Pure and Applied Mathematics Quarterly
Volume 20, Number 3, 1371–1431, 2024

Milnor fibre homology complexes

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Dedicated to Claudio Procesi, good friend, Italian mathematician

Abstract: Let W be a finite Coxeter group. We give an algebraic presentation of what we refer to as "the non-crossing algebra", which is associated to the hyperplane complement of W and to the cohomology of its Milnor fibre. This is used to produce simpler and more general chain (and cochain) complexes which compute the integral homology and cohomology groups of the Milnor fibre F of W. In the process we define a new, larger algebra $\widetilde{A}$, which seems to be "dual" to the Fomin-Kirillov algebra, and in low ranks is linearly isomorphic to it. There is also a mysterious connection between $\widetilde{A}$ and the Orlik-Solomon algebra, in analogy with the fact that the Fomin-Kirillov algebra contains the coinvariant algebra of W. This analysis is applied to compute the multiplicities $\langle \rho, H^k(F,\mathbb{C})\rangle_W$ and $\langle \rho, H^k(M,\mathbb{C})\rangle_W$, where M and F are respectively the hyperplane complement and Milnor fibre associated to W and ρ is a representation of W.

Keywords: Milnor fibre, noncrossing partition lattice, hyperplane arrangement.

1. Introduction

This work is an outgrowth of [Zha23], whose notation we follow, by and large. In particular, W is a finite Coxeter group, M is its corresponding complexified hyperplane complement and F is the corresponding (non-reduced) Milnor fibre, as defined in [DL16, Def. 1] or [Zha23]. Our objective is to construct tractable chain complexes which compute the homology of M (which is known for all W) and that of F (which is poorly understood, even in the case $W = \mathrm{Sym}_n$). Our approach will indicate a connection with the algebra of Fomin-Kirillov [FK99].

In the first two sections, we recall two distinct definitions of the central character in our development, the "non-crossing algebra" $\mathcal{A}$, and prove their

Received October 17, 2022.
2010 Mathematics Subject Classification: 14N20, 20F55, 05E14.

equivalence. This provides us with many properties of the algebra $\mathcal{A}$. In addition, we discuss a number of preliminaries we shall require later. We then introduce a duality theory, by defining a non-degenerate bilinear form on $\mathcal{A}$, and this is applied to study the integral cohomology of F.

Our main purpose here is to show how the algebra $\mathcal{A}$ and its relatives play a crucial role in determining the cohomology of the Milnor fibre F. For example, we give several results similar to the following (see Corollary 5.19 below). Let $\omega = \sum_{t\in T} a_t \in \mathcal{A}$, where T is the set of reflections of W; then $\omega^2 = 0$ in $\mathcal{A}$, and we prove that both left and right multiplication by ω on $\mathcal{A}$ have the same kernel and image. Moreover we have the following isomorphism of graded abelian groups:

$$H^*(F/W;\mathbb{Z})[-1] \cong \frac{\mathcal{A}\omega \cap \omega\mathcal{A}}{\omega\mathcal{A}\omega}, \tag{1.1}$$

where $[-1]$ on the left means that the $\mathbb{Z}$-grading is shifted by -1. This result could be compared with those in [DPSS99], whose ultimate purpose is to compute the left side of the equation (1.1). In principle, the stated result reduces the question to a mechanical computation in $\mathcal{A}$, although in practice, this is not an easy computation.

2. Definitions, notation and preliminaries

2.1. The noncrossing partition lattice

Let (W, S) be a finite Coxeter system of rank n with a geometric representation on the Euclidean space $V := \mathbb{R}^n$, and let $T = \bigcup_{w\in W} wSw^{-1}$ be the set of reflections of W. Denote by $\ell_T(w)$ the number of reflections in a shortest expression for w as a product of reflections, and define a partial order $\leq$ on W by stipulating that $u \leq v$ if and only if $\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$ for $u, v \in W$. Then $(W, \leq)$ is a graded poset whose unique minimal element is the identity e and whose maximal elements are those having no fixed points in $\mathbb{R}^n$, sometimes known as elliptic elements.

Let γ be any Coxeter element, i.e., product of all the simple reflections in some order.

Definition 2.1. Denote by $\mathcal{L}$ the closed interval $\mathcal{L} := [e, \gamma]$ of the poset $(W, \leq)$.

Brady and Watt proved that the closed interval $\mathcal{L}$ is a lattice, which we call the noncrossing partition (NCP) lattice [BW08]. The isomorphism type of the NCP lattice is independent of γ, as all Coxeter elements form a conjugacy class in W. Although all Coxeter elements of W are conjugate in W, we now define a specific Coxeter element in terms of the root system, which has properties we shall find useful later.

Associated with W we have a set Φ of vectors in V, which form a root system (cf. [Bou02, Ch. VI, §1]), and S determines a simple subsystem of Φ, as well as the corresponding set Φ^+ of positive roots. Write $\Pi = \{\alpha_i \mid i \in [n]\}$ for the given simple system. Without loss of generality, we may assume that W is irreducible. Then Π can be written as the disjoint union $\Pi = \Pi_1 \cup \Pi_2$, where $\Pi_1 = \{\alpha_{i_1}, \dots, \alpha_{i_l}\}$ and $\Pi_2 = \{\alpha_{i_l+1}, \dots, \alpha_{i_n}\}$ where the $\alpha_{i_k} \in \Pi_1$ are mutually orthogonal as also are the $\alpha_{i_k} \in \Pi_2$ (see [Ste59]).

The set of positive roots of Φ is in bijection with the set of reflections of W. Recall that W acts faithfully on the Euclidean space $V := \mathbb{R}^n$ whose inner product we denote by $(-,-)$. For any positive root α relative to Π, the corresponding reflection is defined by $t_\alpha(x) := x - 2\frac{(\alpha,x)}{(\alpha,\alpha)}\alpha$ for any $x \in \mathbb{R}^n$. Throughout, we use the following Coxeter element

$$\gamma = (\prod_{\alpha\in\Pi_1} t_\alpha)(\prod_{\alpha\in\Pi_2} t_\alpha),$$

unless otherwise stated. Note that the simple reflections t_α and t_β commute whenever $\alpha, \beta \in \Pi_1$ or $\alpha, \beta \in \Pi_2$.

Now we define a total order on the set of positive roots. Let h be the Coxeter number, i.e. the order of γ. Then the number of positive roots is $nh/2$. It is proved in [Ste59, Theorem 6.3] that the positive roots ρ_k of Φ relative to Π can be produced successively using the following formulae

$$\rho_k = \begin{cases} \alpha_{i_k}, & 1 \le k \le l, \\ -\gamma(\alpha_{i_k}), & l+1 \le k \le n, \\ \gamma(\rho_{k-n}), & n+1 \le k \le \frac{nh}{2}. \end{cases} \tag{2.1}$$

This yields a total order $\preceq$ on the set T of reflections

$$t_{\rho_1} \prec t_{\rho_2} \prec \cdots \prec t_{\rho_{nh/2}}. \tag{2.2}$$

The total order $\preceq$ on T gives rise to an EL-labelling (see [ABW07]) of $\mathcal{L}$. Denote by $\mathcal{E}(\mathcal{L})$ the set of covering relations $u \lessdot v$ of $\mathcal{L}$, that is, relations where there is no third element between u and v. Then we have a natural edge labelling

$$\lambda : \mathcal{E}(\mathcal{L}) \to T, \quad u \lessdot v \mapsto u^{-1}v.$$

Let $\mathbf{c} : x = w_0 < w_1 < \cdots < w_k = y$ be a maximal chain of any closed interval $[x, y]$ of $\mathcal{L}$. We may identify $\mathbf{c}$ with its labelling sequence $\lambda(\mathbf{c}) := (w_0^{-1}w_1, \ldots, w_{k-1}^{-1}w_k)$, where $w_{i-1}^{-1}w_i \in T$ for $1 \le i \le k$. It has been proved that λ is an EL-labelling [ABW07, Bjö80], which means that for every interval $[x, y]$ of $\mathcal{L}$

1. there is a unique increasing maximal chain in $[x, y]$, and
2. this chain is lexicographically smallest among all maximal chains in $[x, y]$.

As $\mathcal{L}$ has an EL-labelling, it is Cohen-Macaulay [Bjö80, Theorem 2.3], i.e. for any $u < v$ of $\mathcal{L}$ we have

$$\widetilde{H}_i(u, v) = 0, \quad \forall i \neq \ell_T(v) - \ell_T(u) - 2,$$

where $\widetilde{H}$ denotes reduced homology of the order complex of (u, v).

If W is a finite Coxeter group, then $\mathcal{L}$ is a direct product of the NCP lattices $\mathcal{L}(W_i)$ over the irreducible components W_i of W. It is a result of [Bjö80] that the EL-labelling is preserved under the direct product. Moreover, any closed interval $[e, w]$ of $\mathcal{L}$ has an EL-labelling given by the natural labelling λ restricted to $[e, w]$.

For any $w \in \mathcal{L}$, denote

$$\mathrm{Rex}_T(w) := \{(t_1, t_2, \ldots, t_k) \,|\, w = t_1 t_2 \cdots t_k \text{ is } T\text{-reduced}\}.$$

With respect to the total order (2.2) on T, we define

$$\mathcal{D}_w := \{(t_1, \ldots, t_k) \in \mathrm{Rex}_T(w) \mid t_1 \succ t_2 \succ \cdots \succ t_k\}. \tag{2.3}$$

In words, $\mathcal{D}_w$ is the set of decreasing labelling sequences for the maximal chains $e < t_1 < t_1t_2 < \cdots < t_1t_2\cdots t_k = w$ of $[e, w]$. Note that any interval $[u, v]$ of $\mathcal{L}$ is isomorphic to $[e, u^{-1}v]$ as posets. The following result can be found in [Zha23, Proposition 2.4].

Proposition 2.2. *For any $w \in \mathcal{L}$ with $1 \le \ell_T(w) = k \le n$, we have*

$$\mathrm{rank}\, \widetilde{H}_{k-2}(e, w) = (-1)^k \mu(w) = |\mathcal{D}_w|,$$

where μ is the Möbius function of $\mathcal{L}$.

Any interval $[e, w]$ of $\mathcal{L}$ has the following important interpretation, due to Bessis [Bes03, Lemma 1.4.3] (see also [Arm09, Proposition 2.6.11]).

Proposition 2.3. *Let $\gamma \in W$ be a Coxeter element and $\mathrm{NC}(W, \gamma)$ the non-crossing partition lattice relative to γ. For any $w \le \gamma$, the interval $[e, w]$ of $\mathrm{NC}(W, \gamma)$ is isomorphic to $\mathrm{NC}(W', w)$ for some parabolic subgroup W' of W.*

2.2. The algebra $\mathcal{A}$

We now give an algebraic definition of the noncrossing algebra in terms of generators and relations.

Definition 2.4. Let $\mathcal{L} := [e, \gamma]$ be the noncrossing partition lattice associated to a finite Coxeter group W and a Coxeter element $\gamma \in W$. We define the noncrossing algebra $\mathcal{A} = \mathcal{A}(W, \gamma)$ to be the graded algebra over $\mathbb{Z}$ generated by homogeneous elements $a_t, t \in T$ of degree 1, subject to the following quadratic relations:

1. $a_t^2 = a_{t_1} a_{t_2} = 0$ for any $t \in T$ and $t_1, t_2 \in T$ with $t_1 t_2 \not\leq \gamma$;
2. $\sum_{(t_1,t_2)\in \mathrm{Rex}_T(w)} a_{t_1} a_{t_2} = 0$ for any $w \in \mathcal{L}$ with $\ell_T(w) = 2$.

Different choices of the Coxeter element produce isomorphic noncrossing algebras, as the Coxeter elements are all conjugate to each other and absolute length is invariant under the conjugation. More precisely, if $\gamma' = w\gamma w^{-1}$ for some $w \in W$, the map $a_t \mapsto a_{wtw^{-1}}$ extends to an algebra isomorphism between $\mathcal{A}(W, \gamma)$ and $\mathcal{A}(W, \gamma')$.

Example 2.5. (Dihedral group) Let $I_2(m)$ be the diheral group defined by

$$I_2(m) := \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_2 s_1)^m = 1 \rangle, \quad m \geq 3.$$

Let $\gamma = s_1 s_2$ be a Coxeter element and let $t_i = s_1(s_2 s_1)^{i-1}$ be the reflections, $1 \leq i \leq m$. Then γ has m reduced reflection factorisations $\gamma = t_1 t_m = t_2 t_1 = \cdots = t_m t_{m-1}$. The noncrossing algebra $\mathcal{A}(I_2(m), \gamma)$ is generated by $a_{t_i}, 1 \leq i \leq m$ subject to the following relations

$$\begin{aligned} a_{t_1} a_{t_i} &= 0, \quad 1 \leq i \leq m-1, \\ a_{t_i} a_{t_j} &= 0, \quad 2 \leq i \leq m, 1 \leq j \neq i-1 \leq m, \\ a_{t_1} a_{t_m} + a_{t_2} a_{t_1} &+ \cdots + a_{t_m} a_{t_{m-1}} = 0. \end{aligned}$$

Because it is relevant for the proof of Proposition 3.4, we set out how the above constructions apply to this case. Note first that, using the relation $\gamma t_i \gamma^{-1} = t_{i+2}$, where the index is taken modulo m, the construction (2.2) leads to the following ordering on the reflections t_i:

$$t_1 (= s_1) \prec t_2 \prec \cdots \prec t_m (= s_2).$$

It follows, taking into account the reduced reflection factorisations of γ given above, that the unique increasing factorisation is $\gamma = t_1 t_m = s_1 s_2$, and all other factorisations are decreasing.

Example 2.6. (Type A_n) Let $W = \mathrm{Sym}_{n+1}$ be the symmetric group generated by the elementary transpositions $(i, i+1), 1 \leq i \leq n$. Let $\gamma = (1, 2, \ldots, n+1)$ be a Coxeter element. We write $a_{ij} := a_{(i,j)}$. Then $\mathcal{A}(W, \gamma)$ of type A is the graded $\mathbb{Z}$-algebra generated by $a_{ij}, 1 \leq i < j \leq n+1$ subject to the following relations:

$$\begin{aligned}
a_{ij}^2 &= 0, && 1 \leq i < j \leq n+1,\\
a_{ik}a_{jl} &= 0, && 1 \leq i < j < k < l \leq n+1,\\
a_{ij}a_{ik} = a_{jk}a_{ij} &= a_{ik}a_{jk} = 0, && 1 \leq i < j < k \leq n+1,\\
a_{ij}a_{kl} + a_{kl}a_{ij} &= a_{il}a_{jk} + a_{jk}a_{il} = 0, && 1 \leq i < j < k < l \leq n+1,\\
a_{ij}a_{jk} + a_{jk}a_{ik} &+ a_{ik}a_{ij} = 0, && 1 \leq i < j < k \leq n+1.
\end{aligned}$$

2.3. The algebra $\mathcal{B}$

We next give what later will turn out to be a combinatorial definition of the noncrossing algebra above, as introduced in [Zha23].

For each $k = 0, \ldots, n$, let $\mathcal{L}_k := \{w \in \mathcal{L} \mid \ell_T(w) = k\}$. Let $C_{k-1}(w)$ be the abelian group with basis all sequences of $\mathrm{Rex}_T(w)$, $w \in \mathcal{L}_k$. Let B_k denote the k-string braid group, with standard generators $\sigma_i, 1 \leq i \leq k-1$ subject to the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq k-2$. The Hurwitz action of B_k on $C_{k-1}(w)$ is defined by

$$\sigma_i.(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, t_{i+2}, \ldots, t_k) := (t_1, \ldots, t_{i-1}, t_{i+1}, t_i^{t_{i+1}}, t_{i+2}, \ldots, t_k)$$

for $1 \leq i \leq k-1$.

Let Sym_k be the symmetric group on k letters with standard generators $s_i = (i, i+1), 1 \leq i \leq k-1$. There is a set-theoretic lifting map:

$$\varphi : \mathrm{Sym}_k \to B_k, \quad \pi = s_{i_1} \cdots s_{i_p} \mapsto \underline{\pi} := \sigma_{i_1} \cdots \sigma_{i_p},$$

where $\pi = s_{i_1} \cdots s_{i_p} \in \mathrm{Sym}_k$ is a reduced expression in the standard generators. This is independent of the choice of reduced expression of π. For any $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathrm{Rex}_T(w)$, we define the following $\mathbb{Z}$-linear map using the above Hurwitz action:

$$\beta : C_{k-1}(w) \to C_{k-1}(w), \quad \mathbf{t} \mapsto \beta_{\mathbf{t}} := \sum_{\pi \in \mathrm{Sym}_k} \mathrm{sgn}(\pi)\, \underline{\pi}.(t_1, t_2, \ldots, t_k),$$

where sgn is the usual sign character of Sym_k. The element $\beta_{\mathbf{t}}$ is viewed as an alternating sum of maximal chains of the interval $(e, w]$, with each sequence

$(t_1, t_2, \ldots, t_k)$ in the sum identified with the following chain

$$t_1 < t_1 t_2 < \cdots < t_1 t_2 \cdots t_k = w.$$

Definition 2.7. For each $w \in \mathcal{L}_k$, we define $\mathcal{B}_w$ to be the abelian group spanned by the elements $\beta_{\mathbf{t}}$ of $\mathrm{Rex}_T(w)$, that is,

$$\mathcal{B}_w := \mathrm{Im}\,\beta = \sum_{\mathbf{t} \in \mathrm{Rex}_T(w)} \mathbb{Z}\beta_{\mathbf{t}} \subseteq C_{k-1}(w).$$

In particular $\mathcal{B}_e := \mathbb{Z}$. Further, we write $\mathcal{B} := \bigoplus_{w \in \mathcal{L}} \mathcal{B}_w$.

We point out a close connection between $\mathcal{B}$ and the homology of $\mathcal{L}$. For any $w \in \mathcal{L}_k$, let $\widetilde{H}_{k-2}(e, w)$ be the top reduced homology group of the open interval (e, w). Define C_{k-1} to be the abelian group freely spanned by the basis $\bigcup_{w \in \mathcal{L}_k} \mathrm{Rex}_T(w)$. Then $C_{k-1} = \bigoplus_{w \in \mathcal{L}_k} C_{k-1}(w)$. For $k \geq 2$, define the linear truncation d_{k-1} by

$$d_{k-1} : C_{k-1} \to C_{k-2}, \quad (t_1, t_2, \ldots, t_{k-1}, t_k) \mapsto (t_1, t_2, \ldots, t_{k-1}), \tag{2.4}$$

and $d_0(t) = 1, \forall t \in T$. We will write $d = d_k$ when there is no danger of confusion. As $\mathcal{B}_w \subseteq C_{k-1}(w)$, we may restrict d to $\mathcal{B}_w$ and define

$$z_{\mathbf{t}} := d(\beta_{\mathbf{t}}), \quad \forall \mathbf{t} \in \mathrm{Rex}_T(w).$$

Proposition 2.8. *[Zha23, Proposition 3.4] Let $w \in \mathcal{L}_k$ with $k \geq 1$. Then for any $\mathbf{t} \in \mathrm{Rex}_T(w)$, we have*

$$z_{\mathbf{t}} = \sum_{i=1}^{k} (-1)^{k-i} \beta_{\mathbf{t}(\hat{i})} \in \widetilde{H}_{k-2}(e, w),$$

where $\mathbf{t}(\hat{i}) := (t_1, \ldots, \hat{t}_i, \ldots, t_k)$ is obtained by removing the i-th entry of $\mathbf{t}$.

Theorem 2.9. *[Zha23, Theorem 4.5] For any $w \in \mathcal{L}$, let $\mathcal{B}_w$ and $\mathcal{D}_w$ be as defined in Definition 2.7 and (2.3), respectively.*

1. *The elements $\beta_{\mathbf{t}}, \mathbf{t} \in \mathcal{D}_w$ constitute a $\mathbb{Z}$-basis for $\mathcal{B}_w$;*
2. *The elements $z_{\mathbf{t}}, \mathbf{t} \in \mathcal{D}_w$ are a $\mathbb{Z}$-basis for $\widetilde{H}_{k-2}(e, w)$, where $k = \ell_T(w) \geq 1$;*
3. *The $\mathbb{Z}$-linear map*

$$d : \mathcal{B}_w \to \widetilde{H}_{k-2}(e, w), \quad \beta_{\mathbf{t}} \mapsto z_{\mathbf{t}},\ \mathbf{t} \in \mathrm{Rex}_T(w)$$

is an isomorphism of free abelian groups.

We now define a multiplicative structure which makes $\mathcal{B}$ into a finite dimensional algebra. For any $\beta_{\mathbf{t}} \in \mathcal{B}_w$ and $\beta_t, t \in T$, define

$$\beta_{\mathbf{t}}\beta_t := \begin{cases} \beta_{(\mathbf{t},t)}, & \text{if } wt \leq \gamma \text{ and } \ell_T(wt) > \ell_T(w), \\ 0, & \text{otherwise,} \end{cases}$$

where $(\mathbf{t}, t)$ denotes the concatenation of $\mathbf{t}$ and t. This multiplication is associative. Clearly, we have $\beta_{\mathbf{t}} = \beta_{t_1}\beta_{t_2}\cdots\beta_{t_k}$ for any T-reduced expression $w = t_1t_2\cdots t_k \in \mathcal{L}$. Therefore, $\mathcal{B}$ is a finite-dimensional $\mathbb{Z}$-graded algebra generated by homogeneous elements $\beta_t, t \in T$ of degree 1.

Proposition 2.10. *[Zha23, Proposition 5.8] We have the following quadratic relations in $\mathcal{B}$:*

1. *$\beta_t^2 = \beta_{t_1}\beta_{t_2} = 0$ for all $t \in T$ and $t_1, t_2 \in T$ with $t_1t_2 \not\leq \gamma$.*
2. *For any $w \in \mathcal{L}_2$, we have*

$$\sum_{(t_1,t_2)\in \mathrm{Rex}_T(w)} \beta_{t_1}\beta_{t_2} = 0.$$

3. An isomorphism between $\mathcal{A}$ and $\mathcal{B}$

3.1. The isomorphism

The main theorem of this section is the following. We are grateful to the referee for pointing out a gap in our original proof.

Theorem 3.1. *Let W be a finite Coxeter group and let T be the set of reflections of W. The assignment $a_t \mapsto \beta_t$ for all $t \in T$ extends to a graded algebra isomorphism $\mathcal{A} \cong \mathcal{B}$.*

The remainder of this section is devoted to proving Theorem 3.1.

We begin with a sketch of the proof. By Proposition 2.10, the assignment $a_t \mapsto \beta_t$ preserves the quadratic relations of $\mathcal{A}$. Since $\mathcal{B}$ is a finite-dimensional algebra generated by β_t, we obtain a surjective algebra homomorphism

$$\phi : \mathcal{A} \to \mathcal{B}, \quad a_t \mapsto \beta_t, \quad t \in T \tag{3.1}$$

from $\mathcal{A}$ to $\mathcal{B}$. To see that this is indeed an isomorphism, we will prove that $\mathcal{A}$ is finite-dimensional and then show that $\operatorname{rank}\mathcal{A} = \operatorname{rank}\mathcal{B}$. It will follow that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

To show $\mathcal{A}$ is finite dimensional, it is crucial to find necessary and sufficient conditions to ensure that $a_{t_1}a_{t_2}\cdots a_{t_k} = 0$. We need the following key lemma.

Lemma 3.2. *(Vanishing property) Let $t_i \in T$ for $1 \leq i \leq k$.*

1. *We have $a_{t_1}a_{t_2}\cdots a_{t_k} = 0$ if $t_1t_2\cdots t_k$ is not T-reduced.*
2. *Let $t_1t_2\cdots t_k$ be a T-reduced expression. Then*

$$a_{t_1}a_{t_2}\cdots a_{t_k} = 0, \quad \textit{if } t_1t_2\cdots t_k \not\leq \gamma.$$

The proof of this lemma is postponed to Section 3.2.

Lemma 3.3. *The element $a_{t_1}a_{t_2}\cdots a_{t_k} \neq 0$ if and only if $t_1t_2\cdots t_k$ is T-reduced and $t_1t_2\cdots t_k \leq \gamma$.*

Proof. The "only if" part is evident from Lemma 3.2. For the "if" part, note that

$$\phi(a_{t_1}a_{t_2}\cdots a_{t_k}) = \beta_{t_1}\beta_{t_2}\cdots\beta_{t_k} = \beta_{\mathbf{t}},$$

where $\mathbf{t} = (t_1, \ldots, t_k)$ is the labelling sequence of the chain $e < t_1 < t_1t_2 < \cdots < t_1t_2\cdots t_k$ of $\mathcal{L}$. By construction $\beta_{\mathbf{t}} \neq 0$ and hence $a_{t_1}a_{t_2}\cdots a_{t_k} \neq 0$. □

Recall that the algebra $\mathcal{A}$ has a natural $\mathbb{Z}$-grading with $\deg(a_t) = 1$ for any $t \in T$. Let $\mathcal{A}_k$ denote the k-th graded component of $\mathcal{A}$. For any $w \in \mathcal{L}_k$, let $\mathcal{A}_w$ be the abelian subgroup of $\mathcal{A}_k$ given by

$$\mathcal{A}_w := \mathrm{Span}_{\mathbb{Z}}\{a_{\mathbf{t}} := a_{t_1}a_{t_2}\cdots a_{t_k} \mid \mathbf{t} \in \mathrm{Rex}_T(w)\}$$

By convention we set $\mathcal{A}_e := \mathbb{Z}$. It follows from Lemma 3.3 that

$$\mathcal{A}_k = \sum_{w\in\mathcal{L}_k} \mathcal{A}_w, \quad 0 \leq k \leq n.$$

Proposition 3.4. *For any $w \in \mathcal{L}$, let $\mathcal{D}_w$ be as in (2.3). Then the elements $a_{\mathbf{t}}, \mathbf{t} \in \mathcal{D}_w$ form a $\mathbb{Z}$-basis for $\mathcal{A}_w$.*

Proof. Since $\phi(a_{\mathbf{t}}) = \beta_{\mathbf{t}}$, the set $\{a_{\mathbf{t}} \mid \mathbf{t} \in \mathcal{D}_w\}$ is $\mathbb{Z}$-linearly independent by Theorem 2.9. By Proposition 2.3, without loss of generality, we may assume $w = \gamma$. It remains to show that every element of $\mathcal{A}_\gamma$ is a $\mathbb{Z}$-linear combination of the decreasing elements $a_{\mathbf{t}}, \mathbf{t} \in \mathcal{D}_\gamma$.

We use induction on $\ell_T(\gamma) = n$. If $n = 1$, then there is nothing to prove. Note that $\mathcal{A}_{n-1} = \sum_{w\in\mathcal{L}_{n-1}} \mathcal{A}_w$. For $n > 1$, we may by induction assume that any $a_{t_1}a_{t_2}\cdots a_{t_{n-1}} \in \mathcal{A}_{n-1}$ can be expressed as a $\mathbb{Z}$-linear combination of the elements $a_{\mathbf{t}}$ for $\mathbf{t} \in \bigcup_{w\in\mathcal{L}_{n-1}} \mathcal{D}_w$.

Consider the following filtration of $\mathcal{A}_n = \mathcal{A}_\gamma$. Recall that T is totally ordered as in (2.2). For each reflection $t_{\rho_i} \in T$, define to be V_{ρ_i} be the abelian subgroup of $\mathcal{A}_\gamma$ spanned by the elements $a_{t_1}\cdots a_{t_{n-1}}a_{t_{\rho_i}}$ for all $(t_1,\ldots,t_{n-1})\in \mathrm{Rex}_T(\gamma t_{\rho_i})$. Then we have a filtration

$$0\subseteq V_{\rho_1}\subseteq\cdots\subseteq\sum_{i=1}^{s}V_{\rho_i}\subseteq\cdots\subseteq\sum_{i=1}^{hn/2}V_{\rho_i}=\mathcal{A}_\gamma.$$

We use induction on s to show that for each s with $1\le s\le \frac{hn}{2}$, $\sum_{i=1}^{s}V_{\rho_i}$ is spanned by the elements $a_{\mathbf{t}},\mathbf{t}\in\mathcal{D}_\gamma$. If $s=1$, then by the minimality of t_{ρ_1} in T and the induction hypothesis on n, any element in V_{ρ_1} can be written as a $\mathbb{Z}$-linear combination of decreasing elements $a_{t_1}\cdots a_{t_{n-1}}a_{t_{\rho_1}}$ for all $(t_1,\ldots,t_{n-1},t_{\rho_1})\in\mathcal{D}_\gamma$.

Now assume $s>1$. For any $a\in V_{\rho_s}$, by the induction hypothesis on n there exist $\lambda_{\mathbf{t}}\in\mathbb{Z}$ such that

$$a=\sum_{\mathbf{t}\in\mathcal{D}_{\gamma t_{\rho_s}}}\lambda_{\mathbf{t}}a_{\mathbf{t}}a_{t_{\rho_s}}=\sum_{\mathbf{t}\in\mathcal{D}^{\succ}_{\gamma t_{\rho_s}}}\lambda_{\mathbf{t}}a_{\mathbf{t}}a_{t_{\rho_s}}+\sum_{\mathbf{t}\in\mathcal{D}^{\prec}_{\gamma t_{\rho_s}}}\lambda_{\mathbf{t}}a_{\mathbf{t}}a_{t_{\rho_s}},$$

where

$$\begin{aligned}\mathcal{D}^{\succ}_{\gamma t_{\rho_s}}&=\{(t_1,\ldots,t_{n-1})\in\mathcal{D}_{\gamma t_{\rho_s}}\mid t_{n-1}\succ t_{\rho_s}\},\\ \mathcal{D}^{\prec}_{\gamma t_{\rho_s}}&=\{(t_1,\ldots,t_{n-1})\in\mathcal{D}_{\gamma t_{\rho_s}}\mid t_{n-1}\prec t_{\rho_s}\}.\end{aligned}$$

Note that $a_{\mathbf{t}}a_{t_{\rho_s}}$ is a decreasing element for any $\mathbf{t}\in\mathcal{D}^{\succ}_{\gamma t_{\rho_s}}$. We claim that

$$a_{\mathbf{t}}a_{t_{\rho_s}}\in V_{\rho_1}+V_{\rho_2}+\cdots+V_{\rho_{s-1}},\quad \forall\mathbf{t}\in\mathcal{D}^{\prec}_{\gamma t_{\rho_s}}. \tag{3.2}$$

Given (3.2), by the induction hypothesis on s, any element $a\in V_{\rho_s}$ is a $\mathbb{Z}$-linear combination of decreasing elements $a_{\mathbf{t}},\mathbf{t}\in\mathcal{D}_\gamma$. Therefore, $\sum_{i=1}^{s}V_{\rho_i}$ is spanned by the elements $a_{\mathbf{t}},\mathbf{t}\in\mathcal{D}_\gamma$ for any positive integer s. In particular, $\mathcal{A}_\gamma=\sum_{i=1}^{hn/2}V_{\rho_i}$ is spanned by the decreasing elements $a_{\mathbf{t}},\mathbf{t}\in\mathcal{D}_\gamma$.

It remains to prove (3.2). Take any $\mathbf{t}=(t_1,\ldots,t_{n-2},t_{n-1})\in\mathcal{D}^{\prec}_{\gamma t_{\rho_s}}$ with $t_{n-1}\prec t_{\rho_s}$. Let $u=t_{n-1}t_{\rho_s}$. Then there exists a poset isomorphism between $[e,u]$ and $[t_1\cdots t_{n-2},\gamma]$ which sends $x\in[e,u]$ to $t_1\cdots t_{n-2}x\in[t_1\cdots t_{n-2},\gamma]$. In particular, this isomorphism preserves the EL-labelling.

By Proposition 2.3, the interval $[e,u]$ is the noncrossing partition lattice of a dihedral group $I_2(m)$ for some integer $m\ge 2$. Let $t_{\tau_1}\prec t_{\tau_2}\prec\cdots\prec t_{\tau_m}$ be the reflections of $[e,u]$ with the total order inherited from (2.2). By the discussion in Example 2.5 the unique increasing maximal chain of $[e,u]$ is labelled by (t_{n-1},t_{ρ_s}) and we have $t_{\tau_1}=t_{n-1}$ and $t_{\tau_m}=t_{\rho_s}$. The other maximal chains in $[e,u]$ are all decreasing. Further, using the defining relation of $\mathcal{A}$, we have

$$a_{t_{n-1}}a_{t_{\rho_s}} = a_{t_{\tau_1}}a_{t_{\tau_m}} = -\sum_{t_{\tau_i}\succ t_{\tau_j}} a_{t_{\tau_i}}a_{t_{\tau_j}}, \tag{3.3}$$

where the sum is over all decreasing labelling sequences of maximal chains in $[e,u]$. For any pair $t_{\tau_i}\succ t_{\tau_j}$ we have

$$t_{\tau_j} \preceq t_{\tau_{m-1}} \preceq t_{\rho_{s-1}} \prec t_{\rho_s} = t_{\tau_m}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$a_{\mathbf{t}}a_{t_{\rho_s}} = a_{t_1}\cdots a_{t_{n-2}}(a_{t_{n-1}}a_{t_{\rho_s}}) = -\sum_{t_{\tau_i}\succ t_{\tau_j}} a_{t_1}\cdots a_{t_{n-2}}a_{t_{\tau_i}}a_{t_{\tau_j}} \in \sum_{i=1}^{s-1} V_{\rho_i}.$$

The statement (3.2) follows, and the proof of Proposition 3.4 is complete. $\square$

Lemma 3.5. *We have the following.*

1. $\mathcal{A}_k = 0$ *for* $k > n = \ell_T(\gamma)$.
2. $\mathcal{A}_k = \bigoplus_{w\in\mathcal{L}_k}\mathcal{A}_w$ *for* $0 \le k \le n = \ell_T(\gamma)$.

Proof. As the maximal rank of the poset $(W,\le)$ is n, any expression $t_1\cdots t_k$ with $k > n$ is not T-reduced. It follows from Lemma 3.3 that $\mathcal{A}_k = 0$ for $k > n$. This proves part (1).

For part (2), recall that $\mathcal{A}_k = \sum_{w\in\mathcal{L}_k}\mathcal{A}_w$ for $0\le k\le n$. For any $u\in\mathcal{L}_k$, take an arbitrary element $x\in\mathcal{A}_u\cap\sum_{u\ne w\in\mathcal{L}_k}\mathcal{A}_w$. Since ϕ restricts to a surjective homomorphism $\phi_w : \mathcal{A}_w \to \mathcal{B}_w$ for any $w\in\mathcal{L}$, we have $\phi(x)\in\mathcal{B}_u\cap\sum_{u\ne w\in\mathcal{L}_k}\mathcal{B}_w = 0$. In particular, $x\in\operatorname{Ker}\phi_u$. On the other hand, for any $w\in\mathcal{L}$ by Proposition 3.4 the subgroup $\mathcal{A}_w$ has a $\mathbb{Z}$-basis $\{a_{\mathbf{t}} \mid \mathbf{t}\in\mathcal{D}_w\}$. Similarly, recall from Theorem 2.9 that $\mathcal{B}_w$ has a $\mathbb{Z}$-basis $\{\beta_{\mathbf{t}} \mid \mathbf{t}\in\mathcal{D}_w\}$. It follows that $\mathcal{A}_w$ is isomorphic to $\mathcal{B}_w$, given by $\phi_w(a_{\mathbf{t}}) = \beta_{\mathbf{t}}$ for any $\mathbf{t}\in\mathcal{D}_w$. This forces $x=0$ and hence $\mathcal{A}_u\cap\sum_{u\ne w\in\mathcal{L}_k}\mathcal{A}_w = 0$ for any $u\in\mathcal{L}_k$. Therefore, the sum $\mathcal{A}_k = \sum_{w\in\mathcal{L}_k}\mathcal{A}_w$ is direct. $\square$

The following is an immediate consequence of Proposition 3.4 and Lemma 3.5.

Corollary 3.6. *The set* $\{a_{\mathbf{t}} \mid \mathbf{t}\in\bigcup_{w\in\mathcal{L}}\mathcal{D}_w,\ \mathbf{t}$ *decreasing*$\}$ *is a* $(\mathbb{Z})$*-basis of* $\mathcal{A}$.

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. The map $\phi : \mathcal{A} \to \mathcal{B}$ defined by $\phi(a_{\mathbf{t}}) = \beta_{\mathbf{t}}$ extends to a surjective graded algebra homomorphism. By the algebra $\mathcal{A}$ has a basis consisting of decreasing elements $a_{\mathbf{t}}$ with $\mathbf{t} \in \bigcup_{w\in\mathcal{L}} \mathcal{D}_w$. It follows from Theorem 2.9 that ϕ is injective and hence $\mathcal{A}$ is isomorphic to $\mathcal{B}$. $\square$

Proposition 3.7. *The algebra $\mathcal{A}$ enjoys the following properties.*

1. *Let $\mathcal{L}$ and $\mathcal{L}'$ be two noncrossing partition lattices. Then as free abelian groups,*
$$\mathcal{A}(\mathcal{L} \times \mathcal{L}') \cong \mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}').$$
2. *Let $\mathcal{L}_w = [e, w]$ be a closed interval in $\mathcal{L}$. Then the inclusion $i : \mathcal{L}_w \hookrightarrow \mathcal{L}$ of posets induces an injective homomorphism $i_{\mathcal{A}} : \mathcal{A}(\mathcal{L}_w) \to \mathcal{A}(\mathcal{L})$ of algebras. In particular,*
$$\mathcal{A}(\mathcal{L}_w)_u = \mathcal{A}(\mathcal{L})_u, \quad \forall u \leq w.$$

Proof. For part (a), let T and T' be the set of reflections of $\mathcal{L}$ and $\mathcal{L}'$ respectively. The algebra $\mathcal{A}(\mathcal{L} \times \mathcal{L}')$ is generated by $a_t, t \in T \cup T'$. By the defining relation we have $a_t a_{t'} = -a_{t'} a_t$ for any $t \in T$ and $t' \in T'$. We have two natural embeddings $i_1 : \mathcal{A}(\mathcal{L}) \to \mathcal{A}(\mathcal{L} \times \mathcal{L}')$ and $i_2 : \mathcal{A}(\mathcal{L}') \to \mathcal{A}(\mathcal{L} \times \mathcal{L}')$, inducing the algebra isomorphism
$$f: \ \mathcal{A}(\mathcal{L}) \otimes \mathcal{A}(\mathcal{L}') \to \mathcal{A}(\mathcal{L} \times \mathcal{L}'),$$
such that $f(a_t \otimes a_{t'}) = i_1(a_t) i_2(a_{t'})$ and $i_1(a_t) i_2(a_{t'}) = -i_2(a_{t'}) i_1(a_t)$ for any $t \in T$ and $t' \in T$.

We turn to the proof of part (b). The set T_w of reflections in $\mathcal{L}_w$ inherits the total order from the totally ordered set T of reflections in $\mathcal{L}$. Since $\mathcal{L}_w \subset \mathcal{L}$, the induced map $i_{\mathcal{A}} : \mathcal{A}(\mathcal{L}_w) \to \mathcal{A}(\mathcal{L})$ defined by $a_t \to a_t, t \in T_w$ preserves the defining relations and hence is an algebraic homomorphism. Moreover, the induced map $i_{\mathcal{A}}$ preserves the decreasing basis elements and thus $i_{\mathcal{A}}$ is injective. $\square$

3.2. The vanishing lemma

In this subsection we prove the vanishing property stated in Lemma 3.2.

3.2.1. *T*-reduced expressions and root systems Consider the following two sets:

(3.5)
$$\mathcal{T}_1 := \{(t_1, t_2, \ldots, t_k) \mid k \in \mathbb{N} \text{ and } t_1 t_2 \cdots t_k \text{ is not } T\text{-reduced}\},$$
$$\mathcal{T}_2 := \{(t_1, t_2, \ldots, t_k) \mid k \in \mathbb{N} \text{ and } t_1 t_2 \cdots t_k \text{ is } T\text{-reduced and } t_1 t_2 \cdots t_k \not\leq \gamma\}.$$

Then Lemma 3.2 can be restated as:

$$a_{\mathbf{t}} = 0, \quad \text{for any } \mathbf{t} \in \mathcal{T}_1 \cup \mathcal{T}_2. \tag{3.6}$$

To prove this, we provide a geometric characterisation for $\mathcal{T}_1 \cup \mathcal{T}_2$ in terms of the root system.

Let $\varrho : W \to \mathrm{GL}(V)$ be the geometric representation of W with $V = \mathbb{R}^n$. Denote by Φ^+ the set of positive roots of W, as determined by S (see the remarks preceding (2.1)). Define

$$\mathrm{Fix}(w) := \mathrm{Ker}\,(\varrho(w) - \mathrm{Id}) \subseteq V$$

to be the vector subspace fixed by $w \in W$. By [Car72, Lemma 2], we have

$$\ell_T(w) = \operatorname{codim} \mathrm{Fix}(w) = n - \dim \mathrm{Fix}(w), \quad \forall w \in W. \tag{3.7}$$

Since the Coxeter element γ fixes no vector in V, the linear map $\gamma - 1$ is an automorphism of V. We define the linear map

$$\vartheta := (\gamma - 1)^{-1} : V \to V.$$

This map satisfies the following properties which come from the proof of [Car72, Lemma 2]; see also [BW08, Corollary 4.2].

Lemma 3.8. *Let $\rho \in \Phi^+$ be a positive root and let ϑ be as defined above.*

1. $(\vartheta(\rho), \rho) = -\frac{1}{2}(\rho, \rho)$*;*
2. $\vartheta(\rho) \in \mathrm{Fix}(t_\rho \gamma)$*.*

Proof. We have $(\gamma - 1)(\vartheta(\rho)) = \rho$, which implies that $\gamma(\vartheta(\rho)) = \vartheta(\rho) + \rho$. Since Coxeter group action preserves the inner product of V, we have $(\gamma(\vartheta(\rho)), \gamma(\vartheta(\rho))) = (\vartheta(\rho), \vartheta(\rho))$ and hence $(\vartheta(\rho), \rho) = -\frac{1}{2}(\rho, \rho)$. It follows that $t_\rho(\vartheta(\rho)) = \vartheta(\rho) + \rho$ and hence $\gamma(\vartheta(\rho)) = t_\rho(\vartheta(\rho))$. This leads to $\vartheta(\rho) \in \mathrm{Fix}(t_\rho \gamma)$. $\square$

The following lemma characterises the reduced T-expressions.

Lemma 3.9. *[Car72, Lemma 3] Let $\rho_1, \rho_2, \ldots, \rho_k \in \Phi^+$. Then the expression $t_{\rho_1} t_{\rho_2} \cdots t_{\rho_k}$ is T-reduced if and only if $\rho_1, \rho_2, \ldots, \rho_k$ are linearly independent.*

This follows directly from the fact that for $w \in W$, $\ell_T(w) = \dim(\mathrm{Im}(w - 1))$ (cf. [HL99, (1.2)]). The following lemma characterises the T-reduced expressions of elements occurring in the lattice $\mathcal{L}$.

Lemma 3.10. *[BW08, Lemma 4.8] Let $t_{\rho_1}t_{\rho_2}\cdots t_{\rho_k}$ be a T-reduced expression. Then the following are equivalent:*

1. *$t_{\rho_1}t_{\rho_2}\cdots t_{\rho_k}\le\gamma$;*
2. *$(\vartheta(\rho_i),\rho_j)=0$ whenever $1\le i<j\le k$.*

3.2.2. Proof of the vanishing lemma To prove the equivalent statement (3.6) of Lemma 3.2, we need a description of the set $\mathcal{T}_1\cup\mathcal{T}_2$ in terms of positive roots.

Proposition 3.11. *Let $\rho_1,\rho_2,\ldots,\rho_k\in\Phi^+$, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be as in (3.5). Then we have $(t_{\rho_1},t_{\rho_2},\ldots,t_{\rho_k})\in\mathcal{T}_1\cup\mathcal{T}_2$ if and only if there exists a pair $i<j$ such that $(\vartheta(\rho_i),\rho_j)\neq0$.*

Proof. Let $\mathbf{t}=(t_{\rho_1},t_{\rho_2},\ldots,t_{\rho_k})\in\mathcal{T}_1\cup\mathcal{T}_2$ and assume for contradiction that we have $(\vartheta(\rho_i),\rho_j)=0$ for any $1\le i<j\le k$. Since the matrix $((\vartheta(\rho_i),\rho_j))_{k\times k}$ is non-singular by part (1) of Lemma 3.8, the positive roots $\rho_i, 1\le i\le k$ are linearly independent. It follows from Lemma 3.9 that $t_{\rho_1}t_{\rho_2}\cdots t_{\rho_k}$ is T-reduced and hence $\mathbf{t}\notin\mathcal{T}_1$, which implies that $\mathbf{t}\in\mathcal{T}_2$. However, if $t_{\rho_1}t_{\rho_2}\cdots t_{\rho_k}$ is T-reduced and $(\vartheta(\rho_i),\rho_j)=0$ for any $1\le i<j\le k$, then by Lemma 3.10 we have $t_{\rho_1}t_{\rho_2}\cdots t_{\rho_k}\le\gamma$, which implies that $\mathbf{t}\notin\mathcal{T}_2$. This contradicts $\mathbf{t}\in\mathcal{T}_1\cup\mathcal{T}_2$. This proves the "only if" part.

For the converse, assume that there exist $i<j$ such that $(\vartheta(\rho_i),\rho_j)\neq0$. If $t_{\rho_1}t_{\rho_2}\cdots t_{\rho_k}$ is not T-reduced, then $(t_{\rho_1},t_{\rho_2},\ldots,t_{\rho_k})\in\mathcal{T}_1$. Otherwise, by Lemma 3.10 $t_{\rho_1}t_{\rho_2}\cdots t_{\rho_k}\not\le\gamma$, and hence $(t_{\rho_1},t_{\rho_2},\ldots,t_{\rho_k})\in\mathcal{T}_2$. This completes the proof. □

Definition 3.12. A sequence of positive roots $(\rho_1,\rho_2,\ldots,\rho_k)$ is called a *vanishing sequence* if the inner product $(\vartheta(\rho_i),\rho_j)=0$ for all $1\le i<j\le k$ except that $(\vartheta(\rho_1),\rho_k)\neq0$.

With this definition, we can refine Proposition 3.11 as follows.

Proposition 3.13. *Let $\rho_1,\rho_2,\ldots,\rho_k\in\Phi^+$. Then $\mathbf{t}=(t_{\rho_1},t_{\rho_2},\ldots,t_{\rho_k})\in\mathcal{T}_1\cup\mathcal{T}_2$ if and only if there exists a pair $i<j$ such that $(\rho_i,\rho_{i+1},\ldots,\rho_j)$ is a vanishing sequence.*

Proof. By Proposition 3.11, we may choose a pair $i<j$ for which $j-i$ is minimal such that $(\vartheta(\rho_i),\rho_j)\neq0$. It follows from the minimality of $j-i$ that $(\vartheta(\rho_s),\rho_t)=0$ for all $i\le s<t\le j$ except for $s=i,t=j$. Therefore, $(\rho_i,\rho_{i+1},\ldots,\rho_j)$ is a vanishing sequence. □

Note that for any $\rho\in\Phi^+$, the sequence (ρ,ρ) is a vanishing sequence as the inner product $(\vartheta(\rho),\rho)\neq0$ by part (1) of Lemma 3.8. If $t_{\rho_1}t_{\rho_2}$ is T-reduced and $t_{\rho_1}t_{\rho_2}\not\le\gamma$, then by Lemma 3.10 the inner product $(\vartheta(\rho_1),\rho_2)\neq0$

and therefore the sequence (ρ_1, ρ_2) is a vanishing sequence. It follows from the defining relations of $\mathcal{A}$ that $a_{t_{\rho_1}} a_{t_{\rho_2}} = a_{t_\rho}^2 = 0$.

In general, we will prove that $a_{t_{\rho_1}} \cdots a_{t_{\rho_k}} = 0$ for any vanishing sequence $(\rho_1, \rho_2, \ldots, \rho_k)$. Before proving this, we need the following observations.

Lemma 3.14. *Let $(\rho_1, \rho_2, \ldots, \rho_k)$ be a vanishing sequence with $k > 2$. Then*

1. *$t_{\rho_1} t_{\rho_2} \cdots t_{\rho_{k-1}} \leq \gamma$ and $t_{\rho_2} t_{\rho_3} \cdots t_{\rho_k} \leq \gamma$ are both T-reduced;*
2. *$t_{\rho_i} t_{\rho_j} \leq \gamma$ is T-reduced for all $i < j$, except $i = 1, j = k$;*
3. *Let $w = t_{\rho_1} t_{\rho_2} \leq \gamma$ and let $\Phi^+(w) = \{\tau_1, \tau_2, \ldots, \tau_m\}$ be the set of all positive roots for which $t_{\tau_i} \leq w$. Then $(\tau_i, \rho_3, \ldots, \rho_k)$ is a vanishing sequence for any $\tau_i \neq \rho_2$.*

Proof. The matrix $((\vartheta(\rho_i), \rho_j))_{(k-1)\times(k-1)}$ with $1 \leq i, j \leq k-1$ is lower triangular with nonzero diagonal entries, thereby $\rho_i, 1 \leq i \leq k-1$ are linearly independent. It follows from Lemma 3.9 and Lemma 3.10 that $t_{\rho_1} t_{\rho_2} \cdots t_{\rho_{k-1}}$ is T-reduced and $t_{\rho_1} t_{\rho_2} \cdots t_{\rho_{k-1}} \leq \gamma$. Similarly, one can prove that $t_{\rho_2} t_{\rho_3} \cdots t_{\rho_k} \leq \gamma$. This completes the proof of part (1), from which part (2) follows.

For part (3), let $V_1 = \mathrm{Fix}(w) \subseteq V$ and let $V_1^\perp$ be the orthogonal subspace such that $V = V_1 \oplus V_1^\perp$. Then by (3.7) $\dim V_1 = n-2$ and $\dim V_1^\perp = 2$. Since w fixes every vector in V_1, so does any expression $t_{\tau_i} t_{\tau_j}$ for w. Therefore, the positive roots $\tau_i \in \Phi^+(w)$ are in $V_1^\perp$. Since $t_{\rho_1} t_{\rho_2}$ is T-reduced, it follows from Lemma 3.9 that $\rho_1, \rho_2 \in V_1^\perp$ are linearly independent and hence constitute a basis for $V_1^\perp$.

For any $\tau_i \in \Phi^+(w) \subseteq V_1^\perp$ we have $\tau_i = \lambda_1 \rho_1 + \lambda_2 \rho_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Note that $\lambda_1 = 0$ if and only if $\tau_i = \rho_2$. Then we have

$$(\vartheta(\tau_i), \rho_j) = \lambda_1 (\vartheta(\rho_1), \rho_j) + \lambda_2 (\vartheta(\rho_2), \rho_j) = 0, \quad 1 \leq i \leq n, 1 \leq j \leq k-1,$$
$$((\vartheta(\tau_i), \rho_k)) = \lambda_1 (\vartheta(\rho_1), \rho_k) + \lambda_2 (\vartheta(\rho_2), \rho_k) = \lambda_1 (\vartheta(\rho_1), \rho_k).$$

Since $(\vartheta(\rho_1), \rho_k) \neq 0$, we have $((\vartheta(\tau_i), \rho_k)) \neq 0$ if and only if $\lambda_1 \neq 0$ if and only if $\tau_i \neq \rho_2$. Therefore the sequence $(\tau_i, \rho_3, \ldots, \rho_k)$ is a vanishing sequence for any $\tau_i \neq \rho_2$. □

Lemma 3.15. *Let $(\rho_1, \rho_2, \ldots, \rho_k)$ be a vanishing sequence with $k \geq 2$. Then*

$$a_{t_{\rho_1}} a_{t_{\rho_2}} \cdots a_{t_{\rho_k}} = 0.$$

Proof. We use induction on k. For the base case $k = 2$, since (ρ_1, ρ_2) is a vanishing sequence we have $(\vartheta(\rho_1), \rho_2) \neq 0$. Then by Proposition 3.11, $(t_{\rho_1}, t_{\rho_2}) \in \mathcal{T}_1 \cup \mathcal{T}_2$. If $(t_{\rho_1}, t_{\rho_2}) \in \mathcal{T}_1$, then $t_{\rho_1} t_{\rho_2}$ is not T-reduced. This implies that $\rho_1 = \rho_2$ and hence $a_{t_{\rho_1}}^2 = 0$. If $(t_{\rho_1}, t_{\rho_2}) \in \mathcal{T}_2$, then we have $t_{\rho_1} t_{\rho_2} \not\leq \gamma$ and hence $a_{t_{\rho_1}} a_{t_{\rho_2}} = 0$.

Now let $(\rho_1, \rho_2, \ldots, \rho_k)$ be a vanishing sequence with $k > 2$. Using part (2) of Lemma 3.14 we have $w = t_{\rho_1} t_{\rho_2} \leq \gamma$. Then by Proposition 2.3 the interval $[e, w]$ is isomorphic to the noncrossing partition lattice of a dihedral group $I_2(m)$ for some $m \geq 2$. Suppose that $\Phi^+(w) = \{\tau_1, \tau_2, \ldots, \tau_m\}$ is set of all positive roots for which $t_{\tau_i} \leq w$. By the defining relation, we have

$$a_{t_{\rho_1}} a_{t_{\rho_2}} = - \sum_{w = t_{\tau_i} t_{\tau_j},\, \tau_j \neq \rho_2} a_{t_{\tau_i}} a_{t_{\tau_j}},$$

where the sum is over all T-reduced expressions $t_{\tau_i} t_{\tau_j}$ of w with $\tau_j \neq \rho_2$. Using the above relation, we obtain

$$a_{t_{\rho_1}} a_{t_{\rho_2}} a_{t_{\rho_3}} \cdots a_{t_{\rho_n}} = - \sum_{w = t_{\tau_i} t_{\tau_j},\, \tau_j \neq \rho_2} a_{t_{\tau_i}} a_{t_{\tau_j}} a_{t_{\rho_3}} \cdots a_{t_{\rho_n}}.$$

It follows from part (3) of Lemma 3.14 that $(\tau_j, \rho_3, \ldots, \rho_n)$ is a vanishing sequence for any $\tau_j \neq \rho_2$, and hence by induction hypothesis $a_{t_{\tau_j}} a_{t_{\rho_3}} \cdots a_{t_{\rho_n}} = 0$. Therefore, we have $a_{t_{\rho_1}} a_{t_{\rho_2}} \cdots a_{t_{\rho_n}} = 0$ $\square$

We are now in a position to prove Lemma 3.2.

Proof of Lemma 3.2. By Proposition 3.13, $\mathbf{t} = (t_{\rho_1}, t_{\rho_2}, \ldots, t_{\rho_k}) \in \mathcal{T}_1 \cup \mathcal{T}_2$ if and only if there exists $i < j$ such that $(\rho_i, \rho_{i+1}, \ldots, \rho_j)$ is a vanishing sequence. Using Lemma 3.15, in this case we have

$$a_{\mathbf{t}} = a_{t_{\rho_1}} \cdots (a_{t_{\rho_i}} a_{t_{\rho_{i+1}}} \cdots a_{t_{\rho_j}}) \cdots a_{t_{\rho_k}} = 0.$$

$\square$

4. Chain complexes for Milnor fibres and hyperplane arrangements

We now begin our discussion of the chain complexes whose homology realise that of the Milnor fibre and of the hyperplane complement associated with W. For any $u, w \in W$, write $u^w := w^{-1} u w$.

4.1. Some acyclic chain complexes

Recall that $\mathcal{B} = \bigoplus_{k=0}^{n} \mathcal{B}_k$ is a $\mathbb{Z}$-graded algebra generated by the β_t for $t \in T$. For any $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathrm{Rex}_T(w)$ with $w \in \mathcal{L}$, i.e., such that

$w = t_1t_2\cdots t_k \in \mathcal{L}$ is a T-reduced expression, we have $\beta_{\mathbf{t}} = \beta_{t_1}\beta_{t_2}\cdots\beta_{t_k}$. Recalling the $\mathbb{Z}$-linear map d from (2.4) and Proposition 2.8, we have

$$d_k : \mathcal{B}_k \to \mathcal{B}_{k-1}, \quad \beta_{\mathbf{t}} \mapsto z_{\mathbf{t}} = \sum_{i=1}^{k}(-1)^{k-i}\beta_{\mathbf{t}(\hat{i})}, \tag{4.1}$$

In particular, $d_1(\beta_t) = 1$ for any $t \in T$. The following results can be found in [Zha23, Lemma 5.11, Proposition 5.13].

Proposition 4.1. *The following properties hold for* $(\mathcal{B}, d)$*.*

1. *We have $d^2 = 0$, whence we have the following chain complex $(\mathcal{B}, d)$:*

$$0 \longrightarrow \mathcal{B}_n \xrightarrow{d_n} \mathcal{B}_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \mathcal{B}_0 \longrightarrow 0.$$

2. *The chain complex $(\mathcal{B}, d)$ is acyclic. More precisely, let $\mathcal{L}_{[k]} := \{w \in \mathcal{L} | 1 \le \ell_T(w) \le k\}$ be a rank-selected subposet of $\mathcal{L}$. If $1 \le k \le n-1$, then*

$$\operatorname{Im} d_k = \widetilde{H}_{k-2}(\mathcal{L}_{[k-1]}), \quad \operatorname{Ker} d_k = \widetilde{H}_{k-1}(\mathcal{L}_{[k]}).$$

Otherwise, $\operatorname{Im} d_n = \widetilde{H}_{n-2}(\mathcal{L}_{[n-1]})$ *and* $\operatorname{Ker} d_n = 0$.

3. *(Leibniz rule) For each $i = 1, 2, \ldots, k-1$, we have*

$$d(\beta_{\mathbf{t}}) = (-1)^{k-i}(d\beta_{(t_1,\ldots,t_i)})\beta_{(t_{i+1},\ldots,t_k)} + \beta_{(t_1\ldots,t_i)}(d\beta_{(t_{i+1},\ldots,t_k)})$$

for any $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathrm{Rex}_T(w)$, where $w \in \mathcal{L}_k$ with $2 \le k \le n$.

In view of Theorem 3.1, the map $\phi : \mathcal{A} \longrightarrow \mathcal{B}$, given by $\phi(a_{t_1}a_{t_2}\cdots a_{t_k}) = \beta_{t_1}\beta_{t_2}\cdots\beta_{t_k}$ for any T-reduced expression $w = t_1t_2\cdots t_k \in \mathcal{L}$, is a graded isomorphism from $\mathcal{A}$ to $\mathcal{B}$. By abuse of notation, for each $k = 1, \ldots, n$ we define the $\mathbb{Z}$-linear map

$$d : \mathcal{A}_k \to \mathcal{A}_{k-1}, \quad a_{t_1}a_{t_2}\cdots a_{t_k} \mapsto \sum_{i=1}^{k}(-1)^{k-i}a_{t_1}\cdots\hat{a}_{t_i}\cdots a_{t_k}. \tag{4.2}$$

Then the properties described in Proposition 4.1 for $(\mathcal{B}, d)$ hold *mutatis mutandem* for $(\mathcal{A}, d)$. In particular, $(\mathcal{A}, d)$ is an acyclic complex.

We proceed next to give another acyclic complex induced by the "Kreweras complement" of the NCP lattice. Now the noncrossing partition lattice $\mathcal{L}$ is self-dual, i.e., $\mathcal{L} \cong \mathcal{L}^{op}$, where $\mathcal{L}^{\mathrm{op}}$ is the set $\mathcal{L}$ with the reverse partial order $<_{\mathrm{op}}$. This isomorphism may be realised explicitly by the Kreweras complement K, defined by

$$K : \mathcal{L} \longrightarrow \mathcal{L}^{\mathrm{op}}, \quad w \mapsto \gamma w^{-1}.$$

The Kreweras map induces an automorphism of $\mathcal{A} = \mathcal{A}(\mathcal{L})$ as a graded algebra. Recall from Lemma 3.3 that an element $a_{t_1} a_{t_2} \cdots a_{t_k}$ of $\mathcal{A}$ is nonzero if and only if $e < t_1 < t_1 t_2 < \cdots < t_1 t_2 \cdots t_k$ is a chain of $\mathcal{L}$. The latter is mapped by K to the chain $\gamma <_{\mathrm{op}} \gamma t_1 <_{\mathrm{op}} \gamma t_2 t_1 <_{\mathrm{op}} \cdots <_{\mathrm{op}} \gamma t_k \cdots t_1$ of $\mathcal{L}^{\mathrm{op}}$, and this corresponds to a nonzero element $a_{t_1} a_{t_2^{t_1}} \cdots a_{t_k^{t_{k-1} \cdots t_1}}$ of $\mathcal{A}(\mathcal{L}^{\mathrm{op}})$. Note further that $\mathcal{A}(\mathcal{L}^{\mathrm{op}})$ is isomorphic to the opposite algebra $\mathcal{A}(\mathcal{L})^{\mathrm{op}}$. Therefore, we have the following composite of linear maps:

$$\kappa : \mathcal{A}(\mathcal{L}) \xrightarrow{\bar{\kappa}} \mathcal{A}(\mathcal{L}^{\mathrm{op}}) \cong \mathcal{A}(\mathcal{L})^{\mathrm{op}} \xrightarrow{\theta} \mathcal{A}(\mathcal{L}),$$

where $\bar{\kappa}$ is induced as above by K, so that $\bar{\kappa}(a_{t_1} a_{t_2} \cdots a_{t_k}) = a_{t_1} a_{t_2^{t_1}} \cdots a_{t_k^{t_{k-1} \cdots t_1}}$ and θ is the anti-isomorphism given by $\theta(a_{t_1} a_{t_2} \cdots a_{t_k}) = a_{t_k} \cdots a_{t_2} a_{t_1}$. Hence the linear automorphism κ of $\mathcal{A}$ is defined explicitly by

$$\kappa(a_{t_1} a_{t_2} \cdots a_{t_k}) = a_{t_k^{t_{k-1} \cdots t_1}} \cdots a_{t_2^{t_1}} a_{t_1}. \tag{4.3}$$

It is clear that κ preserves the defining relations of $\mathcal{A}$.

In addition to the differential (4.2), for each $k = 1, \ldots, n$ we define the following $\mathbb{Z}$-linear map

$$\delta : \mathcal{A}_k \to \mathcal{A}_{k-1}, \quad a_{t_1} a_{t_2} \cdots a_{t_k} \mapsto \sum_{i=1}^{k} (-1)^{i-1} a_{t_1^{t_i}} \cdots a_{t_{i-1}^{t_i}} \hat{a}_{t_i} \cdots a_{t_k}. \tag{4.4}$$

The properties of δ are summarised in the next result.

Proposition 4.2. *Let δ and κ be as defined above.*

1. *(Leibniz rule) For each $k = 2, \ldots, n$, we have*

$$\delta(a_{t_1} a_{t_2} \cdots a_{t_k}) = \delta(a_{t_1} a_{t_2} \cdots a_{t_{k-1}}) a_{t_k} + (-1)^{k-1} a_{t_1^{t_k}} a_{t_2^{t_k}} \cdots a_{t_{k-1}^{t_k}}.$$

   for any $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathrm{Rex}_T(w)$ with $w \in \mathcal{L}_k$.
2. *For each integer $k = 1, \ldots, n$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}_k & \xrightarrow{d} & \mathcal{A}_{k-1} \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} \\ \mathcal{A}_k & \xrightarrow{\delta} & \mathcal{A}_{k-1}. \end{array}$$

Therefore, the complex $(\mathcal{A}, \delta)$ is acyclic.

3. *We have $d\delta = \delta d$.*

Proof. Part (1) follows directly from the definition (4.4). The diagram commutes by straightforward calculation. As κ is an automorphism and $(\mathcal{A}, d)$ is acyclic, the acyclicity of $(\mathcal{A}, \delta)$ follows. It is easily verified directly that d and δ commute with each other. ☐

Remark 4.3. Note that d and δ do not preserve the defining relations of the algebra $\mathcal{A}$. For instance, we have $a_{t_1}a_{t_2} = 0$ for any pair of reflections t_1, t_2 satisfying $t_1t_2 \not\leq \gamma$. However, $d(a_{t_1}a_{t_2}) = -a_{t_2} + a_{t_1} \neq 0$ and $\delta(a_{t_1}a_{t_2}) = -a_{t_2} + a_{t_2t_1t_2} \neq 0$.

Remark 4.4. It follows from Lemma 3.3 that $\mathcal{A}_k$ is spanned by the nonzero elements $a_{t_1}a_{t_2}\cdots a_{t_k}$ as a free abelian group, where $t_1t_2\cdots t_k = w \in \mathcal{L}$ is a T-reduced expression. By the defining relations of $\mathcal{A}$, all linear relations among nonzero elements $a_{t_1}\cdots a_{t_k}$ are generated by the quadratic relation $\sum_{(t_1,t_2)\in \mathrm{Rex}_T(w)} a_{t_1}a_{t_2} = 0$ for any $w \in \mathcal{L}_2$. It is easily verified that $d(\sum_{(t_1,t_2)\in \mathrm{Rex}_T(w)} a_{t_1}a_{t_2}) = 0$ for any $t \in T$, and similarly this holds for δ. Therefore, the linear maps d and δ are well-defined.

4.2. Complexes for the Milnor fibre and hyperplane complement

Let $\mathcal{H}$ be the set of (complexified) reflecting hyperplanes of W and write $M = M_W := V_{\mathbb{C}} \setminus \cup_{H\in\mathcal{H}} H$ for the corresponding hyperplane complement. For $H \in \mathcal{H}$, let $\ell_H \in V_{\mathbb{C}}^*$ be a corresponding linear form, so that $H = \ker(\ell_H)$. Let $Q = \prod_{H\in\mathcal{H}} \ell_H^2$; it is well known that the ℓ_H may be chosen so that Q is W-invariant. The Milnor fibre $F := Q^{-1}(1) = \{v \in V_{\mathbb{C}} \mid Q(v) = 1\}$. Evidently W acts on both M and F, so that we may speak of the orbit spaces M/W and F/W.

4.2.1. Chain complexes for M and F Recall from [Zha23] the following chain complexes which compute the integral homology of the hyperplane complement and of the Milnor fibre. We use the algebra $\mathcal{A}$ instead of $\mathcal{B}$ in the original complexes.

Theorem 4.5. *[Zha23, Theorem 7.2] The integral homology of the hyperplane complement M is isomorphic to the homology of the following chain complex of abelian groups:*

$$0 \longrightarrow \mathbb{Z}W \otimes \mathcal{A}_n \xrightarrow{\partial_n} \cdots \longrightarrow \mathbb{Z}W \otimes \mathcal{A}_1 \xrightarrow{\partial_1} \mathbb{Z}W \otimes \mathcal{A}_0 \longrightarrow 0, \tag{4.5}$$

where the boundary maps are given by

$$\partial_k(w \otimes a_{t_1}a_{t_2}\cdots a_{t_k}) = \sum_{i=1}^{k}(-1)^{i-1} wt_i \otimes a_{t_1^{t_i}} \cdots a_{t_{i-1}^{t_i}} \hat{a}_{t_i} \cdots a_{t_k} \\ - \sum_{i=1}^{k}(-1)^{i-1} w \otimes a_{t_1} \cdots \hat{a}_{t_i} \cdots a_{t_k}$$

for any $w \in W$ and $(t_1, t_2, \ldots, t_k) \in \bigcup_{u \in \mathcal{L}_k} \mathrm{Rex}_T(u)$ with $1 \leq k \leq n$.

Theorem 4.6. *[Zha23, Theorem 6.3] The integral homology of the Milnor fibre $F_Q = Q^{-1}(1)$ is isomorphic to the homology of the following chain complex*

$$0 \to \mathbb{Z}W \otimes d(\mathcal{A}_n) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow \mathbb{Z}W \otimes d(\mathcal{A}_2) \xrightarrow{\partial_1} \mathbb{Z}W \otimes d(\mathcal{A}_1) \to 0, \tag{4.6}$$

where the boundary maps are given by

$$\partial_{k-1}(w \otimes d(a_{t_1}a_{t_2}\cdots a_{t_k})) = \sum_{i=1}^{k}(-1)^{i-1} wt_i \otimes d(a_{t_1^{t_i}} \cdots a_{t_{i-1}^{t_i}} \hat{a}_{t_i} \cdots a_{t_k})$$

for any $w \in W$ and $(t_1, \ldots, t_k) \in \bigcup_{u \in \mathcal{L}_k} \mathrm{Rex}_T(u)$ with $2 \leq k \leq n$.

4.2.2. *W*-action on these complexes Note that W acts on the above chain complexes as follows. For $x \in W$, $x.w \otimes a_{t_1}a_{t_2}\cdots a_{t_k} = (xw) \otimes a_{t_1}a_{t_2}\cdots a_{t_k}$ in the case of (4.5), while in the case of (4.6), $x.w \otimes d(a_{t_1}a_{t_2}\cdots a_{t_k}) = (xw) \otimes d(a_{t_1}a_{t_2}\cdots a_{t_k})$. It is evident that this W-action respects the boundary homomorphisms. Thus these chain complexes are both left W-modules.

Now it is well known that W acts freely on M and F. The quotient spaces M/W and F/W are both $K(\pi, 1)$-spaces. In particular, the fundamental group $\pi_1(M/W) = A(W)$, the Artin group of W. Therefore, we have $H_k(A(W); \mathbb{Z}) = H_k(M/W; \mathbb{Z})$, and this may be computed using $\mathcal{A}$ as follows.

Theorem 4.7. *[Zha23, Theorem 7.5] The integral homology of M/W or the Artin group $A(W)$ is isomorphic to the homology of the following chain complex:*

$$0 \longrightarrow \mathcal{A}_n \xrightarrow{\partial_n} \mathcal{A}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{A}_1 \xrightarrow{\partial_1} \mathcal{A}_0 \longrightarrow 0,$$

where the boundary maps are given by $\partial_k = \delta_k + (-1)^k d_k$, *i.e.*,

(4.7)
$$\partial_k(a_{t_1}\cdots a_{t_k}) = \sum_{i=1}^{k}(-1)^{i-1}a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}\hat{a}_{t_i}\cdots a_{t_k} - \sum_{i=1}^{k}(-1)^{i-1}a_{t_1}\cdots\hat{a}_{t_i}\cdots a_{t_k}$$

for $(t_1, t_2, \ldots, t_k) \in \bigcup_{u\in\mathcal{L}_k}\operatorname{Rex}_T(u)$ *with* $1 \le k \le n$.

Theorem 4.8. *[Zha23, Theorem 6.8] The integral homology of* F/W *is isomorphic to the homology of the following chain complex:*

$$0 \longrightarrow d(\mathcal{A}_n) \xrightarrow{\partial_{n-1}} d(\mathcal{A}_{n-1}) \longrightarrow \cdots \longrightarrow d(\mathcal{A}_2) \xrightarrow{\partial_1} d(\mathcal{A}_1) \longrightarrow 0,$$

where the boundary maps are given by $\partial_k = \delta_k$, *i.e.*,

$$\partial_{k-1}(d(a_{t_1}a_{t_2}\cdots a_{t_k})) = \sum_{i=1}^{k}(-1)^{i-1}d(a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}\hat{a}_{t_i}\cdots a_{t_k}) \tag{4.8}$$

for any $(t_1, \ldots, t_k) \in \bigcup_{w\in\mathcal{L}_k}\operatorname{Rex}_T(w)$ *with* $2 \le k \le n$. *In particular,* $\partial_1 = 0$.

4.3. A new pair of complexes

Let us start with two new complexes defined over $\mathbb{C}$. The first, denoted $\mathcal{C}$, is

$$\mathcal{C} := 0\longrightarrow\mathbb{C}W\otimes\mathcal{A}_n\xrightarrow{\partial_n}\cdots\longrightarrow\mathbb{C}W\otimes\mathcal{A}_1\xrightarrow{\partial_1}\mathbb{C}W\otimes\mathcal{A}_0\longrightarrow 0, \tag{4.9}$$

with the boundary homomorphisms ∂_i defined as in Theorem 4.5. The second, denoted $\mathcal{K}$, is

$$\mathcal{K} := 0\longrightarrow\mathbb{C}W\otimes d(\mathcal{A}_n)\xrightarrow{\partial_{n-1}}\cdots\longrightarrow\mathbb{C}W\otimes d(\mathcal{A}_2)\xrightarrow{\partial_1}\mathbb{C}W\otimes d(\mathcal{A}_1)\longrightarrow 0, \tag{4.10}$$

where the boundary homomorphisms are as defined in Theorem 4.6. Then by Theorem 4.5 and Theorem 4.6, the homology of the complex $\mathcal{C}$ (resp. $\mathcal{K}$) is the homology of the corresponding hyperplane complement $M = M_W$ (resp. Milnor fibre F) with complex coefficients.

4.3.1. Left, right and bi-W-modules In the development below, we shall need to distinguish between left and right $\mathbb{C}W$-modules. Let $\operatorname{Mod}_R(\mathbb{C}W)$ (resp. $\operatorname{Mod}_L(\mathbb{C}W)$) be the category of finite dimensional right (resp. left) $\mathbb{C}W$-modules. We shall use the categorical isomorphism $\lambda_{RL} : U \mapsto U_L$ from $\operatorname{Mod}_R(\mathbb{C}W)\longrightarrow\operatorname{Mod}_L(\mathbb{C}W)$, and similarly the isomorphism $\lambda_{LR} : X \mapsto X_R$

from $\mathrm{Mod}_L(\mathbb{C}W)\longrightarrow\mathrm{Mod}_R(\mathbb{C}W)$ ($U \in \mathrm{Mod}_R(\mathbb{C}W), X \in \mathrm{Mod}_L(\mathbb{C}W)$) where, for $u \in U_L$ and $w \in W$, $w.u := u.w^{-1}$, and for $x \in X_R$, $x.w := w^{-1}.x$.

In addition to the above categories, we shall need the category $\mathrm{Mod}_{LR}(\mathbb{C}W)$ of finite dimensional $\mathbb{C}W$-bimodules. If $Y \in \mathrm{Mod}_{LR}(\mathbb{C}W)$, then for $w_1, w_2 \in W$ and $y \in Y$, we have $(w_1y)w_2 = w_1(yw_2)$. Evidently if $U \in \mathrm{Mod}_R(\mathbb{C}W)$, $X \in \mathrm{Mod}_L(\mathbb{C}W)$, then $X \otimes_{\mathbb{C}} U$ is naturally a W-bimodule. Using this, we have the following formulation of a standard decomposition of a finite group algebra.

Lemma 4.9. *Maintaining the above notation, we have the following isomorphism in the category of W-bimodules.*

$$\mathbb{C}W \cong \oplus_{U\in Irr(\mathrm{Mod}_R(\mathbb{C}W))}(U_L^* \otimes_{\mathbb{C}} U).$$

Here the sum is over the simple modules $U \in \mathrm{Mod}_R(\mathbb{C}W)$ and U_L^* denotes the contragredient of U_L (defined above).

Proof. It is easily verified that for any right $\mathbb{C}W$-module U, the left module U_L^* has the same character as U. This is because $U_L \simeq U$ as vector spaces, and $w \in W$ acts on U_L as w^{-1} does on U. Thus they have complex conjugate traces. But the character of w on U_L^* is the conjugate of its character on U_L, and hence coincides with its character on U. The stated decomposition is now standard.

Note that each of the tensor factors in each summand is referred to as the multiplicity module of the other factor in that summand. □

Corollary 4.10. *In the above notation, we have* $\mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathbb{C}W) \cong U$ *as right W-module. Here the left side indicates homomorphisms of left $\mathbb{C}W$-modules.*

Proof. By Lemma 4.9, we have

$$\mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathbb{C}W) \cong \oplus_{U\in Irr(\mathrm{Mod}_R(\mathbb{C}W))}\mathrm{Hom}_{\mathbb{C}W}(U_L^*, U_L^* \otimes_{\mathbb{C}} U).$$

But for fixed U, $U_L^* \otimes_{\mathbb{C}} U$ is isomorphic to a sum of simple left modules U_L^*. The result follows. □

4.3.2. A new pair of complexes We next define "relative" versions of the complexes (4.9) and (4.10).

Let $U \in \mathrm{Mod}_R(\mathbb{C}W)$. Define complexes $\mathcal{C}(U)$ and $\mathcal{K}(U)$ as follows.

$$\mathcal{C}(U) := 0\longrightarrow U \otimes \mathcal{A}_n \xrightarrow{\partial_n} \cdots \longrightarrow U \otimes \mathcal{A}_1 \xrightarrow{\partial_1} U \otimes \mathcal{A}_0 \longrightarrow 0,$$

where the boundary maps are given (for $u \in U$) by

$$\begin{aligned}\partial(u \otimes a_{t_1}a_{t_2}\cdots a_{t_k}) = &\sum_{i=1}^{k}(-1)^{i-1}ut_i \otimes a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}a_{t_{i+1}}\cdots a_{t_k}\\ &-\sum_{i=1}^{k}(-1)^{i-1}u \otimes a_{t_1}\cdots \widehat{a_{t_i}}\cdots a_{t_k}.\end{aligned} \tag{4.11}$$

$$\mathcal{K}(U) := 0 \longrightarrow U \otimes d(\mathcal{A}_n) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow U \otimes d(\mathcal{A}_2) \xrightarrow{\partial_1} U \otimes d(\mathcal{A}_1) \longrightarrow 0,$$

where the boundary homomorphisms are defined for $u \in U$ by

$$\partial(u \otimes d(a_{t_1}a_{t_2}\cdots a_{t_k})) = \sum_{i=1}^{k}(-1)^{i-1}ut_i \otimes d(a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}a_{t_{i+1}}\cdots a_{t_k}). \tag{4.12}$$

Now as observed in 4.2.2, the complexes $\mathcal{C}$ and $\mathcal{K}$ admit a (left) W-action. This is not generally the case for $\mathcal{C}(U)$ or $\mathcal{K}(U)$. Hence for each integer k, $H_k(\mathcal{C})$ and $H_k(\mathcal{K})$ are left $\mathbb{C}W$-modules. For left $\mathbb{C}W$-modules U_1, U_2 we write $\langle U_1, U_2\rangle_W$ for the multiplicity $\dim \operatorname{Hom}_{\mathbb{C}W}(U_1, U_2)$ as usual.

Recall that for any simple module $U \in \operatorname{Mod}_R(\mathbb{C}W)$, we have a "corresponding" module $U_L^* \in \operatorname{Mod}_L(\mathbb{C}W)$, whose character coincides with that of U.

Theorem 4.11. *For any right W-module U and for each integer $k \geq 0$, we have*

$$\dim H_k(\mathcal{C}(U)) = \langle U_L^*, H_k(M)\rangle \tag{4.13}$$

and

$$\dim H_k(\mathcal{K}(U)) = \langle U_L^*, H_k(F)\rangle. \tag{4.14}$$

Proof. We prove the first equation (4.13); the second equation (4.14) has a similar proof. Since $H_k(M) = H_k(\mathcal{C})$, we have

$$\langle U_L^*, H_k(M)\rangle = \langle U, H_k(\mathcal{C})\rangle = \dim \operatorname{Hom}_W(U_L^*, H_k(\mathcal{C}))$$

By semisimplicity, U_L^* is a flat module, whence the functor $\operatorname{Hom}_{\mathbb{C}}(U_L^*, -)$ is exact. Moreover the k^{th} homology functor commutes with any exact functor. It follows that we have the following isomorphism of left W-modules.

$$H_k(\operatorname{Hom}_{\mathbb{C}}(U_L^*, \mathcal{C})) \cong \operatorname{Hom}_{\mathbb{C}}(U_L^*, H_k(\mathcal{C})). \tag{4.15}$$

Further, the fixed point functor $(-)^W : M \mapsto M^W$ from the left W-module M to the $\mathbb{C}$-module M^W is representable. It is represented by the trivial W-module $\mathbb{C}$, so that $(-)^W$ is naturally isomorphic to the functor $\mathrm{Hom}_W(\mathbb{C}, -)$. Since $\mathbb{C}W$ is a semisimple algebra, the trivial W-module $\mathbb{C}$ is projective. Therefore, $\mathrm{Hom}_W(\mathbb{C}, -) \cong (-)^W$ is an exact functor. It follows that, upon taking W-fixed points in (4.15), we obtain

$$\begin{aligned} H_k(\mathrm{Hom}_{\mathbb{C}}(U_L^*, \mathcal{C}))^W \cong & H_k((\mathrm{Hom}_{\mathbb{C}}(U_L^*, \mathcal{C}))^W) \\ \cong & H_k(\mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathcal{C})) \cong \mathrm{Hom}_{\mathbb{C}W}(U_L^*, H_k(\mathcal{C})). \end{aligned}$$

It now remains only to relate the complex $\mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathcal{C})$ to $\mathcal{C}(U)$. for this, observe that

$$\mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathcal{C})_k \cong \mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathcal{C}_k) \cong \mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathbb{C}W \otimes_{\mathbb{C}} \mathcal{A}_k).$$

Moreover since W acts trivially on $\mathcal{A}$, we have

$$\mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathcal{C})_k \cong \mathrm{Hom}_{\mathbb{C}W}(U_L^*, \mathbb{C}W) \otimes_{\mathbb{C}} \mathcal{A}_k.$$

But by Corollary 4.10, the right side of the above equation is equal to $\mathcal{C}(U)_k = U \otimes_{\mathbb{C}} \mathcal{A}_k$, and the proof is complete. $\square$

4.4. Applications

We give some special cases and applications of Theorem 4.11.

First, consider the case $U = 1_W$, the trivial $\mathbb{C}W$-module. Then $U_L^* = 1_W$ and we have

$$\langle U_L^*, H_k(M) \rangle = \langle 1_W, H_k(M) \rangle = \dim H_k(M)^W.$$

Moreover by the transfer theorem for homology with coefficients in $\mathbb{C}$ (cf. [Bre72, Theorem III.2.4]), $H_k(M)^W \cong H_k(M/W)$. It follows from the first statement in Theorem 4.11 that

$$H_k(M/W) \cong H_k(\mathcal{C}(1_W)).$$

It is readily checked that the complex $\mathcal{C}(1_W)$ coincides with the (complexification of the) complex in Theorem 4.7, and in this way we recover that theorem for complex homology. Note that Theorem 3.7 is stronger, in that it computes the integral homology.

Next, using precisely the same arguments, we deduce that if F is the Milnor fibre as defined above, then

$$H_k(F/W) \cong H_k(\mathcal{K}(1_W)).$$

In this case, one again checks readily that $\mathcal{K}(1_W)$ may be identified with the complexification of the complex in Theorem 4.8, whence in this case we recover Theorem 4.8, again with coefficients in $\mathbb{C}$, by applying Theorem 4.11 in the case of $\mathcal{K}(U)$ with $U = 1_W$.

Consider next the case $U = \varepsilon$, the alternating representation of W. Then $U_L^* \simeq \varepsilon$ and it follows from [Leh96, (1.2)] that

$$\langle H_k(M), \varepsilon \rangle = 0 \text{ for all } k.$$

It follows immediately from the first part of Theorem 4.11 that

The complex $\mathcal{C}(\varepsilon)$ is acyclic.

Now we may think of $\mathcal{C}(U) = \mathcal{C}(\varepsilon)$ as having chain groups with bases $\{b \otimes a_{\mathbf{t}} \mid \mathbf{t} \in \bigcup_{w \in \mathcal{L}} \mathcal{D}_w\}$, where b is the basis element of ε. Using the fact that $bt = -b$ for all reflections t, it follows from (4.11) that $\mathcal{C}(\varepsilon)$ may be identified with the chain complex

$$0 \longrightarrow \mathcal{A}_n \xrightarrow{D_n} \cdots \longrightarrow \mathcal{A}_1 \xrightarrow{D_1} \mathcal{A}_0 \longrightarrow 0, \tag{4.16}$$

where the boundary homomorphism $D_k : \mathcal{A}_k \longrightarrow \mathcal{A}_{k-1}$ is given by

$$\begin{aligned} D_k(b \otimes a_{t_1} \cdots a_{t_k}) = &\sum_{i=1}^{k} (-1)^{i-1}(-b) \otimes a_{t_1^{t_i}} \cdots a_{t_{i-1}^{t_i}} a_{t_{i+1}} \cdots a_{t_k} \\ &- \sum_{i=1}^{k} (-1)^{i-1} b \otimes a_{t_1} \cdots \widehat{a_{t_i}} \cdots a_{t_k}. \end{aligned}$$

It follows that we may identify $\mathcal{C}(\varepsilon)$ with the complex (4.16), where

$$D_k = -\delta_k + \tilde{d}_k,$$

where δ_k is the restriction to $\mathcal{A}_k$ of δ, which is defined in (4.4) and $\tilde{d}_k = (-1)^k d_k$, where $d = \oplus_{k=1}^n d_k$ is defined in (4.2).

It follows from Proposition 4.2 (3) that $\tilde{d}\delta = -\delta\tilde{d}$, and hence that D is a differential. The acyclicity of $(\mathcal{A}, D)$ is not evident from these arguments.

Finally, observe that for any right W-module U we have

$$H_k(\mathcal{K}(U)) = H_k(\mathcal{K}(U\otimes\epsilon)),\quad 0\le k\le n-1, \tag{4.17}$$

as the boundary maps (4.12) of $\mathcal{K}(U)$ and $\mathcal{K}(U\otimes\epsilon)$ differ by -1.

5. Dual complexes

The principal purpose of this section is to obtain sharper results on the integral homology and cohomology of both M, the hyperplane complement, and F, the non-reduced Milnor fibre. With this in mind we begin by defining a $\mathbb{Z}$-bilinear form on $\mathcal{A}$, which will later become a tool for moving between homology and cohomology.

5.1. A bilinear form on $\mathcal{A}$

We will make frequent use of the following $\mathbb{Z}$-linear maps in later sections. For each $t\in T$, we define

$$d_t:\mathcal{A}_k\to\mathcal{A}_{k-1},\qquad a_{t_1}a_{t_2}\cdots a_{t_k}\mapsto\sum_{i=1}^k(-1)^{k-i}\delta_{t,t_i^{t_{i+1}\cdots t_k}}a_{t_1}\cdots\hat{a}_{t_i}\cdots a_{t_k},$$

$$\delta_t:\mathcal{A}_k\to\mathcal{A}_{k-1},\qquad a_{t_1}a_{t_2}\cdots a_{t_k}\mapsto\sum_{i=1}^k(-1)^{i-1}\delta_{t,t_i}a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}\hat{a}_{t_i}\cdots a_{t_k},$$

where $\delta_{t,t_i}=1$ if $t=t_i$ and 0 otherwise. It is clear that in the notation of (4.2) and (4.4), $d=\sum_{t\in T}d_t$ and $\delta=\sum_{t\in T}\delta_t$. Note that δ_t and d_t also satisfy the Leibniz rule, i.e.

$$\begin{aligned}d_t(a_{t_1}a_{t_2}\cdots a_{t_k})&=-d_{t^{t_k}}(a_{t_1}a_{t_2}\cdots a_{t_{k-1}})a_{t_k}+\delta_{t,t_k}a_{t_1}a_{t_2}\cdots a_{t_{k-1}},\\ \delta_t(a_{t_1}a_{t_2}\cdots a_{t_k})&=\delta_t(a_{t_1}a_{t_2}\cdots a_{t_{k-1}})a_{t_k}+(-1)^{k-1}\delta_{t,t_k}a_{t_1^{t_k}}a_{t_2^{t_k}}\cdots a_{t_{k-1}^{t_k}}.\end{aligned}$$

for any $(t_1,t_2,\ldots,t_k)\in\mathrm{Rex}_T(w)$ with $w\in\mathcal{L}_k$. We call d_t and δ_t *skew derivations.*

The linear maps d_t and δ_t are well-defined, for the same reason as in Remark 4.4.

Lemma 5.1. *For any $t,t'\in T$, we have $d_t\delta_{t'}=\delta_{t'}d_t$.*

Proof. We evaluate both sides on $a_{t_1}\cdots a_{t_k}\in\mathcal{A}$ and use induction on k. For any nonzero element $a_{t_1}\cdots a_{t_k}\in\mathcal{A}$, we have

$$\begin{aligned} d_t\delta_{t'}(a_{t_1}\cdots a_{t_k}) =& d_t(\delta_{t'}(a_{t_1}\cdots a_{t_{k-1}})a_{t_k}+(-1)^{k-1}\delta_{t',t_k}a_{t_1^{t_k}}a_{t_2^{t_k}}\cdots a_{t_{k-1}^{t_k}})\\ =& -d_{t^{t_k}}(\delta_{t'}(a_{t_1}\cdots a_{t_{k-1}})a_{t_k}+\delta_{t,t_k}\delta_{t'}(a_{t_1}\cdots a_{t_{k-1}})\\ &+(-1)^{k-1}\delta_{t',t_k}d_t(a_{t_1^{t_k}}a_{t_2^{t_k}}\cdots a_{t_{k-1}^{t_k}}), \end{aligned}$$

while on the other hand,

$$\begin{aligned} \delta_{t'}d_t(a_{t_1}\cdots a_{t_k}) =& \delta_{t'}(-d_{t^{t_k}}(a_{t_1}a_{t_2}\cdots a_{t_{k-1}})a_{t_k}+\delta_{t,t_k}a_{t_1}a_{t_2}\cdots a_{t_{k-1}})\\ =& -\delta_{t'}(d_{t^{t_k}}(a_{t_1}\cdots a_{t_{k-1}}))a_{t_k}+(-1)^{k-1}\delta_{t',t_k}d_t(a_{t_1^{t_k}}\cdots a_{t_{k-1}^{t_k}})\\ &+\delta_{t,t_k}\delta_{t'}(a_{t_1}\cdots a_{t_{k-1}}). \end{aligned}$$

The result is trivial if $k=1$. For $k>1$, by the induction hypothesis and the equations above we have $d_t\delta_{t'}(a_{t_1}\cdots a_{t_k})=\delta_{t'}d_t(a_{t_1}\cdots a_{t_k})$. This proves that $d_t\delta_{t'}=\delta_{t'}d_t$. □

Lemma 5.2. *The skew derivations $\delta_t, t\in T$ satisfy the following relations:*

$$\begin{aligned} \delta_t^2=\delta_{t_2}\delta_{t_1}&=0,\quad \forall t\in T, t_1t_2\not\leq\gamma,\\ \sum_{(t_1,t_2)\in\mathrm{Rex}_T(w)}\delta_{t_2}\delta_{t_1}&=0,\quad \forall w\in\mathcal{L}_2. \end{aligned}$$

Therefore, they describe an action of the opposite algebra $\mathcal{A}^{op}$ on $\mathcal{A}$.

Proof. For any reflections $r_1,r_2\in T$, we have

$$\begin{aligned} \delta_{r_2}\delta_{r_1}(a_{t_1}a_{t_2}\cdots a_{t_k}) =& \delta_{r_2}(\delta_{r_1}(a_{t_1}\cdots a_{t_{k-1}})a_{t_k}+(-1)^{k-1}\delta_{r_1,t_k}a_{t_1^{t_k}}a_{t_2^{t_k}}\cdots a_{t_{k-1}^{t_k}})\\ =& \delta_{r_2}(\delta_{r_1}(a_{t_1}\cdots a_{t_{k-1}}))a_{t_k}+(-1)^{k-2}\delta_{r_2,t_k}\delta_{r_1^{t_k}}(a_{t_1^{t_k}}\cdots a_{t_{k-1}^{t_k}})\\ &+(-1)^{k-1}\delta_{r_1,t_k}\delta_{r_2}(a_{t_1^{t_k}}a_{t_2^{t_k}}\cdots a_{t_{k-1}^{t_k}}), \end{aligned}$$

where $(t_1,t_2,\ldots,t_k)\in\mathrm{Rex}_T(w)$ for some $w\in\mathcal{L}$. We use induction on k to prove the stated relations among $\delta_t, t\in T$ by evaluating the relevant expressions on $a_{t_1}\cdots a_{t_k}$. The base case $k=1$ is obvious. For $k>1$, we prove each relation as follows.

By the induction hypothesis it is easy to see that $\delta_t^2=0$ if $t=r_1=r_2$.

If $r_1r_2\not\leq\gamma$, then there are two cases. Case 1: neither of r_1,r_2 is t_k. By the induction hypothesis, $\delta_{r_2}\delta_{r_1}(a_{t_1}a_{t_2}\cdots a_{t_k})=\delta_{r_2}(\delta_{r_1}(a_{t_1}a_{t_2}\cdots a_{t_{k-1}}))a_{t_k}=0$.

Case 2: exactly one of r_1, r_2 equals t_k. If $r_1 = t_k$, then we have

$$\delta_{r_2}\delta_{r_1}(a_{t_1}a_{t_2}\cdots a_{t_k}) = \delta_{r_2}(\delta_{r_1}(a_{t_1}\cdots a_{t_{k-1}}))a_{t_k} + (-1)^{k-1}\delta_{r_2}(a_{t_1^{t_k}}a_{t_2^{t_k}}\cdots a_{t_{k-1}^{t_k}}).$$

We claim that $\delta_{r_2}(a_{t_1^{t_k}}a_{t_2^{t_k}}\cdots a_{t_{k-1}^{t_k}}) = 0$. Otherwise, by the definition of δ_{r_2}, we have $r_2 = t_i^{t_k}$ for some $1 \leq i \leq k-1$. However, this leads to $r_1 r_2 = t_k t_i^{t_k} = t_i t_k$, which precedes γ by Lemma 3.10 and hence violates our assumption that $r_1 r_2 \not\leq \gamma$. Therefore, $\delta_{r_2}\delta_{r_1}(a_{t_1}a_{t_2}\cdots a_{t_k}) = \delta_{r_2}(\delta_{r_1}(a_{t_1}a_{t_2}\cdots a_{t_{k-1}}))a_{t_k} = 0$ by the induction hypothesis. If $r_2 = t_k$, the proof is similar.

For the last relation, by the induction hypothesis this is equivalent to showing that

$$\sum_{(r_1,r_2)\in \mathrm{Rex}_T(w)} \delta_{r_2,t_k}\delta_{r_1^{t_k}} - \delta_{r_1,t_k}\delta_{r_2} = 0.$$

We may assume $w = s_1 s_m = s_2 s_1 = \cdots = s_m s_{m-1}$ has m T-reduced factorisations. If none of the s_i is equal to t_k, then the above equation holds true. Otherwise, $t_k = s_i$ for some $i = 1, \ldots, m$. In that case we have

$$\begin{aligned}\sum_{(r_1,r_2)\in \mathrm{Rex}_T(w)} \delta_{r_2,t_k}\delta_{r_1^{t_k}} - \delta_{r_1,t_k}\delta_{r_2} &= \sum_{j=1}^{m}(\delta_{s_{j-1},s_i}\delta_{s_j^{s_i}} - \delta_{s_j,s_i}\delta_{s_{j-1}}) \\ &= \delta_{s_{i+1}^{s_i}} - \delta_{s_{i-1}} = 0,\end{aligned}$$

where $s_0 := s_m$ for notational convenience. This completes the proof. $\square$

Definition 5.3. Define the bilinear pairing

$$\langle -, - \rangle : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{Z}$$

by $\langle 1, 1 \rangle = 1$ and

1. $\langle \mathcal{A}_k, \mathcal{A}_\ell \rangle = 0$ for any $0 \leq k \neq \ell \leq n$;
2. For any $x \in \mathcal{A}_k$ and $t_i \in T, i = 1, \ldots, k$,

$$\langle a_{t_1}a_{t_2}\cdots a_{t_k}, x \rangle := \delta_{t_k}\cdots\delta_{t_1}(x).$$

In view of Lemma 5.2, this bilinear form is well-defined.

Lemma 5.4. *For any $t \in T$, and $x, y \in \mathcal{A}$ we have*

$$\langle a_t x, y \rangle = \langle x, \delta_t(y) \rangle, \quad \textit{and} \quad \langle x a_t, y \rangle = \langle x, d_t(y) \rangle.$$

Thus with respect to the bilinear form the skew-derivations δ_t and d_t are right adjoint to left and right multiplication by a_t, respectively.

Proof. We assume $x \in \mathcal{A}_{k-1}$ and $y \in \mathcal{A}_k$ for $1 \leq k \leq n$. The first adjunction follows immediately from the definition. For the second one, we use induction on k. If $k = 1$, then $\langle \lambda a_t, \mu a_{t'} \rangle = \lambda\mu\delta_{t,t'} = \langle \lambda, \mu d_t(a_{t'}) \rangle$ for any $\lambda, \mu \in \mathbb{Z}$. For $k > 1$, we may assume $x = a_{t_1} \cdots a_{t_{k-1}}$. Then

$$\begin{aligned}\langle a_{t_1} \cdots a_{t_{k-1}} a_t, y \rangle &= \langle a_{t_2} \cdots a_{t_{k-1}} a_t, \delta_{t_1}(y) \rangle = \langle a_{t_2} \cdots a_{t_{k-1}}, d_t \delta_{t_1}(y) \rangle \\ &= \langle a_{t_2} \cdots a_{t_{k-1}}, \delta_{t_1} d_t(y) \rangle = \langle a_{t_1} a_{t_2} \cdots a_{t_{k-1}}, d_t(y) \rangle,\end{aligned}$$

where the first and the last equation follow from the adjunction between δ_t and left multiplication by a_t, the second equation follows from the induction hypothesis, and the third equation follows from Lemma 5.1. The proof is complete. □

Proposition 5.5. *The bilinear form $\langle -, - \rangle$ is unimodular, i.e., it induces an isomorphism $\mathcal{A} \cong \mathcal{A}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{A}, \mathbb{Z})$, given by $x \mapsto \langle x, - \rangle$ for any $x \in \mathcal{A}$.*

Proof. We only need to show that the bilinear form induces an isomorphism $\mathcal{A}_k \cong \mathcal{A}_k^*$ for $0 \leq k \leq n$. Recall that $\mathcal{A}_k = \bigoplus_{w \in \mathcal{L}_k} \mathcal{A}_w$. It suffices to prove the following claims:

1. $\langle \mathcal{A}_v, \mathcal{A}_w \rangle = 0$ for any two elements $v \neq w$ of $\mathcal{L}_k$;
2. For any $w \in \mathcal{L}$, the bilinear form induces an isomorphism $\mathcal{A}_w \cong \mathcal{A}_w^*$.

To prove claim (1), we use induction on $\ell_T(v) = \ell_T(w) = k$. It is trivial if $k = 1$. For $k > 1$, assume that $x = x' a_t \in \mathcal{A}_v$. Then for any $y \in \mathcal{A}_w$ with $w \neq v$ we have

$$\langle x, y \rangle = \langle x' a_t, y \rangle = \langle x', d_t(y) \rangle.$$

Suppose that $y = a_{t_1} a_{t_2} \cdots a_{t_k}$ with $w = t_1 t_2 \cdots t_k$ being a T-reduced expression. Then by the definition,

$$d_t(y) = \sum_{i=1}^{k} (-1)^{k-i} \delta_{t, r_i} a_{t_1} \cdots a_{t_{i-1}} a_{t_{i+1}} \cdots a_{t_k},$$

where $r_i = t_i^{t_{i+1} \cdots t_k}$. If $t \neq r_i$ for any i, then $d_t(y) = 0$ and hence $\langle x, y \rangle = 0$. Otherwise, there exists a unique i_0 such that $t = r_{i_0}$. Assume that $x' = a_{t'_1} a_{t'_2} \cdots a_{t'_{k-1}}$ with $v = t'_1 t'_2 \cdots t'_{k-1} t$ being a T-reduced expression. We have

$$\langle x, y \rangle = \langle x', d_t(y) \rangle = (-1)^{k-i_0} \langle a_{t'_1} a_{t'_2} \cdots a_{t'_{k-1}}, a_{t_1} \cdots a_{t_{i_0-1}} a_{t_{i_0+1}} \cdots a_{t_k} \rangle.$$

Assume with a view to obtaining a contradiction that $\langle x, y \rangle \neq 0$. By the induction hypothesis we have $t'_1 t'_2 \cdots t'_{k-1} = t_1 \cdots t_{i_0-1} t_{i_0+1} \cdots t_k$. It follows that

$$v=(t'_1t'_2\cdots t'_{k-1})t=t_1\cdots t_{i_0-1}t_{i_0+1}\cdots t_kr_{i_0}=t_1t_2\cdots t_k=w,$$

which leads to a contradiction. Therefore, we have $\langle x,y\rangle=0$ for any $x\in\mathcal{A}_v$ and $y\in\mathcal{A}_w$ with $w\neq v$.

We proceed next to prove claim (2). To this end, we need to introduce a lexicographical order on the basis of $\mathcal{A}_w$, and then show that the corresponding matrix of the bilinear form is upper triangular with diagonal entries being 1.

Recall from Proposition 3.4 that $\mathcal{A}_w$ has a basis consisting of elements $a_{t_1}a_{t_2}\cdots a_{t_k}$, where $w=t_1t_2\cdots t_k$ is a T-reduced expression and $t_1\succ t_2\succ\cdots\succ t_k$ with respect to the total order of T. We define the lexicographical order on the basis by

$$a_{t_1}a_{t_2}\cdots a_{t_k}<a_{t'_1}a_{t'_2}\cdots a_{t'_k}\iff t_i\prec t'_i, \tag{5.1}$$

where i is the smallest index where the two monomials differ. Let $\{m_1<m_2<\cdots<m_p\}$ be the totally ordered set of the basis of $\mathcal{A}_w$, and let M be the matrix of the bilinear form with the (i,j)-th entry being $\langle m_i,m_j\rangle$. Next we show that M is a upper-triangular matrix with all diagonal entires equal to 1.

To show that M is upper-triangular, we need to prove that

$$\langle a_{t'_1}a_{t'_2}\cdots a_{t'_k},a_{t_1}a_{t_2}\cdots a_{t_k}\rangle=0 \tag{5.2}$$

for any two basis elements $a_{t_1}a_{t_2}\cdots a_{t_k}<a_{t'_1}a_{t'_2}\cdots a_{t'_k}$ of $\mathcal{A}_w$. Note that $t_1\succ t_2\succ\cdots\succ t_k$ and $t'_1\succ t'_2\succ\cdots\succ t'_k$ with respect to the total order on T. Assuming that $t_1=t'_1,\ldots,t_{i-1}=t'_{i-1}$ and $t_i\prec t'_i$, we have

$$\begin{aligned}\langle a_{t'_1}a_{t'_2}\cdots a_{t'_k},a_{t_1}a_{t_2}\cdots a_{t_k}\rangle&=\langle a_{t'_2}\cdots a_{t'_k},\delta_{t'_1}(a_{t_1}a_{t_2}\cdots a_{t_k})\rangle\\&=\langle a_{t'_2}\cdots a_{t'_k},a_{t_2}\cdots a_{t_k}\rangle,\\&\cdots\\&=\langle a_{t'_i}a_{t'_{i+1}}\cdots a_{t'_k},a_{t_i}a_{t_{i+1}}\cdots a_{t_k}\rangle,\end{aligned}$$

where in the first line we have used the adjoint property from Lemma 5.4, and in the third line we repeat the process until we obtain the last equation. Using the adjoint property again, we have

$$\langle a_{t'_i}a_{t'_{i+1}}\cdots a_{t'_k},a_{t_i}a_{t_{i+1}}\cdots a_{t_k}\rangle=\langle a_{t'_{i+1}}\cdots a_{t'_k},\delta_{t'_i}(a_{t_i}a_{t_{i+1}}\cdots a_{t_k})\rangle.$$

Since $t'_i \succ t_i \succ t_{i+1} \succ \cdots \succ t_k$, by the definition of $\delta_{t'_i}$ we have

$$\delta_{t'_i}(a_{t_i}a_{t_{i+1}}a_{t_2}\cdots a_{t_k}) = 0.$$

Hence we have $\langle a_{t'_1}a_{t'_2}\cdots a_{t'_k}, a_{t_1}a_{t_2}\cdots a_{t_k}\rangle = 0$.

Applying the adjunction between δ_t and left multiplication by a_t repeatedly, we have

$$\begin{aligned} \langle a_{t_1}a_{t_2}\cdots a_{t_k}, a_{t_1}a_{t_2}\cdots a_{t_k}\rangle &= \langle a_{t_2}\cdots a_{t_k}, \delta_{t_1}(a_{t_1}a_{t_2}\cdots a_{t_k})\rangle \\ &= \langle a_{t_2}\cdots a_{t_k}, a_{t_2}\cdots a_{t_k}\rangle \\ &= \cdots = 1 \end{aligned} \tag{5.3}$$

for any basis element $a_{t_1}a_{t_2}\cdots a_{t_k}$ of $\mathcal{A}_w$. Hence the diagonal entries of M are all equal to 1. Therefore, the bilinear form is unimodular on $\mathcal{A}_w$. Combing this with claim (1), we complete the proof. $\square$

5.2. Acyclic dual complexes

In this subsection, we introduce some acyclic cochain complexes which are dual to the previous acyclic chain complexes with respect to suitable bilinear forms.

We recall the universal coefficient theorem for cohomology, which will be used later.

Theorem 5.6. *[Hat02, Section 3.1] Let G be an abelian group, and $\mathcal{C}$ be the following chain complex of free abelian groups*

$$0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0.$$

Let $C_k^* = \operatorname{Hom}_{\mathbb{Z}}(C_k, G)$ be the dual of the chain group C_k and $\partial^* : C_{k-1}^* \to C_k^*$ be the dual coboundary map for $1 \leq k \leq n$. Then the cohomology groups $H^k(\mathcal{C}; G)$ of the cochain complex

$$0 \longrightarrow C_0^* \xrightarrow{\partial^*} C_1^* \longrightarrow \cdots \longrightarrow C_{n-1}^* \xrightarrow{\partial^*} C_n^* \longrightarrow 0$$

are determined by split exact sequences

$$0 \longrightarrow \operatorname{Ext}(H_{k-1}(\mathcal{C}), G) \longrightarrow H^k(\mathcal{C}; G) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_k(\mathcal{C}), G) \longrightarrow 0.$$

We are only concerned with the case $G = \mathbb{Z}$. If the homology groups $H_k(\mathcal{C})$ are finitely generated abelian groups with torsion subgroups $T_k \subseteq H_k(\mathcal{C})$, then the homology of the chain complex $(\mathcal{C}, \partial)$ and the cohomology of the dualised chain complex $(\mathcal{C}^*, \partial^*)$ are related by

$$H^k(\mathcal{C}^*; \mathbb{Z}) \cong H_k(\mathcal{C})/T_k \oplus T_{k-1}.$$

In particular, if $(\mathcal{C}, \partial)$ is acyclic then so is $(\mathcal{C}^*, \partial^*)$. If $\mathcal{C}$ consists of finitely generated free abelian groups, then $(\mathcal{C}^{**}, \partial^{**}) \cong (\mathcal{C}, \partial)$.

5.2.1. A pair of acyclic complexes Using the bilinear form on $\mathcal{A}$, we define acyclic cochain complexes which are dual to the acyclic chain complexes $(\mathcal{A}, d)$ and $(\mathcal{A}, \delta)$.

Define the element

$$\omega := \sum_{t \in T} a_t \in \mathcal{A}_1.$$

It is easily verified that $\omega^2 = 0$. Hence it gives rise to a cochain complex whose coboundary maps are right multiplication by ω:

$$0 \longrightarrow \mathcal{A}_0 \xrightarrow{r_\omega} \mathcal{A}_1 \longrightarrow \cdots \xrightarrow{r_\omega} \mathcal{A}_n \longrightarrow 0.$$

We denote this complex by $(\mathcal{A}, r_\omega)$. Similarly, we define the complex $(\mathcal{A}, \ell_\omega)$, where ℓ_ω denotes left multiplication by ω.

Recall from Proposition 5.5 that the bilinear form induces an isomorphism of graded free abelian groups:

$$\psi : \mathcal{A} \to \mathcal{A}^*, \quad x \mapsto \psi(x) := \langle -, x \rangle, \tag{5.4}$$

where $\psi(x)(x') = \langle x', x \rangle$ for any $x, x' \in \mathcal{A}_k$. Therefore, for any linear form $\lambda \in \mathcal{A}^*$, there exists a unique $x \in \mathcal{A}$ such that $\lambda = \psi(x)$. In what follows, a linear map on $\mathcal{A}^*$ will be defined by its action on elements of the form $\psi(x), x \in \mathcal{A}$.

Lemma 5.7. *The linear map ψ induces an isomorphism between chain complexes $(\mathcal{A}, d)$ and $(\mathcal{A}^*, r_\omega^*)$, where r_ω^* is the adjoint map of r_ω defined by*

$$r_\omega^*(\psi(y))(x) := \psi(y)(r_\omega(x)) = \langle x\omega, y \rangle, \quad \forall x \in \mathcal{A}_{k-1}, y \in \mathcal{A}_k.$$

Similarly, ψ induces an isomorphism between chain complexes $(\mathcal{A}, \delta)$ and $(\mathcal{A}^*, \ell_\omega^*)$

Proof. For any integer k, it suffices to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_k & \xrightarrow{d} & \mathcal{A}_{k-1} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \mathcal{A}_k^* & \xrightarrow{r_\omega^*} & \mathcal{A}_{k-1}^*. \end{array}$$

We verify this directly. For any $x \in \mathcal{A}_k, y \in \mathcal{A}_{k-1}$, by the definition of r_ω^* we have

$$((r_\omega^*\psi)(x))(y) = r_\omega^*(\psi(x))(y) = \langle y\omega, x\rangle.$$

On the other hand, we have

$$((\psi d)(x))(y) = \psi(dx)(y) = \langle y, dx\rangle.$$

Using the adjointness property from Lemma 5.4, we have $\langle y\omega, x\rangle = \langle y, dx\rangle$, whence $r_\omega^*\psi = \psi d$. Therefore, ψ is a chain map between $(\mathcal{A}, d)$ and $(\mathcal{A}^*, r_\omega^*)$. As ψ is a graded isomorphism, $(\mathcal{A}, d)$ and $(\mathcal{A}^*, r_\omega^*)$ are isomorphic. The isomorphism between $(\mathcal{A}, \delta)$ and $(\mathcal{A}^*, \ell_\omega^*)$ can be proved similarly. ☐

Proposition 5.8. *The cochain complexes $(\mathcal{A}, r_\omega)$ and $(\mathcal{A}, \ell_\omega)$ are both acyclic.*

Proof. Recall from Proposition 4.1 that the chain complex $(\mathcal{A}, d)$ is acyclic. It follows from Lemma 5.7 that the chain complex $(\mathcal{A}^*, r_\omega^*)$ is acyclic. Since $(\mathcal{A}^*, r_\omega^*)$ is the dual complex of $(\mathcal{A}, r_\omega)$ induced by the isomorphism ψ, the cochain complex $(\mathcal{A}, r_\omega)$ is acyclic by the universal coefficient theorem for cohomology. Similarly, one can prove that $(\mathcal{A}, \ell_\omega)$ is acyclic, using the acyclicity of $(\mathcal{A}, \delta)$ from Proposition 4.2. ☐

5.2.2. Another pair of acyclic complexes Recall from Theorem 4.5 that the boundary maps of the complex $(\mathbb{Z}W \otimes \mathcal{A}_k, \partial_k)$ are given by $\partial_k = \partial'_k + (-1)^k\partial''_k$, where the linear maps $\partial', \partial'' : \mathbb{Z}W \otimes \mathcal{A}_k \to \mathbb{Z}W \otimes \mathcal{A}_{k-1}$ are defined by

$$\begin{aligned} \partial'(w \otimes a_{t_1}a_{t_2}\cdots a_{t_k}) &= \sum_{i=1}^{k}(-1)^{i-1}wt_i \otimes a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}\widehat{a}_{t_i}a_{t_{i+1}}\cdots a_{t_k}, \\ \partial''(w \otimes a_{t_1}a_{t_2}\cdots a_{t_k}) &= \sum_{i=1}^{k}(-1)^{k-i}w \otimes a_{t_1}\cdots \widehat{a}_{t_i}\cdots a_{t_k}. \end{aligned} \tag{5.5}$$

In what follows, whenever dealing with (co)boundary maps, $\mathbb{Z}W$ is viewed as a right $\mathbb{Z}W$-module while any $t \in T$ acts on the right side.

Lemma 5.9. *Let ∂', ∂'' be as in* (5.5). *Then we have*

$$\partial' = \sum_{t\in T} t\otimes\delta_t, \quad \partial'' = \sum_{t\in T} 1\otimes d_t = 1\otimes d.$$

Therefore, ∂', ∂'' satisfy $(\partial')^2 = (\partial'')^2 = 0$ and $\partial'\partial'' = \partial''\partial'$.

Proof. For any nonzero element $a_{t_1}\cdots a_{t_k}$ we have

$$\begin{aligned}\sum_{t\in T} t\otimes\delta_t(w\otimes a_{t_1}a_{t_2}\cdots a_{t_k}) &= \sum_{t\in T}\sum_{i=1}^{k}(-1)^{i-1}wt\otimes\delta_{t,t_i}a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}\widehat{a}_{t_i}a_{t_{i+1}}\cdots a_{t_k}\\ &= \sum_{i=1}^{k}(-1)^{i-1}wt_i\otimes a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}\widehat{a}_{t_i}a_{t_{i+1}}\cdots a_{t_k}.\end{aligned}$$

It follows that $\partial' = \sum_{t\in T} t\otimes\delta_t$, and similarly, $\partial'' = \sum_{t\in T} 1\otimes d_t$. Using the fact that $\delta_t^2 = d_t^2 = 0$ for any $t\in T$, we obtain $(\partial')^2 = (\partial'')^2 = 0$. By Lemma 5.1 d_t and $\delta_{t'}$ commute with each other, we have

$$\partial'\partial'' = \sum_{t,t'\in T} t\otimes\delta_t d_{t'} = \sum_{t,t'\in T} t\otimes d_{t'}\delta_t = \partial''\partial'.$$

$\square$

By Lemma 5.9, we have a pair of chain complexes $(\mathbb{Z}W\otimes\mathcal{A}, \partial')$ and $(\mathbb{Z}W\otimes\mathcal{A}, \partial'')$. We proceed next to define a bilinear form on $\mathbb{Z}W\otimes\mathcal{A}$, which enables us to dualise the chain complexes.

We will always denote by $\langle -,-\rangle$ the bilinear form on a linear space whenever no confusion arises. Recall that there is a standard $\mathbb{Z}$-bilinear form on $\mathbb{Z}W$ defined by

$$\langle -,-\rangle : \mathbb{Z}W\times\mathbb{Z}W\to\mathbb{Z}, \quad \langle v,w\rangle := \delta_{v,w}, \quad \forall v,w\in W,$$

where δ is the Kronecker delta. This together with the bilinear form on $\mathcal{A}$ which we have already defined, gives rise to a $\mathbb{Z}$-bilinear form $\langle -,-\rangle$ on $\mathbb{Z}W\otimes\mathcal{A}$, defined by

$$\langle v\otimes x, w\otimes y\rangle := \langle v,w\rangle\,\langle x,y\rangle, \quad \forall v,w\in W, x,y\in\mathcal{A}.$$

Note that $\mathbb{Z}W\otimes\mathcal{A}$ is a $\mathbb{Z}$-graded algebra, with the grading inherited from $\mathcal{A}$ and multiplication given by $(v\otimes x)(w\otimes y) = vw\otimes xy$. By definition, we have $\langle\mathbb{Z}W\otimes\mathcal{A}_k, \mathbb{Z}W\otimes\mathcal{A}_\ell\rangle = 0$ for any integers $k\neq\ell$.

It follows from Proposition 5.5 that the bilinear form on $\mathbb{Z}W \otimes \mathcal{A}$ is also unimodular. Hence we have the following graded isomorphism of free abelian groups:

$$\psi_W : \mathbb{Z}W \otimes \mathcal{A} \to (\mathbb{Z}W \otimes \mathcal{A})^*, \quad w \otimes x \mapsto \psi_W(w \otimes x) := \langle -, w \otimes x\rangle, \tag{5.6}$$

where $w \in W$ and $x \in \mathcal{A}_k$ for $0 \leq k \leq n$.

We introduce the following elements:

$$\sigma = \sum_{t\in T} t \otimes a_t, \quad \varsigma = \sum_{t \in T} 1 \otimes a_t = 1 \otimes \omega.$$

It is easily verified that $\sigma^2 = \varsigma^2 = 0$ in $\mathbb{Z}W \otimes \mathcal{A}$. Hence σ and ς give rise to two cochain complexes $(\mathbb{Z}W \otimes \mathcal{A}, \ell_\sigma)$ and $(\mathbb{Z}W \otimes \mathcal{A}, r_\varsigma)$, where the coboundary maps are given by

$$\ell_\sigma(w \otimes x) = \sum_{t\in T} wt \otimes a_t x, \quad r_\varsigma(w \otimes x) = \sum_{t\in T} w \otimes x a_t.$$

for any $w \otimes x \in \mathbb{Z}W \otimes \mathcal{A}_k$.

Lemma 5.10. *For any $v, w \in W$ and $x, y \in \mathcal{A}$, we have*

$$\langle \ell_\sigma(v \otimes x), w \otimes y\rangle = \langle v \otimes x, \partial'(w \otimes y)\rangle,$$
$$\langle r_\varsigma(v \otimes x), w \otimes y\rangle = \langle v \otimes x, \partial''(w \otimes y)\rangle.$$

Therefore, with respect to the bilinear form on $\mathbb{Z}W \otimes \mathcal{A}$, ∂' and ∂'' are right adjoint to the linear operators ℓ_σ and r_ς, respectively.

Proof. We only prove the first adjunction; the second one can be treated similarly. Assume that $x \in \mathcal{A}_{k-1}$ and $y \in \mathcal{A}_k$ for some $k = 0, \dots, n$. Then we have

$$\begin{aligned}\langle \ell_\sigma(v \otimes x), w\rangle &= \sum_{t\in T} \langle vt \otimes a_t x, w \otimes y\rangle = \sum_{t \in T}\langle vt, w\rangle \, \langle a_t x, y)\\ &= \sum_{t\in T} \langle v, wt\rangle \, \langle x, \delta_t(y)) = \sum_{t\in T} \langle v \otimes x, (t \otimes \delta_t)(w \otimes y)\rangle \\ &= \langle v \otimes x, \partial'(w \otimes y)\rangle,\end{aligned}$$

where the third equation follows from the adjoint property in Lemma 5.4, and last equation is a consequence of Lemma 5.9. ☐

Proposition 5.11. *The chain complexes $(\mathbb{Z}W \otimes \mathcal{A}, \partial')$ and $(\mathbb{Z}W \otimes \mathcal{A}, \partial'')$ are both acyclic.*

Proof. By Lemma 5.9, we have $\partial'' = 1 \otimes d$. As $\mathbb{Z}W$ is a flat $\mathbb{Z}$-module, the acyclicity of $(\mathbb{Z}W \otimes \mathcal{A}, \partial'')$ follows from that of $(\mathcal{A}, d)$, which is given in Proposition 4.1.

It remains to prove that $(\mathbb{Z}W \otimes \mathcal{A}, \partial')$ is acyclic. Recall from (4.3) the Kereweras automorphism $\kappa : \mathcal{A} \to \mathcal{A}$ whose inverse is given by

$$\kappa^{-1}(a_{t_1}a_{t_2}\cdots a_{t_k}) = a_{t_k}a_{t_{k-1}}^{t_k}\cdots a_{t_1}^{t_2\cdots t_k}.$$

Note that this is a graded automorphism. Applying $1 \otimes \kappa^{-1}$ to the chain complex $(\mathbb{Z}W \otimes \mathcal{A}, \partial')$, we obtain a new chain complex $(\mathbb{Z}W \otimes \mathcal{A}, \widetilde{\partial}')$ whose boundary map satisfies $\widetilde{\partial}'(1 \otimes \kappa^{-1}) = (1 \otimes \kappa^{-1})\partial'$, defined explicitly by

$$\widetilde{\partial}'(w \otimes a_{t_1}a_{t_2}\cdots a_{t_k}) = \sum_{i=1}^{k}(-1)^{k-i}wt_i^{t_{i-1}\cdots t_1} \otimes a_{t_1}\cdots \hat{a}_{t_i}\cdots a_{t_k} \tag{5.7}$$

for any $w \in W$ and nonzero element $a_{t_1}a_{t_2}\cdots a_{t_k} \in \mathcal{A}_k$. Hence we are reduced to proving the acyclicity of $(\mathbb{Z}W \otimes \mathcal{A}, \widetilde{\partial}')$.

For each $k = 0, 1, \ldots, n$, we have the following split exact sequence

$$0 \to \operatorname{Ker}\widetilde{\partial}'_k \to \mathbb{Z}W \otimes \mathcal{A}_k \to \operatorname{Im}\widetilde{\partial}'_k \to 0. \tag{5.8}$$

To prove that the homology $H_k = \operatorname{Ker}\widetilde{\partial}'_k/\operatorname{Im}\widetilde{\partial}'_{k+1}$ is trivial, we shall determine each image $\operatorname{Im}\widetilde{\partial}'_k$, and then use the exact sequence above to show that $\operatorname{Im}\widetilde{\partial}'_{k+1}$ and $\operatorname{Ker}\widetilde{\partial}'_k$ have equal ranks for all k. Finally, we show that the homology is torsion free and hence is trivial.

Let us start with a combinatorial description of the chain group $\mathbb{Z}W \otimes \mathcal{A}_k$. As the chain complex $(\mathcal{A}, d)$ is acyclic, we have decompositions of free abelian groups $\mathcal{A}_k = d(\mathcal{A}_k) \oplus d(\mathcal{A}_{k+1})$ for $0 \le k \le n-1$. It is proved in [Zha23, Theorem 4.8] that $d(\mathcal{A}_k)$ has a $\mathbb{Z}$-basis consisting of elements $d(a_{t_1}\cdots a_{t_k})$ for $1 \le k \le n$, where $(t_1, \ldots, t_k) \in \mathcal{D}_{[k-1]}$ with $\mathcal{D}_{[k-1]}$ defined by

$$\mathcal{D}_{[k-1]} := \left\{(t_1, \ldots, t_k) \;\middle|\; \begin{array}{c} \gamma = t_1t_2\cdots t_n \text{ is } T\text{-reduced and} \\ t_1 \succ \cdots \succ t_{k-1} \succ t_k \prec t_{k+1} \prec \cdots \prec t_n \end{array}\right\}.$$

Then we have

$$\operatorname{rank}(\mathbb{Z}W \otimes \mathcal{A}_k) = |W|(\mathcal{D}_{[k]} + \mathcal{D}_{[k-1]}), \quad 0 \le k \le n.$$

Moreover, recall from that $\mathcal{A}_k$ has a $\mathbb{Z}$-basis $\{a_{\mathbf{t}} = a_{t_1}\cdots a_{t_k} | \mathbf{t} = (t_1, \ldots, t_k) \in \bigcup_{u \in \mathcal{L}_k}\mathcal{D}_u\}$, where $\mathcal{D}_u$ is defined as in (2.3). Therefore, $\{w \otimes a_{\mathbf{t}} | w \in W, \mathbf{t} \in \bigcup_{u \in \mathcal{L}_k}\mathcal{D}_u\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}W \otimes \mathcal{A}_k$.

Now we consider the image $\operatorname{Im}\widetilde{\partial}'_k$. Denote by $B_k \subseteq \mathbb{Z}W \otimes \mathcal{A}_{k-1}$ the free abelian group spanned by the set $S_{[k-1]} := \{\widetilde{\partial}'(w \otimes a_{\mathbf{t}}) \mid w \in W, \mathbf{t} \in \mathcal{D}_{[k-1]}\}$. Then we have $B_k \subseteq \operatorname{Im}\widetilde{\partial}'_k$. We shall prove that the set $S_{[k-1]}$ is $\mathbb{Z}$-linearly independent. To this end, we choose a total order on the basis of $\mathbb{Z}W \otimes \mathcal{A}_k$ for each k such that

$$w \otimes a_{\mathbf{t}} < v \otimes a_{\mathbf{t}'} \quad \text{whenever} \quad a_{\mathbf{t}} < a_{\mathbf{t}'}, \tag{5.9}$$

for any $v, w \in W$ and $\mathbf{t}, \mathbf{t}' \in \bigcup_{u \in \mathcal{L}_k} \mathcal{D}_u$, where $a_{\mathbf{t}} < a_{\mathbf{t}'}$ is the total order defined in (5.1). Then it follows from (5.2) that for any pair of basis elements $w \otimes a_{\mathbf{t}} < v \otimes a_{\mathbf{t}'}$

$$\langle v \otimes a_{\mathbf{t}'}, w \otimes a_{\mathbf{t}} \rangle = 0. \tag{5.10}$$

Now assume that there exist nonzero $\lambda_{w_0,\mathbf{t}_0}, \lambda_{w,\mathbf{t}} \in \mathbb{Z}$ such that

$$\lambda_{w_0,\mathbf{t}_0}\widetilde{\partial}'(w_0 \otimes a_{\mathbf{t}_0}) = \sum_{w_0 \otimes a_{\mathbf{t}_0} > w \otimes a_{\mathbf{t}}} \lambda_{w,\mathbf{t}}\widetilde{\partial}'(w \otimes a_{\mathbf{t}}),$$

for some elements $\widetilde{\partial}'(w \otimes a_{\mathbf{t}})$ and $\widetilde{\partial}'(w_0 \otimes a_{\mathbf{t}_0})$ of the set $S_{[k-1]}$. In view of the definition (5.7), for any element $w \otimes a_{\mathbf{t}} = w \otimes a_{t_1} \cdots a_{t_k}$ with $t_1 \succ \cdots \succ t_k$, the element

$$w \otimes a_{\mathbf{t}}(\hat{k}) := wt_k^{t_{k-1}\cdots t_1} \otimes a_{t_1} \cdots a_{t_{k-1}}$$

is the unique maximal term under the total order (5.9) in the expression of $\widetilde{\partial}'(w \otimes a_{t_1} \cdots a_{t_k})$. Hence by (5.10) and (5.3) we have $\langle w \otimes a_{\mathbf{t}}(\hat{k}), \widetilde{\partial}'(w \otimes a_{\mathbf{t}}) \rangle = 1$. Moreover, observe that if $w_0 \otimes a_{\mathbf{t}_0} > w \otimes a_{\mathbf{t}}$ for $\mathbf{t}, \mathbf{t}_0 \in \mathcal{D}_{[k-1]}$, then $w_0 \otimes a_{\mathbf{t}_0}(\hat{k}) > w \otimes a_{\mathbf{t}}(\hat{k})$ (cf. [Zha23, Proposition 4.7]). This further implies that $\langle w_0 \otimes a_{\mathbf{t}_0}(\hat{k}), \widetilde{\partial}'(w \otimes a_{\mathbf{t}}) \rangle = 0$ by (5.10). Therefore, we obtain

$$\begin{aligned}
\lambda_{w_0,\mathbf{t}_0} &= \langle w_0 \otimes a_{\mathbf{t}_0}(\hat{k}), \lambda_{w_0,\mathbf{t}_0}\widetilde{\partial}'(w_0 \otimes a_{\mathbf{t}_0}) \rangle \\
&= \sum_{w_0 \otimes a_{\mathbf{t}_0} > w \otimes a_{\mathbf{t}}} \langle w_0 \otimes a_{\mathbf{t}_0}(\hat{k}), \lambda_{w,\mathbf{t}}\widetilde{\partial}'(w \otimes a_{\mathbf{t}}) \rangle \\
&= 0,
\end{aligned}$$

which contradicts our assumption that $\lambda_{w,\mathbf{t}_0} \neq 0$. Therefore, $S_{[k-1]}$ is a $\mathbb{Z}$-linearly independent set, which spans the free abelian subgroup $B_k \subseteq \operatorname{Im}\widetilde{\partial}'_k$.

Now by the exact sequence (5.8) we obtain

$$\begin{aligned}\operatorname{rank}\operatorname{Ker}\widetilde{\partial}'_k &= \operatorname{rank}\mathbb{Z}W\otimes\mathcal{A}_k - \operatorname{rank}\operatorname{Im}\widetilde{\partial}'_k \le \operatorname{rank}\mathbb{Z}W\otimes\mathcal{A}_k - \operatorname{rank}B_k\\ &= |W|(\mathcal{D}_{[k]}+\mathcal{D}_{[k-1]}) - |W||\mathcal{D}_{[k-1]}|\\ &= |W||\mathcal{D}_{[k]}|.\end{aligned}$$

On the other hand, since $B_{k+1}\subseteq \operatorname{Im}\widetilde{\partial}'_{k+1}\subseteq \operatorname{Ker}\widetilde{\partial}'_k$, we have

$$\operatorname{rank}\operatorname{Ker}\widetilde{\partial}'_k \ge \operatorname{rank}\operatorname{Im}\widetilde{\partial}'_{k+1} \ge \operatorname{rank}B_{k+1} = |W||\mathcal{D}_{[k]}|.$$

Combing the above two inequalities, for each k we have $\operatorname{rank}\operatorname{Ker}\widetilde{\partial}'_k = |W||\mathcal{D}_{[k]}|$, and

$$\operatorname{rank}\operatorname{Im}\widetilde{\partial}'_k = \operatorname{rank}\mathbb{Z}W\otimes\mathcal{A}_k - \operatorname{rank}\operatorname{Ker}\widetilde{\partial}'_k = |W||\mathcal{D}_{[k-1]}|.$$

It follows that $\operatorname{Im}\widetilde{\partial}'_{k+1}$ and $\operatorname{Ker}\widetilde{\partial}'_k$ have equal rank.

It remains to show that the homology $H_k = \operatorname{Ker}\widetilde{\partial}'_k/\operatorname{Im}\widetilde{\partial}'_{k+1}$ is trivial. We aim to prove that $B_{k+1}=\operatorname{Ker}\widetilde{\partial}'_k=\operatorname{Im}\widetilde{\partial}'_{k+1}$. By the above arguments on ranks, $B_{k+1}\subseteq\operatorname{Ker}\widetilde{\partial}'_k$ is a free abelian subgroup of maximal rank. Therefore, every element of the quotient group $\operatorname{Ker}\widetilde{\partial}'_k/B_{k+1}$ has finite order. For any $\beta\in\operatorname{Ker}\widetilde{\partial}'_k$, there exists a nonzero integer m such that

$$m\beta = \sum_{w\in W,\mathbf{t}\in\mathcal{D}_{[k]}} \lambda_{w,\mathbf{t}}\widetilde{\partial}'_{k+1}(w\otimes a_{\mathbf{t}}) \in B_{k+1},\quad \lambda_{w,\mathbf{t}}\in\mathbb{Z}. \tag{5.11}$$

Suppose that $w_0\otimes\mathbf{t}_0$ is the biggest element under the total order (5.9) such that $m\nmid\lambda_{w_0,\mathbf{t}_0}$. Then

$$\begin{aligned}&m\beta - \sum_{w\otimes\mathbf{t}>w_0\otimes\mathbf{t}_0}\lambda_{w,\mathbf{t}}\widetilde{\partial}'_{k+1}(w\otimes a_{\mathbf{t}})\\ &= \lambda_{w_0,\mathbf{t}_0}\widetilde{\partial}'_{k+1}(w_0\otimes a_{\mathbf{t}_0}) + \sum_{w\otimes\mathbf{t}<w_0\otimes\mathbf{t}_0}\lambda_{w,\mathbf{t}}\widetilde{\partial}'_{k+1}(w\otimes a_{\mathbf{t}}).\end{aligned}$$

Recalling that $\langle w_0\otimes a_{\mathbf{t}_0}(\hat{k}), \widetilde{\partial}'_{k+1}(w\otimes a_{\mathbf{t}})\rangle = 0$ for any $w\otimes\mathbf{t}<w_0\otimes\mathbf{t}_0$ with $\mathbf{t},\mathbf{t}_0\in\mathcal{D}_{[k]}$ and $\langle w_0\otimes a_{\mathbf{t}_0}(\hat{k}), \widetilde{\partial}'_{k+1}(w_0\otimes a_{\mathbf{t}_0})\rangle = 1$, we have

$$\langle w_0\otimes a_{\mathbf{t}_0}(\hat{k}), m\beta - \sum_{w\otimes\mathbf{t}>w_0\otimes\mathbf{t}_0}\lambda_{w,\mathbf{t}}\widetilde{\partial}'_{k+1}(w\otimes a_{\mathbf{t}})\rangle = \lambda_{w_0,\mathbf{t}_0}.$$

By the choice of $w_0\otimes\mathbf{t}_0$, we have $m|\lambda_{w,\mathbf{t}}$ for any $w\otimes\mathbf{t}>w_0\otimes\mathbf{t}_0$. Thus the left hand side is divisible by m, so is $\lambda_{w_0,\mathbf{t}_0}$ on the right hand side. This contradicts our assumption for $\lambda_{w_0,\mathbf{t}_0}$. Therefore, all coefficients $\lambda_{w,\mathbf{t}}$ in (5.11) are divisible by m, and hence $\beta \in B_{k+1}$. This proves that $B_{k+1} = \operatorname{Ker} \widetilde{\partial}'_k$. Similarly, one can prove that $B_{k+1} = \operatorname{Im} \widetilde{\partial}'_{k+1}$. Thus the homology group $H_k = \operatorname{Ker} \widetilde{\partial}'_k / \operatorname{Im} \widetilde{\partial}'_{k+1}$ is trivial. □

Proposition 5.12. *The cochain complexes* $(\mathbb{Z}W \otimes \mathcal{A}, \ell_\sigma)$ *and* $(\mathbb{Z}W \otimes \mathcal{A}, r_\varsigma)$ *are both acyclic.*

Proof. By Lemma 5.10 ∂' is right adjoint to ℓ_σ. Using the same method as in Lemma 5.7, one can show that the isomorphism ψ_W given in (5.6) induces an isomorphism between chain complexes $(\mathbb{Z}W \otimes \mathcal{A}, \partial')$ and $((\mathbb{Z}W \otimes \mathcal{A})^*, \ell_\sigma^*)$. The former is acyclic by Proposition 5.11, so is the latter. Hence $(\mathbb{Z}W \otimes \mathcal{A}, \ell_\sigma)$ is acyclic by the universal coefficient theorem for cohomology. Similarly, one can prove the acyclicity of $(\mathbb{Z}W \otimes \mathcal{A}, r_\varsigma)$. □

5.3. Dual complexes for Milnor fibres and hyperplane complements

In terms of the bilinear forms on $\mathcal{A}$ and $\mathbb{Z}W \otimes \mathcal{A}$, we shall give complexes which are dual to the chain complexes introduced in Section 4.2. These dual complexes have the same integral cohomology as that of the Milnor fibres and hyperplane complements.

5.3.1. Dual complexes for M and M/W Recall from Theorem 4.5 the chain complex which computes the integral homology of the hyperplane complement M. The following gives a cochain complex dual to the chain complex.

Theorem 5.13. *The integral cohomology of the hyperplane complement M is isomorphic to the cohomology of the following cochain complex of free abelian groups:*

$$0 \longrightarrow \mathbb{Z}W \otimes \mathcal{A}_0 \xrightarrow{\partial^0} \mathbb{Z}W \otimes \mathcal{A}_1 \xrightarrow{\partial^1} \cdots \longrightarrow \mathbb{Z}W \otimes \mathcal{A}_n \longrightarrow 0,$$

where the coboundary maps are given by $\partial^k := \ell_\sigma - (-1)^k r_\varsigma$ *for* $0 \le k \le n$, *i.e.*

$$\partial^k(w \otimes x) = \sum_{t \in T} wt \otimes a_t x - (-1)^k \sum_{t \in T} w \otimes x a_t, \quad \forall x \in \mathcal{A}_k, w \in W.$$

Proof. Recall that the chain complex given in Theorem 4.5 computes the integral homology of M. We dualise this chain complex by replacing the k^{th}

chain group with $(\mathbb{Z}W \otimes \mathcal{A}_k)^* \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}W \otimes \mathcal{A}_k, \mathbb{Z})$ and the k^{th} boundary map by its dual coboundary map $\partial^* : (\mathbb{Z}W \otimes \mathcal{A}_{k-1})^* \to (\mathbb{Z}W \otimes \mathcal{A}_k)^*$. Then we obtain the following cochain complex:

$$\mathcal{C}(W)^* : 0 \longrightarrow (\mathbb{Z}W \otimes \mathcal{A}_0)^* \xrightarrow{\partial^{*0}} (\mathbb{Z}W \otimes \mathcal{A}_1)^* \xrightarrow{\partial^{*1}} \cdots \longrightarrow (\mathbb{Z}W \otimes \mathcal{A}_n)^* \longrightarrow 0.$$

By the universal coefficient theorem for cohomology, we have $H^k(\mathcal{C}(W)^*) \cong H^k(M;\mathbb{Z})$ for $0 \le k \le n$. It remains to show that $\mathcal{C}(W)^*$ and the complex given in the theorem have the same cohomology.

Recall the the bilinear form $\langle -, - \rangle$ on $\mathbb{Z}W \otimes \mathcal{A}$ is unimodular. This gives rises to the isomorphism:

$$\psi'_W : \mathbb{Z}W \otimes \mathcal{A} \to (\mathbb{Z}W \otimes \mathcal{A})^*, \quad v \otimes x \mapsto \psi'_W(v \otimes x) := \langle v \otimes x, - \rangle \tag{5.12}$$

for any $v \in W$ and $x \in \mathcal{A}$. Now consider the following diagram:

$$\begin{array}{ccc} \mathbb{Z}W \otimes \mathcal{A}_k & \xrightarrow{\partial^k} & \mathbb{Z}W \otimes \mathcal{A}_{k+1} \\ \big\downarrow{\scriptstyle \psi'_W} & & \big\downarrow{\scriptstyle \psi'_W} \\ (\mathbb{Z}W \otimes \mathcal{A}_k)^* & \xrightarrow{\partial^{*k}} & (\mathbb{Z}W \otimes \mathcal{A}_{k+1})^*. \end{array}$$

For any $v, w \in W$, $x \in \mathcal{A}_k$ and $y \in \mathcal{A}_{k+1}$, we have

$$\begin{aligned} \psi'_W(\partial^k(v \otimes x))(w \otimes y) &= \langle \partial^k(v \otimes x), w \otimes y \rangle \\ &= \langle \ell_\sigma(v \otimes x) - (-1)^k r_\varsigma(v \otimes x), w \otimes y \rangle \\ &= \langle v \otimes x, (\partial'_k + (-1)^{k+1}\partial''_k)(w \otimes y) \rangle, \\ &= \partial^{*k}(\psi'_W(v \otimes x))(w \otimes y). \end{aligned}$$

where the third equation follows from Lemma 5.10 and the last equation follows from the fact that $\partial_k = \partial'_k + (-1)^{k+1}\partial''_k$. Therefore, ψ'_W is a chain isomorphism between $(\mathbb{Z}W \otimes \mathcal{A}, \partial^k)$ and $\mathcal{C}(W)^*$, and hence these two cochain complexes have the same cohomology. $\square$

Theorem 5.14. *The integral cohomology of M/W or the Artin group $A(W)$ is isomorphic to the cohomology of the following cochain complex of free abelian groups:*

$$0 \longrightarrow \mathcal{A}_0 \xrightarrow{\partial^0} \mathcal{A}_1 \xrightarrow{\partial^1} \cdots \longrightarrow \mathcal{A}_n \longrightarrow 0,$$

where $\partial^k = \ell_\omega - (-1)^k r_\omega$ for $0 \le k \le n$ with $\omega = \sum_{t \in T} a_t$, i.e.

$$\partial^k(x) = \omega x - (-1)^k x\omega, \quad \forall x \in \mathcal{A}_k.$$

Proof. Recall from Theorem 4.7 the chain complex which computes the integral homology of M/W or $A(W)$. The theorem can be proved using the same method as in the proof of Theorem 5.13. $\square$

5.3.2. Dual complexes for $\boldsymbol{F}$ and $\boldsymbol{F/W}$ Recall from Theorem 4.6 and Theorem 4.8 the chain complexes which realise the integral homology of the Milnor fibres F and F/W, respectively. Using the bilinear form on $\mathbb{Z}W\otimes\mathcal{A}$, we will construct the cochain complexes which are dual to these chain complexes.

We begin with the following proposition, which identifies $d(\mathcal{A}_{k+1})$ with $\mathcal{A}_k\omega$. The latter is the k^{th} homogeneous component of the left ideal $\mathcal{A}\omega$ of $\mathcal{A}$ generated by ω.

Proposition 5.15. *For each $k=0,\dots,n$, the free abelian group $\mathcal{A}_k$ decomposes as*

$$\mathcal{A}_k = d(\mathcal{A}_{k+1}) \oplus \mathcal{A}_{k-1}\omega.$$

Moreover, we have an isomorphism $\mathcal{A}_{k-1}\omega \cong d(\mathcal{A}_k)$, given by the linear map d.

Proof. First, we prove the isomorphism $\mathcal{A}_{k-1}\omega \cong d(\mathcal{A}_k)$. Since the cochain complex $(\mathcal{A}, r_\omega)$ is acyclic, we have the following short exact sequences

$$0 \to \mathcal{A}_{k-1}\omega \to \mathcal{A}_k \to \mathcal{A}_k\omega \to 0, \quad 0 \le k \le n.$$

Then each short exact sequence splits since $\mathcal{A}_k\omega$ is free, being a subgroup of the free abelian group $\mathcal{A}_{k+1}$. Therefore, we obtain that

$$\mathcal{A}_k \cong \mathcal{A}_{k-1}\omega \oplus \mathcal{A}_k\omega, \quad 0 \le k \le n.$$

Similarly, using the acyclicity of $(\mathcal{A}, d)$ we have

$$\mathcal{A}_k \cong d(\mathcal{A}_k) \oplus d(\mathcal{A}_{k+1}), \quad 0 \le k \le n.$$

Note that $\mathcal{A}_n \cong d(\mathcal{A}_n) \cong \mathcal{A}_{n-1}\omega$. By comparing the isomorphisms above we have $\mathcal{A}_{k-1}\omega \cong d(\mathcal{A}_k)$ for $0 \le k \le n$.

Using the adjoint property in Lemma 5.4, we have $\langle x\omega, d(y)\rangle = \langle x, d^2(y)\rangle = 0$ for any $x \in \mathcal{A}_{k-1}$ and $y \in \mathcal{A}_{k+1}$. Hence $\mathcal{A}_{k-1}\omega$ is orthogonal to $d(\mathcal{A}_{k+1})$ as subgroups of $\mathcal{A}_k$. Moreover, using the above isomorphisms we obtain

$$\operatorname{rank}\mathcal{A}_k = \operatorname{rank} d(\mathcal{A}_k) + \operatorname{rank} d(\mathcal{A}_{k+1}) = \operatorname{rank}\mathcal{A}_{k-1}\omega + \operatorname{rank} d(\mathcal{A}_{k+1}).$$

Therefore, we have the decomposition $\mathcal{A}_k = \mathcal{A}_{k-1}\omega \oplus d(\mathcal{A}_{k+1})$ for all k. $\square$

Theorem 5.16. *The integral cohomology of the Milnor fibre F is isomorphic to the cohomology of the following cochain complex of free abelian groups:*

$$0 \longrightarrow \mathbb{Z}W \otimes \mathcal{A}_0\omega \xrightarrow{\ell_\sigma} \mathbb{Z}W \otimes \mathcal{A}_1\omega \xrightarrow{\ell_\sigma} \cdots \longrightarrow \mathbb{Z}W \otimes \mathcal{A}_{n-1}\omega \longrightarrow 0,$$

where $\omega = \sum_{t\in T} a_t$ and the coboundary maps are given by ℓ_σ, i.e.

$$\ell_\sigma(w \otimes x\omega) = \sum_{t\in T} wt \otimes a_t x\omega, \quad \forall x \in \mathcal{A}_k, w \in W.$$

Proof. For each $k = 0, 1, \ldots, n-1$, let $(\mathbb{Z}W \otimes \mathcal{A}_k\omega)^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}W \otimes \mathcal{A}_k\omega, \mathbb{Z})$. Consider the following chain complex $((\mathbb{Z}W \otimes \mathcal{A}\omega)^*, \ell_\sigma^*)$:

$$0 \longrightarrow (\mathbb{Z}W \otimes \mathcal{A}_{n-1}\omega)^* \xrightarrow{\ell_\sigma^*} (\mathbb{Z}W \otimes \mathcal{A}_{n-2}\omega)^* \xrightarrow{\ell_\sigma^*} \cdots \longrightarrow (\mathbb{Z}W \otimes \mathcal{A}_0\omega)^* \longrightarrow 0.$$

We will show that this chain complex is isomorphic to the chain complex $(\mathbb{Z}W \otimes d(\mathcal{A}_k), \partial_{k-1})$ given in Theorem 4.6. Thus, these two chain complexes both compute the integral homology of F. By the universal coefficient theorem for cohomology (cf. Theorem 5.6), the cochain complex $(\mathbb{Z}W \otimes \mathcal{A}\omega, \ell_\sigma)$ given in the theorem, which is the dual complex of $((\mathbb{Z}W \otimes \mathcal{A}\omega)^*, \ell_\sigma^*)$, has the integral cohomology of F.

Now we only need to prove that $(\mathbb{Z}W \otimes d(\mathcal{A}_k), \partial_{k-1})$ and $((\mathbb{Z}W \otimes \mathcal{A}\omega)^*, \ell_\sigma^*)$ are isomorphic. We start by constructing an isomorphism between $\mathbb{Z}W \otimes d(\mathcal{A}_k)$ and $(\mathbb{Z}W \otimes \mathcal{A}_{k-1}\omega)^*$ for each $k = 1, \ldots, n$. By Proposition 5.15, we have

$$\mathbb{Z}W \otimes \mathcal{A}_k\,\omega \cong \mathbb{Z}W \otimes d(\mathcal{A}_{k+1}), \quad w \otimes x \mapsto \partial''(w \otimes x) = u \otimes d(x)$$

for any $w \in W$ and $x \in \mathcal{A}_k\omega$. Moreover, the linear map ψ_W defined in (5.6) restricts to the following isomorphism:

$$\mathbb{Z}W \otimes \mathcal{A}_k\,\omega \cong (\mathbb{Z}W \otimes \mathcal{A}_k\,\omega)^*, \quad w \otimes x \mapsto \psi_W(w \otimes x) = \langle -, w \otimes x\rangle.$$

Combining the above two isomorphisms, we obtain that

$$\xi_k : \mathbb{Z}W \otimes d(\mathcal{A}_{k+1}) \to (\mathbb{Z}W \otimes \mathcal{A}_k\,\omega))^*$$

is an isomorphism given by

$$\xi_k(w \otimes d(x)) := \psi_W(w \otimes x) = \langle -, w \otimes x\rangle$$

for any $w \in W$ and $x \in \mathcal{A}_k\,\omega$. This is well-defined, since if $d(x) = d(y)$ for some $x, y \in \mathcal{A}_k\,\omega$, then we have $w \otimes (x - y) \in \mathrm{Ker}(1 \otimes d) = \mathrm{Ker}\,\partial''$ and

$$\xi_k(w \otimes d(x-y))(v \otimes z\omega) = \langle v \otimes z\omega, w \otimes (x-y)\rangle = \langle v \otimes z, \partial''(w \otimes (x-y))\rangle = 0,$$

for any $w, v \in W$ and $z \in \mathcal{A}_k$, where the last equation follows from Lemma 5.10.

We now prove that ξ_k is an isomorphism between chain complexes. It suffices to show that the following diagram commutes for each k:

$$\begin{tikzcd}
\mathbb{Z}W \otimes d(\mathcal{A}_{k+1}) \arrow[r, "\partial_k"] \arrow[d, "\xi_k"] & \mathbb{Z}W \otimes d(\mathcal{A}_k) \arrow[d, "\xi_{k-1}"] \\
(\mathbb{Z}W \otimes \mathcal{A}_k\omega)^* \arrow[r, "\ell_\sigma^*"] & (\mathbb{Z}W \otimes \mathcal{A}_{k-1}\omega)^*.
\end{tikzcd}$$

Recalling that in terms of notation (5.5), we have $\partial_k = \partial'_k$ and $1 \otimes d_k = \partial''_k$. For any $v, w \in W$ and $y \in \mathcal{A}_{k-1}\,\omega$, we have

$$\begin{aligned}
\xi_{k-1}\partial_k(v \otimes d(x))(w \otimes y) &= \xi_{k-1}(\partial'\partial''(v \otimes x))(w \otimes y) \\
&= \xi_{k-1}(\partial''\partial'(v \otimes x))(w \otimes y) \\
&= \langle w \otimes y, \partial'(v \otimes x)\rangle,
\end{aligned}$$

where the second equation follows from Lemma 5.9. On the other hand,

$$\ell_\sigma^*\xi_k(v \otimes d(x))(w \otimes y) = \xi_k(v \otimes d(x))(\ell_\sigma(w \otimes y)) = \langle \ell_\sigma(w \otimes y), v \otimes x\rangle.$$

By the adjoint property in Lemma 5.10, we have $\xi_{k-1}\partial_k = \ell_\sigma^*\xi_k$. Therefore, ξ_k is an isomorphism between the chain complexes $(\mathbb{Z}W \otimes d(\mathcal{A}_k), \partial_k)$ and $((\mathbb{Z}W \otimes \mathcal{A}\omega)^*, \ell_\sigma^*)$. This completes the proof. □

Corollary 5.17. *Let $\mathscr{A} := \mathbb{Z}W \otimes \mathcal{A}$ be the tensor product equipped with the usual multiplicative structure. Then we have the following $\mathbb{Z}$-graded isomorphism of abelian groups:*

$$H^*(F; \mathbb{Z})[-1] = (\mathscr{A}\varsigma \cap \sigma\mathscr{A})/\sigma\mathscr{A}\varsigma,$$

where $\sigma = \sum_{t \in T} t \otimes a_t$ and $\varsigma = \sum_{t \in T} 1 \otimes a_t$, and $A[-1]_n := A_{n-1}$ for the $\mathbb{Z}$-graded abelian group A.

Proof. Recall from Proposition 5.12 that the complex $(\mathbb{Z}W \otimes \mathcal{A}, \ell_\sigma)$ is acyclic. Denote $\mathscr{A}_k := \mathbb{Z}W \otimes \mathcal{A}_k$ for $0 \le k \le n$. Then the coboundary map $\ell_\sigma : \mathscr{A}_{k+1} \to \mathscr{A}_{k+2}$ has the kernel $\sigma\mathscr{A}_k$. Using Theorem 5.16, we obtain

$$H^k(F; \mathbb{Z}) \cong \mathrm{Ker}(\mathscr{A}_k\varsigma \xrightarrow{\ell_\sigma} \mathscr{A}_{k+1}\varsigma)/\sigma\mathscr{A}_{k-1}\varsigma = (\sigma\mathscr{A}_k \cap \mathscr{A}_k\varsigma)/\sigma\mathscr{A}_{k-1}\varsigma.$$

This completes the proof. $\square$

Theorem 5.18. *The integral cohomology of the Milnor fibre F/W is isomorphic to the cohomology of the following cochain complex of free abelian groups:*

$$0\longrightarrow\mathcal{A}_0\omega\xrightarrow{\ell_\omega}\mathcal{A}_1\omega\xrightarrow{\ell_\omega}\cdots\xrightarrow{\ell_\omega}\mathcal{A}_{n-1}\omega\longrightarrow 0,$$

where the coboundary maps are given by left multiplication by $\omega=\sum_{t\in T}a_t$.

Proof. The proof is similar to that of Theorem 5.16. Recall from Theorem 4.8 the chain complex which computes the integral homology of F/W. Using the bilinear form (5.4), one can identify $d(\mathcal{A}_{k+1})$ and $(\mathcal{A}_k\omega)^*$ and prove that the complexes $(d(\mathcal{A}_{k+1}),\partial_k)$ and $((\mathcal{A}\omega)^*,\ell_\omega^*)$ are isomorphic. Thus the complex $(\mathcal{A}\omega,\ell_\omega)$ computes the integral cohomology of F/W. $\square$

Corollary 5.19. *Let $\mathcal{A}\omega$ (resp. $\omega\mathcal{A}$) be the left (resp. right) ideal of $\mathcal{A}$ generated by ω. Then we have the following $\mathbb{Z}$-graded isomorphism of abelian groups:*

$$H^*(F/W;\mathbb{Z})[-1]\cong(\mathcal{A}\omega\cap\omega\mathcal{A})/\omega\mathcal{A}\omega,$$

where $A[-1]_n:=A_{n-1}$ for the $\mathbb{Z}$-graded abelian group A.

Proof. By Proposition 5.8, the cochain complex $(\mathcal{A},\ell_\omega)$ is acyclic. It follows that the coboundary map $\ell_\omega:\mathcal{A}_{k+1}\to\mathcal{A}_{k+2}$ has the kernel $\omega\mathcal{A}_k$. Using Theorem 5.18, we have

$$H^k(F/W;\mathbb{Z})\cong\operatorname{Ker}(\mathcal{A}_k\omega\xrightarrow{\ell_\omega}\mathcal{A}_{k+1}\omega)/\omega\mathcal{A}_{k-1}\omega=(\omega\mathcal{A}_k\cap\mathcal{A}_k\omega)/\omega\mathcal{A}_{k-1}\omega.$$

This completes the proof. $\square$

5.4. A pair of dual complexes with complex coefficients

We shall introduce a pair of cochain complexes which are dual to the complexes $\mathcal{C}(U)$ and $\mathcal{K}(U)$. In this subsection, we work over $\mathbb{C}$.

Let U be any finite dimensional right $\mathbb{C}W$-module. We define the dual complex of $\mathcal{C}(U)$ by

$$\mathcal{C}^*(U):=0\longrightarrow U\otimes\mathcal{A}_0\xrightarrow{\partial^*}\cdots\longrightarrow U\otimes\mathcal{A}_{n-1}\xrightarrow{\partial^*}U\otimes\mathcal{A}_n\longrightarrow 0,$$

where the coboundary maps are given by

$$\partial^*(u\otimes x)=\sum_{t\in T}ut\otimes a_tx-(-1)^ku\otimes x\omega,\quad\forall x\in\mathcal{A}_k,u\in U.$$

It is straightforward to verify that $(\partial^*)^2 = 0$. Similarly, define the dual complex of $\mathcal{K}(U)$ by

$$\mathcal{K}^*(U) := 0 \longrightarrow U \otimes \mathcal{A}_0\,\omega \xrightarrow{\partial_0^*} \cdots \longrightarrow U \otimes \mathcal{A}_{n-2}\,\omega \xrightarrow{\partial_{n-2}^*} U \otimes \mathcal{A}_{n-1}\,\omega \longrightarrow 0,$$

where $\mathcal{A}_k\,\omega$ is the subspace of $\mathcal{A}_{k+1}$ linearly spanned by elements of the form $a_{t_1} \cdots a_{t_k}\omega$, and the coboundary maps are defined by

$$\partial^*(u \otimes x\omega) = \sum_{t \in T} ut \otimes a_t x\omega, \quad \forall x \in \mathcal{A}_k, u \in U.$$

The following is a cohomology version of Theorem 4.11.

Theorem 5.20. *For any right W-module U and for each integer $k \geq 0$, we have:*

$$\dim H^k(\mathcal{C}^*(U)) = \langle U_L^*, H^k(M) \rangle$$

and

$$\dim H^k(\mathcal{K}^*(U)) = \langle U_L^*, H^k(F) \rangle.$$

Proof. Use Theorem 5.13 and Theorem 5.16. The proof is similar to that of Theorem 4.11. □

6. Complements on a covering algebra $\widetilde{\mathcal{A}}$ of $\mathcal{A}$

In this section we define an algebra $\widetilde{\mathcal{A}}$, whose presentation is simpler than that of $\mathcal{A}$, and which in fact has $\mathcal{A}$ as a homomorphic image. The algebra $\widetilde{\mathcal{A}}$ has some remarkable similarities to the Fomin-Kirillov algebra $\mathcal{E}_n$ (in type A_{n-1}) (cf. [FK99], and we conjecture that the two algebras have the same Hilbert-Poincaré series (always in type A_n). They are in some sense "dual" to each other, but are not Koszul. Our purpose for including this discussion is that both $\widetilde{\mathcal{A}}$ and $\mathcal{E}_n$ have multiple connections to other branches of mathematics, and have clear similarities to each other. Another motivation is that numerous authors have observed analogies between the flag variety and the hyperplane complement associated to a complex Lie group [AB02]. The two algebras could provide a pathway to understanding this hitherto mysterious connection.

In Section 6.1 we define a braided Hopf algebra and show that there exists a surjective algebra homomorphism from the braided Hopf algebra to the noncrossing algebra. We show also that $\widetilde{\mathcal{A}}$ has the structure of a W-graded Hopf algebra, that it has a braiding, and more generally belongs to a category of Yetter-Drinfeld modules over $\mathbb{C}W$. In Section 6.2 we define some differential operators on the braided Hopf algebra, and determine some of their adjoint properties, which are similar to those of $\mathcal{A}$. We also prove that there is a bilinear form on $\widetilde{\mathcal{A}}$, and that this form descends to the one we already have on $\mathcal{A}$. Throughout this section, we work over the complex field $\mathbb{C}$.

6.1. A new braided Hopf algebra

We introduce a cover of the noncrossing algebra, which is analogous to the Fomin-Kirillov algebra [FK99]. This new algebra is a Yetter-Drinfeld module over the group algebra $\mathbb{C}W$, and has a braided Hopf algebra structure. We refer to [AS02] for background concerning Yetter-Drinfeld modules.

6.1.1. Definition and examples

Definition 6.1. Let W be any finite Coxeter group and T be the set of reflections of W. Define $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}(W)$ to be the associative algebra over $\mathbb{C}$ generated by $\alpha_t, t \in T$, subject to the following quadratic relations:

$$\alpha_t^2 = 0, \quad \text{for any } t \in T\ , \tag{6.1}$$

$$\sum_{(t_1,t_2)\in \mathrm{Rex}_T(w)} \alpha_{t_1}\alpha_{t_2} = 0, \quad \text{for any } w \in W \text{ with } \ell_T(w) = 2. \tag{6.2}$$

The algebra $\widetilde{\mathcal{A}}$ is a $\mathbb{Z}$-graded algebra with the $\mathbb{Z}$-grading $\deg(\alpha_t) = 1$ for all $t \in T$. In this section, we denote by $\mathcal{A}$ the noncrossing algebra over the complex field $\mathbb{C}$. Theses two algebras are related by the following lemma.

Lemma 6.2. *We have a surjective algebra homomorphism $\pi : \widetilde{\mathcal{A}} \to \mathcal{A}$, given by $\alpha_t \mapsto a_t$ for all $t \in T$.*

Proof. It suffices to check that the relations (6.2) are preserved in $\mathcal{A}$. If $w \leq \gamma$, then relation (6.2) is sent to the defining relation of $\mathcal{A}$ under the map π. Otherwise, for any two reflections t_1, t_2 such that $t_1t_2 \not\leq \gamma$, we have $\pi(\alpha_{t_1}\alpha_{t_2}) = a_{t_1}a_{t_2} = 0$, and hence for any $w \not\leq \gamma$ we have $\sum_{(t_1,t_2)\in \mathrm{Rex}_T(w)} \pi(\alpha_{t_1}\alpha_{t_2}) = 0$. $\square$

Example 6.3. The new algebra $\widetilde{\mathcal{A}}(\mathrm{Sym}_n)$ of type A_{n-1} is generated by $\alpha_{ij} = \alpha_{ji}$ for $1 \leq i < j \leq n$ with the following relations:

$$\begin{gathered} \alpha_{ij}^2 = 0, \\ \alpha_{ij}\alpha_{kl} + \alpha_{kl}\alpha_{ij} = 0, \quad \text{for distinct } i,j,k,l \\ \alpha_{ij}\alpha_{jk} + \alpha_{jk}\alpha_{ki} + \alpha_{ki}\alpha_{ij} = 0, \quad \text{for distinct } i,j,k \end{gathered} \tag{6.3}$$

In type A_2, these relations read:

$$\begin{aligned}\alpha_{12}^2 = \alpha_{13}^2 = \alpha_{23}^2 = 0,\\ \alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{13} + \alpha_{13}\alpha_{12} = 0,\\ \alpha_{23}\alpha_{12} + \alpha_{12}\alpha_{13} + \alpha_{13}\alpha_{23} = 0.\end{aligned}$$

This algebra looks similar to the Fomin-Kirillov algebra $\mathcal{E}_n$ [FK99], which is generated by $x_{ij} = -x_{ji}$ for $1 \le i < j \le n$ with relations:

$$\begin{aligned}x_{ij}^2 &= 0,\\ x_{ij}x_{kl} - x_{kl}x_{ij} &= 0, \quad \text{for distinct } i,j,k,l\\ x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} &= 0, \quad \text{for distinct } i,j,k.\end{aligned}$$

For any $\mathbb{Z}$-graded algebra $A = \bigoplus_{k\in\mathbb{Z}} A_k$, we denote by $H_A(t) = \sum_{k\in\mathbb{Z}} \dim A_k\, t^k$ the Hilbert-Poincaré series of A. Using computational software, we have verified that $\widetilde{\mathcal{A}}(\mathrm{Sym}_n)$ and $\mathcal{E}_n$ have the same Hilbert-Poincaré series for $n \le 5$:

$$\begin{aligned}&n = 1 : H_{\widetilde{\mathcal{A}}(\mathrm{Sym}_n)}(t) = H_{\mathcal{E}_n}(t) = 1,\\ &n = 2 : H_{\widetilde{\mathcal{A}}(\mathrm{Sym}_n)}(t) = H_{\mathcal{E}_n}(t) = [2] = 1 + t,\\ &n = 3 : H_{\widetilde{\mathcal{A}}(\mathrm{Sym}_n)}(t) = H_{\mathcal{E}_n}(t) = [2]^2[3] = 1 + 3t + 4t^2 + 3t^3 + t^4,\\ &n = 4 : H_{\widetilde{\mathcal{A}}(\mathrm{Sym}_n)}(t) = H_{\mathcal{E}_n}(t) = [2]^2[3]^2[4]^2,\\ &n = 5 : H_{\widetilde{\mathcal{A}}(\mathrm{Sym}_n)}(t) = H_{\mathcal{E}_n}(t) = [4]^4[5]^2[6]^4,\end{aligned}$$

where we have used the notation $[k] := 1+t+\cdots+t^{k-1}$. These Hilbert-Poincaré series have symmetric coefficients. In particular, the top homogeneous component has dimension 1. It is unknown whether $\mathcal{E}_n$ is finite-dimensional for $n \ge 6$.

Conjecture 6.4. *The algebra* $\widetilde{\mathcal{A}}(\mathrm{Sym}_n)$ *and the Fomin-Kirillov algebra* $\mathcal{E}_n$ *have the same Hilbert-Poincaré series.*

Remark 6.5. Recall that the Orlik-Solomon algebra associated to the reflection arrangement of Sym_n is generated by elements $e_{ij} = e_{ji}$ for $1 \le i < j \le n$, subject to the following relations:

$$\begin{aligned}e_{ij}e_{kl} &= -e_{kl}e_{ij}, &&1 \le i < j \le n, 1 \le k \le l \le n,\\ e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} &= 0, &&1 \le i,j,k \le n.\end{aligned}$$

This appears as a quotient of $\widetilde{\mathcal{A}}(\mathrm{Sym}_n)$ by imposing the anti-commutative relations $\alpha_{ij}\alpha_{kl} = -\alpha_{kl}\alpha_{ij}$ for $i < j$ and $k < l$, that is, we allow $\{i, j\} \cap \{k, l\} \neq \emptyset$ in the second relation of (6.3).

6.1.2. Braided Hopf algebra structure We now take a Hopf-theoretic point of view to the covering algebra $\widetilde{\mathcal{A}}$ [AG99, MS00, AS02].

Let us recall relevant definitions. The group algebra $\mathbb{C}W$ has a Hopf algebra structure with the comultiplication $\Delta(w) = w \otimes w$, counit $\epsilon(w) = 1$ and antipode $S(w) = w^{-1}$ for any $w \in W$. A *Yetter-Drinfeld module* A over $\mathbb{C}W$ is a W-graded vector space $A = \bigoplus_{w \in W} A_w$, which is a W-module such that $w.A_u \subseteq A_{wuw^{-1}}$ for all $u, w \in W$.

The algebra $\widetilde{\mathcal{A}}$ is a Yetter-Drinfeld module over $\mathbb{C}W$, as we now describe. In addition to the natural $\mathbb{Z}$-grading, $\widetilde{\mathcal{A}}$ has a grading with respect to W such that the W-degree of the generator α_t is $t \in T$ and this is extended to all monomials by multiplication. As the defining relations of $\widetilde{\mathcal{A}}$ are homogeneous with respect to the W-degree, this gives a W-grading of $\widetilde{\mathcal{A}} = \bigoplus_{w \in W} \widetilde{\mathcal{A}}_w$, where $\widetilde{\mathcal{A}}_w$ is spanned by monomials $\alpha_{t_1} \cdots \alpha_{t_k}$ such that $t_1 t_2 \cdots t_k = w$. Note that $w = t_1 t_2 \cdots t_k$ is not necessarily a reduced expression with respect to reflections of T or simple reflections of S.

The W-module structure on $\widetilde{\mathcal{A}}$ is defined by

$$w.\alpha_t := (-1)^{\ell(w)} \alpha_{wtw^{-1}}, \quad , \forall w \in W, \tag{6.4}$$

where $\ell(w)$ is the usual length of w with respect to the generating set S of W. Clearly, this action preserves the defining relations (6.1) and (6.2) of $\widetilde{\mathcal{A}}$, and is compatible with the W-grading, i.e. $w.\widetilde{\mathcal{A}}_u \subseteq \widetilde{\mathcal{A}}_{wuw^{-1}}$ for all $u, w \in W$. Therefore, $\widetilde{\mathcal{A}}$ is a Yetter-Drinfeld module over $\mathbb{C}W$.

We denote by ${}^W_W\mathcal{YD}$ the category of Yetter-Drinfeld modules over $\mathbb{C}W$. A morphism $f : A \to B$ of the category ${}^W_W\mathcal{YD}$ is a homomorphism of W-modules which preserves the W-grading.

An important ingredient of the Yetter-Drinfeld category is the canonical braiding. For any $A, B \in {}^W_W\mathcal{YD}$, the canonical braiding $c : A \otimes B \to B \otimes A$ is defined by

$$c(a \otimes b) = b \otimes (w^{-1}.a), \quad \forall a \in A, b \in B_w. \tag{6.5}$$

The tensor product $A \otimes B$ is an object of ${}^W_W\mathcal{YD}$, with the W-grading $(A \otimes B)_w = \bigoplus_{ab=w} A_a \otimes B_b$ and the W-action $w.(a \otimes b) = w.a \otimes w.b$ for any $w \in W$, $a \in A$ and $b \in B$. In particular, we have $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}} \in {}^W_W\mathcal{YD}$. Moreover, the tensor product $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$ is still an algebra, with multiplication defined via the canonical braiding:

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1x_2 \otimes (w^{-1}.y_1)y_2, \quad \forall x_2 \in \widetilde{\mathcal{A}}_w,\ x_1, y_1, y_2 \in \widetilde{\mathcal{A}}. \tag{6.6}$$

More concisely, $\mu_{\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{A}}} = (\mu_{\widetilde{\mathcal{A}}} \otimes \mu_{\widetilde{\mathcal{A}}})(1 \otimes c \otimes 1)$, where $\mu_A : A \otimes A \to A$ denotes the multiplication map of the algebra A.

Recall that a *braided bialgebra* A in ${}^W_W\mathcal{YD}$ is a collection $(A, \mu, \eta, \Delta, \epsilon)$ such that (A, μ, η) is an algebra in ${}^W_W\mathcal{YD}$, (A, Δ, ϵ) is a coalgebra in ${}^W_W\mathcal{YD}$ and $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{C}$ are morphisms of algebras (here $A \otimes A$ is an algebra in ${}^W_W\mathcal{YD}$ with multiplication defined via the braiding c). We call A a *braided Hopf algebra* if in addition there is an antipode $S : A \to A$ in ${}^W_W\mathcal{YD}$ such that $(1 \otimes S)\Delta = (S \otimes 1)\Delta = \eta\epsilon$.

Proposition 6.6. *The algebra $\widetilde{\mathcal{A}}$ is a braided Hopf algebra in ${}^W_W\mathcal{YD}$ with the coproduct Δ, the counit ϵ and the antipode S defined on the generators $\alpha_t, t \in T$ by*

$$\begin{aligned} &\Delta(\alpha_t) = \alpha_t \otimes 1 + 1 \otimes \alpha_t, \\ &\epsilon(\alpha_t) = 0, \quad S(\alpha_t) = -\alpha_t. \end{aligned} \tag{6.7}$$

Proof. We need to check that Δ, ϵ and S are well-defined, and then check that they satisfy Hopf algebra axioms. Straightforward calculations show that:

$$\begin{aligned} &\Delta(\alpha_t^2) = \alpha_t^2 \otimes 1 + 1 \otimes \alpha_t^2, \quad S(\alpha_t^2) = \alpha_t^2, \quad \epsilon(\alpha_t^2) = 0, \\ &\Delta(R_w) = R_w \otimes 1 + 1 \otimes R_w, \\ &S(R_w) = R_w, \quad \epsilon(R_w) = 0, \end{aligned}$$

where we have used the notation $R_w := \sum_{(t_1,t_2)\in \mathrm{Rex}_T(w)} \alpha_{t_1}\alpha_{t_2}$ for any $w \in W$ with $\ell_T(w) = 2$. Therefore, Δ, ϵ and S are all well-defined.

Next we check the Hopf algebra axioms on the generators of $\widetilde{\mathcal{A}}$: (1) Coassociativity:

$$\begin{aligned} (\Delta \otimes 1)(\Delta(\alpha_t)) &= (\Delta \otimes 1)(\alpha_t \otimes 1 + 1 \otimes \alpha_t) \\ &= \alpha_t \otimes 1 \otimes 1 + 1 \otimes \alpha_t \otimes 1 + 1 \otimes 1 \otimes \alpha_t \\ &= (1 \otimes \Delta)(\Delta(\alpha_t)). \end{aligned}$$

(2) The counit axiom:

$$\begin{aligned} (\epsilon \otimes 1)(\Delta(\alpha_t)) = (\epsilon \otimes 1)(\alpha_t \otimes 1 + 1 \otimes \alpha_t) = \alpha_t, \\ (1 \otimes \epsilon)(\Delta(\alpha_t)) = (1 \otimes \epsilon)(\alpha_t \otimes 1 + 1 \otimes \alpha_t) = \alpha_t. \end{aligned}$$

(3) The antipode axiom:

$$\mu(1\otimes S)(\Delta(\alpha_t)) = \mu(1\otimes S)(\alpha_t\otimes 1 + 1\otimes\alpha_t) = \alpha_t - \alpha_t = 0 = \epsilon(\alpha_t),$$
$$\mu(S\otimes 1)(\Delta(\alpha_t)) = \mu(S\otimes 1)(\alpha_t\otimes 1 + 1\otimes\alpha_t) = -\alpha_t + \alpha_t = 0 = \epsilon(\alpha_t). \quad \square$$

We can extend (6.7) to any monomials in the generators of $\widetilde{\mathcal{A}}$. Clearly, $\epsilon(\alpha_{t_1}\cdots\alpha_{t_k}) = 0$ for $k\geq 1$. We next give explicit formulas for S and Δ.

Proposition 6.7. *The antipode S is given explicitly by*

$$S(\alpha_{t_1}\cdots\alpha_{t_k}) = \varepsilon(t_1,\dots,t_k)\,\alpha_{t_k}\alpha_{t_{k-1}^{t_k}}\cdots\alpha_{t_1^{t_2\cdots t_k}}, \tag{6.8}$$

where $\varepsilon(t_1,\dots,t_k) = (-1)^k\prod_{i=2}^k(-1)^{\ell(t_i\cdots t_k)}$.

Proof. Recall that in ${}^W_W\mathcal{YD}$ we have $S\mu = \mu(S\otimes S)c$. This follows from the fact that both $S\mu$ and $\mu(S\otimes S)c$ are the inverse of μ under the convolution product in the algebra $\mathrm{Hom}(\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{A}},\widetilde{\mathcal{A}})$; refer to [AG99, Lemma 1.2.2]. Using this equation and induction on k, we have

$$\begin{aligned} S(\alpha_{t_1}\alpha_{t_2}\cdots\alpha_{t_k}) &= S(\alpha_{t_2}\cdots\alpha_{t_k})S((t_k\cdots t_2).\alpha_{t_1})\\ &= -(-1)^{\ell(t_2\cdots t_k)}S(\alpha_{t_2}\cdots\alpha_{t_k})\alpha_{t_1^{t_2\cdots t_k}}. \end{aligned}$$

The formula follows by induction hypothesis. $\square$

Proposition 6.8. *The comultiplication of $\widetilde{\mathcal{A}}$ is given explicitly by*

$$\Delta(\alpha_{t_1}\cdots\alpha_{t_k}) = \sum_{j=0}^k\sum_{1\leq i_1<i_2<\cdots<i_j\leq k}\alpha_{t_{i_1}}\cdots\alpha_{t_{i_j}}\otimes E_{t_{i_j}}\cdots E_{t_{i_1}}(\alpha_{t_1}\cdots\alpha_{t_k}),$$

where $E_{t_{r_i}}(\alpha_{t_{r_1}}\cdots\alpha_{t_{r_s}}) := t_{r_i}.(\alpha_{t_{r_1}}\cdots\alpha_{t_{r_{i-1}}})\alpha_{t_{r_{i+1}}}\cdots\alpha_{t_{r_s}}$, *for any monomial* $\alpha_{t_{r_1}}\cdots\alpha_{t_{r_s}}\in\widetilde{\mathcal{A}}$ *and* $1\leq i\leq s$.

Proof. Use induction on k. The formula is trivial if $k=1$. For $k>1$, by induction hypothesis we have

$$\begin{aligned} &\Delta(\alpha_{t_1}\cdots\alpha_{t_k}) = \Delta(\alpha_{t_1}\cdots\alpha_{t_{k-1}})\Delta(\alpha_{t_k})\\ =&\sum_{j=0}^{k-1}\sum_{1\leq i_1<\cdots<i_j\leq k-1}(\alpha_{t_{i_1}}\cdots\alpha_{t_{i_j}}\otimes E_{t_{i_j}}\cdots E_{t_{i_1}}(\alpha_{t_1}\cdots\alpha_{t_{k-1}}))(1\otimes\alpha_{t_k}+\alpha_{t_k}\otimes 1)\\ =&\sum_{j=0}^{k-1}\sum_{1\leq i_1<\cdots<i_j\leq k-1}\alpha_{t_{i_1}}\cdots\alpha_{t_{i_j}}\otimes(E_{t_{i_j}}\cdots E_{t_{i_1}}(\alpha_{t_1}\cdots\alpha_{t_{k-1}}))\alpha_{t_k} \end{aligned}$$

$$+\sum_{j=0}^{k-1}\sum_{1\le i_1<\cdots<i_j\le k-1}\alpha_{t_{i_1}}\cdots\alpha_{t_{i_j}}\alpha_{t_k}\otimes t_k.(E_{t_{i_j}}\cdots E_{t_{i_1}}(\alpha_{t_1}\cdots\alpha_{t_{k-1}})).$$

Note that in the above equation, we have

$$\begin{aligned}(E_{t_{i_j}}\cdots E_{t_{i_1}}(\alpha_{t_1}\cdots\alpha_{t_{k-1}}))\alpha_{t_k} &= E_{t_{i_j}}\cdots E_{t_{i_1}}(\alpha_{t_1}\cdots\alpha_{t_{k-1}}\alpha_{t_k}),\\ t_k.(E_{t_{i_j}}\cdots E_{t_{i_1}}(\alpha_{t_1}\cdots\alpha_{t_{k-1}})) &= E_{t_k}E_{t_{i_j}}\cdots E_{t_{i_1}}(\alpha_{t_1}\cdots\alpha_{t_k}).\end{aligned}$$

Then we obtain the formula of $\Delta(\alpha_{t_1}\cdots\alpha_{t_k})$ as desired. □

Example 6.9. Using Proposition 6.8, we have

$$\begin{aligned}\Delta(\alpha_{t_1}\alpha_{t_2}) =& 1\otimes\alpha_{t_1}\alpha_{t_2}+\alpha_{t_1}\otimes E_{t_1}(\alpha_{t_1}\alpha_{t_2})+\alpha_{t_2}\otimes E_{t_2}(\alpha_{t_1}\alpha_{t_2})\\ &+\alpha_{t_1}\alpha_{t_2}\otimes E_{t_2}E_{t_1}(\alpha_{t_1}\alpha_{t_2})\\ =& 1\otimes\alpha_{t_1}\alpha_{t_2}+\alpha_{t_1}\otimes\alpha_{t_2}-\alpha_{t_2}\otimes\alpha_{t_2t_1t_2}+\alpha_{t_1}\alpha_{t_2}\otimes 1.\end{aligned}$$

It follows that $\widetilde{\mathcal{A}}$ is not cocommutative.

Proposition 6.10. *The noncrossing algebra $\mathcal{A}$ is a subcoalgebra of $\widetilde{\mathcal{A}}$.*

Proof. By Lemma 6.2 $\mathcal{A}$ can be lifted as a subspace of $\widetilde{\mathcal{A}}$. Note that if $a_{t_1}\cdots a_{t_k}$ is a nonzero element of $\mathcal{A}$, that is, $w=t_1t_2\cdots t_k\in\mathcal{L}$ is a T-reduced expression, then $E_{t_i}(a_{t_1}\cdots a_{t_k})=(-1)^{i-1}a_{t_1^{t_i}}\cdots a_{t_{i-1}^{t_i}}a_{t_{i+1}}\cdots a_{t_k}$ is still a nonzero element of $\mathcal{A}$. In view of Proposition 6.8, $\mathcal{A}$ is closed under the comultiplication Δ of $\widetilde{\mathcal{A}}$. In addition, $\mathcal{A}$ is clearly closed under the counit ϵ of $\widetilde{\mathcal{A}}$. Therefore, $\mathcal{A}$ is a subcoalgebra of $\widetilde{\mathcal{A}}$. □

6.2. Skew-derivations on $\widetilde{\mathcal{A}}$

Recall that the noncrossing algebra $\mathcal{A}$ has skew-derivations δ_t and d_t for any $t\in T$. We shall show that these skew-derivations can be lifted to the algebra $\widetilde{\mathcal{A}}$ with similar properties. The difference is that these skew-derivations are defined using the braided Hopf algebra structure of $\widetilde{\mathcal{A}}$.

For any integer $k\ge 0$ let $\pi_k:\widetilde{\mathcal{A}}\to\widetilde{\mathcal{A}}_k$ be the projection of $\widetilde{\mathcal{A}}$ onto its k-th homogeneous component $\widetilde{\mathcal{A}}_k$. We denote by

$$\Delta_{i,j}:\widetilde{\mathcal{A}}_{i+j}\xrightarrow{\Delta}\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{A}}\xrightarrow{\pi_i\otimes\pi_j}\widetilde{\mathcal{A}}_i\otimes\widetilde{\mathcal{A}}_j$$

the (i,j)-th component of the comultiplication Δ.

For any $t \in T$, we define the linear map $\nabla_t : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ as follows: let $\nabla_t(1) = 0$, and for any $x \in \widetilde{\mathcal{A}}_k$ define $\nabla_t(x) \in \widetilde{\mathcal{A}}_{k-1}$ by

$$\Delta_{1,n-1}(x) = \sum_{t \in T} \alpha_t \otimes \nabla_t(x). \tag{6.9}$$

Similarly, we define $D_t : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ by $D_t(1) = 0$ and $\Delta_{n-1,1}(x) = \sum_{t \in T} D_t(x) \otimes \alpha_t$. It is clear that $\nabla_{t_1}(\alpha_{t_2}) = D_{t_1}(\alpha_{t_2}) = \delta_{t_1,t_2}$ (Kronecker delta).

Proposition 6.11. *For any $t \in T$, let ∇_t, D_t be as above.*

1. *For any $x, y \in \widetilde{\mathcal{A}}$, we have*

$$\nabla_t(xy) = \nabla_t(x)y + (t.x)\nabla_t(y),$$
$$D_t(xy) = (|y|.D_t)(x)y + xD_t(y),$$

where $|y| \in W$ denotes the W-grading of y, i.e. $y \in \widetilde{\mathcal{A}}_{|y|}$, and $|y|.D_t := (-1)^{\ell(|y|)} D_{|y|t|y|^{-1}}$.

2. *For any $\alpha_{t_1}\alpha_{t_2}\cdots\alpha_{t_k} \in \widetilde{\mathcal{A}}$, we have*

$$\nabla_t(\alpha_{t_1}\alpha_{t_2}\cdots\alpha_{t_k}) = \sum_{i=1}^{k}(-1)^{i-1}\delta_{t,t_i}\alpha_{t_1^{t_i}}\cdots\alpha_{t_{i-1}^{t_i}}\alpha_{t_{i+1}}\cdots\alpha_{t_k},$$
$$D_t(\alpha_{t_1}\alpha_{t_2}\cdots\alpha_{t_k}) = \sum_{i=1}^{k}(-1)^{k-i}\delta_{t,r_i}\alpha_{t_1}\cdots\alpha_{t_{i-1}}\alpha_{t_{i+1}}\cdots\alpha_{t_k},$$

where $r_i = t_i^{t_{i+1}\cdots t_k}$, and δ is the Kronecker delta.

3. *The linear operators ∇_t, D_t preserve the defining relations of $\widetilde{\mathcal{A}}$.*

Proof. For part (1), note that

$$\begin{aligned}\Delta(xy) &= \Delta(x)\Delta(y)\\ &= (1 \otimes x + \sum_{t\in T}\alpha_t \otimes \nabla_t(x) + \cdots)(1 \otimes y + \sum_{t \in T}\alpha_t \otimes \nabla_t(y) + \cdots)\\ &= 1 \otimes xy + \sum_{t\in T}\alpha_t \otimes (t.x)\nabla_t(y) + \sum_{t \in T}\alpha_t \otimes \nabla_t(x)y + \cdots .\end{aligned}$$

It follows that $\nabla_t(xy) = \nabla_t(x)y + (t.x)\nabla_t(y)$. Similarly, using the expression

$\Delta(x) = x \otimes 1 + \sum_{t \in T} D_t(x) \otimes \alpha_t + \cdots$, we have

$$\begin{aligned}\Delta(xy) &= (x \otimes 1 + \sum_{t \in T} D_t(x) \otimes \alpha_t + \cdots)(y \otimes 1 + \sum_{t \in T} D_t(y) \otimes \alpha_t + \cdots) \\ &= xy \otimes 1 + \sum_{t \in T} x D_t(y) \otimes \alpha_t + \sum_{t \in T} D_t(x) y \otimes |y|^{-1}.\alpha_t + \cdots .\end{aligned}$$

Note that

$$\begin{aligned}\sum_{t \in T} D_t(x) y \otimes |y|^{-1}.\alpha_t &= \sum_{t \in T} (-1)^{\ell(|y|)} D_t(x) y \otimes \alpha_{|y|^{-1} t |y|} \\ &= \sum_{t \in T} (-1)^{\ell(|y|)} D_{|y| t |y|^{-1}}(x) y \otimes \alpha_t\end{aligned}$$

Therefore, we have $D_t(xy) = (|y|.D_t)(x)y + xD_t(y)$. Part (2) is a consequence of part (1), and part (3) follows immediately from the formulae in part (2). □

Remark 6.12. The linear operators ∇_t, D_t are called skew-derivations of the braided Hopf algebra $\widetilde{\mathcal{A}}$ [AG99, AS02].

Proposition 6.13. *The skew-derivations $\nabla_t, t \in T$ satisfy the following relations:*

$$\begin{aligned}\nabla_t^2 = 0, &\quad \forall t \in T, \\ \sum_{(t_1,t_2) \in \mathrm{Rex}_T(w)} \nabla_{t_1} \nabla_{t_2} = 0, &\quad \forall w \in W \textit{ with } \ell_T(w) = 2,\end{aligned}$$

Therefore, they describe an action of $\widetilde{\mathcal{A}}$ on itself.

Proof. We evaluate these relations on $x \in \widetilde{\mathcal{A}}$ and use induction on the $\mathbb{Z}$-degree of x. It is trivial if $\deg(x) = 1$. In general, assume that $x = \alpha_{t_1} \cdots \alpha_{t_k}$. For any $r_1, r_2 \in T$, using the formula from part (1) of Proposition 6.11 we have

$$\begin{aligned}&\nabla_{r_1} \nabla_{r_2}(\alpha_{t_1} \cdots \alpha_{t_k}) \\ =&\nabla_{r_1}(\nabla_{r_2}(\alpha_{t_1} \alpha_{t_2} \cdots \alpha_{t_{k-1}}) \alpha_{t_k} + (-1)^{k-1} \delta_{r_2, t_k} \alpha_{t_1^{t_k}} \alpha_{t_2^{t_k}} \cdots \alpha_{t_{k-1}^{t_k}}) \\ =&\nabla_{r_1}(\nabla_{r_2}(\alpha_{t_1} \cdots \alpha_{t_{k-1}})) \alpha_{t_k} + (-1)^{k-2} \delta_{r_1, t_k} \nabla_{r_2^{t_k}}(\alpha_{t_1^{t_k}} \cdots \alpha_{t_{k-1}^{t_k}}) \\ &+ (-1)^{k-1} \delta_{r_2, t_k} \nabla_{r_1}(\alpha_{t_1^{t_k}} \alpha_{t_2^{t_k}} \cdots \alpha_{t_{k-1}^{t_k}}),\end{aligned}$$

If $r_1 = r_2 = t$, then by the induction hypothesis we have $\nabla_{r_1} \nabla_{r_2}(\alpha_{t_1} \cdots \alpha_{t_k}) = 0$, proving the first relations.

For the second relation, by the induction hypothesis it is equivalent to proving that

$$\sum_{(r_1,r_2)\in \mathrm{Rex}_T(w)} \delta_{r_1,t_k}\nabla_{t_k r_2 t_k} - \delta_{r_2,t_k}\nabla_{r_1} = 0$$

for any $w \in W$ with $\ell_T(w) = 2$. If $t_k \not\prec w$, then the above equation holds trivially. Otherwise, we have two T-reduced expressions $w = t_k t = (t_k t t_k)t_k$ for $t = t_k^{-1}w \in T$, which leads to the above equation. $\square$

We do not know whether this action of $\widetilde{\mathcal{A}}$ on itself faithful. Compare [FK99, §9]. Next we define a bilinear form on $\widetilde{\mathcal{A}}$ in terms of the skew-derivations.

Definition 6.14. Define the bilinear pairing

$$\langle -, - \rangle : \widetilde{\mathcal{A}} \times \widetilde{\mathcal{A}} \longrightarrow \mathbb{C}$$

by $\langle 1, 1\rangle = 1$ and

1. $\langle \widetilde{\mathcal{A}}_k, \widetilde{\mathcal{A}}_\ell\rangle = 0$ for any $0 \leq k \neq \ell$;
2. For any $x \in \widetilde{\mathcal{A}}_k$ and $t_i \in T, i = 1, \ldots, k$,

$$\langle \alpha_{t_1}\alpha_{t_2}\cdots\alpha_{t_k}, x\rangle := \nabla_{t_1}\nabla_{t_2}\cdots\nabla_{t_k}(x).$$

The bilinear form on $\widetilde{\mathcal{A}}$ is well-defined in view of Proposition 6.13. Note that we do not reverse the order of ∇_{t_i} in the above definition. This is different from that in Definition 6.14. However, one can prove the following the properties which are similar to those given for $\mathcal{A}$.

Proposition 6.15. *We have the following properties.*

1. *For any $t \in T$ and $x \in \widetilde{\mathcal{A}}$, we have*

$$\nabla_t(x) = \sum_{(x)} \langle \alpha_t, x_{(1)}\rangle\, x_{(2)},$$
$$D_t(x) = \sum_{(x)} x_{(1)}\, \langle x_{(2)}, \alpha_t\rangle,$$

   where we have used the Sweedler's notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ for the comultiplication of $\widetilde{\mathcal{A}}$.
2. *For any $t, t' \in T$, we have $\nabla_t D_{t'} = D_{t'}\nabla_t$.*

3. *For any $x, y \in \widetilde{\mathcal{A}}$, we have*

$$\langle x\alpha_t, y\rangle = \langle x, \nabla_t(y)\rangle, \quad \text{and} \quad \langle \alpha_t x, y\rangle = \langle x, D_t(y)\rangle.$$

Therefore, with respect to the bilinear form the skew-derivations ∇_t and D_t are right adjoint to right and left multiplication by α_t, respectively.

Proof. Part (1) follows from the definitions of ∇_t, D_t (see (6.9)) and the bilinear form. For part (2), for any $x \in \widetilde{\mathcal{A}}$ we have

$$\nabla_t D_{t'}(x) = \nabla_t(\sum_{(x)} x_{(1)} \langle x_{(2)}, \alpha_{t'}\rangle) = \sum_{(x)} \langle \alpha_t, x_{(1)}\rangle\, x_{(2)} \langle x_{(3)}, \alpha_{t'}\rangle.$$

Similarly, we can express $D_{t'}\nabla_t(x)$ and obtain that $\nabla_t D_{t'} = D_{t'}\nabla_t$. The proof of part (3) is similar to that of Lemma 5.4. $\square$

Remark 6.16. We do not know whether the bilinear form on $\widetilde{\mathcal{A}}$ is non-degenerate. Note that by Lemma 6.2 $\mathcal{A}$ and its opposite $\mathcal{A}^{op}$ can be lifted to $\widetilde{\mathcal{A}}$ as vector spaces. It follows from Proposition 5.5 that the restriction $\langle -, -\rangle : \mathcal{A}^{op} \times \mathcal{A} \to \mathbb{C}$ is non-degenerate.

We define

$$\widetilde{\omega} := \sum_{t \in T} \alpha_t.$$

By the defining relations of $\widetilde{\mathcal{A}}$ we have $\widetilde{\omega}^2 = 0$. Hence $(\widetilde{\mathcal{A}}, r_{\widetilde{\omega}})$ (resp. $(\widetilde{\mathcal{A}}, \ell_{\widetilde{\omega}})$) is a cochain complex, where $r_{\widetilde{\omega}}$ (resp. $\ell_{\widetilde{\omega}}$) is given by right (resp. left) multiplication by $\widetilde{\omega}$.

Proposition 6.17. *We have the following:*

1. *Let $\nabla = \sum_{t\in T} \nabla_t$ and $D = \sum_{t \in T} D_t$. Then we have*

$$\langle x\widetilde{\omega}, y\rangle = \langle x, \nabla(y)\rangle, \quad \text{and} \quad \langle \widetilde{\omega} x, y\rangle = \langle x, D(y)\rangle.$$

2. *The complexes $(\widetilde{\mathcal{A}}, D)$ and $(\widetilde{\mathcal{A}}, r_{\widetilde{\omega}})$ are acyclic.*
3. *The complexes $(\widetilde{\mathcal{A}}, \nabla)$ and $(\widetilde{\mathcal{A}}, \ell_{\widetilde{\omega}})$ are acyclic.*

Proof. Part (1) is a consequence of Proposition 6.15. For part (2), we have

$$Dr_{\widetilde{\omega}}(x) = D(x\widetilde{\omega}) = \sum_{t\in T} D(x\alpha_t) = \sum_{t \in T}(-D(x)\alpha_t + x) = -r_{\widetilde{\omega}} D(x) + Nx.$$

Therefore, we have $Dr_{\widetilde{\omega}} + r_{\widetilde{\omega}} D = N\mathrm{id}$, which implies that $(\widetilde{\mathcal{A}}, D)$ and $(\widetilde{\mathcal{A}}, r_{\widetilde{\omega}})$ are acyclic and part (2) follows. Part (3) can be proved similarly. $\square$

Appendix A. Computational results on the multiplicity

In this appendix, we tabulate some computational results on the cohomology $H^k(\mathcal{K}^*(U))$ for the simple $\mathbb{C}W$-module U, which by Theorem 5.20 counts the multiplicity of the contragredient U_L^* in the cohomology $H^k(F;\mathbb{C})$ of the Milnor fibre. The homology $H_k(\mathcal{K}(U))$ returns the same result; see Theorem 4.11. All calculations have been done with the computational algebra system Magma.

We only focus on the case of the symmetric group $W = \mathrm{Sym}_{n+1}$. Then the Milnor fibre F is an algebraic variety defined by

$$F := \{(x_1, \dots, x_{n+1}) \in \mathbb{C}^{n+1} \mid \prod_{1\le i<j\le n+1} (x_i - x_j)^2 = 1\}.$$

The reduced Milnor fibre F_0 is defined by

$$F_0 := \{(x_1, \dots, x_{n+1}) \in \mathbb{C}^{n+1} \mid \prod_{1\le i<j\le n+1} (x_i - x_j) = 1\}.$$

The symmetric group Sym_{n+1} acts on F by permuting coordinates. Hence it induces a linear (left) action on the cohomology $H^k(F;\mathbb{C})$ for $0 \le k \le n-1$. As vector spaces, $H^k(F;\mathbb{C}) \cong H^k(F_0;\mathbb{C}) \oplus H^k(F_0;\mathbb{C})$; see [DL16].

The simple right modules S_λ of Sym_{n+1} are indexed by partitions $\lambda = (\lambda_1^{m_1}, \dots, \lambda_p^{m_p})$ of $n+1$, where $\lambda_i^{m_i}$ means that λ_i repeats m_i times and $\sum_{i=1}^p m_i\lambda_i = n+1$. Let λ' be the conjugate partition of λ. Then we have $S_{\lambda'} \cong S_\lambda \otimes \epsilon$, where ϵ is the alternating representation associated to (1^{n+1}). It follows from (4.17) that

$$H^k(\mathcal{K}^*(S_\lambda)) \cong H^k(\mathcal{K}^*(S_{\lambda'})), \quad 0 \le k \le n-1$$

for any conjugate pair λ, λ' of partitions.

Let $(S_\lambda)_L$ denote the simple left module of Sym_{n+1} associated to the partition λ. It is well known that the contragredient $(S_\lambda)_L^*$ is isomorphic to $(S_\lambda)_L$ as left Sym_{n+1}-module. Therefore, using Theorem 5.20 we have

$$\langle (S_\lambda)_L, H^k(F,\mathbb{C})\rangle = \dim H^k(\mathcal{K}^*(S_\lambda)). \tag{A.1}$$

The Poincaré polynomial $P(t)$ of the Milnor fibre F can be computed by

$$P(t) = \sum_{k=0}^{n-1} \dim H^k(F;\mathbb{C}) t^k = \sum_{k=0}^{n-1} \sum_{\lambda\vdash n+1} \dim S_\lambda \langle (S_\lambda)_L, H^k(F,\mathbb{C})\rangle t^k. \tag{A.2}$$

Note that the Poincaré polynomial $P_0(t)$ of F_0 is $P_0(t) = P(t)/2$.

We tabulate computational results on (A.1) and (A.2) for all simple modules of Sym_{n+1} for $2 \le n \le 7$ in Tables 1–6. Each conjugate pair of partitions is listed in the same row as they produce the same cohomology.

Table 1: Sym_3, $P(t) = 2 + 8t$

	H^0	H^1
$(3), (1^3)$	1	2
(2,1)	0	2

Table 2: Sym_4, $P(t) = 2 + 14t + 36t^2$

	H^0	H^1	H^2
$(4), (1^4)$	1	2	2
$(3,1), (2,1,1)$	0	1	4
$(2,2)$	0	2	4

Table 3: Sym_5, $P(t) = 2 + 18t + 56t^2 + 160t^3$

	H^0	H^1	H^2	H^3
$(5), (1^5)$	1	0	2	4
$(4,1), (2,1^3)$	0	1	1	4
$(3,2), (2^2,1)$	0	1	2	6
$(3,1,1)$	0	0	4	10

Table 4: Sym_6, $P(t) = 2 + 28t + 146t^2 + 412t^3 + 1012t^4$

	H^0	H^1	H^2	H^3	H^4
$(6), (1^6)$	1	0	2	4	2
$(5,1), (2,1^4)$	0	1	1	3	8
$(4,2), (2^2,1^2)$	0	1	1	6	15
$(3,3), (2,2,2)$	0	0	1	7	11
$(4,1,1), (3,1^3)$	0	0	2	5	13
$(3,2,1)$	0	0	4	6	18

Table 5: Sym_7, $P(t) = 2 + 40t + 314t^2 + 1240t^3 + 2572t^4 + 6648t^5$

	H^0	H^1	H^2	H^3	H^4	H^5
$(7), (1^7)$	1	0	2	0	2	6
$(6,1), (2,1^5)$	0	1	1	3	3	6
$(5,2), (2^2,1^3)$	0	1	1	4	8	18
$(5,1,1), (3,1^4)$	0	0	2	3	10	24
$(4,3), (2^3,1)$	0	0	1	4	7	18
$(4,2,1), (3,2,1^2)$	0	0	2	8	17	46
$(3,3,1), (3,2^2)$	0	0	1	5	11	28
$(4,1^3)$	0	0	0	6	8	22

Table 6: Sym_8, $P(t) = 2+54t+590t^2+3330t^3+10212t^4+17744t^5+50644t^6$

	H^0	H^1	H^2	H^3	H^4	H^5	H^6
$(8), (1^8)$	1	0	0	0	2	6	4
$(7,1), (2,1^6)$	0	1	1	1	1	3	10
$(6,2), (2^2,1^4)$	0	1	1	2	4	10	28
$(6,1^2), (3,1^5)$	0	0	2	3	3	7	26
$(5,3), (2^3,1^2)$	0	0	1	3	6	10	34
$(5,2,1), (3,2,1^3)$	0	0	2	5	16	25	76
$(5,1^3), (4,1^4)$	0	0	0	3	10	22	50
$(4,4), (2^4)$	0	0	0	1	4	19	30
$(4,3,1), (3,2^2,1)$	0	0	1	6	18	27	84
$(4,2^2), (3^2,1^2)$	0	0	0	3	16	31	74
$(4,2,1^2)$	0	0	0	8	22	36	112
$(3,3,2)$	0	0	0	4	10	16	52

Remark A.1. Note that the results tabulated here are consistent with those appearing in [DL16], up to type A_4, where $W = \mathrm{Sym}_5$. However the cases computed in *loc. cit.* include the action of the monodromy group on the cohomology, in the sense that the structure of $H^*(F,\mathbb{C})$ is described as a Γ-module, where $\Gamma = \mathrm{Sym}_{n+1} \times \mu_{n(n+1)}$. In the present work, although the original Brady-Falk-Watt model of F does come with an action of the monodromy on the CW complex describing F (see [BFW18] or [Zha20, Section 3.4]), our analysis of the model has not been able to preserve the monodromy action, except to the following extent. In general, the group $\langle\gamma\rangle$ acts (like any subgroup of W) on F, and hence on $H^*(F)$. But it is known that $\langle\gamma\rangle$ may be identified with a quotient (or subgroup) of the monodromy μ, and this action is easily identifiable in our model [Zha23, Remark 6.4].

Remark A.2. Settepanella computed the cohomology $H^k(PB_{n+1}, \mathbb{Q}[q,q^{-1}])$ of the pure braid group PB_{n+1} with coefficients in the Laurent polynomial ring $\mathbb{Q}[q,q^{-1}]$ for $n \leq 7$ [Set09, Table 2]. This is related to the Milnor fibre F_0 by

$$H^{k+1}(PB_{n+1}, \mathbb{Q}[q,q^{-1}]) \cong H^k(F_0, \mathbb{Q}).$$

Hence Settepanella's results give rise to the Poincaré polynomials $P_0(t)$ in type A_n for $n \leq 7$. The Poincaré polynomials $P_0(t) = P(t)/2$ given in the tables below coincide with those of Settepanella up to type A_6. However, for type A_7 our cohomology groups $H^k(F_0;\mathbb{Q})$ agree with Settepanella's except for $k = 5, 6$, but the Euler characteristic remains the same.

Remark A.3. Apart from the monodromy action, the major issue untouched by our work is the mixed Hodge structure on each cohomology group $H^i(F,\mathbb{C})$. We hope to return to this theme later.

References

[AB02] M. Atiyah, and R. Bielawski. Nahm's equations, configuration spaces and flag manifolds. *Bull. Braz. Math. Soc. (N.S.)*, **33**(2): 157–176, 2002. MR1940347

[AG99] N. Andruskiewitsch, and M. Graña. Braided Hopf algebras over non-abelian finite groups. Colloquium on Operator Algebras and Quantum Groups (Spanish) (Vaquerias, 1997). *Bol. Acad. Nac. Cienc. (Córdoba)*, **63**: 45–78, 1999. MR1714540

[AS02] N. Andruskiewitsch, and H. J. Schneider. Pointed Hopf algebras. New directions in Hopf algebras, *Math. Sci. Res. Inst. Publ.*, 43. Cambridge Univ. Press, Cambridge, 1–68, 2002. MR1913436

[Arm09] D. Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups, *Mem. Amer. Math. Soc.*, 202. x+159 pp, 2009. MR2561274

[ABW07] C. Athanasiadis, T. Brady, and C. Watt. Shellability of noncrossing partition lattices. *Proc. Amer. Math. Soc.*, **135**(4): 939–949, 2007. MR2262893

[Bes03] D. Bessis. The dual braid monoid. *Ann. Sci. Ecole Norm. Sup.*, **36**: 647–683, 2003. MR2032983

[Bjö80] A. Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, **260**(1): 159–183, 1980. MR0570784

[Bou02] N. Bourbaki. Lie groups and Lie algebras. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley. *Elements of Mathematics (Berlin)*. Springer-Verlag, Berlin, 2002. MR1890629

[BFW18] T. Brady, M. J. Falk, and C. Watt. Noncrossing partitions and Milnor fibers. *Algebr. Geom. Topol.*, **18**(7): 3821–3838, 2018. MR3892232

[BW02] T. Brady and C. Watt. A partial order on the orthogonal group. *Comm. Algebra*, **30**(8): 3749–3754, 2002. MR1922309

[BW08] T. Brady and C. Watt. Non-crossing partition lattices in finite real reflection groups. *Trans. Amer. Math. Soc.*, **360**(4): 1983–2008, 2008. MR2366971

[Bre72] G. E. Bredon. Introduction to compact transformation groups, *Pure Appl. Math.*, 46. Academic Press, New York-London, 1972. MR0413144

[Car72] R. W. Carter. Conjugacy classes in the Weyl group. *Compositio Math.*, **25**: 1–59, 1972. MR0318337

[DPSS99] C. De Concini, C. Procesi, M. Salvetti, and F. Stumbo. Arithmetic properties of the cohomology of Artin groups. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **28**(4): 695–717, 1999. MR1760537

[DL16] A. Dimca, and G. I. Lehrer. Cohomology of the Milnor fibre of a hyperplane arrangement with symmetry. Configuration spaces, *Springer INdAM Ser.*, 14. Springer, [Cham], 233–274, 2016. MR3615735

[FK99] S. Fomin, and A. N. Kirillov. Quadratic algebras, Dunkl elements, and Schubert calculus. Advances in geometry, *Progr. Math.*, 172. Birkhäuser Boston, Boston, MA, 147–182, 1999. MR1667680

[Hat02] A. Hatcher. Algebraic topology. Cambridge University Press, Cambridge. xii+544 pp. 2002. MR1867354

[HL99] R. B. Howlett, and G. I. Lehrer. On reflection length in reflection groups, *Arch. Math. (Basel)*, **73**(5): 321–326, 1999. MR1712091

[Leh96] G. I. Lehrer. A vanishing theorem for hyperplane cohomology, *Bull. Austral. Math. Soc.*, **53**(3): 361–368, 1996. MR1388584

[MS00] A. Milinski, and H. J. Schneider. Pointed indecomposable Hopf algebras over Coxeter groups. New trends in Hopf algebra theory (La Falda, 1999), *Contemp. Math.*, 267. Amer. Math. Soc., Providence, RI, 215–236, 2000. MR1800714

[Set09] S. Settepanella. Cohomology of pure braid groups of exceptional cases. *Topology Appl.*, **156**(5): 1008–1012, 2009. MR2498934

[Ste59] R. Steinberg. Finite reflection groups. *Trans. Amer. Math. Soc.*, **91**: 493–504, 1959. MR0106428

[Zha20] Y. Zhang. Combinatorics of Milnor fibres of reflection arrangements. PhD thesis, University of Sydney, 2020.

[Zha23] Y. Zhang. On the homology of the noncrossing partition lattice and the Milnor fibre. *Adv. Math.*, **415**: 108897, 2023. MR4543452

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