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Catalan numbers and noncommutative Hilbert schemes*

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Abstract: We find an explicit S_n-equivariant bijection between the integral points in a certain zonotope in $\mathbb{R}^n$, combinatorially equivalent to the permutahedron, and the set of m-parking functions of length n. This bijection restricts to a bijection between the regular S_n-orbits and (m,n)-Dyck paths, the number of which is given by the Fuss-Catalan number $A_n(m,1)$. Our motivation came from studying tilting bundles on noncommutative Hilbert schemes. As a side result we use these tilting bundles to construct a semi-orthogonal decomposition of the derived category of noncommutative Hilbert schemes.

1. Introduction

1.1. Some combinatorial results

In this section we state some purely combinatorial results which give a new interpretation of parking functions and Fuss-Catalan numbers [21, 22] in terms of lattice points in a certain polytope related to the permutathedron. In the next section we will give the motivation behind these results.

Let $m, n \in \mathbb{N}$ and let $(e_i)_{i=1,\ldots,n}$ be the standard basis for $\mathbb{Z}^n \subset \mathbb{R}^n$. We let the symmetric group S_n act on $\mathbb{Z}^n$ and $\mathbb{R}^n$ by permutations. Let $\Delta^{m,n}$ be the S_n-invariant zonotope which is the Minkowski sum of the intervals

$$[0, e_i], \quad 1 \leq i \leq n,$$

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$$\left[0, \frac{m}{2}(e_i - e_j)\right], \quad 1 \leq i \neq j \leq n.$$

For $\nu := \sum_{i=1}^n e_i$ and $\tau \in \mathbb{R}$ put $\Delta_\tau^{m,n} = \Delta^{m,n} + \tau\nu$.

Definition 1.1. We say that τ is *admissible* if $\tau - m(n-1)/2$ is not a rational number with denominator $\leq n$.

The significance of this condition is the following:

Lemma 1.2. *(see §3.1) τ is admissible if and only if $\partial(\Delta_\tau^{m,n}) \cap \mathbb{Z}^n = \emptyset$.*

In this note we prove the following result.

Proposition 1.3. *(see §3.2) Assume τ is admissible. Let $L = (mn+1)\mathbb{Z}^n + \mathbb{Z}\nu$. Then*

$$\Delta_\tau^{m,n} \cap \mathbb{Z}^n \to \mathbb{Z}^n/L : a \mapsto \bar{a} \tag{1.1}$$

is an S_n-equivariant bijection.

This result follows very quickly from the fact that $\Delta_\tau^{m,n}$ is equivalent, in a suitable sense, to the permutahedron and hence is space tiling, see Proposition 3.1.

Proposition 1.3 allows one to relate the lattice points in $\Delta_\tau^{m,n}$ for admissible τ to *parking functions*. Recall that an (m,n)-parking function is a sequence of natural numbers[1] $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ such that its weakly increasing rearrangement $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_n}$ satisfies $a_{i_j} \leq m(j-1)$. Note that S_n acts on parking functions by permuting indices. Below we denote the set of (m,n)-parking functions by $\mathcal{Q}^m$. According to [20, NOTE in §3] or [3, §5.1] the map

$$\mathcal{Q}^m \to \mathbb{Z}^n/L : a \mapsto \bar{a} \tag{1.2}$$

is an S_n-equivariant bijection. Combining Proposition 1.3 with (1.2) yields:

Corollary 1.4. *If τ is admissible then there is an explicit S_n-equivariant bijection between lattice points in $\Delta_\tau^{m,n}$ and (m,n)-parking functions.*

If $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is weakly increasing then we say that a is an (m,n)-*Dyck path* if $a_j \leq (m-1)(j-1)$. The number of (m,n)-Dyck paths is

$$A_n(m,1) := \frac{1}{mn+1}\binom{mn+1}{n} = \frac{1}{(m-1)n+1}\binom{mn}{n} \tag{1.3}$$

[1] We assume $0 \in \mathbb{N}$.

and is called the (m,n)-*Fuss-Catalan number*. The following is clear:

Lemma 1.5. *There is a bijection between regular orbits of (m,n)-parking functions and (m,n)-Dyck paths which sends the orbit representative $a = (a_1,\dots,a_n) \in \mathbb{N}^n$ with $a_1 < \dots < a_n$ to $(a_1, a_2 - 1, \dots, a_n - (n-1))$.*

We thus obtain a new interpretation of the Fuss-Catalan numbers.

Corollary 1.6. *If τ is admissible then there is an explicit S_n-equivariant bijection between regular S_n-orbits in $\Delta_\tau^{m,n} \cap \mathbb{Z}^n$ and (m,n)-Dyck paths. In particular the number of such regular orbits is $A_n(m,1)$.*

The claim in Corollary 1.6 concerning $A_n(m,1)$ was first observed by us as a consequence of the properties of the "noncommutative Hilbert scheme". This is explained in §1.2 below.

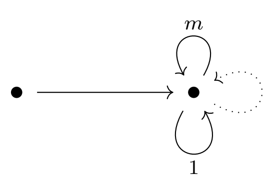
In an appendix we will also give a second combinatorial proof of the claim about $A_n(m,1)$. The basic idea is that if τ is admissible then, since $\Delta_\tau^{m,n}$ is a zonotope such that $\partial(\Delta_\tau^{m,n}) \cap \mathbb{Z}^n = \emptyset$, we have $|\Delta_\tau^{m,n} \cap \mathbb{Z}^n| = \mathrm{Vol}(\Delta_\tau^{m,n})$ by Proposition A.2 and moreover there is an explicit formula for $\mathrm{Vol}(\Delta_\tau^{m,n})$ in terms of spanning trees in a suitable graph (see §8). Using an appropriate inclusion/exclusion argument we may upgrade this to a count of the regular orbits in $\Delta_\tau^{m,n} \cap \mathbb{Z}^n$.

1.2. The noncommutative Hilbert scheme

The Hilbert scheme of length n-sheaves on $\mathbb{A}^n$ may be viewed as the moduli space of cyclic modules of dimension n over the polynomial algebra $\mathbb{C}[x_1,\dots,x_m]$. It is then natural to define the corresponding *noncommutative Hilbert scheme* $H_{m,n}$ as the moduli space of cyclic modules of dimension n over the free algebra $\mathbb{C}\langle x_1,\dots,x_m\rangle$. We recall:

Proposition 1.7 ([15, 23])**.** *$H_{m,n}$ has a stratification consisting of affine spaces and the number of strata is given by the Fuss-Catalan number $A_n(m,1)$.*

It is clear that $H_{m,n}$ can be described as the moduli space of stable (or equivalently semi-stable) representations with dimension vector $(1,n)$ and stability condition $(-n,1)$ [8, Definition 1.1] of the following quiver $Q_{m,n}$:



It follows from loc. cit. that $H_{m,n}$ can also be described as a GIT quotient for the group $(\mathbb{C}^* \times \mathrm{GL}_n(\mathbb{C}))/\{\text{center}\} \cong \mathrm{GL}_n(\mathbb{C})$. More precisely we get $H_{m,n} = W^{ss,\chi}/G$ where $G = \mathrm{GL}_n(\mathbb{C})$, $W = \mathrm{End}(\mathbb{C}^n)^{\oplus m} \oplus \mathbb{C}^n$ and $W^{ss,\chi} \subset W$ is the semi-stable locus associated to the determinant character χ.

Using the GIT description $H_{m,n}$ we will show using [7, 18] that $H_{m,n}$ admits a family of tilting bundles. Let $\Delta_\tau^{m,n} \subset \mathbb{R}^n$ be as in §1.1. We identify $\mathbb{Z}^n$ with the character group of the diagonal torus $(\mathbb{C}^*)^n$ in $\mathrm{GL}_n(\mathbb{C})$. Let $(\mathbb{Z}^n)^+$ be the "dominant" part of $\mathbb{Z}^n$, i.e. those $(c_1, \ldots, c_n) \in \mathbb{Z}^n$ such that $c_1 \geq \cdots \geq c_n$. For $\xi \in (\mathbb{Z}^n)^+$ let $V(\xi)$ be the irreducible $\mathrm{GL}_n(\mathbb{C})$ representation with highest weight ξ and let $\mathcal{V}(\xi)$ be the equivariant vector bundle on $H_{m,n}$ corresponding to the $\mathrm{GL}_n(\mathbb{C})$-equivariant vector bundle $V(\xi) \otimes_k \mathcal{O}_{W^{ss,\chi}}$ on $\mathcal{O}_{W^{ss,\chi}}$. Put

$$\hat{\rho} = \frac{1}{2}\sum_{i<j}(e_i - e_j) + \frac{1}{2}(n-1)\nu = (n-1, n-2, \ldots, 1, 0).$$

Proposition 1.8. *(see §4) Let τ be admissible. Then*

$$\mathcal{T}_\tau := \bigoplus_{\xi \in (\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})} \mathcal{V}(\xi) \tag{1.4}$$

is a tilting bundle on $H_{m,n}$.

We refer the reader to Appendix B for tables with tilting bundles on $H_{2,2}$, $H_{3,2}$ and $H_{4,2}$.

Comparing the ranks of $K_0(H_{m,n})$ obtained from Propositions 1.7 and 1.8 yields the identity

$$|(\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})| = A_n(m,1). \tag{1.5}$$

Sending $a \mapsto a + \hat{\rho}$ defines a bijection between $(\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})$ and the regular orbits in $\mathbb{Z}^n \cap \Delta_\tau^{m,n}$. This yields a "geometric" proof of the claim about $A_n(m,1)$ in Corollary 1.6.

1.3. A semi-orthogonal decomposition of the non-commutative Hilbert scheme

We will use the tilting bundles defined in (1.4) to obtain an interesting side result on the structure of $\mathcal{D}(H_{m,n}) := D_{\mathrm{Qch}}(H_{m,n})$. Recall that if X is a noetherian scheme and $\mathcal{A}$ is a sheaf of Azumaya algebras on X of rank n^2 then the Brauer-Severi scheme $Y = \mathrm{BS}(\mathcal{A}, X)$ is defined as the moduli space of left ideals of codimension n in $\mathcal{A}$ (using the opposite convention from [14, §8.4]). The definition of the Brauer-Severi variety was extended to singular schemes in [2] and in [23] (see also [10, 16]) it was shown that $H_{m,n}$ is the Brauer-Severi scheme of the so-called "trace ring of m generic $n \times n$-matrices $\mathbb{T}_{m,n}$".

Trace rings were studied by Artin [1] and Procesi [13], we refer the reader to [18, §1.4.5] for a short introduction. Recall that the commutative trace ring $Z_{m,n}$ of m generic $n\times n$-matrices is equal to $\Gamma(W_0)^G$ with $W_0 = M_n(\mathbb{C})^{\oplus m}$ and $G = \mathrm{GL}_n(\mathbb{C})$, acting by conjugation. The noncommutative trace ring $\mathbb{T}_{m,n}$ is the $Z_{m,n}$-*algebra of covariants* [18, §4.1] $M(M_n(\mathbb{C})) := (M_n(\mathbb{C}) \otimes \Gamma(W_0))^G$. It follows from the definitions that $H_{m,n} \to \operatorname{Spec} Z_{m,n}$ is a standard Brauer-Severi scheme when restricted to the Azumaya locus of $\mathbb{T}_{m,n}$. This locus is non-empty and dense when $m \geq 2$.

If $\xi \in \mathbb{Z}^n$ is a weight for G then we define its *color* as $c(\xi) = \sum_i \xi_i$; i.e. it is the weight of ξ when restricted to the center of G. For $c \in \mathbb{Z}$ we put

$$V_\tau(c) := \bigoplus_{\xi \in (\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho}), c(\xi) = c} V(\xi) \tag{1.6}$$

Proposition 1.9. *Assume that τ is admissible and $m \geq 2$. There is a semi-orthogonal decomposition (depending on τ)*

$$\mathcal{D}(H_{m,n}) = \langle D(R_{\tau,u}), \dots, D(R_{\tau,u+n-1}) \rangle \tag{1.7}$$

linear over $Z_{m,n}$ with $u = \lceil n\tau \rceil - n(n-1)/2$ such that

$$R_{\tau,c} = M(\operatorname{End}(V_\tau(c))).$$

The restriction of $R_{\tau,c}$ to the Azumaya locus of $\mathbb{T}_{m,n}$ is Morita equivalent to the restriction of $\mathbb{T}_{m,n}^{\otimes c}$.

The proof is based on partitioning the tilting bundle $\mathcal{T}_\tau$ into semi-orthogonal parts.

Remark 1.10. The reader will guess that the restriction of (1.7) to the Azumaya locus of $\mathbb{T}_{m,n}$ is a rotation of the usual semi-orthogonal decomposition of a Brauer-Severi scheme (see [5, Theorem 5.1]). One may show that this guess is correct.

Remark 1.11. From (1.7) we get a τ-dependent decomposition

$$A_n(m,1) = \operatorname{rk} K_0(H_{m,n}) = \sum_{c=u}^{u+n-1} \operatorname{rk} K_0(R_{\tau,c}).$$

It would be interesting to see if this decomposition can be made concrete using some of the many combinatorial interpretations of $A_n(m, 1)$.

1.3.1. Related work In [12] the authors construct a semi-orthogonal decomposition of $\mathcal{D}(H_{3,n})$. Their decomposition is more refined as it also involves categories generated by suitable $\mathcal{V}(\xi)$ on approprate smaller noncommutative Hilbert schemes. Moreover, they construct a semi-orthogonal decomposition of the category of matrix factorisations on $H_{3,n}$ with a super-potential whose critical locus is the Hilbert scheme of points.

2. Zonotopes

2.1. Generalities

A zonotope Z in a finite dimensional $\mathbb{R}$-vector space V is a subset of the form $t + \sum_{i=1}^{u} [\beta_i, \gamma_i]$. The vectors $\gamma_i - \beta_i$ are the *defining vectors* of the zonotope. The faces of a zonotope are easy to compute (see e.g. [11, §2]). They are of the form

$$t + \sum_{\langle \lambda, \gamma_i - \beta_i \rangle > 0} \gamma_i + \sum_{\langle \lambda, \gamma_i - \beta_i \rangle < 0} \beta_i + \sum_{\langle \lambda, \gamma_i - \beta_i \rangle = 0} [\beta_i, \gamma_i] \tag{2.1}$$

for $\lambda \in V^*$. In particular if Z is full dimensional then the facets correspond to those λ such that the defining vectors in the kernel of λ span a hyperplane in V. Note that the facets come in parallel pairs, and the companion to a facet corresponding to λ is the one corresponding to $-\lambda$.

Remark 2.1. The defining vectors of a zonotope are of course not unique. However if we restrict to the case that no two defining vectors are parallel then the defining vectors are unique – up to sign. Indeed it follows from (2.1) that they are given by the edges of the zonotope. We may always reduce to the case that no two defining vectors are parallel by taking the sum of all groups of parallel intervals.

For use below we define the *tiling lattice* as the subgroup of V spanned by the vectors

$$t_\lambda = \sum_{\langle \lambda, \gamma_i - \beta_i \rangle > 0} (\gamma_i - \beta_i) - \sum_{\langle \lambda, \gamma_i - \beta_i \rangle < 0} (\gamma_i - \beta_i) \tag{2.2}$$

where λ runs through the $\lambda \in V^*$ defining facets (as explained above).

2.2. Space tiling zonotopes

Let $Z \subset V$ be a full dimensional zonotope where $\dim V = n$. We say that Z is *space tiling* if there is a lattice $L \subset V$ such that $V = \bigcup_{l\in L}(l+Z)$ and such that $Z \cap (l+Z)$ for $l \in L$ is a face in both Z and $l+Z$. It is easy to see that L must be equal to the tiling lattice of Z.

The following result gives an easy way of recognizing space tiling zonotopes.

Proposition 2.2 ([11, §2.II])**.** *A full dimensional zonotope is space tiling if and only if every $n-2$-dimensional subspace of V spanned by the defining vectors is contained in 2 or 3 (and not more) hyperplanes spanned by the defining vectors.*

Lemma 2.3. *Assume that $M \subset V$ is a lattice and that Z is a translation of a zonotope with vertices in M such that in addition $M \cap \partial Z = \emptyset$. Assume furthermore that Z is space tiling with tiling lattice L (in particular $L \subset M$). Then the map*

$$Z \cap M \mapsto M/L : m \mapsto \bar{m}$$

is a bijection.

Proof. By the definition of space tiling we have an L-equivariant decomposition:

$$M = \coprod_{l\in L}(l + M \cap Z).$$

Quotienting out the L-action on both sides yields the lemma. ☐

3. The structure of $\Delta_\tau^{m,n}$

One may apply (2.1) to determine the facets of $\Delta_\tau^{m,n}$. One verifies using (2.1) that the facets of $\Delta_\tau^{m,n}$ are defined by $\pm w\lambda_k$ for $w \in S_n$, for $k = 1, \ldots, n$, $\lambda_k = (1^k, 0^{n-k})$. More precisely the facets associated to $\pm w\lambda_k$ are respectively of the form

$$\begin{aligned}
F_{w,k}^{+} &= \tau\nu + (1/2)m \sum_{(w\lambda_k)_i=1,(w\lambda_k)_j=0} (e_i - e_j) + \sum_{(w\lambda_k)_i=1} e_i \\
&\quad + (1/2)m \sum_{(w\lambda_k)_i=(w\lambda_k)_j} [0, e_i - e_j] + \sum_{(w\lambda_k)_i=0} [0, e_i], \\
F_{w,k}^{-} &= \tau\nu + (1/2)m \sum_{(w\lambda_k)_i=0,(w\lambda_k)_j=1} (e_i - e_j) \\
&\quad + (1/2)m \sum_{(w\lambda_k)_i=(w\lambda_k)_j} [0, e_i - e_j] + \sum_{(w\lambda_k)_i=0} [0, e_i].
\end{aligned} \tag{3.1}$$

Proposition 3.1. *The zonotope $\Delta_\tau^{m,n}$ is space tiling with tiling lattice $(mn+1)\mathbb{Z}^n + \mathbb{Z}(1,\ldots,1)$.*

Proof. We identify $\mathbb{R}^n$ with the hyperplane in $\mathbb{R}^{n+1}$ given by $x_0+\cdots+x_n=0$. Then $\Delta_\tau^{m,n}$ is a translation of the zonotope with the defining vectors e_i-e_0, $m(e_j-e_i)$, $1\le i<j\le n$. On the other hand, the permutahedron is the Minkowski sum of $[e_i,e_j]$, $0\le i<j\le n$; i.e. it is a zonotope with the defining vectors e_j-e_i, $0\le i<j\le n$ (see e.g. [4, Theorem 9.4]). It is known that the permutahedron is space tiling, see e.g. [4, Ex. 9.12]. By Proposition 2.2, the space tiling property does not depend on the length of the defining vectors. Thus, $\Delta_\tau^{m,n}$ is also space tiling.

Now we compute the tiling lattice. As indicated above the facets of $\Delta_k^{m,n}$ are determined by $\pm w\lambda_k$. Furthermore in the formula (2.2) we have $t_{-\lambda}=-t_\lambda$ so it suffices to compute the lattice spanned by $(t_{w\lambda_k})_{w,k}$ Let $S=\{i\mid (w\lambda_k)_i=1\}$ and hence $S^c=\{1,\ldots,n\}\setminus S=\{i\mid (w\lambda_k)_i=0\}$. Note that $|S|=k$, $|S^c|=n-k$. We write δ_{iS} for the characteristic function of S; i.e. $\delta_{iS}=1$ if $i\in S$ and $\delta_{iS}=0$ otherwise. With $(\delta_{iS})_{1\le i\le n}$ we denote $(\delta_{1S},\ldots,\delta_{nS})$. Then

$$\begin{aligned}
t_{w\lambda_k} &= (m/2)\sum_{i\in S, j\in S^c}(e_i-e_j) - (m/2)\sum_{i\in S^c, j\in S}(e_i-e_j) + \sum_{i\in S} e_i\\
&= m\sum_{i\in S, j\in S^c}(e_i-e_j) + \sum_{i\in S} e_i\\
&= (m(n-k)+1)(\delta_{iS})_{1\le i\le n} - mk(\delta_{iS^c})_{1\le i\le n}\\
&= (mn+1)(\delta_{iS})_{1\le i\le n} - mk(1,\ldots,1).
\end{aligned}$$

Setting $k=n$ we obtain $t_{w\lambda_n}=(1,\ldots,1)$. Adding a suitable multiple of $t_{w\lambda_n}$ to $t_{w\lambda_k}$ we then obtain that the lattice is generated by $(1,\ldots,1)$, $(mn+1)(\delta_{iS})_{1\le i\le n}$ for $S\subset\{1,\ldots,n\}$, which easily implies our claim. □

3.1. Proof of Lemma 1.2

Fix $1\le k\le n$. We have to understand when $F_{w,k}^{\pm}$ contains an integral point for $w\in S_n$. Renumbering the $(e_i)_i$ we may assume that $w=\mathrm{id}$ where id is the identity in S_n. Moreover it follows from (3.1) that $F_{\mathrm{id},k}^-$ contains a rational point if and only if $F_{\mathrm{id},k}^+$ does too. Hence we are reduced to understanding to when $F_k := F_{\mathrm{id},k}^+$ contains an integral point.

Let $L_k=\sum_{i>k}\mathbb{Z}e_i+\sum_{i,j\le k \text{ or } i,j>k}\mathbb{Z}(e_i-e_j)$. Then $(L_k)_\mathbb{R}$ is the hyperplane through the origin which is parallel to F_k. Sending e_i to 1 for $i\le k$

and to 0 for $i > k$ identifies $\mathbb{Z}^n/L_k$ with $\mathbb{Z}$ and $\mathbb{R}^n/(L_k)_\mathbb{R}$ with $\mathbb{R}$. We regard $\mathbb{Z}^n/L_k$ as being contained in $\mathbb{R}^n/(L_k)_\mathbb{R}$.

Let $\overline{F_k}$ be the image of F_k in $\mathbb{R}^n/(L_k)_\mathbb{R}$. Note that $\overline{F_k}$ is a singleton.

Claim 1. F_k contains an integral point if and only if $\overline{F_k} \subset \mathbb{Z}^n/L_k$.

One direction is obvious. To see the other direction assume $x \in \mathbb{Z}^n$ is such that $(x + (L_k)_\mathbb{R}) \cap F_k \neq \emptyset$. Then $-x + F_k$ is a translation of

$$(1/2)m \sum_{i,j \le k \text{ or } i,j > k} [0, e_i - e_j] + \sum_{i>k} [0, e_i] \tag{3.2}$$

inside $(L_k)_\mathbb{R}$. Now (3.2) is itself a translation of a full dimensional lattice polytope and any translation of a full dimensional lattice polytope contains an integral point (e.g. by Proposition A.2). Hence there exists some $y \in \mathbb{Z}^n$ such that $y \in -x + F_k$. We conclude $x + y \in F_k \cap \mathbb{Z}^n$.

Claim 2. $\overline{F_k}$ is contained in $\mathbb{Z}^n/L_k$ if and only if $\tau - m(n-1)/2 \in (1/k)\mathbb{Z}$,

To prove this we use the identifications $\mathbb{Z}^n/L_k \cong \mathbb{Z}$, $\mathbb{R}^n/(L_k)_\mathbb{R} \cong \mathbb{R}$ given above. By (3.1) $\overline{F_k}$ is the singleton $\{k\tau + mk(n-k)/2 + k\}$. This is contained in $\mathbb{Z}$ if and only if $\tau + m(n-k)/2 \in (1/k)\mathbb{Z}$ which may be rewritten as $\tau - m(n-1)/2 \in -m(k-1)/2 + (1/k)\mathbb{Z} = (1/k)\mathbb{Z}$.

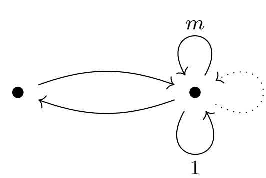
Combining Claims 1 and 2 finishes the proof.

3.2. Proof of Proposition 1.3

The fact that (1.1) is a bijection follows directly from Lemma 2.3 combined with Proposition 3.1. The fact that (1.1) is S_n-equivariant is clear.

4. Proof of Proposition 1.8

Let the setting be as in §1.2 in the introduction. Our aim is to apply [7] but the representation W is not "quasi-symmetric" (see [18]). So the theory of [7] does not apply on the nose. We circumvent this by relating $H_{m,n}$ to the moduli space $\tilde{H}_{m,n}$ of stable (or equivalently semi-stable) representations with dimension vector $(1, n)$ and stability condition $(-n, 1)$ of the quiver $\tilde{Q}_{m,n}$:



This quiver is symmetric so the corresponding GIT setting is symmetric. Moreover it is easy to see that a representation of $\tilde{Q}_{m,n}$ is semi-stable if and only if its restriction ot $Q_{m,n}$ is semi-stable. It then follows by a descent argument that the map $\pi : \tilde{H}_{m,n} \to H_{m,n}$, obtained by forgetting the left pointing arrow, is a vector bundle.

We now assume that τ is admissible. By Lemma 1.2, $\partial(\Delta_\tau^{m,n}) \cap \mathbb{Z}^n = \emptyset$. For $\xi \in \mathbb{Z}^n$ let $\tilde{\mathcal{T}}_\tau$ be defined as $\mathcal{T}_\tau$ but using the quiver $\tilde{Q}_{m,n}$. Then $\tilde{\mathcal{T}}_\tau$ is a tilting bundle on $\tilde{H}_{m,n}$ by [7, Corollary 4.2].[2]

Lemma 4.1. *Let $\pi : \tilde{X} \to X$ be a morphism of quasi-compact quasi-separated schemes with a section $i : X \to \tilde{X}$. Let $\mathcal{T} \in \mathrm{Perf}(X)$ such that $L\pi^*\mathcal{T}$ is a tilting object in $D_{\mathrm{Qch}}(\tilde{X})$. Then $\mathcal{T}$ is itself tilting.*

Proof. We first need to prove that $\mathcal{T}$ is a generator of $D_{\mathrm{Qch}}(X)$, i.e. that $\mathcal{T}^\perp = 0$. Let $\mathcal{N} \in D_{\mathrm{Qch}}(X)$ be such that $\mathrm{RHom}_X(\mathcal{T}, \mathcal{N}) = 0$. We compute

$$\begin{aligned} \mathrm{RHom}_X(\mathcal{T}, \mathcal{N}) &= \mathrm{RHom}_X(\mathcal{T}, R\pi_* Ri_* \mathcal{N}) \\ &= \mathrm{RHom}_{\tilde{X}}(L\pi^*\mathcal{T}, Ri_*\mathcal{N}). \end{aligned}$$

Since $(L\pi^*\mathcal{T})^\perp = 0$ deduce $Ri_*\mathcal{N} = 0$ and hence $\mathcal{N} = R\pi_* Ri_* \mathcal{N} = 0$. So $\mathcal{T}^\perp = 0$. We also have

$$\mathrm{RHom}_{\tilde{X}}(L\pi^*\mathcal{T}, L\pi^*\mathcal{T}) = \mathrm{RHom}_X(\mathcal{T}, R\pi_* L\pi^*\mathcal{T}).$$

Since $\tilde{X} \to X$ is split, the unit map $\mathcal{T} \to R\pi_* L\pi^*\mathcal{T}$ is also split. It follows that $\mathrm{RHom}_X(\mathcal{T}, \mathcal{T})$ is a direct summand of $\mathrm{RHom}_{\tilde{X}}(L\pi^*\mathcal{T}, L\pi^*\mathcal{T})$. Hence if $L\pi^*\mathcal{T}$ has no non-zero self-extensions then neither has $\mathcal{T}$. □

5. A semi-orthogonal decomposition for the noncommutative Hilbert scheme

We use the same notations as in §4. Let $\tilde{W}$ be the GIT setting corresponding to $\tilde{Q}_{m,n}$, i.e. $(\tilde{W}, G)$ with $\tilde{W} = \mathbb{C}^n \oplus (\mathbb{C}^n)^* \oplus M_n(\mathbb{C})^{\oplus m}$. We will consider the $\mathbb{C}^*$-action on $\tilde{W}$ obtained by scaling the right pointing arrow in $\tilde{Q}_{m,n}$. This

[2]The polytope used in [7] to construct tilting objects is denoted by $(1/2)\bar{\Sigma} - \rho + \delta$ in loc. cit., where ρ is half the sum of the positive roots and (in our setting) $\delta = \tau'\nu$ for suitable τ'. See [7, Lemma 2.9, Corollary 4.2]. Using the definition of Σ in [7, §2] one checks that $\Delta_\tau^{m,n} - \hat{\rho} = (1/2)\bar{\Sigma} - \rho + \tau'\nu$ where $\tau = \tau' - 1 + n/2$, so that we may indeed use the results of [7]. Note that the exact relation between τ and τ' is not important here.

action commutes with the G-action and we have $\tilde{W}/\!\!/\mathbb{C}^* = W$, $\tilde{H}_{m,n}/\!\!/\mathbb{C}^* = H_{m,n}$.

Below we will define some notions for the GIT setting (W, G). Similar notion related to the GIT setting $(\tilde{W}, G)$ will be decorated with a tilde.

Recall that $H_{m,n} = W^{ss,\chi}/G$. For $\tau \in \mathbb{R}$ let $\mathcal{M}_\tau \subset D_{\mathrm{Qch}}(W/G)$ be the smallest cocomplete subcategory of $D_{\mathrm{Qch}}(W/G)$ containing the G-equivariant $\mathcal{O}_W$-modules $V(\xi) \otimes \mathcal{O}_W$ for $\xi \in (\Delta_\tau^{m,n} - \hat{\rho}) \cap (\mathbb{Z}^n)^+$.

Lemma 5.1. *Assume that τ is admissible. The restriction map*

$$\mathrm{res} : D_{\mathrm{Qch}}(W/G) \to D_{\mathrm{Qch}}(W^{ss,\chi}/G)$$

restricts to an equivalence $\mathcal{M}_\tau \cong D_{\mathrm{Qch}}(W^{ss,\chi}/G)$.

Proof. Using the fact that $\mathcal{T}_\tau$ (cfr Proposition 1.8) is a tilting bundle it suffices to prove that when $\xi, \xi' \in (\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})$ the restriction map defines an isomorphism

$$\mathrm{Hom}_W(V(\xi) \otimes \mathcal{O}_W, V(\xi') \otimes \mathcal{O}_W)^G \to \mathrm{Hom}_{H_{m,n}}(\mathcal{V}(\xi), \mathcal{V}(\xi')).$$

This follows by applying $(-)^{\mathbb{C}^*}$ to

$$\mathrm{Hom}_{\tilde{W}}(V(\xi) \otimes \mathcal{O}_{\tilde{W}}, V(\xi') \otimes \mathcal{O}_{\tilde{W}})^G \to \mathrm{Hom}_{\tilde{H}_{m,n}}(\tilde{\mathcal{V}}(\xi), \tilde{\mathcal{V}}(\xi'))$$

together with [7, Theorem 3.2]. □

Put

$$\mathcal{T}_{\tau,c} := \bigoplus_{\xi \in (\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho}), c(\xi) = c} \mathcal{V}(\xi)$$

Lemma 5.2. *Assume τ that is admissible.*

1. *We have $\mathcal{T}_{\tau,c} = 0$ unless $c \in \{u, \dots, u+n-1\}$ for $u = \lceil n\tau \rceil - n(n-1)/2$.*
2. *Assume that $m \geq 2$. Then $\mathcal{T}_{\tau,c} \neq 0$ for $c \in \{u, \dots, u+n-1\}$ for $u = \lceil n\tau \rceil - n(n-1)/2$.*
3. *We have $\mathrm{Hom}_{H_{m,n}}(\mathcal{T}_{\tau,c}, \mathcal{T}_{\tau,c'}) = 0$ if $c > c'$.*
4. *We have $\mathrm{Hom}_{H_{m,n}}(\mathcal{T}_{\tau,c}, \mathcal{T}_{\tau,c}) = (\mathrm{End}(V_\tau(c)) \otimes \Gamma(W_0))^{\bar{G}}$ where $\bar{G} = G/Z(G) = \mathrm{PGL}_n(\mathbb{C})$ and where $V_\tau(c)$ was defined in (1.6).*

Proof. 1. If $\xi \in \Delta_\tau^{m,n} - \hat{\rho}$ then it follows from the definition of $\Delta_\tau^{m,n}$ that $c(\xi) = \sum_i \xi_i = v + n\tau - c(\hat{\rho}) = v + n\tau - n(n-1)/2$ for $v \in \{0, \dots, n\}$. By admissibility $n\tau$ is non-integral. It follows that $c(\xi)$ can only be an integer when it is as in the statement of the lemma.

2. It suffices (by (1)) to find n points $(\xi^i)_{i=1}^n$ in $(\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})$ whose c-values give all congruence classes modulo n. We will construct such $(\xi^i)_i$ such that each $\xi = \xi^i$ satisfies the following additional condition:
   (I) ξ does not belong to any facet of $\Delta_{\tau'}^{m,n} - \hat{\rho}$ for any $\tau' \in \mathbb{R}$ except for possibly the "extreme" facets $F^+_{\mathrm{id},n} - \hat{\rho}, F^-_{\mathrm{id},n} - \hat{\rho}$, cf. §3.

   Assume that for a particular τ we have constructed $(\xi^i)_i$, covering n congruence classes, such that in addition (I) is satisfied for each $\xi = \xi^i$. Then changing τ by crossing an non-admissible value τ_0 to τ', we either still have $\xi \in \Delta_{\tau'}^{m,n} - \hat{\rho}$ or $\xi \notin \Delta_{\tau'}^{m,n} - \hat{\rho}$ and ξ lies on one of the two extreme facets of $\Delta_{\tau_0}^{m,n} - \hat{\rho}$. In this case $\xi + \sum_i e_i$ or $\xi - \sum_i e_i$ lies in $\Delta_{\tau_0}^{m,n} - \hat{\rho}$ on the opposite facet and hence belongs to $(\mathbb{Z}^n)^+ \cap (\Delta_{\tau'}^{m,n} - \hat{\rho})$ and its c-value is congruent to $c(\xi)$ modulo n. Hence we keep n points in $(\mathbb{Z}^n)^+ \cap (\Delta_{\tau'}^{m,n} - \hat{\rho})$ whose c-values give all congruence classes modulo n. Thus it is sufficient to construct the $(\xi^i)_i$ satisfying (I) and covering n congruence classes under the assumption that $0 < \tau < 1/n$.
   For $0 < \tau < 1/n$ and $0 \leq l \leq n-1$ let

$$\xi = (n-1, n-3, \ldots, n-2l+1, n-2l, n-2l-2, \ldots, -n+2) - \hat{\rho}.$$

   One verifies that $\xi \in (\mathbb{Z}^n)^+$. We claim that $\xi \in \Delta_\tau^{m,n} - \hat{\rho}$ and that ξ satisfies (I). Since $c(\xi) = n - l - c(\hat{\rho})$, the possible ξ cover n congruence classes and hence we are done if we can prove this claim.
   We will verify both parts of the claim simultaneously. To show that ξ satisfies condition (I) we must understand the facets of $\Delta_{\tau'}^{m,n}$ for all $\tau' \in \mathbb{R}$. So as before let $F^\pm_{w,k}$ be the facets of $\Delta_{\tau'}^{m,n}$ and let $f^\pm \in F^\pm_{w,k}$. By (3.1) we obtain:

$$\begin{aligned} \langle w\lambda_k, f^+\rangle &= \tau' k + (1/2)mk(n-k) + k, \\ \langle -w\lambda_k, f^-\rangle &= -\tau' k + (1/2)mk(n-k). \end{aligned} \tag{5.1}$$

   These equations define the supporting hyperplanes of $\Delta_{\tau'}^{m,n}$. On the other hand, direct verification using the definition of ξ yields:

$$\begin{aligned} \langle w\lambda_k, \xi + \hat{\rho}\rangle &\leq \begin{cases} k(n-k) & \text{if } k \leq l, \\ k(n-k) + k - l & \text{if } k > l, \end{cases} \\ &= k(n-k) + \max\{0, k-l\} \\ \langle -w\lambda_k, \xi + \hat{\rho}\rangle &\leq \begin{cases} k(n-k) - k & \text{if } k \leq n-l, \\ k(n-k) - (n-l) & \text{if } k > n-l \end{cases} \end{aligned} \tag{5.2}$$

$$= k(n-k) - \min\{k, n-l\}$$

with equalities only possible if $w = \mathrm{id}$. Using our hypotheses $m \geq 2$ and $0 < \tau < 1/n$, comparing (5.2) with (5.1) for $\tau' = \tau$ we obtain $\xi + \hat{\rho} \in \Delta_\tau^{m,n}$. It remains to show condition (I), i.e. that $\xi + \hat{\rho} \notin F_{w,k}^{\pm}$ for $(w, k) \neq (\mathrm{id}, n)$. By applying $c(-)$ to $\xi + \hat{\rho}$, $\Delta_{\tau'}^{m,n}$ we note that if $\xi+\hat{\rho} \in \Delta_{\tau'}^{m,n}$ then $n\tau' \leq n-l \leq n\tau'+n$, or $-l/n \leq \tau' \leq (n-l)/n$. If this holds then we get from (5.1) and (5.2), together with the assumption $m \geq 2$:

$$\begin{aligned}
\langle w\lambda_k, f^+\rangle &\geq -\frac{l}{n}k + (1/2)mk(n-k) + k \\
&\geq k(n-k) + \max\{0, k-l\} \geq \langle w\lambda_k, \xi + \hat{\rho}\rangle, \\
\langle -w\lambda_k, f^-\rangle &\geq -\frac{n-l}{n}k + (1/2)mk(n-k) \\
&\geq k(n-k) - \min\{k, n-l\} \geq \langle -w\lambda_k, \xi + \hat{\rho}\rangle.
\end{aligned}$$

If one of these chained inequalities is actually an equality (which would be the case if $\xi + \hat{\rho} \in F_{w,k}^+ \cup F_{w,k}^-$) then it follows that $(w, k) = (\mathrm{id}, n)$. Indeed in both equations the middle inequality can only be an equality if $k = n$, while the last inequality can only be an equality if $w = \mathrm{id}$, as mentioned after (5.2). Hence the claim follows.

3. We need to prove that $\mathrm{Hom}_{H_{m,n}}(\mathcal{V}(\xi), \mathcal{V}(\xi')) = 0$ when $\xi, \xi' \in (\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})$ and $c(\xi) > c(\xi')$. By Lemma 5.1 we have

$$\begin{aligned}
\mathrm{Hom}_{H_{m,n}}(\mathcal{V}(\xi), \mathcal{V}(\xi')) &= \mathrm{Hom}_W(V(\xi) \otimes \mathcal{O}_W, V(\xi') \otimes \mathcal{O}_W)^G \\
&= (\mathrm{Hom}(V(\xi), V(\xi')) \otimes \Gamma(W))^G \\
&\subset (\mathrm{Hom}(V(\xi), V(\xi')) \otimes \Gamma(W))^{Z(G)} \\
&= 0
\end{aligned}$$

where in the fourth line we use that fact that the weights of $\Gamma(W)$ with respect to $Z(G) = \mathbb{C}^*$ are ≤ 0 and by hypothesis $c(\xi) > c(\xi')$.

4. This is proved by a similar computation. Assume $c(\xi) = c(\xi')$

$$\begin{aligned}
\mathrm{Hom}_{H_{m,n}}(\mathcal{V}(\xi), \mathcal{V}(\xi')) &= \mathrm{Hom}_W(V(\xi) \otimes \mathcal{O}_W, V(\xi') \otimes \mathcal{O}_W)^G \\
&= (\mathrm{Hom}(V(\xi), V(\xi')) \otimes \Gamma(W))^G \\
&= (\mathrm{Hom}(V(\xi), V(\xi')) \otimes \Gamma(W)^{Z(G)})^{\bar{G}} \\
&= (\mathrm{Hom}(V(\xi), V(\xi')) \otimes \Gamma(W_0))^{\bar{G}}. \qquad \square
\end{aligned}$$

Recall that if a reductive group acts on a variety X then the stable locus $X^s \subset X$ is the set of points with closed orbit and finite stabilizer.

Lemma 5.3. *The inverse image in W_0 of the Azumaya locus U of $\mathbb{T}_{m,n}$ in $\operatorname{Spec} Z_{m,n}$ is equal to the $\bar{G}$-stable locus W_0^s in W_0. The stabilizer of every point in W_0^s is trivial.*

Proof. By Artin's Theorem [1, Theorem (8.3)] the Azumaya locus of $\operatorname{Spec} Z_{m,n}$ corresponds to the simple m-dimensional representations of the n-loop quiver. It is well-known that this is precisely the stable locus [9]. If V is a representation of a quiver Q, considered as a point in the representation space of Q, then its stabilizer is connected (see [6, Propositon 2.2.1]). It follows that a stable representation has trivial stabilizer. □

Remark 5.4. Note that Lemma 5.3 and the results below that depend on it, only have content for $m \geq 2$ since for $m = 0, 1$ we have $U = \emptyset$.

Let $c \in \mathbb{Z}$. We say that a representation of G has color c if $Z(G)$ acts with character c. Note that if $\xi \in (\mathbb{Z}^n)^+$ has color c then so does $V(\xi)$.

Lemma 5.5. *Let V, V' be two non-zero G-representations with the same color. Then the restrictions to U of $(\operatorname{End}(V) \otimes \Gamma(W_0))^{\bar{G}}$ and $(\operatorname{End}(V') \otimes \Gamma(W_0))^{\bar{G}}$ are Morita equivalent.*

Proof. There is a $\bar{G}$-equivariant nondegenerate Morita context between $\operatorname{End}(V) \otimes \Gamma(W_0)$ and $\operatorname{End}(V') \otimes \Gamma(W_0)$ given by $\operatorname{Hom}(V, V') \otimes \Gamma(W_0)$ and $\operatorname{Hom}(V', V) \otimes \Gamma(W_0)$. When restricted to W_0^s this descends to a non-degenerate Morita context between the restrictions of $(\operatorname{End}(V) \otimes \Gamma(W_0))^{\bar{G}}$ and $(\operatorname{End}(V') \otimes \Gamma(W_0))^{\bar{G}}$. □

Corollary 5.6. *Assume that V is a G-representation with color c. Then the restriction of $(\operatorname{End}(V) \otimes \Gamma(W_0))^{\bar{G}}$ to U is Morita equivalent to the restriction of $\mathbb{T}_{m,n}^{\otimes c}$ to U.*

Proof. This follows from the fact that $(\mathbb{C}^n)^{\otimes c}$ has color c. □

Proof of Proposition 1.9. The proof follows by combining Proposition 2.2, Lemma 5.2 and Corollary 5.6. □

Appendix A. A second proof of the numerical claim in Corollary 1.6

We recall some results on the combinatorics of zonotopes.

A.1. Volumes of zonotopes and lattice points

We first recall that any zonotope can be tiled by elementary (cubical) zonotopes.

Proposition A.1. *[17, §5] Assume Z is a full dimensional zonotope in V of the form $t + \sum_{i\in H}[0, \beta_i]$ with $t \in V$ and $(\beta_i)_i \in V$. Then Z can be tiled by elementary zonotopes of the form $t + \sum_{i\in S} \beta_i + \sum_{j\in B}[0, \beta_j]$ for $S, B \subset H$, with every B such that $(\beta_i)_{i\in B}$ is a basis of V occurring exactly once and such that moreover every facet of an elementary zonotope that appears in the tiling is contained in the translation of a facet of Z by a composition of translations by $\pm\beta_k$ for $k \in H$.*

From this we can deduce a formula for the number of lattice points of Z in case there are no lattice points on the boundary.

Proposition A.2. *Let $M \subset V$ be a lattice and let $Z \subset V$ be a translation of a full dimensional zonotope with vertices in M. Then*

$$|Z \cap M| \geq \mathrm{Vol}_M(Z)$$

with equality if $\partial Z \cap M = \emptyset$.

Proof. By shifting Z slightly (which does not increase $|Z\cap M|$) we may ensure that M does not intersect any facet of the tiling elementary zonotopes. Then Proposition A.1 reduces us to the case that Z is an elementary zonotope which is trivial. □

Combining this with Stanley's formula [19, Theorem 2.2] for $\mathrm{Vol}_M(Z)$ we obtain a formula for $|Z \cap M|$ in case $\partial Z \cap M = \emptyset$.

Corollary A.3. *[19, Theorem 2.2] Let the setting be as in Proposition A.2. Then $|Z \cap M| = \sum_S h(S)$ where S ranges over all maximal linearly independent subsets of $\{\beta_i \mid i \in H\}$ and $h(S)$ is (the absolute value of) the volume (with respect to M) of the parallelepiped spanned by $\beta_i \in S$.*

A.2. Group action on zonotopes

Lemma A.4. *Let $Z = t + \sum_{i\in I}[0, \beta_i]$ be a zonotope in V. Let G be a finite group of affine automorphisms of V which preserves Z and let G^0 be the underlying linear group. The invariant polytope Z^G is the zonotope in V^G given by*

$$Z^G = \bar{t} + \sum_{i\in I}[0, \hat{\beta}_i] \tag{A.1}$$

where $\bar{?}$, $\hat{?}$ *denotes averaging for respectively* G *and* G^0*; i.e.* $\bar{u} = (1/|G|) \times \sum_{\sigma\in G} \sigma(u)$. $\hat{u} = (1/|G|) \sum_{\sigma\in G^0} \sigma(u)$.

Proof. Since Z is preserved by G and convex it is closed under averaging. Hence

$$Z^G = \{\bar{u} \mid u \in Z\}.$$

We now use the linearity properties of averaging to obtain (A.1). □

Corollary A.5. *Fix a lattice* M *in* V *and let* Z *be a translation of a lattice zonotope in* V*. Let* G *be a finite group of affine automorphisms of* V *which preserves* Z*. Then* Z^G *is the translation of a lattice zonotope.*

Proof. Let γ_i be the vectors corresponding to the edges of Z. By definition $\gamma_i \in M$. The γ_i are only determined up to sign but they yield a canonical multiset $\{\beta_j \mid j \in J\} := \{\pm\gamma_i \mid i \in I\} \subset M$. It follows from Remark 2.1 that Z may be written as:

$$Z = t + \sum_{j\in J} [0, \beta_j/2]$$

for suitable t. Note that $\{\beta_j \mid j \in J\}$ is preserved by G^0 (since the set of edges is preserved by G). It follows that t is preserved by G. By Lemma A.4 we have

$$Z^G = t + \sum_{j\in J} [0, \hat{\beta}_j/2]. \tag{A.2}$$

We consider G^0 as acting on J. The right-hand side of (A.2) can be written as

$$Z^G = t + \sum_{j\in J/G^0} [0, \tilde{\beta}_j/2]$$

where $\tilde{\beta}_j = \sum_{\beta_k\in G^0(\beta_j)} \beta_k \in M$. We have $\widetilde{(-\beta_j)} = -\tilde{\beta}_j$ and if $G^0(-\beta_j) = G^0(\beta_j)$ then $\tilde{\beta}_j = 0$. Thus we obtain

$$Z^G = s + \sum_{j\in(J/G^0)/\pm} [0, \tilde{\beta}_j]$$

for suitable s. Hence Z^G is indeed a translated lattice zonotope. □

A.3. The Möbius function of set partitions

Let S be a finite set. We consider the poset Π_S of partitions $\mathcal{S} = \{S_1, \dots, S_t\}$ of S, ordered by refinement (e.g. $\{S', S \setminus S'\} < \{S\}$ for $S' \subset S$). Part of the corresponding Möbius function [21, p.7] is given by

$$\mu(\mathcal{S}) := \mu(\{\{1\}, \dots, \{n\}\}, \mathcal{S}) = \prod_{S \in \mathcal{S}} \mu(S)$$

with

$$\mu(S) = (-1)^{|S|-1}(|S| - 1)!.$$

Or summarizing

$$\mu(\mathcal{S}) = (-1)^{n-|\mathcal{S}|} \prod_{S \in \mathcal{S}} (|S| - 1)!.$$

A.4. Lattice points in $\Delta_\tau^{m,n}$ and spanning trees

In this section and the next we assume that τ is admissible (see Definition 1.1). Let $\mathcal{V}$ be the multiset $\{(e_i - e_j)^m \mid 1 \leq j < i \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}$ (i.e. each $e_i - e_j$, $1 \leq j < i \leq n$ in the multiset appears m times). Then $\Delta_\tau^{m,n}$ is a translation of the lattice zonotope $\sum_{f \in \mathcal{V}} [0, f]$. Hence we may appy Corollary A.3 to compute $|\Delta_\tau^{m,n} \cap \mathbb{Z}^n|$. Thus we need to find all subsets of n linearly independents elements from $\mathcal{V}$.

To this end we create an undirected graph G with vertices $\{0, \dots, n\}$ with an edge between i and j for each vector $e_i - e_j$ in $\mathcal{V}$ and an edge between 0 and i for each vector e_i in $\mathcal{V}$.

Put $[n]$=$\{1, \dots, n\}$. So summarizing G is the graph with vertices $\{0, \dots, n\}$ and edges

- m edges between i and j for every $i \neq j \in [n]$,
- 1 edge between 0 and i for every $i \in [n]$.

We obtain the following correspondence that follows by construction and the definition of spanning trees.

Lemma A.6. *The above correspondence between the elements in $\mathcal{V}$ and edges in G gives a bijective correspondence between the subsets of n linearly independent elements from $\mathcal{V}$ and the set of spanning trees in G.*

For a graph H we denote by $\tau(H)$ the set of its spanning trees.

Corollary A.7. *We have*

$$|\tau(G)| = |\Delta_\tau^{m,n} \cap \mathbb{Z}^n|.$$

Proof. We apply Corollary A.3. We may do this since $\partial(\Delta_\tau^{m,n}) \cap \mathbb{Z}^n = \emptyset$. Note that the volumes of the parallelepipeds (given by suitable minors) are all equal to 1. Hence Lemma A.6 implies the equality. □

For every set partition $\mathcal{S}$ of $[n]$ we denote by $G/\mathcal{S}$ the contracted graph, obtained by shrinking all subgraphs connecting vertices in the same $S \in \mathcal{S}$ to a point (note the special role of the vertex 0, which is never contracted).

A.5. Regular orbits and spanning trees

Recall that S_n acts on $\Delta_\tau^{m,n}$. We extend Corollary A.7 to $G/\mathcal{S}$. Denote by $H_\mathcal{S}$ the stabilizer of $\mathcal{S}$, i.e. $H_\mathcal{S} = \{g \in S_n \mid gS = S, S \in \mathcal{S}\}$. We write $(-)^\mathcal{S}$ and $(-)_\mathcal{S}$ for respectively the invariants and the coinvariants under $H_\mathcal{S}$. Concretely if $L_\mathcal{S} = \sum_{i,j \in S \in \mathcal{S}} \mathbb{Z}(e_i - e_j)$ then $\mathbb{R}^n_\mathcal{S} = \mathbb{R}^n/(L_\mathcal{S})_\mathbb{R}$, $\mathbb{Z}^n_\mathcal{S} = \mathbb{Z}^n/L_\mathcal{S}$. We denote by $(\Delta_\tau^{m,n})_\mathcal{S}$ the image of $\Delta_\tau^{m,n}$ in $\mathbb{R}^n_\mathcal{S}$. Finally we put ${}_\mathcal{S}(\Delta_\tau^{m,n}) = \{x \in \Delta_\tau^{m,n} \mid \mathrm{Stab}(x) = H_\mathcal{S}\}$.

Lemma A.8. *We have*

$$|(\Delta_\tau^{m,n})_\mathcal{S} \cap (\mathbb{Z}^n)_\mathcal{S}| = \left(\prod_{S \in \mathcal{S}} |S|\right) |(\Delta_\tau^{m,n})^\mathcal{S} \cap (\mathbb{Z}^n)^\mathcal{S}|.$$

Proof. Since $\partial((\Delta_\tau^{m,n})^\mathcal{S}) \subset \partial(\Delta_\tau^{m,n})$ and τ is admissible we have $\partial((\Delta_\tau^{m,n})^\mathcal{S}) \cap (\mathbb{Z}^n)^\mathcal{S} = \emptyset$. Likewise using the proof of Lemma 1.2 in §3.1 we see that $\partial((\Delta_\tau^{m,n})_\mathcal{S}) \cap \mathbb{Z}^n_\mathcal{S} = \emptyset$ whenever $\tau - m(n-1)/2$ can be written as a fraction whose denominator is not a sum of cardinalities of elements of $\mathcal{S}$. Since we have assumed that τ is admissible, this holds in our case.

We apply Corollary A.3 to conclude that $|(\Delta_\tau^{m,n})_\mathcal{S} \cap (\mathbb{Z}^n)_\mathcal{S}| = \mathrm{Vol}_{(\mathbb{Z}^n)_\mathcal{S}}((\Delta_\tau^{m,n})_\mathcal{S})$. We note that $(\Delta_\tau^{m,n})^\mathcal{S}$ is a translate lattice polytope by Corollary A.5. Hence by applying Corollary A.3 again we obtain $|(\Delta_\tau^{m,n})^\mathcal{S} \cap (\mathbb{Z}^n)^\mathcal{S}| = \mathrm{Vol}_{(\mathbb{Z}^n)^\mathcal{S}}((\Delta_\tau^{m,n})^\mathcal{S})$.

We note that $(\Delta_\tau^{m,n})^\mathcal{S} \to (\Delta_\tau^{m,n})_\mathcal{S}$ obtained from $(\mathbb{R}^n)^\mathcal{S} \to (\mathbb{R}^n)_\mathcal{S}$ is a bijection, while the cokernel of $(\mathbb{Z}^n)^\mathcal{S} \hookrightarrow (\mathbb{Z}^n)_\mathcal{S}$ has order $\prod_{S\in\mathcal{S}} |S|$. (Indeed, the cokernel is isomorphically mapped to $\prod_{S\in\mathcal{S}}(\mathbb{Z}/|S|\mathbb{Z})$ where the unit e_i in the i-th component maps to $\prod_{S\in\mathcal{S}} \delta_{iS}$.) Hence

$$|(\Delta_\tau^{m,n})_\mathcal{S} \cap (\mathbb{Z}^n)_\mathcal{S}| = \mathrm{Vol}_{(\mathbb{Z}^n)_\mathcal{S}}((\Delta_\tau^{m,n})_\mathcal{S}) =$$

$$\left(\prod_{S\in\mathcal{S}}|S|\right)\mathrm{Vol}_{(\mathbb{Z}^n)^{\mathcal{S}}}((\Delta_\tau^{m,n})^{\mathcal{S}}) = \left(\prod_{S\in\mathcal{S}}|S|\right)|(\Delta_\tau^{m,n})^{\mathcal{S}}\cap(\mathbb{Z}^n)^{\mathcal{S}}|. \quad \square$$

Corollary A.9. *We have*

$$|\tau(G/\mathcal{S})| = \left(\prod_{S\in\mathcal{S}}|S|\right)\sum_{\mathcal{S}'\geq\mathcal{S}}|_{\mathcal{S}'}(\Delta_\tau^{m,n})\cap\mathbb{Z}^n|.$$

Proof. During the proof of Lemma A.8 we have shown that $\partial((\Delta_\tau^{m,n})_{\mathcal{S}})\cap\mathbb{Z}^n_{\mathcal{S}} = \emptyset$. It then follows as in Corollary A.7 that $\tau(G/\mathcal{S}) = |(\Delta_\tau^{m,n})_{\mathcal{S}}\cap(\mathbb{Z}^n)_{\mathcal{S}}|$ so that we must prove

$$|(\Delta_\tau^{m,n})_{\mathcal{S}}\cap(\mathbb{Z}^n)_{\mathcal{S}}| = \left(\prod_{S\in\mathcal{S}}|S|\right)\sum_{\mathcal{S}'\geq\mathcal{S}}|_{\mathcal{S}'}(\Delta_\tau^{m,n})\cap\mathbb{Z}^n|.$$

This follows from Lemma A.8. $\square$

If X is a set with an S_n-action then we write $\mathrm{reg}(X)$ for the set of regular orbits. Applying Möbius inversion we obtain the following formula for $|\mathrm{reg}(\Delta_\tau^{m,n}\cap\mathbb{Z}^n)|$.

Corollary A.10. *We have*

$$|\mathrm{reg}(\Delta_\tau^{m,n}\cap\mathbb{Z}^n)| = \frac{1}{n!}\sum_{\mathcal{S}\in\Pi_{[n]}}\frac{1}{\prod_{S\in\mathcal{S}}|S|}\mu(\mathcal{S})|\tau(G/\mathcal{S})|. \tag{A.3}$$

Proof. We have $|\mathrm{reg}(\Delta_\tau^{m,n}\cap\mathbb{Z}^n)| = (1/n!)|_{\mathcal{S}_0}(\Delta_\tau^{m,n})\cap\mathbb{Z}^n|$ with $\mathcal{S}_0 = \{\{1\},\dots,\{n\}\}$. Now we apply Möbius inversion to

$$f,g:\Pi_{[n]}\to\mathbb{Q},\quad f:\mathcal{S}\mapsto\frac{1}{\prod_{S\in\mathcal{S}}|S|}|\tau(G/\mathcal{S})|,\quad g:\mathcal{S}\mapsto|_{\mathcal{S}}(\Delta_\tau^{m,n})\cap\mathbb{Z}^n|$$

using Corollary A.9. $\square$

A.6. Computing the sum over graphs

In this section we complete the numeric claim in Corollary 1.6 by computing the sum on right-hand side of (A.3).

Theorem A.11. *We have*

$$\frac{1}{n!}\sum_{\mathcal{S}\in\Pi_{[n]}}\frac{1}{\prod_{S\in\mathcal{S}}|S|}\mu(\mathcal{S})|\tau(G/\mathcal{S})| = \frac{1}{(m-1)n+1}\binom{mn}{n}. \tag{A.4}$$

Proof. We use Kirchhoff's theorem which says that the number of spanning trees of a graph is the absolute value of an arbitrary cofactor in the Laplacian matrix. The Laplacian matrix of G is the matrix of size $(n+1)\times(n+1)$ given by

$$\begin{pmatrix} -n & 1 & 1 & \cdots & 1 \\ 1 & -(n-1)m-1 & m & \cdots & m \\ 1 & m & -(n-1)m-1 & \cdots & m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & m & m & \cdots & -(n-1)m-1 \end{pmatrix} \tag{A.5}$$

We will look at the cofactor consisting of the rows $[1:n]$ and columns $[2:(n+1)]$. By substracting m times the top row $(1,\ldots,1)$ from the other rows one finds that it is (up to sign) the determinant of the square submatrix with rows $[1:n]$ in the following (rectangular) $(n+1)\times n$ matrix.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ -nm-1 & 0 & \cdots & 0 \\ 0 & -nm-1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -nm-1 \end{pmatrix} \tag{A.6}$$

which is (up to sign)

$$(nm+1)^{n-1}.$$

Now we evaluate the corresponding cofactor for $G/\mathcal{S}$ where $\mathcal{S}$ is the set partition

$$\{\{1,\ldots,n_1\},\{n_1+1,\ldots,n_1+n_2\},\ldots\}$$

(thus the sizes of the parts are $n_1,n_2,\ldots,n_t$). This amounts to replacing the top row and leftmost column in (A.5) by

$$(-n,n_1,\ldots,n_t).$$

In the lower left $n\times n$-matrix, the $n_i\times n_j$ blocks are replaced by their sum. We obtain the following matrix

$$\begin{pmatrix} -n & n_1 & n_2 & \cdots & n_t \\ n_1 & -n_1((n-n_1)m+1) & n_1n_2m & \cdots & n_1n_tm \\ n_2 & n_2n_1m & -n_2((n-n_2)m+1) & \cdots & n_2n_tm \\ \vdots & \vdots & \vdots & & \vdots \\ n_t & n_tn_1m & n_tn_2m & \cdots & -n_t((n-n_t)m+1) \end{pmatrix}$$

The matrix corresponding to (A.6) becomes

$$\begin{pmatrix} n_1 & n_2 & \cdots & n_t \\ -n_1(nm+1) & 0 & \cdots & 0 \\ 0 & -n_2(nm+1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -n_t(nm+1) \end{pmatrix}$$

and now we have to calculate the determinant of the top t rows. We find (up to sign)

$$n_1 \cdots n_t (nm+1)^{t-1}.$$

Hence (A.4) becomes

$$\begin{aligned} &\frac{1}{n!} \sum_{\mathcal{S} \in \Pi_{[n]}} \frac{1}{\prod_{S \in \mathcal{S}} |S|} \mu(\mathcal{S})(mn+1)^{|\mathcal{S}|-1} \prod_{S \in \mathcal{S}} |S| \\ &\qquad = \frac{(-1)^n}{n!} \sum_{\mathcal{S} \in \Pi_{[n]}} (-1)^{|\mathcal{S}|}(mn+1)^{|\mathcal{S}|-1} \prod_{S \in \mathcal{S}} (|S|-1)! \\ &\qquad = \frac{(-1)^n}{n!} \sum_{n_1+\ldots+n_t=n} (-1)^t \frac{1}{t!} \binom{n}{n_1 \cdots n_t} (mn+1)^{t-1} \prod_i (n_i-1)! \\ &\qquad = (-1)^n \sum_{n_1+\ldots+n_t=n} (-1)^t \frac{1}{t!} (mn+1)^{t-1} \prod_{i=1}^t \frac{1}{n_i} \end{aligned}$$

(the factor $1/t!$ comes from the choice in enumerating the elements of $\mathcal{S}$). Putting $X = mn+1$ in the following lemma then finishes the proof of Theorem A.11. □

Lemma A.12. *We have*

$$(-1)^n \sum_{n_1+\ldots+n_t=n} (-1)^t \frac{1}{t!} X^{t-1} \prod_{i=1}^t \frac{1}{n_i} = \frac{1}{n!}(X-(n-1))\cdots(X-1). \tag{A.7}$$

Proof. To compute the lefthand side we need the coefficient of u^n in the sum

$$\begin{aligned} \sum_{t\geq 1} \sum_{(n_i)_i \in \mathbb{N}^t_{>0}} (-1)^t (-u)^{\sum_{i=1}^t n_i} \frac{1}{t!} X^{t-1} \prod_{i=1}^t \frac{1}{n_i} &= \sum_{t\geq 1} \frac{X^{t-1}}{t!} \sum_{(n_i)_i \in \mathbb{N}^t_{>0}} \prod_{i=1}^t \left(-\frac{(-u)^{n_i}}{n_i}\right) \\ &= \sum_{t\geq 1} \frac{X^{t-1}}{t!} \prod_{i=1}^t \sum_{n\geq 1} \left(-\frac{(-u)^n}{n}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{t\geq 1} \frac{X^{t-1}}{t!} \log(1+u)^t \\
&= X^{-1}(\exp(X\log(1+u)) - 1) \\
&= X^{-1}((1+u)^X - 1) \\
&= \sum_{n\geq 1} u^n \frac{(X-1)\cdots(X-(n-1))}{n!}.
\end{aligned}
$$

Hence the sought coefficient is precisely the righthand side of (A.7). This finishes the proof. □

A.7. Second proof of the numeric claim in Corollary 1.6

The claim that $|\mathrm{reg}(\Delta_\tau^{m,n} \cap \mathbb{Z}^n)| = A_n(m,1)$ follows by combining (1.3), Corollary A.10 and Theorem A.11.

Appendix B. Tables of tilting bundles

If in (1.4) we replace τ by $\tau + u$, $u \in \mathbb{Z}$ then this amounts to tensoring $\mathcal{T}_\tau$ with the line bundle $\mathcal{V}(u,\dots,u)$ which does not change its endomorphism ring. Moreover if $[\tau,\tau']$ does not intersect the inadmissible locus then $\mathcal{T}_\tau = \mathcal{T}_{\tau'}$. Hence without loss of generality we may assume

$$\tau = \varepsilon + t + m(n-1).$$

with $0 < \varepsilon \ll 1$ and

$$t = -\frac{p}{k} \qquad 1 \leq k \leq n,\ 0 \leq p < k.$$

Below we list the weights $\xi \in (\mathbb{Z}^n)^+ \cap (\Delta_\tau^{m,n} - \hat{\rho})$ that determine the tilting bundle $\mathcal{T}_\tau$ for varying t and $n = 2,3,4$, $m = 2$. They were computed with a computer program.

B.1. Tilting bundles for $(n,m) = (2,2)$

$t = -0/1$	$[1,0]$	$[1,1]$
$t = -1/2$	$[0,0]$	$[1,0]$

B.2. Tilting bundles for $(n, m) = (3, 2)$

$t = -0/1$	$[1,1,1]$	$[2,1,0]$	$[2,1,1]$	$[2,2,0]$	$[2,2,1]$
$t = -1/3$	$[1,1,0]$	$[1,1,1]$	$[2,1,0]$	$[2,1,1]$	$[2,2,0]$
$t = -1/2$	$[1,1,0]$	$[2,0,0]$	$[1,1,1]$	$[2,1,0]$	$[2,1,1]$
$t = -2/3$	$[1,0,0]$	$[1,1,0]$	$[2,0,0]$	$[1,1,1]$	$[2,1,0]$

B.3. Tilting bundles for $(n, m) = (4, 2)$

$t = -0/1$

$[3,1,1,1]$	$[2,2,1,1]$	$[3,2,1,0]$	$[2,2,2,0]$	$[3,2,1,1]$	$[2,2,2,1]$	$[3,3,1,0]$
$[3,2,2,0]$	$[3,3,1,1]$	$[3,2,2,1]$	$[2,2,2,2]$	$[3,3,2,0]$	$[3,3,2,1]$	$[3,2,2,2]$

$t = -1/4$

$[2,1,1,1]$	$[2,2,1,0]$	$[3,1,1,1]$	$[2,2,1,1]$	$[3,2,1,0]$	$[2,2,2,0]$	$[3,2,1,1]$
$[2,2,2,1]$	$[3,3,1,0]$	$[3,2,2,0]$	$[3,3,1,1]$	$[3,2,2,1]$	$[2,2,2,2]$	$[3,3,2,0]$

$t = -1/3$

$[2,1,1,1]$	$[3,1,1,0]$	$[2,2,1,0]$	$[3,1,1,1]$	$[2,2,1,1]$	$[3,2,1,0]$	$[2,2,2,0]$
$[3,2,1,1]$	$[2,2,2,1]$	$[3,3,1,0]$	$[3,2,2,0]$	$[3,3,1,1]$	$[3,2,2,1]$	$[2,2,2,2]$

$t = -1/2$

$[1,1,1,1]$	$[2,1,1,0]$	$[2,2,0,0]$	$[2,1,1,1]$	$[3,1,1,0]$	$[2,2,1,0]$	$[3,2,0,0]$
$[3,1,1,1]$	$[2,2,1,1]$	$[3,2,1,0]$	$[2,2,2,0]$	$[3,2,1,1]$	$[2,2,2,1]$	$[3,2,2,0]$

$t = -2/3$

$[1,1,1,1]$	$[2,1,1,0]$	$[3,1,0,0]$	$[2,2,0,0]$	$[2,1,1,1]$	$[3,1,1,0]$	$[2,2,1,0]$
$[3,2,0,0]$	$[3,1,1,1]$	$[2,2,1,1]$	$[3,2,1,0]$	$[2,2,2,0]$	$[3,2,1,1]$	$[2,2,2,1]$

$t = -3/4$

$[1,1,1,0]$	$[2,1,0,0]$	$[1,1,1,1]$	$[2,1,1,0]$	$[3,1,0,0]$	$[2,2,0,0]$	$[2,1,1,1]$
$[3,1,1,0]$	$[2,2,1,0]$	$[3,2,0,0]$	$[3,1,1,1]$	$[2,2,1,1]$	$[3,2,1,0]$	$[2,2,2,0]$

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work with Yokinobu Toda in which, among others, a semi-orthogonal decomposition of $\mathcal{D}(H_{3,n})$ is constructed. We thank him for sharing this work with us [12].

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