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Supertranslation invariance of angular momentum at null infinity in double null gauge*

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Abstract: The supertranslation invariance of the Chen-Wang-Yau (CWY) angular momentum in the Bondi-Sachs formalism/gauge was ascertained by the authors in [12, 13]. In this article, we study the corresponding problem in the double null gauge. In particular, supertranslation ambiguity of this gauge is identified and the CWY angular momentum is proven to be free of this ambiguity. A similar result is obtained for the CWY center of mass integral.

1.

The friendship of Professor Christodoulou and Yau started in 1981. In the first months of his commute from Syracuse to the Institute of Advanced Study, Christodoulou stayed in Yau's apartment. He also sat in Yau's course on minimal surface theory and applications for two years [11]. Christodoulou eventually applied geometric analysis to hyperbolic partial differential equations, culminating first in the monumental proof of stability of the Minkowski spacetime with Klainerman [7] and later in the formation of black holes in general relativity [8] and the formation of shocks in fluid mechanics [9]. We take this opportunity to celebrate Christodoulou's achievement and the friendship that has lasted more than 40 years.

In a series of papers in the 1960s, Bondi [3], Bondi-van der Burg-Metzner [4], and Sachs [26] provided one of the first convincing theoretical evidences for gravitational radiation. Assuming an asymptotically Minkowskian isolated system admits a coordinate system adapted to an optical function u and a luminosity distance r, Bondi et al. were able to solve the vacuum Einstein

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equation in power series of r^{-1}, defined the energy for the u = const. null hypersurfaces

$$E(u) = \frac{1}{4\pi}\int_{S^2} m,$$

and derived the *Bondi mass loss formula*[1]

$$\partial_u E = -\frac{1}{32\pi}\int_{S^2} N_{AB}N^{AB} \leq 0,$$

which is interpreted as saying an isolated system emits gravitational waves that carries mass away from the system.

The Bondi-Sachs coordinate-based approach (henceforth referred to as Bondi-Sachs formalism [23]) was incorporated into Penrose's conformal treatment of null infinity. However, it is difficult to construct a large class of spacetimes that admit conformal compactifications (or Bondi-Sachs coordinates). It is in the work of stability of Minkowski spacetime [7] that Christodoulou-Klainerman give a rigorous treatment of null infinity in the setting of initial value problems: a detailed asymptotic decay estimate for Ricci coefficients and curvature components are derived, the Bondi mass is defined by the limit of Hawking mass, and the mass loss formula is rigorously proved. The work also leads to the discovery of the Christodoulou memory effect [5].

Defining angular momentum for gravitational fields turns out to be more subtle. In special relativity, associated with a particle (represented by a curve γ in $\mathbb{R}^{3,1}$) are the conserved quantities

1. energy $E = \langle \gamma', \frac{\partial}{\partial t}\rangle$
2. linear momentum $P^i = \langle \gamma', \frac{\partial}{\partial x^i}\rangle$
3. angular momentum $J_{ij} = \langle \gamma', x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}\rangle$
4. center of mass $C^i = \langle \gamma', t\frac{\partial}{\partial x^i} + x^i\frac{\partial}{\partial t}\rangle$

If one changes the coordinate system by a translation $\bar{t} = t + \alpha^0, \bar{x}^i = x^i + \alpha^i$, energy and linear momentum remain the same but the angular momentum and center of mass transform as

$$\begin{aligned} \bar{J}_{ij} &= J_{ij} + \alpha^i P^j - \alpha^j P^i, \\ \bar{C}^i &= C^i + \alpha^0 P^i + \alpha^i E. \end{aligned} \tag{1}$$

We conclude that the angular momentum and center of mass have a "4-dimensional translation ambiguity" that comes from the choice of the origin.

[1]The function m and 2-tensor N_{AB} appear in the Taylor expansion of metric coefficients. See Section 4.1 for their definition.

In Bondi-Sachs formalism, the symmetry group of null infinity is given by coordinate transformations that preserve the form of metric tensors and asymptotics. Although structurally similar to the Poincaré group, it is the semi-direct product of the Lorentz group and an infinite-dimensional abelian group—*supertranslations*—instead of the 4-dimensional group of translations. The transformation law of all existing proposals of angular momentum under supertranslations bears no resemblance to (1) and physicists have yet to find an interpretation. This is the "supertranslation ambiguity" that has puzzled researchers in the field since the 1960s.

The purpose of this article is to explain the supertranslation invariance of Chen-Wang-Yau angular momentum, and center of mass integral [12, 13] with the main results presented in the double null gauge rather than the Bondi-Sachs formalism as in [12, 13].

In Section 2, the definitions of the Chen-Wang-Yau quasi-local angular momentum and center of mass integral are given. They are extensions of the Wang-Yau quasi-local mass [28, 29] and defined for spacelike 2-surfaces in spacetimes. The key idea is to isometrically embed the surface into the Minkowski spacetime and then pull back the Killing vector fields along the surface. This resolves two difficulties encountered in the Hamiltonian approach to define conserved quantities in general relativity: the lack of background coordinate systems and the absence of symmetries. We take the limits, as $r \to \infty$ of quasi-local Chen-Wang-Yau angular momentum and center of mass on $r =$ const. surfaces to get the corresponding Chen-Wang-Yau quantities for each $u =$ const. null hypersurface. The Bondi-Sachs formalism is recalled in Section 2.2, In particular, we define supertranslations in the Bondi-Sachs formalism.

For the rest of the article, we focus on the double null gauge, which is familiar to the readers of Christodoulou's work. In Section 3, we discuss angular momentum at null infinity in double null gauge. We first present A. Rizzi's definition of angular momentum proposed in Rizzi's thesis supervised by Christodoulou. Then we define the Chen-Wang-Yau angular momentum in this context.

In Section 4, we study the effect of supertranslation on the total flux of angular momentum. In Section 4.1, we define supertranslations, compute the transformations of Ricci coefficients, and discuss Rizzi's attempt to restore the 4-dimensional dependence of origin. The definition of supertranslation in double null gauge relies on the construction of suitable optical functions. We prove the existence of such optical functions in Appendix A. In Section 4.2, we compute the supertranslation ambiguity for the total flux of Rizzi's angular momentum and show that it agrees with the (classical) Ashtekar-Streubel definition [2]. In the final Section 4.3, we show that the Chen-Wang-Yau angular momentum and center of mass integral transform according to (1) under supertranslations, which is the main result in [12, 13] and is presented here in the double null gauge.

Definition 1.1. Throughout this article, we denote the standard metric on S^2 by σ. More precisely, $\sigma = d\theta^2 + \sin^2\theta d\phi^2$ in spherical coordinates. We raise and lower indices with respect to σ. Let ∇ and Δ be the covariant derivative and Laplace operator with respect to σ. If the volume form is omitted in the integrals $\int_{S^2} F$, it is taken with respect to the volume form of σ. We also need the spherical harmonics decomposition. Let $\mathcal{H}_{\ell\leq 1}$ denote the space of functions spanned by $1, \widetilde{X}^1 = \sin\theta\cos\phi, \widetilde{X}^2 = \sin\theta\sin\phi, \widetilde{X}^3 = \cos\theta$ and $\mathcal{H}_{\ell\geq 2}$ denote the space of functions supported in $\ell \geq 2$ modes.

2. Chen-Wang-Yau angular momentum and center of mass

2.1. Quais-local conserved quantities

Given a spacelike 2-surface Σ in the spacetime, we get data $(\not{g}, |H|, \alpha_H)$ where $\not{g}$ is the induced metric, $|H|$ the norm of the mean curvature vector (which is assumed to be spacelike), α_H the connection 1-form of the normal bundle of Σ. In terms of Ricci coefficients [5, page 1486],

$$|H| = \sqrt{-\operatorname{tr}\chi\operatorname{tr}\underline{\chi}}, \quad \alpha_H = \zeta - \frac{1}{2}d\log\operatorname{tr}\chi + \frac{1}{2}d\log(-\operatorname{tr}\underline{\chi})$$

are independent of the scaling $L \to aL, \underline{L} \to a^{-1}\underline{L}$ of the two null normal vector fields along Σ.

Consider an isometric embedding $X : (\Sigma, \not{g}) \to \mathbb{R}^{3,1}$ into the Minkowski spacetime. For the image surface $X(\Sigma)$, we can compute the norm of the mean curvature vector H_0 and connection 1-form α_{H_0}. For a constant timelike unit vector field T_0, viewed as an observer, let $\tau = -\langle X, T_0\rangle$. Define the mass density function and momentum density one-form by

$$\begin{aligned}
\boldsymbol{\rho} &= \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}} \\
\boldsymbol{j} &= -\boldsymbol{\rho}\nabla\tau + \nabla\left[\sinh^{-1}\left(\frac{\boldsymbol{\rho}\Delta\tau}{|H_0||H|}\right)\right] + \alpha_{H_0} - \alpha_H
\end{aligned}$$

Note that the sign convention of $\boldsymbol{j}$ is opposite to [15, 20] but coincides with [14, Proposition 7.1].

There are many such choices of isometric embedding X and observer T_0 in the Minkowski spacetime. We consider only those pairs (X, T_0) with associated data satisfying the optimal isometric embedding equation

$$d\!/\!iv\,\boldsymbol{j} = 0. \tag{2}$$

We write $K_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$ for the rotation Killing vector fields and $K_i = t\frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t}$ for the boost Killing vector fields; together they form a basis of the Lorentz algebra. Here is the definition of Chen-Wang-Yau quasi-local conserved quantities [14, Definition 2.2].

Definition 2.1. The quasi-local conserved quantity of Σ with respect to an optimal isometric embedding (X, T_0) and a Killing field K in Minkowski spacetime is

$$E(\Sigma, X, T_0, K) = \frac{1}{8\pi} \int_\Sigma -\langle K, T_0 \rangle \boldsymbol{\rho} + \boldsymbol{j}(K^\top) d\Sigma \tag{3}$$

where $K^\top$ denotes the projection of a Lorentz Killing field K onto the tangent space of the image $X(\Sigma)$.

Suppose $T_0 = A\left(\frac{\partial}{\partial t}\right)$ for a Lorentz transformation, then the quasi-local conserved quantities corresponding to $A(K_{ij})$ are called the quasi-local angular momentum with respect to T_0 and the quasi-local conserved quantities corresponding to $A(K_i)$ are called the quasi-local center of mass integrals with respect to T_0.

Chen-Wang-Yau conserved quantities satisfy the following properties:

1. They vanish for any spacelike 2-surface in Minkowski spacetime.
2. The quasi-local angular momentum coincides with the Komar integral for an axially symmetric 2-surface in an axially symmetric spacetime.

Given an asymptotically flat initial data set (M, g, k), the limits of Chen-Wang-Yau conserved quantities of the coordinate spheres Σ_r give rise to the *total Chen-Wang-Yau conserved quantities*. On the one hand, the total energy-momentum recovers the ADM energy-momentum 4-vector [30]. On the other hand, the total angular momentum and center of mass integrals agree with ADM angular momentum and Regge-Teitelboim center of mass for harmonic asymptotes when the linear momentum vanishes but are different in general [16].

Moreover, the following properties also hold:

1. The total angular momentum of any spacelike hypersurface of Kerr spacetime is an invariant.
2. Let $(M, g(t), k(t))$ be a solution for the vacuum Einstein equation with lapse function $N = 1 + O(r^{-1})$ and shift vector $\gamma = \gamma^{(-1)} + O(r^{-2})$, center of mass integrals $C^i(t)$ and the total angular momentum $J^i(t)$ of $(M, g(t), k(t))$ satisfy

$$\partial_t C^i(t) = \frac{p^i}{e}, \quad \partial_t J^i(t) = 0 \tag{4}$$

where (e, p^i) is the ADM energy-momentum 4-vector.

In his lecture given at ETH, Christodoulou proposed definitions of angular momentum and center of mass integral for *strongly* asymptotically flat initial data sets [10, Section 3.3, 3.4] with the dynamical formula (4) proved. While his angular momentum agrees with ADM's, the center of mass integral is new.

2.2. Bondi-Sachs formalism, total conserved quantities and supertranslation

We briefly review the Bondi-Sachs formalism and refer the readers to the excellent survey [23] for more details.

If we set $u = t - r$, the metric tensor of the Minkowski spacetime becomes

$$-du^2 - 2dudr + r^2\sigma_{AB}dx^A dx^B.$$

Similarly, the Schwarzschild metric in Eddington-Finkelstein coordinates is

$$-(1 - \frac{2m}{r})du^2 - 2dudr + r^2\sigma_{AB}dx^A dx^B$$

Taking the above two examples as models, Bondi and his collaborators postulated that the spacetime admits a coordinate system (Bondi-Sachs coordinates) (u, r, x^A) where u is an optical function, r is the "luminosity distance" from the source, and $x^A, A = 2, 3$ are the coordinates of the spherical section; $r \in (r_0, \infty), u \in (u_0, u_1)$. In Bondi-Sachs coordinates, the metric tensor takes the form

$$-UV du^2 - 2Ududr + r^2 h_{AB}(dx^A + W^A du)(dx^B + W^B du). \tag{5}$$

Assuming the spacetime is asymptotically Minkowskian, the metric coefficients satisfy

$$U \to 1, V \to 1, h_{AB} \to \sigma_{AB}, W^A \to 0 \tag{6}$$

as $r \to \infty$.

Bondi and his collaborators make two more assumptions. The first is a determinant condition $\det(h_{AB}) = \det(\sigma_{AB})$. The second is the "outgoing radiation condition" that all metric coefficients can be expanded into power series of $\frac{1}{r}$ with coefficients being functions of u, x^A.

The null vacuum Einstein constraint equations then enjoy a remarkable hierarchy and all metric coefficients can be determined term-by-term

$$\begin{aligned}
U &= 1 - \frac{1}{16r^2}|C|^2 + O(r^{-3}),\\
V &= 1 - \frac{2m}{r} + \frac{1}{r^2}\left(\frac{1}{3}\nabla^A N_A + \frac{1}{4}\nabla^A C_{AB}\nabla_D C^{BD} + \frac{1}{16}|C|^2\right) + O(r^{-3}),\\
W^A &= \frac{1}{2r^2}\nabla_B C^{AB} + \frac{1}{r^3}\left(\frac{2}{3}N^A - \frac{1}{16}\nabla^A|C|^2 - \frac{1}{2}C^{AB}\nabla^D C_{BD}\right) + O(r^{-4}),\\
h_{AB} &= \sigma_{AB} + \frac{C_{AB}}{r} + \frac{1}{4r^2}|C|^2\sigma_{AB} + O(r^{-3})
\end{aligned}$$

where $m = m(u, x^A)$ is the *mass aspect*, $N_A = N_A(u, x^A)$ is the *angular momentum aspect* and $C_{AB} = C_{AB}(u, x^A)$ is the *shear tensor* of this Bondi-Sachs coordinate system.

Bondi-Sachs extract physical information from the coefficients of expansion; the Bondi-Sachs energy momentum 4-vector (E, P^k) for a $u = const.$ hypersurface is given by

$$\begin{aligned}
E(u) &= \frac{1}{4\pi}\int_{S^2} m \\
P^k(u) &= \frac{1}{4\pi}\int_{S^2} m\widetilde{X}^k, k = 1, 2, 3
\end{aligned} \tag{7}$$

where $\widetilde{X}^k, k = 1, 2, 3$ are the standard coordinate functions on $\mathbb{R}^3$ restricted to the unit sphere S^2.

Let $N_{AB} = \frac{\partial}{\partial u}C_{AB}$ denote the *news tensor*. From the vacuum Einstein equations,

$$\partial_u m = -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}\nabla^A\nabla^B N_{AB}, \tag{8}$$

one obtains the famous Bondi mass loss formula

$$\partial_u E = -\frac{1}{32\pi}\int_{S^2} N_{AB}N^{AB}. \tag{9}$$

Bondi-Sachs formalism is a coordinate-based approach and one should take all coordinate transformations that preserve the form of the metric tensor (5) and the asymptotic conditions (6) into consideration. One such coordinate transformation is *supertranslation*: any smooth function $f(x)$ on S^2 gives rise to a coordinate transformation

$$\begin{aligned} \bar{u} &= u + f(x) + \frac{\bar{u}^{(-1)}(u,x)}{r} + o(r^{-1}), \\ \bar{r} &= r + \bar{r}^{(0)}(u,x) + o(1), \\ \bar{x}^A &= x^A + \frac{\bar{x}^{A(-1)}(u,x)}{r} + o(r^{-1}) \end{aligned} \tag{10}$$

where $\bar{u}^{(-1)}, \bar{r}^{(0)}, \bar{x}^{A(-1)}$ are determined by f [4]. The mass aspect, angular momentum aspect, and shear tensor transform nontrivially under a supertranslation. It can be shown that (see for example [27]) the Bondi-Sachs energy momentum 4-vector (7) is supertranslation invariant, but the "supertranslation ambiguity" for the definition of angular momentum has persisted for decades until the discovery of the Chen-Wang-Yau angular momentum in the Bondi-Sachs formalism in [12, 13]. In the rest of this article, we address the supertranslation ambiguity of angular momentum in the double null gauge.

3. Angular momentum at null infinity in double null gauge

Suppose the spacetime metric takes the form

$$g = -4\Omega^2 du dv + \not{g}_{AB}(dx^A - b^A dv)(dx^B - b^B dv) \tag{11}$$

in a double null gauge where we assume $v \in (v_0, +\infty)$ and $u \in (-\infty, +\infty)$.

The u, v level sets $C_u, \underline{C}_v$ are null hypersurfaces intersecting along 2-surfaces $S_{u,v}$. Let

$$r(u,v) = \sqrt{\frac{\text{Area}(S_{u,v})}{4\pi}}$$

be the areal radius.

Associated to the double null gauge is a frame

$$e_A = \frac{\partial}{\partial x^A}, \quad e_3 = \Omega^{-1}\frac{\partial}{\partial u}, \quad e_4 = \Omega^{-1}\left(\frac{\partial}{\partial v} + b^A \frac{\partial}{\partial x^A}\right). \tag{12}$$

Let D denote the Levi-Civita connection of the spacetime metric. We follow the convention of Riemann curvature tensors in [8]

$$\begin{aligned} R(X,Y)Z &= \mathrm{D}_X \mathrm{D}_Y Z - \mathrm{D}_Y \mathrm{D}_X Z - \mathrm{D}_{[X,Y]} Z, \\ R(X,Y,W,Z) &= g(R(X,Y)Z,W). \end{aligned}$$

We introduce the Ricci coefficients

$$\begin{aligned} \chi_{AB} &= g(\mathrm{D}_{e_A} e_4, e_B), \quad \underline{\chi}_{AB} = g(\mathrm{D}_{e_A} e_3, e_B), \\ \zeta_A &= \frac{1}{2} g(\mathrm{D}_{e_A} e_4, e_3), \end{aligned}$$

the curvature components

$$\begin{aligned} \underline{\alpha}_{AB} &= R(e_3, e_A, e_3, e_B), \\ \beta_A &= \frac{1}{2} R(e_A, e_4, e_3, e_4), \quad \underline{\beta}_A = \frac{1}{2} R(e_A, e_3, e_3, e_4), \\ \rho &= \frac{1}{4} R(e_3, e_4, e_3, e_4), \quad \sigma = \frac{1}{4} {}^* R(e_3, e_4, e_3, e_4), \end{aligned}$$

and

$$\widehat{\chi}_{AB} = \chi_{AB} - \frac{1}{2} (\mathrm{tr}_{\not{g}}\, \chi) \not{g}_{AB}, \quad \widehat{\underline{\chi}}_{AB} = \underline{\chi}_{AB} - \frac{1}{2} (\mathrm{tr}_{\not{g}}\, \underline{\chi}) \not{g}_{AB}.$$

Motivated by the result of [7], we assume that along each null hypersurface C_u, the following limits exist:

$$\lim_{r\to\infty} \hat{\chi}_{AB} = \Sigma_{AB}, \quad \lim_{r\to\infty} r^{-1} \hat{\underline{\chi}}_{AB} = \Xi_{AB}, . \tag{13}$$

Let $Y_{(i)}, i = 1,2,3$ be the rotation Killing vector fields on S^2, [25, Figure 3]. Rizzi's definition of angular momentum reads [25, (2)]

$$L(Y_{(i)}) = \frac{1}{8\pi} \lim_{v\to\infty} \int_{S_{u,v}} \zeta_A Y^A_{(i)} dS_{\not{g}}. \tag{14}$$

See [24, Footnote 7 on page 28] for the well-definedness, which is assumed in this article.

Next we defined the mass aspect function.

Definition 3.1. The mass aspect function is given by the following limit:

$$2m(u,x) = \lim_{r\to\infty} r^3 \left(K + \frac{1}{4} \mathrm{tr}\, \chi\, \mathrm{tr}\, \underline{\chi} \right). \tag{15}$$

Motivated by the Chen-Wang-Yau angular momentum at null infinity in the Bondi-Sachs formalism [12, 13], we define the Chen-Wang-Yau angular momentum at null infinity in double null gauge as

$$L_{CWY}(Y_{(i)}) \equiv L(Y_{(i)}) - \frac{1}{8\pi}\int_{S^2} Y^A_{(i)}\mathfrak{s}\nabla_A m$$

where the function $\mathfrak{s}$ is the potential in the Hodge decomposition

$$\Sigma_{AB} = \nabla_A\nabla_B\mathfrak{s} - \frac{1}{2}\Delta\mathfrak{s}\,\sigma_{AB} + (\text{co-closed part}).$$

We assume $\mathfrak{s}$ is supported in $\ell \geq 2$ modes and is thus unique. This nonlocal term appears when solving the optimal isometric embedding equation [20].

Remark 3.2. To give a dictionary between Bondi-Sachs formalism and the double null gauge, the limits of the Ricci coefficients with respect to null frame

$$e_4 = \partial_r, \quad e_3 = \frac{2}{U}\left(\partial_u - W^D\partial_D - \frac{V}{2}\partial_r\right)$$

are given by [12, Appendix A]

$$\begin{aligned}\Sigma_{AB} &= -\frac{1}{2}C_{AB},\\ \Xi_{AB} &= N_{AB}.\end{aligned} \tag{16}$$

Remark 3.3. For the corresponding coefficients in the Bondi-Sachs formalism, (13), (15), and (14) follow from the expansions of the metric coefficients discussed in Section 2.2. However, (13), (15), and (14) are well-defined for spacetimes considered in [7, 6] and likely for more general spacetimes. In this article, we assume (13) and (15) hold at pointwise level, and (14) holds at integral level.

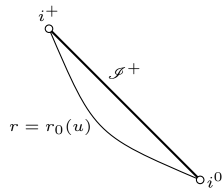
4. Supertranslation and angular momentum in double null gauge

4.1. Supertranslation in double null gauge

Suppose the spacetime metric takes the form

$$g = -4\Omega^2 du dv + \not{g}_{AB}(dx^A - b^A dv)(dx^B - b^B dv) \tag{17}$$

in a double null gauge that covers a neighborhood of null infinity $\{r \geq r_0(u), -\infty < u < \infty\} \times S^2$.



Assumptions on the metric coefficients. We assume that

$$\Omega = 1 + o(1), \tag{18}$$

$$\not{g}_{AB} = r^2 \sigma_{AB} + o(r^2) \tag{19}$$

$$b^A = O(r^{-2}) \tag{20}$$

along each outgoing null hypersurface C_u. We assume the remainders satisfy

$$\mathcal{Z}\Omega = o(1), \mathcal{Z}\not{g} = o(r^2), \mathcal{Z}b = O(r^{-2}) \tag{21}$$

where $\mathcal{Z}$ consists of the derivative ∂_u and covariant derivative ∇ on S^2 but with[2] a stronger

$$\frac{\partial b^A}{\partial u} = o(r^{-2}). \tag{22}$$

Remark 4.1. We prove (in the appendix) the existence of suitable optical functions in order to define supertranslations in double null gauge. To this end, we assume also (54) and (55) on the derivatives of the metric coefficients.

Given a smooth function f on S^2, consider a change of coordinate

$$u = \hat{u} + \frac{f(x)}{2} \tag{23}$$

and v, x^A remain the same. The metric tensor becomes

$$g = -4\Omega^2 d\hat{u}dv - 4\Omega^2 \frac{\partial f}{\partial x^A} dx^A dv + \not{g}_{AB}(dx^A - b^A dv)(dx^B - b^B dv).$$

Note that $\hat{u}$ is only an approximate optical function since

$$g^{\hat{u}\hat{u}} = \Omega^{-2} b^A \partial_A f + \not{g}^{AB} \partial_A f \partial_B f = O(r^{-2}).$$

[2] In view of $\frac{\partial b^A}{\partial u} = 4\Omega^2(\not{g}^{-1})^{AB}\zeta_B$ [8, (1.199)], it is equivalent to $\zeta_A = o(1)$ or $\zeta = o(r^{-1})$.

In the appendix, we show that there exists an optical function $\widetilde{u}$ such that $\widetilde{u} = \hat{u} + O(r^{-1})$. One can then consider the new double null gauge $\widetilde{u}, \widetilde{v} = v, \widetilde{x}^A = x^A$ with[3]

$$u = \widetilde{u} + \frac{f(x)}{2} + O(r^{-1}). \tag{24}$$

We henceforth referred to (24) as *supertranslation in double null gauge.*

In the new double null gauge the metric tensor is given by

$$g = -4\widetilde{\Omega}^2 d\widetilde{u}d\widetilde{v} + \widetilde{\not{g}}_{AB}(d\widetilde{x}^A - \widetilde{b}^A d\widetilde{v})(d\widetilde{x}^B - \widetilde{b}^B d\widetilde{v})$$

and we compute how the Ricci coefficients transform.

Proposition 4.2. *Under a supertranslation in double null gauge (24), the limits of the traceless second fundamental forms (13) transform as*

$$\widetilde{\Sigma}_{AB}(\widetilde{u}, x) = \Sigma_{AB}\left(\widetilde{u} + \frac{f(x)}{2}, x\right) + \nabla_A\nabla_B f - \frac{1}{2}\Delta f\, \sigma_{AB}, \tag{25}$$

$$\widetilde{\Xi}_{AB}(\widetilde{u}, x) = \Xi_{AB}\left(\widetilde{u} + \frac{f(x)}{2}, x\right). \tag{26}$$

If, additionally,

$$\lim_{r\to\infty} r^3\rho = P, \tag{27}$$

$$\lim_{r\to\infty} r\underline{\beta}_A = \underline{B}_A, \tag{28}$$

$$\lim_{r\to\infty} r^{-1}\underline{\alpha}_{AB} = \underline{A}_{AB},, \tag{29}$$

then P transforms as

$$\widetilde{P}(\widetilde{u}, x) = P - \nabla^A f \underline{B}_A + \frac{1}{4}\nabla^A f \nabla^B f \underline{A}_{AB} \tag{30}$$

where P, $\underline{B}_A$ and $\underline{A}_{AB}$ on right-hand side are evaluated at $\left(\widetilde{u} + \frac{f}{2}, x\right)$.

Proof. We compute

$$du = d\widetilde{u} + \frac{\partial_A f}{2} d\widetilde{x}^A + O(r^{-1})$$

[3] The discrepancy of (24) and (10) comes from the fact the optical function u used in double null gauge (resp. Bondi-Sachs) is equal to $\frac{t-r}{2}$ (resp. $t-r$) in Minkowski spacetime.

$$\begin{aligned} dv &= d\widetilde{v} \\ dx^A &= d\widetilde{x}^A. \end{aligned}$$

Thus the coordinate vector fields transform as

$$\begin{aligned} \frac{\partial}{\partial \widetilde{u}} &= (1 + O(r^{-1}))\frac{\partial}{\partial u} \\ \frac{\partial}{\partial \widetilde{v}} &= (1 + O(r^{-1}))\frac{\partial}{\partial v} \\ \frac{\partial}{\partial \widetilde{x}^A} &= \left(\frac{\partial_A f}{2} + O(r^{-1})\right)\frac{\partial}{\partial u} + \frac{\partial}{\partial x^A} \end{aligned}$$

and the frame (12) transforms as

$$\begin{aligned} \widetilde{e}_A &= \left(\frac{\partial_A f}{2} + O(r^{-1})\right)\Omega e_3 + e_A \\ \widetilde{e}_3 &= \left(\widetilde{\Omega}^{-1}\Omega + O(r^{-1})\right) e_3 \\ \widetilde{\Omega}\widetilde{e}_4 &= \Omega e_4 - b^A e_A + \widetilde{b}^A \widetilde{e}_A \\ &= \left(\Omega + O(r^{-1})\right) e_4 + \left(\widetilde{b}^A - b^A + O(r^{-3})\right) e_A \\ &\quad + \left(\widetilde{b}^A \frac{\partial_A f}{2}\Omega + O(r^{-3})\right) e_3. \end{aligned}$$

Recall we identify $\widetilde{v}, \widetilde{x}^A$ with v, x^A. From the relation of coordinate vector fields, the metric coefficients are related by

$$\begin{aligned} \widetilde{\Omega} &= \Omega + O(r^{-1}) \\ \widetilde{\not{g}}_{AB} &= \not{g}_{AB} \\ \widetilde{\not{g}}_{AB}\widetilde{b}^B &= \Omega^2 \partial_A f + \not{g}_{AB} b^B + O(r^{-1}) \end{aligned} \tag{31}$$

where the left-hand side are evaluated at $(\widetilde{u}, v, x)$ and the right-hand side are evaluated at $\left(\widetilde{u} + \frac{f}{2}, v, x\right)$.

We have

$$\langle D_{e_3} e_3, e_B \rangle = -\langle e_3, D_{e_3} e_B \rangle = -\langle e_3, [e_3, e_B] \rangle = \partial_B \Omega^{-1} \langle e_3, \frac{\partial}{\partial u} \rangle = 0.$$

Hence,

$$\left\langle D_{\widetilde{e}_A} \widetilde{e}_3, \widetilde{e}_B \right\rangle = \left(\widetilde{\Omega}^{-1}\Omega + O(r^{-1})\right) \langle D_{e_A} e_3, e_B \rangle$$

and take limit to get $\widetilde{\Xi}_{AB} = \Xi_{AB}$.

Next we compute

$$\begin{aligned}
&\left\langle D_{\widetilde{e}_A}\widetilde{e}_B, \widetilde{\Omega}\widetilde{e}_4\right\rangle \\
&= \left(\left(\frac{\partial_A f}{2} + O(r^{-1})\right)\Omega e_3 + e_A\right)\left(\left(\frac{\partial_B f}{2} + O(r^{-1})\right)\Omega\right)\langle e_3, \widetilde{\Omega}\widetilde{e}_4\rangle \\
&\quad + \left(\frac{\partial_A f}{2} + O(r^{-1})\right)\Omega\left\langle D_{\left(\frac{\partial_A f}{2}+O(r^{-1})\right)\Omega e_3 + e_A} e_3, \widetilde{\Omega}\widetilde{e}_4\right\rangle \\
&\quad + \left\langle D_{\left(\frac{\partial_A f}{2}+O(r^{-1})\right)\Omega e_3 + e_A} e_B, \widetilde{\Omega}\widetilde{e}_4\right\rangle.
\end{aligned}$$

Denote the first line on the right-hand side by I and the rest by II. We have

$$I = -\partial_A\partial_B f + o(1).$$

On the other hand, we have by (22)

$$\begin{aligned}
\langle D_{e_3}e_3, e_4\rangle &= o(1), \\
\langle D_{e_A}e_3, e_4\rangle, \langle D_{e_3}e_B, e_4\rangle &= o(1),
\end{aligned}$$

and hence

$$\begin{aligned}
II &= \langle D_{e_A}e_B, \Omega e_4 + (\widetilde{b}^C - b^C)e_C\rangle + o(1) \\
&= \langle D_{e_A}e_B, \Omega e_4\rangle + \not{g}^{CD}\partial_D f\langle D_{e_A}e_B, e_C\rangle + o(1)
\end{aligned}$$

where we used (31) in the second equality. Putting these together, we have

$$\left\langle D_{\widetilde{e}_A}\widetilde{e}_B, \widetilde{\Omega}\widetilde{e}_4\right\rangle = -\nabla_A\nabla_B f + \langle D_{e_A}e_B, \Omega e_4\rangle + o(1).$$

and hence

$$\widetilde{\Sigma}_{AB} = \Sigma_{AB} + \nabla_A\nabla_B f - \frac{1}{2}\Delta f\,\sigma_{AB}$$

by taking the limit of the traceless part. Finally, we compute the transformation of ρ

$$\begin{aligned}
\widetilde{\rho} &= \frac{1}{4}R\left(\widetilde{e}_3, \widetilde{e}_4, \widetilde{e}_3, \widetilde{e}_4\right) \\
&= \frac{1}{4}R(e_3, e_4, e_3, e_4) + \frac{1}{2}(\widetilde{b}^A - b^A)R(e_3, e_4, e_3, e_A)
\end{aligned}$$

$$+\frac{1}{4}(\widetilde{b}^A - b^A)(\widetilde{b}^B - b^B)R(e_3, e_A, e_3, e_B) + O(r^{-4})$$
$$= \rho - \frac{1}{r^2}\nabla^A f \underline{\beta}_A + \frac{1}{4r^4}\nabla^A f \nabla^B f \underline{\alpha}_{AB} + o(r^{-3})$$

and take limit. ◻

The supertranslation can be used to kill the closed part[4] of the shear tensor Σ.

Theorem 4.3. *Suppose*

$$\sup_{(-\infty,\infty)\times S^2} \Xi_{AB}\Xi^{AB} < 48. \tag{32}$$

Then for a fixed $\widetilde{u} = \widetilde{u}_0$ *and any* $\psi_{sl} \in \mathcal{H}_{\ell\le 1}$, *there exists a unique* $\Psi \in \mathcal{H}_{\ell\ge 2}$ *such that with* $\psi = \psi_{sl} + \Psi$, *the symmetric traceless 2-tensor*

$$\widetilde{\Sigma}_{AB}(\widetilde{u}) \equiv \Sigma_{AB}(\widetilde{u} + \frac{\psi(x)}{2}, x) + \nabla_A\nabla_B\psi - \frac{1}{2}\Delta\psi\sigma_{AB} \tag{33}$$

has no closed part.

Since the vector space $\mathcal{H}_{\ell\le 1}$ is 4-dimensional, the result singles out a cut, with a 4-dimensional degree of freedom, of null infinity. In Section 4.2 of [24], Rizzi considers the lapse transformation $L \to a^{-1}L, \underline{L} \to a\underline{L}$ with $\lim_{r\to\infty} a = \psi$ along each null hypersurface. This leads to a linear equation

$$\Delta(\Delta+2)\psi = \nabla^A\nabla^B(\psi\Xi_{AB}),$$

which is the linearized equation of (34), whose solution is used to describe a procedure that retains the 4-dimensional dependence of origins in his definition. See Chapter 4 and 5 of [24]. The original proof of Rizzi's version [24, Theorem 1], due to Christodoulou, requires the upper bound 16. We follow his argument with improved estimates.

Proof. Since $\nabla_A\nabla_B h - \frac{1}{2}\Delta h\sigma_{AB} = 0$ for any $h \in \mathcal{H}_{\ell\le 1}$ and

$$\nabla^A\nabla^B(\nabla_A\nabla_B h' - \frac{1}{2}\Delta h'\sigma_{AB}) = \frac{1}{2}\Delta(\Delta+2)h'$$

for any smooth function h', the equation to be solved is

$$\Delta(\Delta+2)\Psi = -2\nabla^A\nabla^B\left(\Sigma_{AB}(\widetilde{u} + \frac{\Psi+\psi_{sl}}{2}, x)\right). \tag{34}$$

[4] It is called the electric part in physics literature.

Set up an iterative equation

$$\Delta(\Delta+2)\Psi_{n+1} = -2\nabla^A\nabla^B\left(\Sigma_{AB}(\widetilde{u}+\frac{\Psi_n+\psi_{sl}}{2},x)\right).$$

and start with $\Psi_0 = 0$. We will show that Ψ_n converge to Ψ in L^2 and this proves the existence.

Let $h_{n+1} = \Psi_{n+1} - \Psi_n$ and we have

$$\begin{aligned}&\Delta(\Delta+2)h_{n+1}\\&= -2\nabla^A\nabla^B\left(\Sigma_{AB}(\widetilde{u}+\frac{\Psi_n+\psi_{sl}}{2},x) - \Sigma_{AB}(\widetilde{u}+\frac{\Psi_{n-1}+\psi_{sl}}{2},x)\right).\end{aligned}$$

Let us analyze the equation

$$\Delta(\Delta+2)g = \nabla^A\nabla^B\eta_{AB} \tag{35}$$

for g supported in $\ell \geq 2$ modes and a symmetric traceless 2-tensor η_{AB}. Multiplying by g and integrating, we get

$$\int_{S^2}\Delta g(\Delta+2)g = \int_{S^2}\eta_{AB}\left(\nabla^A\nabla^B g - \frac{1}{2}\Delta g\sigma^{AB}\right).$$

Decompose g into spherical harmonics $g = \sum_{\ell=2}^{\infty} g_l$. Because of the orthogonality of g_l's, we have

$$\int_{S^2}\Delta g(\Delta+2)g = \sum_{\ell=2}^{\infty}[-\ell(\ell+1)][2-\ell(\ell+1)]\int_{S^2}g_\ell^2 \geq 24\int_{S^2}g^2. \tag{36}$$

On the other hand, for any smooth function f on S^2, integration by parts yields the identity

$$\int_{S^2}\left(\nabla_A\nabla_B f - \frac{1}{2}\Delta f\sigma_{AB}\right)\left(\nabla^A\nabla^B f - \frac{1}{2}\Delta f\sigma^{AB}\right) = \frac{1}{2}\int_{S^2}\Delta f(\Delta+2)f.$$

By Cauchy-Schwarz and Hölder inequality, we get

$$\int_{S^2}\eta_{AB}\left(\nabla^A\nabla^B g - \frac{1}{2}\Delta g\sigma^{AB}\right) \leq \sqrt{\int_{S^2}|\eta|^2\cdot\int_{S^2}|\nabla^2 g - \frac{1}{2}(\Delta g)\sigma|^2}.$$

Putting these together, we obtain

$$24\int_{S^2}g^2 \leq \frac{1}{2}\int_{S^2}|\eta|^2.$$

By the Mean Value Theorem, for each $x \in S^2$

$$\left|\Sigma_{AB}(\widetilde{u} + \frac{\Psi_n + \psi_{sl}}{2}(x), x) - \Sigma_{AB}(\widetilde{u} + \frac{\Psi_{n-1} + \psi_{sl}}{2}(x), x)\right| \leq \sup_{(-\infty,\infty)\times\{x\}} |\Xi| \cdot \frac{1}{2}|\Psi_n - \Psi_{n-1}|(x),$$

and integrating over S^2 we get

$$\int_{S^2} h_{n+1}^2 \leq \frac{1}{48} \sup_{(-\infty,\infty)\times S^2} |\Xi|^2 \int_{S^2} h_n^2.$$

If $\sup|\Xi|^2 < 48$, then h_n converge to 0 and hence Ψ_n converge to Ψ in L^2.

The uniqueness follows similarly. Suppose ϕ_1, ϕ_2 solve the equation with the same ψ_{sl}. Then $\phi = \phi_2 - \phi_1$ solves $\Delta(\Delta + 2)\phi = -2\nabla^A\nabla^B(\Sigma_{AB}(\widetilde{u} + \frac{\psi_{sl}+\phi_2}{2}, x) - \Sigma_{AB}(\widetilde{u} + \frac{\psi_{sl}+\phi_1}{2}, x))$. The above analysis implies that

$$\int_{S^2} \phi^2 \leq \frac{1}{48} \sup|\Xi|^2 \int_{S^2} \phi^2$$

and hence $\phi = 0$ by (32). ☐

4.2. Total flux of angular momentum and the supertranslation ambiguity

We first recall the following evolution formulae for the mass aspect function and Rizzi's angular momentum. First we state the mass loss formula:

Lemma 4.4. *We have the following mass loss formula:*

$$\partial_u m = -\frac{1}{4}|\Xi|^2 + \frac{1}{2}\nabla^B\nabla^C\Xi_{BC}. \tag{37}$$

Proof. The mass loss formula is derived in various formulations of null infinity. We sketch the proof for double gauge using the null structure equation in [8]. We start with the Gauss equation

$$K + \frac{1}{4}\operatorname{tr}\chi\operatorname{tr}\underline{\chi} = -\rho + \frac{1}{2}(\hat{\chi}, \hat{\underline{\chi}}).$$

We differentiate ρ using the Bianchi identity [8, Proposition 1.2] and replace $\underline{\beta}$ via the null Codazzi equation [8, (1.152)]. For the second term on the right-hand side, we apply [8, (1.171)] and [8, (1.59)]. The lemma follows from the asymptotically flat assumption. ☐

The evolution of Rizz's angular momentum is given by [25, (4)]

$$\frac{\partial L(Y_{(k)})}{\partial u} = \frac{1}{4\pi}\int_{S^2}\left\{-\Xi_{AB}\nabla_C\Sigma^{CB} + \frac{1}{2}\left(\Sigma_{AB}\nabla_C\Xi^{CB} - \Sigma_B^C\nabla^B\Xi_{CA}\right)\right\}Y^A_{(k)}.$$

Remark 4.5. While [25, (4)] is stated with strong asymptotic conditions. The evolution formula is valid with much weaker asymtotics using the Bianchi identity for $D_{e_3}\beta$.

Remark 4.6. Note that the factor is $\frac{1}{4\pi}$ rather than $\frac{1}{8\pi}$. The reason is that the parameter u used in [5] or [25] (and used in Bondi-Sachs formalism) is equal to $t-r$ plus some constant in Minkowski spacetime. When citing their formula, the derivative $2\frac{\partial}{\partial u}$ should be replaced by $\frac{\partial}{\partial u}$. For one more example, equation (5) of [5] now reads

$$\frac{\partial\Sigma}{\partial u} = -\Xi. \tag{38}$$

For the rest of the paper, we assume

$$\Xi_{AB} = O(|u|^{-1-\varepsilon}) \tag{39}$$

for some $\varepsilon > 0$ as u approaches $\pm\infty$.

We fix the rotation Killing vector $Y^A = \epsilon^{AB}\nabla_B\widetilde{X}^k$ for some $k \in \{1,2,3\}$ for definiteness. Integrating by parts the last term and then integrating from $u=-\infty$ to $u=+\infty$, we obtain the *total flux* of Rizzi's angular momentum

$$\delta L = \frac{1}{8\pi}\cdot \int_{-\infty}^{+\infty}\left(\int_{S^2} Y^A\left(\Sigma_{AB}\nabla_C\Xi^{CB} - \Xi_{AB}\nabla_C\Sigma^{CB}\right) + \epsilon^{AB}\widetilde{X}^k\left(\Sigma_A^C\Xi_{CB}\right)\right)du.$$

The integral is finite because of (39). In view of (16), the result, up to a minus sign[5], coincides with the classical Ashtekhar-Streubel flux of angular momentum [2], see [12, Theorem 1.2].

The total flux of angular momentum changes when computing in different double null gauges.

[5]Firstly, the factor $\frac{1}{8\pi}$ is omitted in [12]. Secondly, take into account the discrepancy in u. Lastly, for Kerr spacetime Rizzi's definition gives ma while Chen-Wang-Yau gives $-ma$.

Theorem 4.7. *Consider a supertranslation in a double null gauge* $u = \widetilde{u} + \frac{f(x)}{2}$ *and let* δL_f *denote the total flux of angular momentum computed in* $(\widetilde{u}, v, x)$ *gauge. Suppose* (39) *holds, then*

$$\delta L_f - \delta L = \frac{1}{4\pi}\int_{S^2} fY^A\nabla_A\left(m^+ - m^-\right) \tag{40}$$

where

$$m^{\pm} = \lim_{u\to\pm\infty} m(u,x).$$

Proof. We sketch the proof. For details, see Section 5.1 of [12]. First rewrite δL as

$$\frac{1}{8\pi}\int_{-\infty}^{+\infty}\left(\int_{S^2} -\nabla_C Y^A\Sigma_{AB}\Xi^{CB} - Y^A\nabla_C\Sigma_{AB}\Xi^{CB} - Y^A\Xi_{AB}\nabla_C\Sigma^{CB}\right.$$
$$\left. + \epsilon^{AB}\widetilde{X}^k\Sigma^C_A\Xi_{CB}\right) du$$

For δL_f, simply replace Σ, Ξ, du by $\widetilde{\Sigma}, \widetilde{\Xi}, d\widetilde{u}$. Applying the chain rule to (25) yields

$$\begin{aligned}
\nabla_C\widetilde{\Sigma}_{AB}(\widetilde{u},x) &= -\Xi_{AB}(\widetilde{u}+\frac{f}{2},x)\frac{\nabla_C f}{2} + (\nabla_C\Sigma_{AB})(\widetilde{u}+\frac{f}{2},x)\\
&\quad + \nabla_C(\nabla_A\nabla_B f - \frac{1}{2}\Delta f\sigma_{AB}),\\
\nabla_C\widetilde{\Sigma}^{CB}(\widetilde{u},x) &= -\Xi^{CB}(\widetilde{u}+\frac{f}{2},x)\frac{\nabla_C f}{2} + \left(\nabla_C\Sigma^{CB}\right)(\widetilde{u}+\frac{f}{2},x)\\
&\quad + \frac{1}{2}\nabla^B(\Delta+2)f,
\end{aligned} \tag{41}$$

where we used (38).

After a series of change of variables and integration by parts, the difference of total flux is put into the following simple form

$$\delta L_f - \delta L = -\frac{1}{16\pi}\int_{-\infty}^{+\infty}\left(\int_{S^2} fY^A\nabla_A\left(|\Xi|^2 - 2\nabla^B\nabla^C\Xi_{BC}\right)\right) du.$$

The conclusion follows from the mass loss formula (37). ☐

When f is supported in $\ell \leq 1$ modes, the difference is equal to the total flux of linear momentums. However, if f has $\ell \geq 2$ modes, the formula bears no resemblance to (1) and physicists have yet to find an interpretation. This is the so-called "supertranslation ambiguity". For more on supertranslation ambiguity, we refer the readers to Ashtekar-De Lorenzo-Khera [1].

4.3. Supertranslation invariance of the Chen-Wang-Yau angular momentum and center of mass integral

Following the argument of [12], the total flux of the Chen-Wang-Yau angular momentum δJ is given by

$$\delta J = \delta L - \frac{1}{8\pi}\left[\int_{S^2} Y^A \mathfrak{s} \nabla_A m\right]_{-\infty}^{+\infty}$$

where the function $\mathfrak{s}(u, x)$ is the potential in the Hodge decomposition

$$\Sigma_{AB} = \nabla_A \nabla_B \mathfrak{s} - \frac{1}{2}\Delta \mathfrak{s}\, \sigma_{AB} + (\text{co-closed part}).$$

The main result in [12, 13] is that the total flux of the Chen-Wang-Yau angular momentum transforms according to (1) under supertranslations.

Theorem 4.8. *Consider a supertranslation in double null gauge (24) and let δJ_f denote the total flux of the Chen-Wang-Yau angular momentum computed in $(\tilde{u}, v, x)$ gauge. Suppose (39) holds, then*

$$\delta J_f - \delta J = -\varepsilon^{ik}{}_l \alpha_i \delta P^l \tag{42}$$

where $f_{\ell \leq 1} = \alpha_0 + \alpha_i \widetilde{X}^i$, ε is the Levi-Civita symbol, and

$$\delta P^l = \frac{1}{4\pi}\int_{S^2} \widetilde{X}^l (m^+ - m^-)$$

is the total flux of linear momentum.

Proof. By (25), we have

$$\tilde{\mathfrak{s}}^{\pm}(x) = \mathfrak{s}^{\pm}(x) + f_{\ell \geq 2}. \tag{43}$$

It can be argued from (30) (see Remark 4.11 below) that

$$\widetilde{m}^{\pm}(x) = m^{\pm}(x). \tag{44}$$

Putting these together, we obtain

$$\delta J_f - \delta J = \frac{1}{8\pi}\int_{S^2} f_{\ell \leq 1} Y^A \nabla_A (m^+ - m^-).$$

Recall that $Y^A = \epsilon^{AB}\nabla_B \widetilde{X}^k$ and we have

$$Y^A \nabla_A f_{\ell\leq 1} = \varepsilon^{ik}{}_l \alpha_i \widetilde{X}^l.$$

□

Next, we turn to the effect of supertranslation on the center of mass integral. In double null gauge, the total flux of Ashtekar-Streubel center of mass integral $\delta\widetilde{C}^k$ is given by

$$\begin{aligned}\delta\widetilde{C}^k =& \frac{1}{8\pi}\cdot \\ & \int_{-\infty}^{+\infty}\left[\int_{S^2}\nabla^A\widetilde{X}^k\left(-\frac{u}{2}\nabla_A|\Xi|^2 + \Sigma_{AB}\nabla_D\Xi^{BD} - \Xi_{AB}\nabla_D\Sigma^{BD}\right)\right]du,\end{aligned}$$

up to a minus sign. Again, compare [12, Theorem 1.2].

Theorem 4.9. *Consider a supertranslation in double null gauge (24) and let $\delta\widetilde{C}_f$ denote the total flux of Ashtekar-Streubel center of mass integral computed in $(\tilde{u}, v, x)$ gauge. Suppose (39) holds, then*

$$\delta\widetilde{C}^k_f - \delta\widetilde{C}^k = \frac{1}{8\pi}\int_{S^2}\left(-6\widetilde{X}^k(m^+ - m^-) + 2\nabla^A\widetilde{X}^k\nabla_A(m^+ - m^-)\right)f.$$

Proof. First of all, note that the first term of $\delta\widetilde{C}^k$ is equal to $\int_{S^2} -u\widetilde{X}^k|\Xi|^2$. For $\delta\widetilde{C}_f$, simply replace Σ, Ξ, du by $\widetilde{\Sigma}, \widetilde{\Xi}, d\tilde{u}$. Applying the chain rule to (26), we get

(45)
$$\nabla_D\widetilde{\Xi}^{BD}(\tilde{u}, x) = \partial_u\Xi^{BD}\left(\tilde{u} + \frac{f(x)}{2}, x\right)\frac{\nabla_D f}{2} + \nabla_D\Xi^{BD}\left(\tilde{u} + \frac{f(x)}{2}, x\right).$$

Combining with (41), we obtain

$$\begin{aligned}&\delta\widetilde{C}^k_f - \delta\widetilde{C}^k \\ &= \frac{1}{8\pi}\int_{-\infty}^{+\infty}\left[\int_{S^2}\frac{f}{2}\widetilde{X}^k|\Xi|^2\right. \\ &\qquad\left. + \nabla^A\widetilde{X}^k(\nabla_A\nabla_B f - \frac{1}{2}\Delta f\sigma_{AB})(\partial_u\Xi^{BD}\frac{\nabla_D f}{2} + \nabla_D\Xi^{BD})\right]du \\ &\quad + \frac{1}{8\pi}\int_{-\infty}^{+\infty}\left[\int_{S^2}\nabla^A\widetilde{X}^k\left(\Sigma_{AB}\partial_u\Xi^{BD}\frac{\nabla_D f}{2}\right.\right.\end{aligned}$$

$$
-\Xi_{AB}(-\Xi^{CB}\frac{\nabla_C f}{2}+\frac{1}{2}\nabla^B(\Delta+2)f)\Big)\Big]\,du
$$

$$
=\frac{1}{8\pi}\int_{-\infty}^{+\infty}\left[\int_{S^2}\frac{f}{2}\widetilde{X}^k|\Xi|^2+\nabla^A\widetilde{X}^k(\nabla_A\nabla_B f-\frac{1}{2}\Delta f\sigma_{AB})\nabla_D\Xi^{BD}\right]du
$$

$$
+\frac{1}{8\pi}\int_{-\infty}^{+\infty}\left[\int_{S^2}\nabla^A\widetilde{X}^k\left(|\Xi^2|\frac{\nabla_A f}{2}-\frac{1}{2}\Xi_{AB}\nabla^B(\Delta+2)f\right)\right]du
$$

where we integrate by parts in u and use (38) together with the identity $\Xi_{AB}\Xi^{BC}=\frac{1}{2}\delta_A^C|\Xi|^2$ in the second equality. Further integration by parts on S^2 (see the proof of Theorem 5.3 [12]) leads to

$$
\begin{aligned}
&\delta\widetilde{C}^k_f-\delta\widetilde{C}^k\\
&=\frac{1}{8\pi}\int_{-\infty}^{+\infty}\Bigg[\int_{S^2}f\widetilde{X}^k\left(\frac{|\Xi|^2}{2}-\nabla^A\nabla^B\Xi_{AB}\right)\\
&\qquad\qquad+\nabla^A\widetilde{X}^k\left(\frac{|\Xi|^2}{2}-\nabla^B\nabla^C\Xi_{BC}\right)\nabla_A f\Bigg]du\\
&=\frac{1}{8\pi}\int_{S^2}\left(-6\widetilde{X}^k(m^+-m^-)+2\nabla^A\widetilde{X}^k\nabla_A(m^+-m^-)\right)f
\end{aligned}
$$

□

On the other hand, the total flux of the Chen-Wang-Yau center of mass-integral transforms according to (1) under supertranslations.

Theorem 4.10. *Consider a supertranslation in a double null gauge* $u=\widetilde{u}+\frac{f(x)}{2}$ *and let* δC_f *denote the total flux of the Chen-Wang-Yau center of mass integral computed in* $(\widetilde{u},v,x)$ *gauge. Suppose* (39) *holds,*

$$
\delta C^k_f-\delta C^k=-\alpha_0\delta P^k-\alpha_k\delta E \tag{46}
$$

where $f_{\ell\le 1}=\alpha_0+\alpha_i\widetilde{X}^i$, *and* $\delta E=\frac{1}{4\pi}\int_{S^2}m^+-m^-$ *is the total flux of energy.*

Proof. The Chen-Wang-Yau center of mass integral has the correction term on the total flux

$$
\delta C^k=\delta\widetilde{C}^k+\frac{1}{8\pi}\int_{S^2}-6\widetilde{X}^k\mathfrak{s}m+2\nabla^A\widetilde{X}^k\mathfrak{s}\nabla_A m\Big|_{-\infty}^{+\infty}.
$$

Recall that under the supertranslation,

$$
\widetilde{m}^\pm(x)=m^\pm(x),\quad \widetilde{\mathfrak{s}}^\pm(x)=\mathfrak{s}^\pm(x)+f_{\ell\ge 2},
$$

and we obtain

$$\begin{aligned}\delta C^k_f - \delta C^k &= \frac{1}{8\pi}\int_{S^2}\left(-6\widetilde{X}^k(m^+ - m^-) + 2\nabla^A\widetilde{X}^k\nabla_A(m^+ - m^-)\right) f_{\ell\le 1}\\ &= \frac{1}{8\pi}\int_{S^2} -2\alpha_0\widetilde{X}^k(m^+ - m^-) - 2\alpha_k(m^+ - m^-),\end{aligned}$$

where we used the identity $\nabla^A\widetilde{X}^k\nabla_A\widetilde{X}^i = \delta^{ik} - \widetilde{X}^k\widetilde{X}^i$. □

We remark that for non-radiative spacetimes, namely $\Xi \equiv 0$, the Chen-Wang-Yau angular momentum and center of mass integral are constant in u and transform according to (1) under supertranslations. See [12, Theorem 6.2] or [13, Theorem 2 and Section 5]. Regarding the effect of Lorentz transformations (defined in Bondi-Sachs formalism), the total fluxes of the Chen-Wang-Yau angular momentum and center of mass integral transform equivariantly; similarly, the Chen-Wang-Yau angular momentum and center of mass integral themselves transform equivariantly in non-radiative spacetimes. They are established in Theorem 4.1 and 4.4 of [17].

Remark 4.11. In the following, we show that (44) holds under the assumptions of [5]. By (3) of [5], we have

$$2m = -P + \frac{1}{2}\Sigma_{AB}\Xi^{AB}. \tag{47}$$

Moreover, by virtue of equations (2) and (6) of [5],

$$\nabla^D\Xi_{AD} = B_A, \quad \frac{\partial}{\partial u}\Xi_{AB} = -\underline{A}_{AB},$$

(30) and the assumption on the decay of Ξ (39) imply $\widetilde{P}^{\pm} = P^{\pm}$. In the Bondi-Sachs formalism, (44) is established through [13, equation (15)]. Both (43) and (44) should hold for more general spacetimes.

Appendix A. Construction of optical function

The goal of this appendix is to construct an optical function in a neighborhood of the null infinity with prescribed asymptotics in order to define supertranslations in double null gauge.

In double null gauge, the spacetime metric takes the form

$$g = -4\Omega^2 du dv + \not{g}_{AB}(dx^A - b^A dv)(dx^B - b^B dv) \tag{48}$$

with nonzero components of the inverse metric given by

$$g^{uv} = -\frac{1}{2}\Omega^{-2}, g^{uA} = g^{Au} = -\frac{1}{2}\Omega^{-2}b^A, g^{AB} = \not{g}^{AB}. \tag{49}$$

We assume that the double null gauge covers a region

$$\{r \geq r_0(u)\} \times S^2$$

on which the metric coefficients are smooth and have asymptotics

$$\Omega = 1 + o(1) \tag{50}$$

$$\not{g}_{AB} = r^2\sigma_{AB} + o(r^2) \tag{51}$$

$$b = O(r^{-2}). \tag{52}$$

along each outgoing null hypersurface C_u. We also assume decay of the derivatives

$$\mathcal{Z}(\Omega^2\not{g}^{-1}) = o(r^{-2}), \mathcal{Z}b = O(r^{-2}) \tag{53}$$

and

$$\mathcal{Z}^2(\Omega^2\not{g}^{-1}) = o(r^{-2}), \mathcal{Z}^2b = O(r^{-2}) \tag{54}$$

where $\mathcal{Z}$ consists of the derivative ∂_u and covariant derivative ∇ on S^2. Lastly, we assume[6]

$$\frac{\partial r}{\partial v} = 1 + o(1). \tag{55}$$

The convergence assumption means that for any ε there exists a $\mathfrak{r}(u, \varepsilon)$ depending continuously on u such that $|\Omega - 1|, |r^{-2}\not{g} - \sigma|, |r^2\mathcal{Z}(\Omega^2\not{g}^{-1})|, |r^2\mathcal{Z}^2(\Omega^2\not{g}^{-1})| < \varepsilon$ provided $r > \mathfrak{r}(u, \varepsilon)$. Moreover, $b, \mathcal{Z}b, \mathcal{Z}^2b = O(r^{-2})$ means that $|b|, |\mathcal{Z}b|, |\mathcal{Z}^2b| \leq \frac{C_1}{r^2}$ on $\{r \geq r_0(u)\} \times S^2$ for some constant C_1. Note that we compute norm of tensors on S^2 by σ.

Remark A.1. We do not need (22) in proving Theorem A.2.

The main theorem of the appendix is the following existence result for optical functions:

[6] This follows from the more geometrical assumption $\operatorname{tr}\chi = \frac{2}{r} + o(r^{-1})$ and the first variation formula of area.

Theorem A.2. *For any C^2 function $f(x)$ on S^2, there exists an optical function $\widetilde{u}$ in a neighborhood of null infinity $\{r \geq r_*(u)\} \times S^2$ such that $\widetilde{u} = u - f(x) + O(r^{-1})$.*

Observe that it is easy to construct such an optical function on the Minkowski spacetime which admits a smooth compactification. Indeed, the Minkowski metric is $\eta = -4du^2 - 4dudr + r^2\sigma$ with $u = \frac{t-r}{2}$ and the unphysical conformal metric is given by

$$\hat{\eta} = r^{-2}\eta = -4s^2du^2 + 4duds + \sigma$$

where $s = \frac{1}{r}$. Consider level sets of $u - f(x)$ on the boundary $s = 0$ and the null hypersurfaces emanating from them. One gets an optical function $\widetilde{u}$ whose level sets are these null hypersurfaces. Moreover, $\widetilde{u}$ exists on some region $\{0 \leq s \leq s_0\} \times S^2$ as the (unphysical) metric is independent of u. Since the eikonal equation is conformal invariant, $\widetilde{u}$ is the desired optical function on the Minkowski spacetime.

To simplify notation, we will henceforth omit the factor S^2 in the description of regions. For example, $\{v_0 \leq v < \infty, u_1 \leq u \leq u_2\}$ stands for $\{v_0 \leq v < \infty, u_1 \leq u \leq u_2\} \times S^2$.

For any C^2 function $f(x)$ on S^2, the function $u - f(x)$ is an approximate optical function as direct computation shows

$$|\mathrm{grad}(u - f)|^2 = O(r^{-2}).$$

We would like to add a small correction U so that $u - f(x) + U$ becomes a genuine optical function. U will be solved in regions of the form

$$\{r \geq r_2(u), u_* \leq u \leq u^*\} \tag{56}$$

Suggested by [22, Section 10], we will first solve optical functions U_T with

$$g^{\alpha\beta}\partial_\alpha(u - f + U_T)\partial_\beta(u - f + U_T) = 0, \tag{57}$$

$$U_T = 0 \text{ on } \{v = T\} \tag{58}$$

and then show that U_T converges to U in C^1 as $T \to \infty$. The solutions U obtained in two regions of the form (56) will thus coincide on the intersection.

Fix two constants $-\infty < u_* < u^* < \infty$. Consider the level sets of $u - f(x)$ on the cylinder $\{v = T, u_* - \max_{S^2}|f| \leq u \leq u^* + \max_{S^2}|f|\}$. Because the spacetime under consideration is close to the Minkowski spacetime, for sufficiently large T the null hypersurfaces emanating from these

2-surfaces have no focal points and the (disjoint) union of them contains a region $\{r \geq r_1(u), u_* \leq u \leq u^*\}$ where $r_1(u)$ is independent of T. U_T is uniquely determined by (58) and by requiring $u - f(x) + U_T$ are constants along these null hypersurfaces.

Let $\widetilde{u} = u - f(x) + U$ be an optical function. By (49), the eikonal equation $-\Omega^2 g^{\alpha\beta}\partial_\alpha \widetilde{u}\partial_\beta \widetilde{u} = 0$ reads

$$\begin{aligned}(1+\partial_u U)\partial_v U + b^A(1+\partial_u U)(-\partial_A f + \partial_A U)\\ -\Omega^2 \not{g}^{AB}(-\partial_A f + \partial_A U)(-\partial_B f + \partial_B U) = 0.\end{aligned} \tag{59}$$

We rewrite it as a transport equation

$$\mathcal{V}U = b^A\partial_A f + \Omega^2 \not{g}^{AB}\partial_A f \partial_B f \tag{60}$$

with the vector field

$$\begin{aligned}\mathcal{V} = (1+\partial_u U)\partial_v + b^A(-\partial_A f + \partial_A U)\partial_u + b^A\partial_A - 2\Omega^2 \not{g}^{AB}\partial_A f \partial_B\\ -\Omega^2 \not{g}^{AB}\partial_A U \partial_B.\end{aligned} \tag{61}$$

The next lemma provide the necessary estimate up to second derivatives.

Lemma A.3. *There exist a constant C_2 and a function $r_2(u)$ that are both independent of T such that on $\{v \leq T, r \geq r_2(u), u_* \leq u \leq u^*\}$ we have*

$$|U_T| \leq \frac{C_2}{r} \tag{62}$$

$$|\mathcal{Z}U_T| \leq \frac{C_2}{r}, |\partial_v U_T| \leq \frac{C_2}{r^2} \tag{63}$$

$$|\mathcal{Z}^2 U_T| \leq \frac{C_2}{r}, |\partial_v \mathcal{Z} U_T| \leq \frac{C_2}{r^2} \tag{64}$$

where $\mathcal{Z}$ consists of the derivative ∂_u and covariant derivative ∇ on S^2.

Proof. We write $U = U_T$ in the proof. For convenience, assume $\|f\|_{C^2(S^2)} \leq 1$.

We have $\partial_u U = \nabla U = 0$ on $\{v = T\}$ by definition. The eikonal equation (59) then implies $|\partial_v U| \leq \frac{3C_1}{r^2}$ on $\{v = T\}$ algebraically. This enables us to make the bootstrap assumption

$$\begin{aligned}|\partial_u U|, |\nabla U| &\leq \frac{15C_1}{r}\\ |\partial_v U| &\leq \frac{15C_1}{r^2}\end{aligned}$$

in the region $\mathcal{D} = \{v \le T, r \ge r_2(u), u_* \le u \le u^*\}$ where $r_2(u)$ presumably depends on T.

We estimate $\mathcal{Z}U$ by the transport equation of it. Differentiating (60) by ∂_u, we obtain

$$\begin{aligned}\widetilde{\mathcal{V}}\left(\partial_u U\right) = {} & \partial_u b^A \partial_A f + \partial_u \left(\Omega^2 \not{g}^{AB}\right) \partial_A f \partial_B f \\ & + \partial_u b^A \partial_A U - \partial_u b^A \partial_A f \partial_u U + \partial_u b^A \partial_A U \partial_u U \\ & + 2\partial_u(\Omega^2 \not{g}^{AB})\partial_A f \partial_B U - \partial_u(\Omega^2 \not{g}^{AB})\partial_A U \partial_B U\end{aligned} \tag{65}$$

with

$$\widetilde{\mathcal{V}} = \mathcal{V} + \partial_v U \partial_u + b^A \partial_u U \partial_A - \Omega^2 \not{g}^{AB} \partial_B U \partial_A.$$

Differentiating (60) by ∂_C, we obtain

$$\begin{aligned}\widetilde{\mathcal{V}}(\partial_C U) = {} & \partial_C \left(b^A \partial_A f + \Omega^2 \not{g}^{AB} \partial_A f \partial_B f\right) \\ & + \partial_C b^A \partial_A U - \partial_C (b^A \partial_A f) \partial_u U + \partial_C b^A \partial_A U \partial_u U \\ & + 2\partial_C(\Omega^2 \not{g}^{AB})\partial_A f \partial_B U - \partial_C(\Omega^2 \not{g}^{AB})\partial_A U \partial_B U\end{aligned} \tag{66}$$

Suppose γ is the integral curve of $\widetilde{\mathcal{V}}$ that goes from $(u_1, v_1, x_1) \in \mathcal{D}$, the point we want to estimate ∂U, to (u_2, T, x_2) and let s be the affine parameter of γ, $\mathcal{V}s = 1$, that is equal to s_1 and T at the initial and terminal points respectively. By (55), we have

$$\widetilde{\mathcal{V}} r = 1 + o(1).$$

As a consequence we bound the integrals of r^{-k} along γ:

$$\int_\gamma r^{-k} = \int_{s_1}^{T} r^{-k} ds \le \int_{r_1}^{r_2} 2r^{-k} dr < \frac{2}{k-1}(r_1)^{-k+1}$$

where $r_1 = r(u_1, v_1)$ is the area radius at the initial point.

By the assumptions on the metric coefficients and the bootstrap assumption, each term on the right-hand side of (65) is bounded by $\frac{C_1}{r^2}$ as long as $r_2(u)$ is sufficiently large depending on $C_1, \mathfrak{r}(u, \varepsilon)$. Hence $|\widetilde{\mathcal{V}}(\partial_u U)| \le \frac{7C_1}{r^2}$. Therefore $|\partial_u U|(u_1, v_1, x_1) \le \frac{14C_1}{r_1}$ by integration. Similarly, we obtain the estimate of $|\nabla U|(u_1, v_1, x_1) \le \frac{14C_1}{r_1}$ (here we choose the normal coordinates centered at x_1 on S^2; therefore the partial derivative ∂_C is equivalent to covariant derivative and the projection of γ on S^2 lies entirely in this coordinate neighborhood since $\widetilde{\mathcal{V}} = \partial_v + O(r^{-2})$). Finally estimate $|\partial_v U|(u_1, v_1, x_1) \leq \frac{2C_1}{(r_1)^2}$ from the eikonal equation (59) and the estimate of $\mathcal{Z}U$. We thus justify the bootstrap assumption and obtain the estimate in $\mathcal{D}$ with $r_2(u)$ now independent of T.

The pointwise estimate (62) is obtained by integrating the transport equation (60) and using (63).

The estimate of second derivatives proceeds identically. First note that $\mathcal{Z}^2 U_T = 0$ on $\{v = T\}$ by definition. Estimate of $\mathcal{Z}^2 U$ follows from their transport equations and a bootstrap argument. Estimate of $\partial_v \mathcal{Z} U$ then follows from (65) and (66) algebraically. □

Proof of Theorem A.2. By (55), there exists a function $r_3(u) \geq r_1(u)$ so that

$$0.9r \leq v \leq 1.1r \tag{67}$$

on the region $\{u_* \leq u \leq u^*, r \geq r_3(u)\}$. Let $\mathcal{R}$ be a compact set of this region. We would like to show that U_T converges in $C^1(\mathcal{R})$ to U. Then $u - f(x) + U$ is the desired optical function.

We start with the C^0 convergence. Let $T_1 < T_2$ and write $U_1 = U_{T_1}$ and $U_2 = U_{T_2}$. $U_2 - U_1$ satisfies the transport equation

$$\mathcal{V}_0(U_2 - U_1) = 0 \tag{68}$$

with

$$\begin{aligned}\mathcal{V}_0 = \partial_v + \partial_u U_2 \partial_v + \partial_v U_1 \partial_u - b^A \partial_A f \partial_u + b^A \partial_A + b^A \partial_u U_2 \partial_A + b^A \partial_A U_1 \partial_u \\ + \Omega^2 (2 \not{g}^{AB} \partial_B f \partial_A - \not{g}^{AB} \partial_A U_2 \partial_B - \not{g}^{AB} \partial_A U_1 \partial_B).\end{aligned}$$

By (62) and (67), $|U_2 - U_1| \leq \frac{C_2}{T_1}$ on $\{v = T_1\}$ and hence by (68) $|U_2 - U_1| \leq \frac{C_2}{T_1}$ on $\mathcal{R}$. This proves that $\{U_T\}$ forms a Cauchy sequence in $C^0(\mathcal{R})$.

Next we show the convergence in $C^1(\mathcal{R})$. Let $T \leq T_1 < T_2$. Subtracting the transport equations of $\partial_u U_2$ and $\partial_u U_1$ in (65), we obtain the transport equation of $\partial_u(U_2 - U_1)$:

$$\begin{aligned}&\left(\mathcal{V}_2 + \partial_v U_2 \partial_u + b^A \partial_u U_2 \partial_A - \Omega^2 \not{g}^{AB} \partial_B U_2 \partial_A\right)(\partial_u(U_2 - U_1)) \\ &+ (\mathcal{V}_2 - \mathcal{V}_1)\partial_u U_1 + \underline{\partial_v(U_2 - U_1)\partial_u^2 U_1} \\ &+ b^A \partial_u(U_2 - U_1)\partial_A \partial_u U_1 - \Omega^2 \not{g}^{AB} \partial_B(U_2 - U_1)\partial_A \partial_u U_1 \\ &= \partial_u b^A \partial_A(U_2 - U_1) - \partial_u b^A \partial_A f \partial_u(U_2 - U_1) \\ &\quad + \partial_u b^A \left(\partial_A(U_2 - U_1)\partial_u U_2 + \partial_A U_1 \partial_u(U_2 - U_1)\right) \\ &\quad + 2\partial_u(\Omega^2 \not{g}^{AB})\partial_A f \partial_B(U_2 - U_1)\end{aligned}$$

$$- \partial_u(\Omega^2 \not{g}^{AB}) \left(\partial_A(U_2 - U_1)\partial_B U_2 + \partial_A U_1 \partial_B(U_2 - U_1) \right).$$

Here $\mathcal{V}_2$ (resp. $\mathcal{V}_1$) is the vector field $\mathcal{V}$ with U replaced by U_2 (resp. U_1). We denote the vector field in the first line by $\mathcal{V}_2'$ and compute $\mathcal{V}_2 - \mathcal{V}_1 = \underline{\partial_u(U_2 - U_1)\partial_v} + b^A\partial_A(U_2 - U_1)\partial_u - \Omega^2\not{g}^{AB}\partial_A(U_2 - U_1)\partial_B$.

By (63), we have $|\partial_u(U_2-U_1)|, |\nabla(U_2-U_1)| \leq \frac{2C_2}{r}$ and $|\partial_v(U_2-U_1)| \leq \frac{2C_2}{r^2}$ on $\{v = T\}$ which, together with (67), enables us to make the bootstrap assumption

$$|\partial_u(U_2 - U_1)|, |\nabla(U_2 - U_1)| \lesssim \frac{1}{T}$$
$$|\partial_v(U_2 - U_1)| \lesssim \frac{1}{Tr}$$

in the region $\{r \geq r_3(u), v \leq T_1, u_* \leq u \leq u^*\}$ for some $r_3(u)$ presumably depending on T_1, T_2.

By the bootstrap assumption, the assumption on metric coefficients, and (64), we see that $|\mathcal{V}_2' \left(\partial_u(U_2 - U_1)\right)| \lesssim \frac{1}{Tr^2}$ if $r_3(u)$ is sufficiently large depending on $C_1, \mathfrak{r}(u, \varepsilon)$. Here we used the fact that v-derivatives (underlined terms) decay faster by a factor of $\frac{1}{r}$ in contrast with $\mathcal{Z}$-derivatives. Integration yields

$$|\partial_u(U_2 - U_1)| \lesssim \frac{1}{Tr}$$

and we get an improved bound if r is sufficiently large. Similarly, we obtain an improved bound $|\nabla(U_2 - U_1)| \lesssim \frac{1}{Tr}$ by its transport equation and finally an improved bound $|\partial_v(U_2 - U_1)| \lesssim \frac{1}{Tr^2}$ from (68). This justifies the bootstrap assumption and we conclude that $\{U_T\}$ is a Cauchy sequence in $C^1(\mathcal{R})$. ☐

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