12}$$

with the following initial conditions

$$a)\quad h_{0,0} = 1 \qquad\qquad b)\quad h_{n,k} = 0 \quad for \quad n<k\,. \tag{3.13}$$

Then

$$h_{n,k} = J(x;c,\lambda)\, Q_k^*\Big|_{x^{n+k}}, \tag{3.14}$$

where

$$Q_k^*(x) = x^kQ_k(1/x)\,. \tag{3.15}$$

In particular, the moment μ_n *of 1.2 is given by the identity*

$$\mu_n = h_{n,0} \tag{3.16}$$

Proof. We begin with the recursion in 1.4 written in the form

$$xQ_k \;=\; \lambda_k Q_{k-1} \;+\; c_k Q_k \;+\; Q_{k+1}. \tag{3.17}$$

This given, we derive that

$$h_{n,k} = \big\langle x^{n-1}, xQ_k\big\rangle \;=\; \lambda_k\big\langle x^{n-1}, Q_{k-1}\big\rangle \;+\; c_k\big\langle x^{n-1}, Q_k\big\rangle \;+\; \big\langle x^{n-1}, Q_{k+1}\big\rangle$$

or better

$$h_{n,k} \;=\; \lambda_k\, h_{n-1,k-1} \;+\; c_k\, h_{n-1,k} \;+\; h_{n-1,k+1},$$

this proves (3.12). Since we have $h_{0,0} = \big\langle x^0, Q_0\big\rangle = 1$ because of 1) of (1.3) and we also have $\big\langle x^n, Q_k\big\rangle = 0$ when $k > n$, we see that the conditions in (3.13) are also satisfied. We also see that since $Q_0 = 1$ then $h_{n,0} = \big\langle x^n\big\rangle = \mu_n$, this proves (3.16) as well. The proof of (3.12) and (3.13) shows that the sequence $\big\{\widetilde{h}_{n,k}\big\}_{n\geq k}$ of Proposition 3.1 satisfies the same recursion and the same initial conditions as the sequence $\big\{h_{n,k}\big\}_{n\geq k}$. Thus we must have the equality

$$\widetilde{h}_{n,k} \;=\; h_{n,k} \qquad\qquad \text{(for all } n \geq k) \tag{3.18}$$

Combining (3.16) with (3.2) for $k = 0$ we derive that

$$\mu_n \;=\; \tilde{h}_{n,0} \;=\; \sum_{\pi\in\Pi_{0,0}^{=n}} w(\pi)\Big|_{\substack{a_{i-1}\to\lambda_i,\\ c_i\to c_i, b_i\to 1}}$$

and (3.11) becomes

$$J(x;c,\lambda) \;=\; \sum_{n\geq 0} x^n\mu_n. \tag{3.19}$$

Now since $Q_0 = 1$ then by (3.13) we also have $Q_0^* = 1$. Thus for $k = 0$ the identity in (3.14) reduces to $h_{n,0} = J(x;c,\lambda)\Big|_{x^n}$, which is true by (3.16) and (3.19). This not only shows the expansion in (3.19) but also proves the identity

$$\mu_n = \sum_{\pi\in\Pi_{0,0}^{=n}} w(\pi), \qquad\qquad \text{(for all } n \geq 0). \tag{3.20}$$

In particular, we derive that each μ_n is a polynomial in the variables c_i and λ_i. For instance the computer gives

$$\mu_4 \,=\, c_0^4 \,+\, 3\,c_0^2\,\lambda_1 \,+\, 2\,c_0\,c_1\,\lambda_1 \,+\, c_1^2\,\lambda_1 \,+\, \lambda_1^2 \,+\, \lambda_1\,\lambda_2.$$

It remains to verify (3.14) for $0 < k \leq n$. Using (3.19), 1) of (1.3) and (3.15) the identity in (3.14) becomes

(3.21)

$$h_{n,k} = \Big(\sum_{r\geq 0}\mu_r x^r\Big)\Big(x^n + \sum_{s=1}^{k-1} a_{k,s}x^{k-s}\Big)\Big|_{x^{n+k}} = \mu_{n+k} \,+\, \sum_{s=1}^{k-1} a_{k,s}\,\mu_{n+s} = \langle x^n, Q_k\rangle.$$

This completes our proof of the theorem.

This shows that the three term recurrence uniquely determines the scalar product with respect to which the polynomials Q_k have to be orthogonal. The essential part of Theorem 3.1 is given by the equality in (3.14), which, in particular, states that the moment sequence μ_n is given by (3.19).

According to the original definition (1.1) and (1.2) we have for two polynomials $A(x) = \sum_{i=0}^{d_a} a_i x^i$ and $B(x) = \sum_{j=0}^{d_b} b_j x^j$ with real coefficients

$$\langle x^n\rangle \,=\, \mu_n \,\rightarrow \qquad \langle A(x), B(x)\rangle = \sum_{i=0}^{d_a}\sum_{j=0}^{d_b} a_i b_j\,\mu_{i+j}. \tag{3.22}$$

From (1.3) we have an orthogonal basis of monic polynomials $\{Q_n(x)\}_{n\geq 0}$ generated by the three term recursion

$$Q_{n+1} \,=\, (x - c_n)Q_n - \lambda_n\,Q_{n-1}, \tag{3.23}$$

with initial conditions $Q_{-1} = 0$ and $Q_0 = 1$. The orthogonality together with the monic condition implies

$$\begin{aligned} Q_n(x) &= \frac{1}{d_{n-1}} det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix} \quad \text{with} \\ d_n &= det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}. \end{aligned} \tag{3.24}$$

This given, we have

Theorem 3.2.

$$(3.25)\qquad a)\quad \frac{d_n}{d_{n-1}} = \lambda_1\lambda_2\cdots\lambda_n, \qquad b)\quad c_n = \frac{\chi_n}{d_n} - \frac{\chi_{n-1}}{d_{n-1}}$$

with

$$(3.26)\qquad \begin{aligned} &a)\quad d_n = det\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}, \\ &b)\quad \chi_n = det\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{pmatrix}. \end{aligned}$$

Proof. Since there is only one way for a Motzkin path to reach height n from height 0 in n steps (*North-East* all the way), formula (3.20) for $k = n$ reduces to (here we use (3.2) and (3.18))

$$(3.27)\qquad \langle Q_n, Q_n\rangle = \langle Q_n, x^n\rangle = a_0a_1\cdots a_{n-1}\,|_{a_i=\lambda_{i+1}} = \lambda_1\lambda_2\cdots\lambda_n\ .$$

On the other hand, using (3.24) we get

$$(3.28)\qquad \langle Q_n\ ,\ x^n\rangle = \frac{1}{d_{n-1}}\,det\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix} = \frac{d_n}{d_{n-1}}.$$

Combining (3.27) and (3.28) we deduce that

$$(3.29)\qquad \frac{d_n}{d_{n-1}} = \lambda_1\lambda_2\cdots\lambda_n,$$

this proves (3.25) a). To prove (3.25) b) we will proceed purely combinatorially.

We start by observing that a Motzkin path can reach height n from height 0 in $n+1$ steps if and only if it takes i successive *North-East* steps followed by a single *East* step and then finish up with $n-i$ successive *North-East* steps (for $i=0,1,\ldots,n$). By summing all these possibilities (removing non-commutativity) and using the identity in (3.2) (for $k=n$ and $n\to n+1$) we obtain

$$h_{n+1,n} \;=\; \sum_{\pi\in\Pi_{0,n}^{=n+1}} w(\pi)\,\Big|_{\substack{a_i=\lambda_{i+1}\\ b_i=1}} = (c_0+c_1+\cdots+c_n)\,\lambda_1\lambda_2\cdots\lambda_n. \tag{3.30}$$

Now (3.24) and (3.26) b) give

$$\langle x^{n+1},Q_n\rangle \;=\; \frac{\chi_n}{d_{n-1}} \tag{3.31}$$

and by combining (3.30) with (3.31) we get

$$(c_0+c_1+\cdots+c_n)\,\lambda_1\lambda_2\cdots\lambda_n \;=\; \frac{\chi_n}{d_{n-1}}. \tag{3.32}$$

Finally a use of (3.25) a) yields

$$c_0+c_1+\cdots+c_n \;=\; \frac{\chi_n}{d_n},$$

proving (3.25) b) and completing our proof of Theorem 3.2.

Theorem 3.3. *The matrix $\|h_{n,k}/(\lambda_1\cdots\lambda_k)\|$ is the inverse of the matrix $\|a_{n,k}\|$ of the coefficients of the polynomials Q_n.*

Proof. Since $\{Q_n(x)\}_{n\geq 0}$ is an orthogonal basis, we have

$$\sum_{k=0}^{n}\frac{\langle y^n,Q_k(y)\rangle}{\langle Q_k,Q_k\rangle}\,Q_k(x) \;=\; x^n. \tag{3.33}$$

Expanding the polynomial $Q_k(x)$ we get

$$\sum_{k=0}^{n}\frac{\langle y^n,Q_k(y)\rangle}{\langle Q_k,Q_k\rangle}\sum_{s=0}^{k} a_{k,s}\,x^s \;=\; x^n.$$

Equating the coefficients of x^s on both sides of this identity we obtain

$$\sum_{k=0}^{n}\frac{\langle y^n,Q_k(y)\rangle}{\langle Q_k,Q_k\rangle}\,a_{k,s} \;=\; \chi(s=n).$$

In view of the definition of $h_{n,k}$ and (3.27), this identity is none other than

$$\sum_{k=0}^{n} \frac{h_{n,k}}{\lambda_1\lambda_2\cdots\lambda_k}\, a_{k,s} \;=\; \chi(s=n), \tag{3.35}$$

but that is exactly what the theorem states.

This proves the desired result the hard way. The simpler but equivalent way is to prove instead

$$\sum_{k=0}^{n} a_{n,k}\, h_{k,s} \;=\; \chi(s=n)\lambda_1\lambda_2\cdots\lambda_s.$$

This was proved purely combinatorially for $s = n$, (see the proof of (3.27)). But for $s < n$, using the definition of $h_{k,s}$ given in Theorem 3.1, we have

$$\sum_{k=0}^{n} a_{n,k}\, h_{k,s} \;=\; \Big\langle \sum_{k=0}^{n} a_{n,k}\, x^k, Q_s \Big\rangle \;=\; \langle Q_n, Q_s\rangle \;=\; 0.$$

This completes our derivation of the heaps identities we need to complete our work on Fibonacci polynomials. However, we will include in this section some additional identities since they might be of interest. Proofs of these identities can be found in the heaps lecture notes [5].

The finite continued fraction

$$J^{(n)}(x;c,\lambda) \;=\; \cfrac{1}{1-c_0x-\cfrac{\lambda_1x^2}{1-c_1x-\cfrac{\lambda_2x^2}{1-c_2x-\cfrac{\cdots}{\cdots\cfrac{\lambda_nx^2}{1-c_nx}}}}} \tag{3.36}$$

is usually referred to as the n^{th} *convergent* of $J(x;c,\lambda)$.

The next result relates $J^{(n)}(x;c,\lambda)$ to the polynomials $Q_n\,'s$. To state it we need further notation. Given a polynomial $\phi(x;c_0,c_1,\ldots;\lambda_1,\lambda_2,\ldots)$, let $S\phi$ denote the polynomial obtained by replacing in ϕ each c_i by c_{i+1} and each λ_i by λ_{i+1}.

This given we have:

Theorem 3.4.

$$J^{(n)}(x;c,\lambda) \;=\; \frac{SQ_n^*(x)}{Q_{n+1}^*(x)} \tag{3.37}$$

where

$$(3.38) \qquad 1) \quad Q_n^*(x) = x^n Q_n(1/x), \qquad\qquad 2) \quad Q_{n+1}^*(x) = x^{n+1} Q_{n+1}(1/x).$$

Theorem 3.5.

$$(3.39) \qquad J^{(n)}(x;c,\lambda) \;-\; J^{(n-1)}(x;c,\lambda) \;=\; \frac{\lambda_1\lambda_2\cdots\lambda_n\, x^{2n}}{Q_n^*(x)Q_{n+1}^*(x)}$$

An immediate corollary of (3.39) is that the rational function $J^{(n)}(x;c,\lambda)$ does converge to $J(x;c,\lambda)$ at least in the formal power series sense. In fact we see that the coefficient of x^n in the Taylor series expansion of $J^{(m)}(x;c,\lambda)$ is the same as that of $J(x;c,\lambda)$ itself as soon as $2m > n$. This shows that μ_n may be directly computed from $J^{(\frac{n-1}{2})}(x;c,\lambda)$ if n is odd and from $J^{(\frac{n}{2})}(x;c,\lambda)$ if n is even. In any case, we see that (3.36) defines it to be a polynomial in $c_0, c_1, \ldots, c_m$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$ where m is the largest integer in $n/2$. Nevertheless, the computation of μ_n by means of one of the convergents $J^{(m)}(x;c,\lambda)$, requires (in view of (3.17)) the calculation of the Taylor series of the rational inverse of Q_{m+1}^*. To avoid this step Stieltjes devised an algorithm for the recursive computation of the moments μ_n. Stieltjes' result is a proof of the identity (3.16). We proved this identity together with other identities in Theorem 3.1.

We are now ready to apply our results to two classical substitutes for the Fibonacci polynomials.

4. Heap identities for the Catalan polynomials

The simplest substitute is the basis $\{Q_n(x)\}_{n\geq 0}$ constructed by the recursion

$$(4.1) \qquad Q_{n+1} = x\,Q_n - Q_{n-1},$$

and initial conditions

$$(4.2) \qquad 1) \quad Q_{-2}(x) \;=\; 0\,, \qquad\qquad 2) \quad Q_{-1}(x) \;=\; 1.$$

Firstly, by comparison with the general case in (2.23) which is

$$(4.3) \qquad Q_{n+1} \;=\; (x-c_n)Q_n - \lambda_n\, Q_{n-1},$$

we see that in (4.1) we have

$$(4.4) \qquad 1) \quad \lambda_n = 1 \quad (\text{for all } n \geq 1), \quad \text{and} \quad 2) \quad c_n = 0 \quad (\text{for all } n \geq 0).$$

Combining the definition $h_{n.k} = \langle x_n \, , \, Q_k \rangle$ with the result in (3.14) we next obtain

$$\mu_n \;=\; \langle x^n \, , \, Q_0 \rangle \;=\; \langle x^n, 1 \rangle. \tag{4.5}$$

But the identity in (3.2) specialized for $k = 0$ gives

$$\mu_n \;=\; \sum_{\pi \in \Pi_{0,0}^{=n}} w(\pi) \Big|_{\substack{a_i=\lambda_{i+1}=1 \\ c_i=0,\, b_i=1}} \tag{4.6}$$

where $\Pi_{0,0}^{=n}$ denotes the collection of Dyck paths that go from height 0 to height 0 in n steps.

Since in (4.6) we are reduced to counting Dyck paths of length n we derive that

$$\mu_n \;=\; \begin{cases} \frac{1}{m+1}\binom{2\,m}{m} & \text{if n=2m,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

For the same reason identity a) of (3.25) forces $d_n = d_{n-1}$ for all $n \geq 1$. But the second of (3.24) gives $d_0 = \mu_0 = 1$. Thus (3.24) yields the following two identities

$$\begin{aligned} &a) \quad Q_n(x) = \det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}, \\ &b) \quad \det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix} = 1. \end{aligned} \tag{4.8}$$

Likewise an easy induction, based on b) of (3.25) yields

$$\det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0. \tag{4.9}$$

Finally we obtain that, in view of (4.7), (3.10) reduces to

(4.10)
$$L(x;a,b,c)\Big|_{\substack{a_ib_{i+1}=1\\ c_i=0}} = J(x,0,1) = \cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-\cfrac{x^2}{\cdots}}}} = \sum_{m\geq 0} x^{2m}\tfrac{1}{m+1}\tbinom{2\,m}{m}$$

Due to this identity we will call this basis $\{Q_n(x)\}_{n\geq 0}$ the "*Catalan Polynomials*".

5. Proofs of Fibonacci polynomials identities

The substitute we use here is the basis $\{Q_n(x)\}_{n\geq 0}$ which satisfies the recursion

$$Q_{n+1}(x) \;=\; x\,Q_n(x) - \lambda_n\,Q_{n-1}(x), \tag{5.1}$$

and initial conditions

$$1)\quad Q_{-2}(x) \;=\; 0\,, \qquad\qquad 2)\quad Q_{-1}(x) \;=\; 1. \tag{5.2}$$

Comparing with the general recursion

$$Q_{n+1}(x) \;=\; (x-c_n)Q_n(x) - \lambda_n\,Q_{n-1}(x), \tag{5.3}$$

we see that the only difference is that

$$c_n = 0, \qquad\qquad \big(\text{for all } n\geq 0\big). \tag{5.4}$$

Here we start by applying to the basis $\{Q_n(x)\}_{n\geq 0}$ all the identities of the general theory. In particular, the corresponding moment sequence will be given by the formula (3.20) for $k=0$ and all $c_n=0$. This is the language of all Dyck paths of length n and we obtain

$$\mu_n \;=\; \begin{cases} \displaystyle\sum_{\pi\in\Pi_{0,0}^{=n}} w(\pi)\,\Big|_{\substack{a_i=\lambda_{i+1}\\ b_i=1,c_i=0}} & \text{if n=2m,}\\[2ex] 0 & \text{otherwise.}\end{cases} \tag{5.5}$$

To make sure that the meaning of (5.5) is well understood we will illustrate below the case $n = 6$

$$\mu_6 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_2 + \lambda_1\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_1 + \lambda_1\lambda_1\lambda_1. \tag{5.6}$$

Passing to commutative variables from (3.10), (5.4) and (5.5) we get

$$J(x;0,\lambda) = \cfrac{1}{1-\cfrac{\lambda_1x^2}{1-\cfrac{\lambda_2x^2}{1-\cfrac{\lambda_3x^2}{\cdots}}}} = \sum_{m\geq 0}\mu_{2m}\,x^{2m} \ . \tag{5.7}$$

In fact, we can use (5.7) for μ_n and specialize all the identities given by (3.24), (3.25) and (3.26).

Carrying this out yields

$$\begin{aligned} Q_n(x) &= \frac{1}{d_{n-1}} det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix} \quad \text{with} \\ d_n &= det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}. \end{aligned} \tag{5.8}$$

$$a)\ \ \frac{d_n}{d_{n-1}} = \lambda_1\lambda_2\cdots\lambda_n, \qquad\qquad b)\ \ \frac{\chi_n}{d_n} = \frac{\chi_{n-1}}{d_{n-1}} \tag{5.9}$$

where we have as in (4.9)

$$\chi_n = det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{pmatrix} = 0. \tag{5.10}$$

Let us now recall that the Fibonacci basis $\{P_n(x)\}_{n\geq 0}$ satisfies the recursion

$$P_{n+1}(x) \;=\; xP_n(x) \;+\; P_{n-1}(x) \tag{5.11}$$

and initial conditions

$$1)\quad P_{-1}(x) = 0, \qquad\qquad 2)\quad P_0(x) = 1, \tag{5.12}$$

Comparing (5.11), (5.12) with (5.1), (5.2) we see that to obtain the Fibonacci basis from our present substitute basis we have to make the replacements $\lambda_i \to -1$ for all $i \geq 1$. Thus all the identities we have established from the classical theory for our substitute basis must remain valid for the Fibonacci basis after this substitution.

Now the first identity is (5.5), under this substitution (see also (5.6)) becomes

(5.13)
$$\mu_n = \begin{cases} \displaystyle\sum_{\pi\in\Pi_{0,0}^{=n}} w(\pi)\Big|_{\substack{a_i=\lambda_{i+1}\\ b_i=1,c_i=0}} & \text{if n=2m,}\\[2ex] 0 & \text{otherwise,}\end{cases} \;\to\; \nu_n = \begin{cases} \frac{(-1)^m}{m+1}\binom{2\,m}{m} & \text{if n=2m,}\\[2ex] 0 & \text{otherwise.}\end{cases}$$

This proves (I.4). Making the same substitutions on (5.7) gives

$$J(x;0,\lambda) = \cfrac{1}{1-\cfrac{\lambda_1 x^2}{1-\cfrac{\lambda_2 x^2}{1-\cfrac{\lambda_3 x^2}{\cdots}}}} = \sum_{m\geq 0}\mu_{2m}\,x^{2m} \;\to$$

$$J(x,0,-1) = 1 \;+\; \cfrac{1}{1+\cfrac{x^2}{1+\cfrac{x^2}{1+\cfrac{x^2}{\cdots}}}} = \sum_{m\geq 0}\tfrac{(-1)^m}{m+1}\tbinom{2\,m}{m}\,x^{2m}$$

This proves (I.6).

Since $\mu_0 = 1$ and (5.9) a) gives the recursion

$$d_n \;=\; (-1)^n \times d_{n-1}, \tag{5.14}$$

which is easily seen to be periodic with period 2 (after $d_0 = 1$) thus (5.9) a) becomes

$$d_n \;=\; det\begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_n \\ \nu_1 & \nu_2 & \cdots & \nu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \nu_{n-1} & \nu_n & \cdots & \nu_{2n-1} \\ \nu_n & \nu_{n+1} & \cdots & \nu_{2n} \end{pmatrix} \;=\; (-1)^{\lceil n/2 \rceil}; \tag{5.15}$$

this proves the second of (I.5). From (5.15) and (5.8) we derive that

$$P_n(x) = (-1)^{\lceil \frac{n-1}{2} \rceil}\, det\begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_n \\ \nu_1 & \nu_2 & \cdots & \nu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \nu_{n-1} & \nu_n & \cdots & \nu_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}; \tag{5.16}$$

this proves the first equality of (I.5).

Moreover, using (5.13) the identity in (5.10) becomes

$$det\begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_n \\ \nu_1 & \nu_2 & \cdots & \nu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \nu_{n-1} & \nu_n & \cdots & \nu_{2n-1} \\ \nu_{n+1} & \nu_{n+2} & \cdots & \nu_{2n+1} \end{pmatrix} \;=\; 0. \tag{5.17}$$

We terminate with an expansion result in terms of Fibonacci polynomials which can be stated as a separate

Proposition 5.1. *For any polynomial $P(x)$ of degree d we have*

$$P(x) \;=\; \sum_{k=0}^{d} (-1)^k \langle P\,, P_k\rangle\, P_k(x)\,, \tag{5.18}$$

with a non degenerate scalar product.

Proof. Since $\{P_k(x)\}_{k\geq 0}$ is a basis we can certainly have the expansion

$$P(x) \;=\; \sum_{k=0}^{d} \frac{\langle P\,, P_k\rangle}{\langle P_k\,, P_k\rangle}\, P_k(x). \tag{5.19}$$

However from (3.27) we derive that

$$\langle P_k\,, P_k\rangle \;=\; (-1)^k,$$

this proves (5.18). To show that the quadratic form is non-degenerate, the $(n+1)\times(n+1)$ relevant matrix is

$$A_n \;=\; \big\|\nu_{r+s}\big\|_{r,s=0}^{n}\,.$$

Since addition is commutative this is a symmetric matrix, thus diagonalizable. In particular its determinant gives the product of the eigenvalues. But we have seen in (5.15) that $\det(A_n) = (-1)^{\lceil n/2\rceil}$, so none of these eigenvalues can vanish. This completes our proof of the proposition.

We purposely programmed on the computer the expansion in (5.18) to obtain x^n. What came out is a rather challenging problem. For instance we got

$$\begin{aligned} x^7 &= 1 - 14P_1 + 14P_3 - 6P_5 + P_7 \qquad \text{and} \\ x^8 &= 1 + 14P_0 - 28P_2 + 20P_4 - 7P_6 + P_8\,. \end{aligned} \tag{5.20}$$

We will leave it as a challenge to prove a general formula giving these computer generated identities.

We terminate this section with a list of what is known and what may be new.

Of course it is well known that the Fibonacci polynomials satisfy the recurrence

$$P_{n+1}(x) \;=\; xP_n(x) \;+\; P_{n-1}(x)\,. \tag{5.21}$$

The generating function identity

$$\sum_{n\ge 0} t^n P_n(x) \;=\; \frac{1}{1-xt-t^2}\,, \tag{5.22}$$

(as we have already seen in the introduction) is an immediate consequence of (5.21).

The formula

$$P_n(x) \;=\; \frac{\left(\frac{x+\sqrt{x^2+4}}{2}\right)^{n+1} - \left(\frac{x-\sqrt{x^2+4}}{2}\right)^{n+1}}{\frac{x+\sqrt{x^2+4}}{2} - \frac{x-\sqrt{x^2+4}}{2}} \tag{5.23}$$

follows from (5.22) by the following identities

$$(5.24) \qquad 1 - xt - t^2 \;=\; (1-at)(1-bt) \;=\; 1-(a+b)t+abt^2,$$

thus $a+b=x$ and $ab=-1$. Solving these two identities for a and b yields

$$(5.25) \qquad 1) \quad a = \frac{x+\sqrt{x^2+4}}{2}, \qquad\qquad 2) \quad b = \frac{x-\sqrt{x^2+4}}{2}.$$

On the other hand from (5.22) and (5.24) we also have

$$(5.26) \qquad \frac{1}{1-xt-t^2} \;=\; \sum_{r\geq 0} t^r \sum_{n+m=r} a^n b^{r-n} \;=\; \sum_{r\geq 0} t^r \frac{a^{r+1}-b^{r+1}}{a-b}.$$

This proves (5.23).

Now from what Qi and Guo show in [8] we can derive that

$$(5.27) \qquad \frac{1}{n+1}\binom{2n}{n} \;=\; 4^{n+1}\frac{\int_0^1 x^n \sqrt{\frac{1-x}{x}}\,dx}{2\pi}.$$

This amazing identity proves that if we set

$$(5.28) \qquad \alpha(u) \;=\; \begin{cases} \dfrac{2}{\pi}\displaystyle\int_0^u \sqrt{\frac{1-x}{x}}dx & \text{if } u \geq 0 \\ 0 & \text{if } u < 0, \end{cases}$$

then the unit measure $d\alpha(x)$ has the density $\frac{2}{\pi}\sqrt{\frac{1-x}{x}}$ and (4.7) becomes (using the notation of section 1)

$$(5.29) \qquad \mu_n^\alpha \;=\; \begin{cases} 4^{n+1}\frac{\int_0^1 x^n \sqrt{\frac{1-x}{x}}\,dx}{2\pi} \;=\; \displaystyle\int_{-\infty}^{+\infty} x^n\,d\alpha & \text{if n=2m,} \\ 0 & \text{otherwise.} \end{cases}$$

and the right hand side of (5.13) becomes

$$(5.30) \qquad \nu_n \;=\; \begin{cases} (-1)^m 4^{n+1}\frac{\int_0^1 x^n \sqrt{\frac{1-x}{x}}dx}{2\pi} & \text{if n=2m,} \\ 0 & \text{otherwise.} \end{cases}$$

This should be a new identity. Likewise all the identities (except (I.5) [2]) related to the Fibonacci basis $\{P_n(x)\}_{n\geq 0}$ stated in the introduction, along with (5.17), should be new. The expansion result stated in Proposition 5.1 should also be new as well as the non degeneracy of the scalar product.

We should give at least an idea of what was known to the mathematicians who developed the classical theory, such as Jacobi, Rogers, and Stieltjes, to base their definition of the scalar product by means of moments. To simplify our arguments we will work in a very special setting.

Proposition 5.2. *Suppose that $f(x) \geq 0$ is a continuous function in the interval $[0,1]$ such that*

$$\int_0^1 f(x)\,dx \;=\; 1. \tag{5.31}$$

Set

$$\mu_r \;=\; \int_0^1 x^r\, f(x)\,dx, \qquad \text{(for all } r \geq 0\text{)}. \tag{5.32}$$

Let for each $n \geq 0$

$$A_n \;=\; \Big\| \mu_{i+j} \Big\|_{i,j=0}^{n}, \tag{5.33}$$

then A_n has only positive eigenvalues.

Proof. It will be sufficient to carry out our argument in the 3×3 case.

Let $\mathbf{u} = [u_1, u_2, u_3]^T$ be an eigenvector of A_3 with eigenvalue λ, then

$$A_3 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 + \mu_1 u_2 + \mu_2 u_3 \\ \mu_1 u_1 + \mu_2 u_2 + \mu_3 u_3 \\ \mu_2 u_1 + \mu_3 u_2 + \mu_4 u_3 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{5.34}$$

thus

$$[u_1, u_2, u_3]^T\, A_3 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \;=\; \lambda\,(u_1^2 + u_2^2 + u_3^2) \;=\; \lambda\, \|\mathbf{u}\|^2 \tag{5.35}$$

and

(5.36)

$$\lambda ||\mathbf{u}||^2 = \int_0^1 \Big(x^0 u_1 (x^0 u_1 + x^1 u_2 + x^2 u_3) + x^1 u_2 (x^0 u_1 + x^1 u_2 + x^2 u_3)$$

$$+ x^2u_3(x^0u_1 + x^1u_2 + x^2u_3)\Big)f(x)d\alpha$$
$$= \int_0^1 (x^0u_1 + x^1u_2 + x^2u_3)^2 f(x)d\alpha > 0.$$

This completes our proof.

Another result, shown in section 4, is that the Catalan polynomials are close to the Fibonacci polynomials. We find this fact as well as (I.4), namely the identity

(5.37)
$$\nu_n = \begin{cases} \frac{(-1)^m}{m+1}\binom{2\,m}{m} = (-1)^m\, 4^{m+1} \frac{\int_0^1 x^m \sqrt{\frac{1-x}{x}}\, dx}{2\pi} & \text{if n=2m,} \\ 0 & \text{otherwise} \end{cases} \qquad (\text{for all } n \geq 0).$$

as somewhat unexpected.

The expansion result and the non degeneracy of the scalar product of Fibonacci polynomials suggests that the classical theory can be extended to include arbitrary real values for the parameters $\{c_i\}_{i\geq 0}$ and $\{\lambda_i\}_{i\geq 1}$.

The non-vanishing of the determinants of all the Hankel matrices of the corresponding moments should remain valid even in this extended case.

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