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Null Penrose inequality in a perturbed Schwarzschild spacetime

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Abstract: In this paper, we review the proof of the null Penrose inequality in a perturbed Schwarzschild spacetime. The null Penrose inequality conjectures that, on an incoming null hypersurface, the Hawking mass of the outmost marginally trapped surface is not greater than the Bondi mass at past null infinity. An approach to prove the null Penrose inequality is to construct a foliation on the null hypersurface starting from the marginally trapped surface to past null infinity, on which the Hawking mass is monotonically nondecreasing. However to achieve a proof, there arises an obstacle on the asymptotic geometry of the foliation at past null infinity. In order to overcome this obstacle, Christodoulou and Sauter proposed a strategy by varying the hypersurface to search for another null hypersurface where asymptotic geometry of the foliation becomes round. This strategy leads us to study the perturbation of null hypersurfaces systematically. Applying the perturbation theory of null hypersurfaces in a perturbed Schwarzschild spacetime, we carry out the strategy of Christodoulou and Sauter successfully. We find a one-parameter family of null hypersurfaces on which the null Penrose inequality holds. This paper gives a overview of our proof.

Keywords: Black hole, Schwarzschild spacetime, Penrose inequality, Null hypersurface.

1. A brief review of the Penrose inequality

1.1. Schwarzschild black hole spacetime

Soon after Einstein's discovery of his field equations [E1] [E2], Schwarzschild found the spherically symmetric static vacuum solution of the Einstein equations in 1916 [Sc]. The Schwarzschild black hole spacetime is named after him.

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Its metric reads as follows:

$$\mathrm{d}s^2 = -(1-2m/r)\mathrm{d}t^2 + (1-2m/r)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2).$$

The parameter m in the metric has the physical meaning of the mass of the black hole. When $m = 0$, the metric becomes the flat Minkowski metric. At first sight, $r = 0$ and $r = 2m$ look like values for which the metric is singular. It was realised by Lemaître 1933 [Le] that only $r = 0$ is a true singularity but $r = 2m$ is merely a coordinate singularity, which can be removed by coordinate transformations. Such a coordinate transformation was already found by Eddington 1924 [Ed] but without realising the removal of $r = 2m$ coordinate singularity, and then rediscovered by Finkelstein 1958 [Fi]. In the Eddington-Finkelstein coordinates $\{v, r, \theta, \phi\}$, the Schwarzschild metric takes the form

$$\mathrm{d}s^2 = -(1-2m/r)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2).$$

Synge 1950 [Sy], Kruskal 1960 [Kr] and Szekeres 1960 [Sz] provided coordinate systems that cover the maximal analytic extension of the Schwarzschild metric. In the Kruskal-Szekeres coordinate system $\{u, v, \theta, \phi\}$, the Schwarzschild metric is written as

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{16m^2}{r}\exp(\frac{-r}{2m})\mathrm{d}u\mathrm{d}v + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \\ uv &= -(r-2m)\exp\frac{r}{2m}. \end{aligned}$$

Figure 1 visualises the Schwarzschild spacetime in the Kruskal-Szekeres coordinate system. In the figure, each point represents a sphere of radius r. The shadowed region is the interior of the black hole, consisting of spheres of radius $r \leq 2m$. The boundary of the shadowed region is the event horizon. The sphere on the event horizon has the radius $r = 2m$.

1.2. Concepts of null infinity and a closed trapped surface

Before the above complete figure of the Schwarzschild spacetime was obtained, there were important works done in the meantime. In 1939 [OS], Oppenheimer and Snyder studied the gravitational collapse of a dust ball with uniform density, based on the previous work of Tolman 1934 [T]. This is the first work on relativistic gravitational collapse, and provides the intuition leading to the concept of a *future event horizon*, and the concept of a *black hole*.

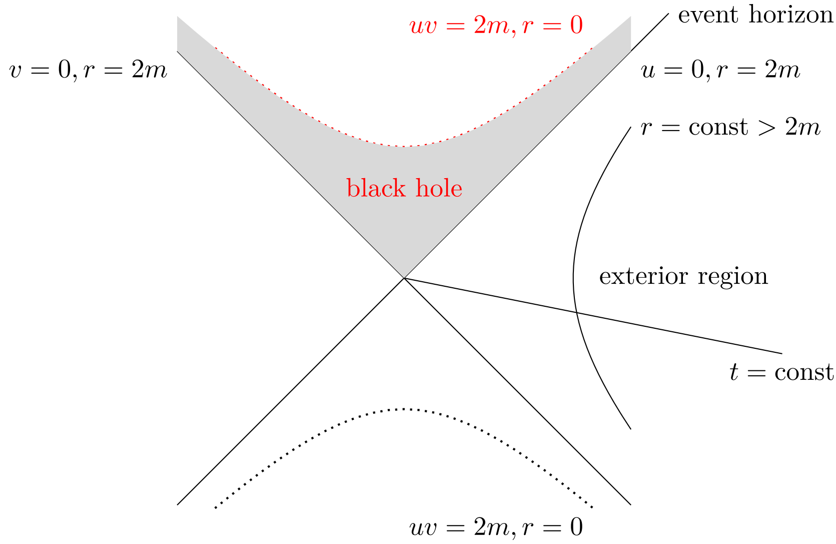
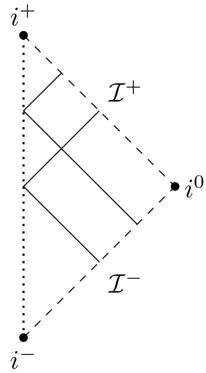


Figure 1: Schwarzschild spacetime in the Kruskal-Szekeres coordinate system.

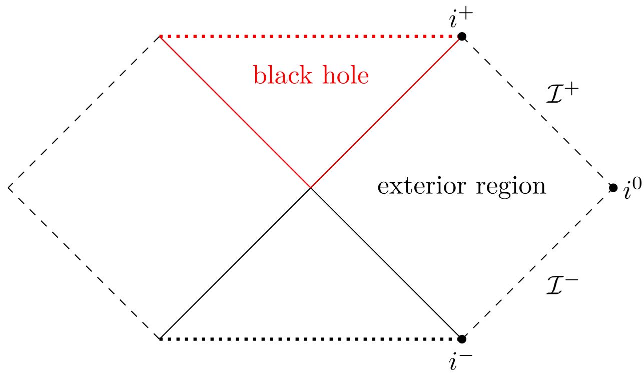
In 1964 [P1], Penrose introduced the idea of *conformal compactification* and the *Penrose diagram*, to define the concept of *null infinity*. With this concept, he could mathematically define the black hole region to be the complement of the past of the future null infinity, and the boundary of the black hole region as the future event horizon. Figure 2 shows the conformal compactifications of the Minkowski spacetime and the Schwarzschild spacetime.

Examining the derivative of the radius r in the Schwarzschild spacetime, one can easily find that, in the exterior region, r increases in the outgoing future null direction, but decreases in the incoming future null direction. While in the black hole region, r decreases in both future null directions. Such spheres in the black hole region are called *trapped*. On the event horizon, r is a constant along the tangential null direction, but decreases along the transversal future null direction. We call these spheres on the event horizon *marginally trapped*.

In 1965 [P2], Penrose introduced the general concept of a *closed trapped surface* by requiring the area element decreases along both future null normal directions. Let Σ be a 2-dim closed spacelike surface embedded in a 4-dimensional time-oriented Lorentzian manifold (M, g). At each point, choose a conjugate null frame $\{L, \underline{L}\}$ in the normal bundle of Σ, where $L, \underline{L}$ are both future null vector and satisfy the conjugate condition $g(L, \underline{L}) = -2$. Then for each future null direction, one can define the corresponding second



(a) The Minkowski spacetime



(b) The Schwarzschild spacetime

Figure 2: The Penrose diagram.

fundamental form: let $\{e_1, e_2\}$ be a unit orthogonal tangential frame on Σ,

$$\chi_{AB} = g(\nabla_{e_A} L, e_B), \quad \underline{\chi}_{AB} = g(\nabla_{e_A} \underline{L}, e_B).$$

The mean curvatures in two future null directions are the traces of χ and $\underline{\chi}$ in the tangent space of Σ, denoted by $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$,

$$\mathrm{tr}\chi = \chi_{11} + \chi_{22}, \quad \mathrm{tr}\underline{\chi} = \underline{\chi}_{11} + \underline{\chi}_{22}.$$

$\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$ are called *future null expansions* of Σ. They measure the rate of change of the area element of Σ in future null directions. The definition of a closed trapped surface introduced by Penrose is as follows.

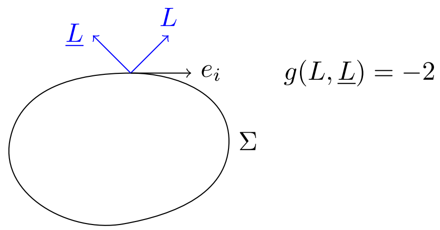


Figure 3: Conjugate null frame $\{L, \underline{L}\}$ on Σ.

Definition 1.1. *A closed 2-dimensional spacelike surface Σ in a 4-dimensional time-oriented spacetime is called trapped if both future null expansions on Σ are everywhere negative. Σ is called marginally trapped if one of its future null expansion vanishes everywhere and the other one is everywhere negative.*

Based on the concepts of null infinity and a closed trapped surface, Penrose proved his celebrated incompleteness theorem: *If a spacetime $(\mathcal{M}, g)$ satisfies the following three assertions:*

i. *the null convergence condition:* $\mathrm{Ric}(N, N) \geq 0$ *for all null vectors N;*
ii. *there is a non-compact Cauchy hypersurface in $\mathcal{M}$;*
iii. *there is a closed trapped surface in $\mathcal{M}$;*

then (M, g) cannot be future null geodesically complete. Roughly speaking, the theorem tells us that under reasonable physical assumptions, assuming the future null infinity is complete, the existence of a closed trapped surface implies the existence of a future event horizon, of which the future interior contains the trapped surface. The future interior region of the future event horizon, the region unable to communicate with future null infinity, is given the term black hole (introduce by Wheeler in 1967, see [Wh]).

1.3. Bondi mass, irreducible mass and event horizon area

Einstein studied approximated integration of the field equations of gravitation in 1916 [E3], and predicted the existence of gravitational waves which can transport energy.

In 1962 [BBM], Bondi et al. considered the gravitational waves in axi-symmetric solutions of Einstein's equations. For such solutions, Bondi defined the *Bondi mass aspect function* $M(u, \theta)$ and the *Bondi news function* $\partial_u c(u, \theta)$ at future null infinity, where θ is the angle variable in the spherical coordinates

(θ, ϕ), and ϕ is the angle around the axis of symmetry. The integration of the Bondi mass aspect function gives the *Bondi mass* at future null infinity

$$m(u) = \frac{1}{2} \int_0^\pi M(u, \theta) \sin \theta \mathrm{d}\theta.$$

Moreover it is showed that the Bondi mass satisfies the *Bondi mass loss formula*:

$$\frac{\mathrm{d}m}{\mathrm{d}u}(u) = -\frac{1}{2} \int_0^\pi |\partial_u c|^2(u, \theta) \sin \theta \mathrm{d}\theta \leq 0.$$

By this formula, Bondi concluded *"The mass of a system is constant if and only if there is no news. If there is news, the mass decreases monotonically as long as the news continues."*

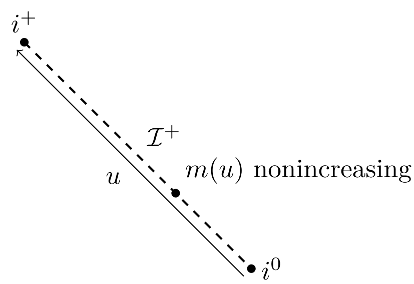


Figure 4: Bondi mass at future null infinity.

The Bondi mass $m(u)$ measures the total mass of the spacetime at a retarded time u at future null infinity, while what is the mass of a black hole inside the spacetime? In 1970 [C1], Christodoulou introduced the *irreducible mass* m_{ir} and the concepts of *reversible and irreversible transformations* for black holes. The irreducible mass m_{ir} of a stationary black hole of total mass m and angular momentum L is given as follows:

$$m^2 = m_{ir}^2 + \frac{L^2}{4m_{ir}^2}.$$

His most significant results by these concepts are that *"an irreversible transformation is characterised by an increase in the irreducible mass of the black hole and there exists no process which will decreased the irreducible mass".*

A decisive development arrived in 1971 [H2], where Hawking proved the Hawking's area theorem that the event horizon area cannot decrease in any process. He showed that even when two black holes capture each other to form a single black hole, the area of the resulting black hole cannot be smaller than the sum of two initial areas. For a stationary black hole, its irreducible mass m_{ir} and event horizon area A_{eh} are related by the simple identity

$$A_{eh} = 16\pi m_{ir}^2.$$

The results of Christodoulou [C1] and of Hawking [H2] motived Bekenstein to choose the area of a black hole as a measure of its entropy in 1972 [Be], leading to the celebrated black hole thermodynamics.

1.4. Penrose inequality, heuristic arguments and Riemannian Penrose inequality

Based on the Bondi mass loss formula, Hawking's area theorem and other reasonable physical assumptions, Penrose heuristically derived an inequality relating the mass of black holes and the mass of the spacetime in 1973 [P3]. Figure 5 illustrates the circumstance where Penrose derived his inequality. The heuristic argument follows a chain of inequalities as follows: assume that

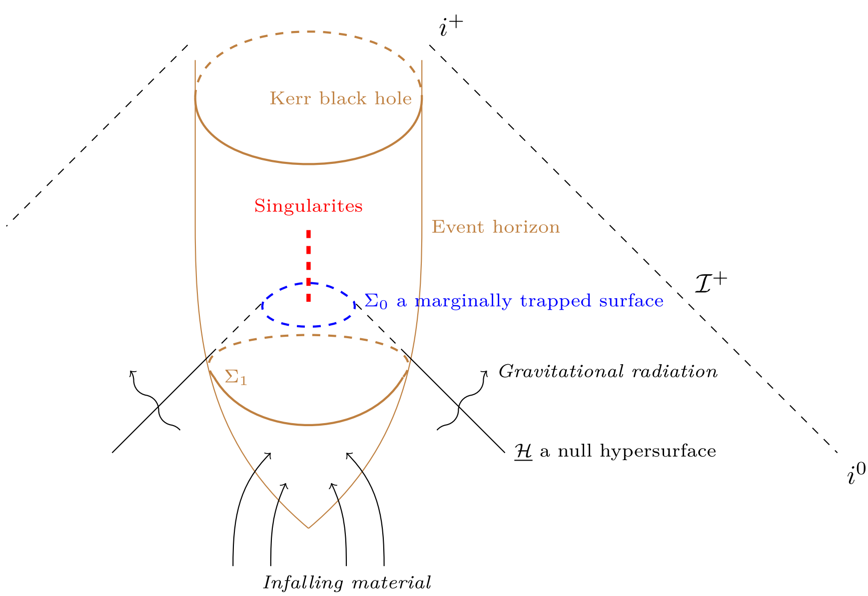


Figure 5: Gravitational collapse to a black hole.

no naked singularity exists, then

$$|\Sigma\underbrace{{}_0| \leq |}_{\text{i.}}\underbrace{\Sigma_1| \leq A_{eh}}_{\text{ii.}}\underbrace{(i^+) = 16\pi\left[m_{ir}(i^+)\right]^2 \leq 16\pi\left[m(i^+)\right]^2}_{\text{iii.}}$$
$$\underbrace{\leq 16\pi}_{\text{iv.}}\underbrace{\left[m(i^0)\right]^2 = 16\pi}_{\text{v.}} m_{ADM}^2.$$

i. $\mathrm{tr}\underline{\chi} < 0$ between Σ_0 and Σ_1;
ii. Hawking's area theorem;
iii. the black hole eventually settles down to a Kerr black hole at future timelike infinity;
iv. Bondi mass loss formula;
v. Bondi mass converges to the ADM mass at spatial infinity.

The inequality between the farmost two terms in the above chain is called the Penrose inequality

$$|\Sigma_0| \leq 16\pi m_{ADM}^2.$$

A breakthrough in the study of the Penrose inequality is the confirmation of an important case on a time-symmetric spacelike hypersurface, which is called the Riemannian Penrose inequality. The proof is given by two groups of mathematicians, Huisken and Ilmanen 2001 [HI], and Bray 2001 [Br], with different methods. Both proofs are built on the magnificent toolkit of geometric analysis, which was first applied successfully to general relativity in the monumental proof of the positive mass theorem by Schoen and Yau [SY1], [SY2] and later by Witten [Wi] in a different way.

Huisken and Ilmanen adopted the *inverse mean curvature flow* in their proof. In 1973 [G], Geroch discovered that the *Hawking mass* along an inverse mean curvature flow is monotone nondecreasing provided that the scalar curvature of the manifold is nonnegative. This idea was later extended by Jang and Wald 1977 [JW] to prove the Riemannian Penrose inequality heuristically. However since the inverse mean curvature flow generally develops singularities, this approach didn't succeed until Huisken and Ilmanen [HI] introduced the *weak inverse mean curvature flow*, established its global existence and confirmed the *Geroch monotonicity formula of Hawking mass* for the weak inverse mean curvature flow. Their theorem of the Riemannian Penrose inequality states as follows.

Theorem 1.2 (Huisken and Ilmanen 2001 [HI]). *Let* $(\mathcal{M}, g)$ *be a complete, connected 3-manifold. Suppose that:*

(i) $(\mathcal{M}, g)$ *has nonnegative scalar curvature.*
(ii) $(\mathcal{M}, g)$ *is asymptotically flat, with* ADM *mass* m.
(iii) The boundary of $\mathcal{M}$ *is compact and consists of minimal surfaces, and* $(\mathcal{M}, g)$ *contains no other compact minimal surfaces.*

Then

$$m \geq \sqrt{\frac{|N|}{16\pi}},$$

where $|N|$ *is the area of any connected component of* $\partial\mathcal{M}$. *Equality holds if and only if* $\mathcal{M}$ *is isometric to one-half of the spatial Schwarzschild manifold.*

Bray improved Huisken and Ilmanen's result by generalising the area of any outermost minimal surface to the total area of all outermost minimal surfaces.

Theorem 1.3 (Bray 2001 [Br]). *Let* $(\mathcal{M}, g)$ *be a complete, smooth, asymptotically flat* 3*-manifold with nonnegative scalar curvature and total mass* m *whose outermost minimal spheres have total surface area* A. *Then*

$$m \geq \sqrt{\frac{A}{16\pi}},$$

with equality if and only if $(\mathcal{M}, g)$ *is isometric to a Schwarzschild manifold outside their respective outermost horizons.*

Instead of using the inverse mean curvature flow in the fixed Riemannian manifold, Bray constructed a flow of the metric g_t by *conformal deformation*. He showed that the total area of all outermost minimal spheres remains constant, while the total mass of the manifold is non-increasing. As t goes to ∞, the manifold (M, g_t) approaches a spatial Schwarzschild manifold.

2. Null Penrose inequality

2.1. General null Penrose inequality

Analogue to the Riemannian Penrose inequality on a spacelike hypersurface, the null Penrose inequality is a case of the Penrose inequality on a null hypersurface. Figure 6 illustrates the circumstance of the null Penrose inequality. Instead of comparing the area of Σ_0 with the ADM mass at spatial infinity

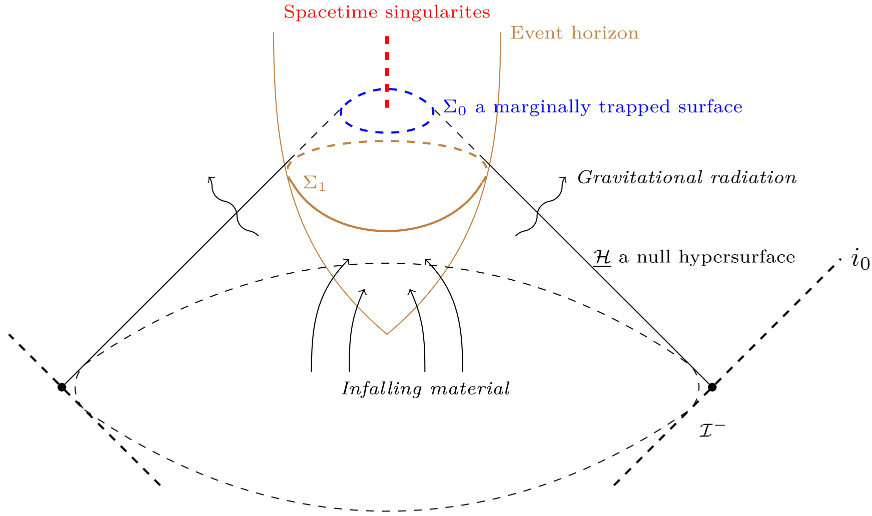


Figure 6: The null Penrose inequality.

i_0, the null Penrose inequality concerns the Bondi mass measured on $\underline{\mathcal{H}}$ at past null infinity. It conjectures that on a null hypersurface $\underline{\mathcal{H}}$, the area of the marginally trapped surface Σ_0 is bounded by the Bondi mass as follows

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} \leq m_{Bondi}(\underline{\mathcal{H}}).$$

We present here a heuristic argument for the null Penrose inequality from the Penrose inequality. At past null infinity $\mathcal{I}^-$, by the Bondi mass loss formula, the Bondi mass is monotone non-decreasing, and remains constant if there is no incoming gravitational radiation. The non-decreasing monotonicity of the Bondi mass at past null infinity implies that the Penrose inequality follows from the null Penrose inequality, since

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} \leq \underbrace{m_{Bondi}(\underline{\mathcal{H}}) \leq m(i_0)}_{\text{Bondi mass loss formula}} = m_{ADM}.$$

It seems that the null Penrose inequality is stronger than the Penrose inequality from the above argument, however we note that if there is no incoming gravitational wave after $\underline{\mathcal{H}}$, then the Bondi mass $m_{Bondi}(\underline{\mathcal{H}})$ should be equal

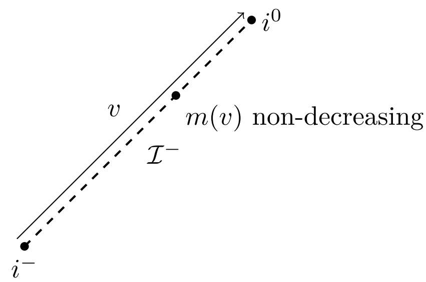


Figure 7: Bondi mass at past null infinity.

to $m(i_0)$, thus the Penrose inequality will imply the null Penrose inequality,

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} \leq m_{ADM} = \underbrace{m(i_0) = m_{Bondi}(\underline{\mathcal{H}})}_{\substack{\text{no incoming} \\ \text{gravitational radiation}}}.$$

Thus if assuming the existence of such a spacetime with no incoming gravitational radiation after $\underline{\mathcal{H}}$, the null Penrose inequality follows from the Penrose inequality.

2.2. Constant mass aspect function foliation

A natural approach to prove the null Penrose inequality is to construct a foliation starting from the marginally trapped surface Σ_0 and approaching past null infinity, along which the Hawking mass is monotonically non-decreasing.

In 1968 [H1], where Hawking introduced the Hawking mass, he already realised that under certain physical assumptions, by choosing the foliation parameter as a "luminosity parameter", the Hawking mass will be monotonically non-decreasing. As showed in figure 8, $\{\Sigma_v\}$ is a foliation on $\underline{\mathcal{H}}$ parameterised by v, then one can define the associated conjugate null frame along the foliation by

$$\underline{L}v = 1, \quad g(L, \underline{L}) = 2.$$

The foliation parameterised by a "luminosity parameter" proposed by Hawking is equivalent to the constant null expansion condition, i.e. the null expansion in the direction of $\underline{L}$ is a constant function on each Σ_v,

$$\mathrm{tr}\underline{\chi}|_{\Sigma_v} \equiv \text{constant}.$$

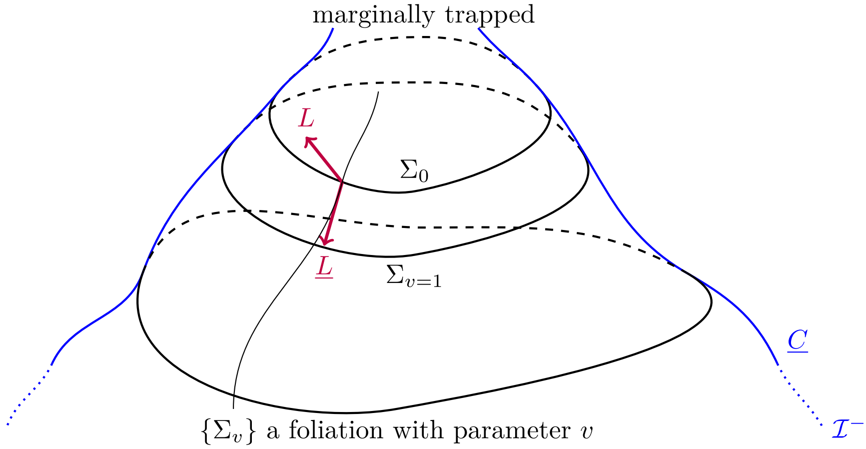


Figure 8: A foliation from a marginally trapped surface Σ_0 to past null infinity $\mathcal{I}^-$ on a null hypersurface $\underline{\mathcal{H}}$.

In 2008 [S], Sauter constructed two kinds of foliations on a class of nearly spherically symmetric null hypsurfaces, along which the Hawking mass is monotonically non-decreasing towards past null infinity:

i. *Constant null expansion foliation*, where the null expansion $\mathrm{tr}\underline{\chi}$ in the direction of $\underline{L}$ is constant on each Σ_v. As we mentioned before, this is the foliation parameterised by a "luminosity parameter" proposed by Hawking;

ii. *Constant mass aspect function foliation*, where the mass aspect function μ is constant on each Σ_v. The mass aspect function μ for the foliation $\{\Sigma_v\}$ is defined by

$$\mu = K - \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \mathrm{d}\!\!/\mathrm{iv}\,\eta,$$

where K is the Gauss curvature of Σ_v, $\mathrm{tr}\chi$ is the null expansion of Σ_v in the direction of L, η is the torsion 1-from of the conjugate null frame $\{L, \underline{L}\}$ on Σ_v defined by

$$\eta(X) = \frac{1}{2}g(\nabla_X \underline{L}, L), \quad \forall X \in T\Sigma_v,$$

and $\mathrm{d}\!\!/\mathrm{iv}$ is the intrinsic divergence operator on the surface $(\Sigma_v, g\!\!/_v = g|_{\Sigma_v})$. This foliation was proposed by Christodoulou, see 2008 [C3] [S].

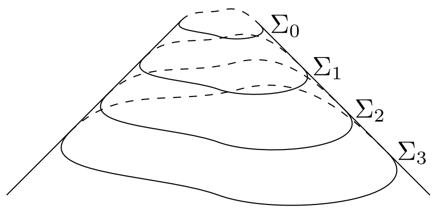


Figure 9: The foliation is *not asymptotically round* at $\mathcal{I}^-$. $\lim_{v\to+\infty} m_H(\Sigma_v)$ has no clear physical meaning.

The mass aspect function was first introduced by Christodoulou in 1991 [C2], and applied in the monumental proof of the global nonlinear stability of Minkowski spacetime by Christodoulou and Klainerman in 1993 [CK]. An analogue of the constant mass aspect function foliation on spacelike hypersurface was introduced in [CK], playing an important role in the proof. A simple calculation shows the relation between the mass aspect function and the Hawking mass: by the Gauss-Bonnet theorem and the Stokes theorem,

$$\int_{\Sigma_v} \mu \mathrm{dvol}_{\not{g}_v} = \frac{8\pi}{r_v} m_H(\Sigma_v),$$

where r_v is the area radius of Σ_v, i.e. $|\Sigma_v| = 4\pi r_v^2$.

2.3. Obstacle towards proving null Penrose inequality by foliations

By Sauter's construction in [S], he obtained that along each of the two above foliations, the Hawking mass is monotonically non-decreasing, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}s} m_H(\Sigma_v) \geq 0 \Rightarrow \sqrt{\frac{|\Sigma_0|}{16\pi}} = m_H(\Sigma_0) \leq \lim_{v\to+\infty} m_H(\Sigma_v) \overset{??}{\longleftrightarrow} m_{Bondi}(\underline{\mathcal{H}}).$$

If the limit Hawking mass $\lim_{v\to+\infty} m_H(\Sigma_v)$ was equal to the Bondi mass on $\underline{\mathcal{H}}$ at past null infinity, then this would prove the null Penrose inequality. However contrary to the weak inverse mean curvature flow on a spacelike hypersurface, where the Hawking mass converges to the ADM mass at spatial infinity as proved in [HI], there is no definite relation between $\lim_{v\to+\infty} m_H(\Sigma_v)$ and $m_{Bondi}(\underline{\mathcal{H}})$. The obstacle comes from the asymptotic geometry of the foliation at past null infinity.

Figures 9 and 10 explain the main issue of the obstacle. In general, the foliations constructed by Sauter are not asymptotically round at past null

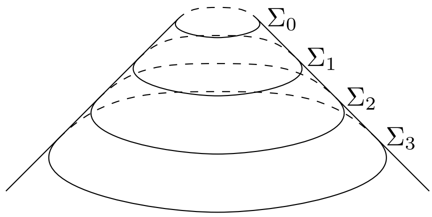


Figure 10: The foliation is *asymptotically round* at $\mathcal{I}^-$, then it defines an *asymptotic reference frame.* $\lim_{v\to+\infty} m_H(\Sigma_v)$ is the *Bondi energy* measured w.r.t. this frame on $\underline{\mathcal{H}}$ at $\mathcal{I}^-$.

infinity, thus the Hawking mass does not converges to the Bondi energy measured w.r.t. some asymptotic reference frame on $\underline{\mathcal{H}}$, let alone the Bondi mass on $\underline{\mathcal{H}}$.

2.4. Strategy to overcome the obstacle

The obstacle to prove the null Penrose inequality by foliations is the asymptotic geometry of the foliation at past null infinity, while there is no room to deform the foliation's asymptotic geometry on a fixed null hypersurface, since the foliation is determined by the marginally trapped surface Σ_0.

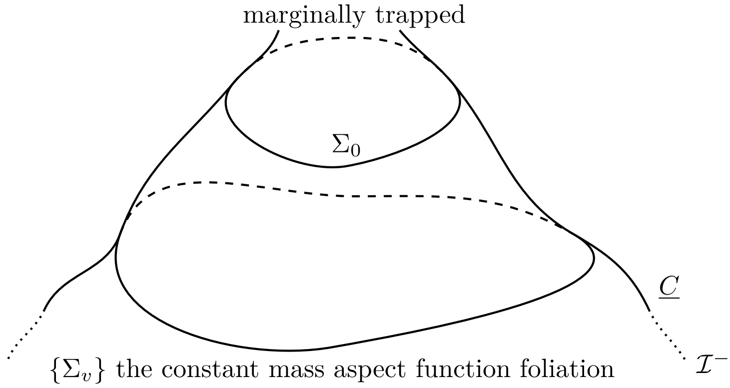


Figure 11: $\{\Sigma_v\}$ is not asymptotically round at $\mathcal{I}^-$.

Christodoulou and Sauter in [S] proposed the strategy to vary the whole null hypersurface inside the spacetime, and hope that one could find a nearby null hypersurface and an embedded marginally trapped surface $\bar{\Sigma}_0$, such that

$|\bar{\Sigma}_0| = |\Sigma_0|$ and the foliation starting from $\bar{\Sigma}_0$ is asymptotically round, hopefully even defines the center-of-mass reference frame at past null infinity. Figure 12 visualises this strategy. It is the approach to prove the null Penrose inequality adopted by us in [L2].

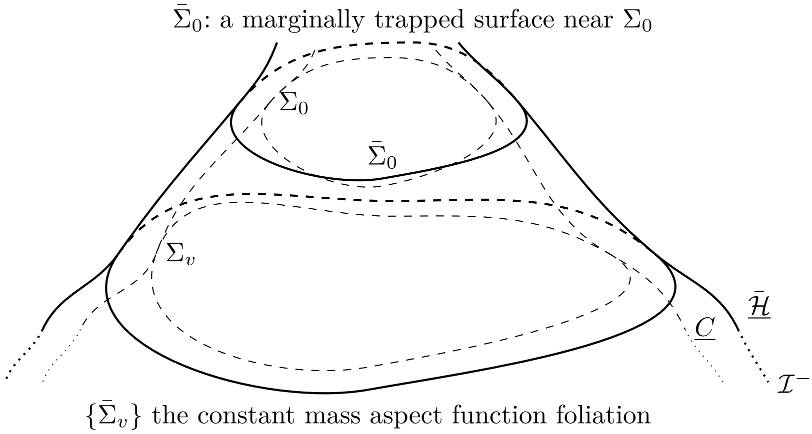


Figure 12: Strategy by varying the null hypersurface: $\{\bar{\Sigma}_v\}$ on a nearby null hypersurface $\underline{\bar{\mathcal{H}}}$ is asymptotically round at $\mathcal{I}^-$.

3. Null Penrose inequality in a perturbed Schwarzschild spacetime

3.1. Assumptions on a perturbed Schwarzschild spacetime

Recall the Schwarzschild metric in the Kruskal-Szekeres coordinates,

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{16m^2}{r}\exp(\frac{-r}{2m})\mathrm{d}u\mathrm{d}v + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \\ uv &= -(r-2m)\exp\frac{r}{2m}. \end{aligned}$$

As we mentioned before, the Kruskal-Szekeres coordinates has the advantage for covering the maximal analytic extension of the Schwarzschild spacetime. However, when considering the null Penrose inequality, we only concerns the spacetime along and near a null hypersurface. Therefore we introduce the following coordinate transformation $\{s, \underline{s}\} \to \{u, v\}$: let $r_0 = 2m$,

$$\begin{cases} u(s) = -\dfrac{s}{v_0} \exp \dfrac{s + r_0}{r_0}, \\ v(\underline{s}) = v_0 \exp \dfrac{\underline{s}}{r_0}. \end{cases}$$

Then the null hypersurface $\{v = v_0\}$ in the Kruskal-Szekeres coordinate system is the null hypersurface $\{\underline{s} = 0\}$ in the $\{s, \underline{s}\}$ coordinate system. In this new double null coordinate system $\{\underline{s}, s, \theta, \phi\}$, the Schwarzschild metric takes the form

$$\begin{aligned} g_{Sch} = &\frac{2(s + r_0)}{r} \exp \frac{s + \underline{s} + r_0 - r}{r_0} \left(\mathrm{d}s \otimes \mathrm{d}\underline{s} + \mathrm{d}\underline{s} \otimes \mathrm{d}s\right) \\ &+ r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \\ (r - r_0) \exp \frac{r}{r_0} &= s \exp \frac{s + \underline{s} + r_0}{r_0}. \end{aligned}$$

We shall consider a vacuum perturbed Schwarzschild metric in a neighbourhood of the null hypersurface $\underline{C}_{\underline{s}=0} = \{\underline{s} = 0, s \geq 0\}$.

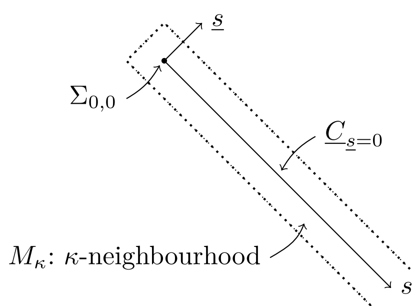


Figure 13: $M_\kappa = \{p : s(p) > -\kappa r_0, |\underline{s}| < \kappa r_0\}$.

A general vacuum perturbed Schwarzschild metric in M_κ takes the form

$$\begin{aligned} g = 4\Omega^2 \mathrm{d}s\mathrm{d}\underline{s} + \not{g}_{\theta\theta} \left(\mathrm{d}\theta - b^\theta \mathrm{d}s\right)^2 + 2\not{g}_{\theta\phi} \left(\mathrm{d}\theta - b^\theta \mathrm{d}s\right) \left(\mathrm{d}\phi - b^\phi \mathrm{d}s\right) \\ + \not{g}_{\phi\phi} \left(\mathrm{d}\phi - b^\phi \mathrm{d}s\right)^2. \end{aligned}$$

The coordinate system $\{s, \underline{s}, \theta, \phi\}$ inherited from the Schwarzschild spacetime is still a double null coordinate system for the perturbed metric. Let $\Sigma_{s,\underline{s}}$

denote the $(s, \underline{s})$-surface in the coordinate system. There is a natural choice of a null frame on $\Sigma_{s,\underline{s}}$ as follows

$$\underline{L} = \partial_s, \quad L = \partial_{\underline{s}} + b^\theta \partial_\theta + b^\phi \partial_\phi.$$

Associated with this null frame, one can define the corresponding structure coefficients, i.e. the corresponding second fundamental forms, the torsion and the accelerations: for $X, Y \in T\Sigma_{s,\underline{s}}$,

$$\begin{aligned}
&\underline{\chi}(X,Y) = g(\nabla_X \underline{L}, Y), \quad \chi(X,Y) = g(\nabla_X L, Y),\\
&\eta(X) = \frac{1}{2\Omega^2} g(\nabla_X \underline{L}, L),\\
&\omega = \frac{1}{4\Omega^2} g(\nabla_L L, \underline{L}), \quad \underline{\omega} = \frac{1}{4\Omega^2} g(\nabla_{\underline{L}} \underline{L}, L).
\end{aligned}$$

The curvature tensor can also be decomposed into components regarding to the null frame $\{L, \underline{L}\}$ as follows: let $\{e_1, e_2\}$ be a unit orthonormal frame on $\Sigma_{s,\underline{s}}$, and latin indices A, B denote $1, 2$,

$$\begin{aligned}
&\underline{\alpha}_{AB} = \mathrm{R}_{\underline{L}AB\underline{L}}, && \alpha_{AB} = \Omega^{-4}\mathrm{R}_{LABL},\\
&\underline{\beta}_A = \frac{1}{2\Omega^2}\mathrm{R}_{A\underline{L}L\underline{L}}, && \beta_A = \frac{1}{2\Omega^4}\mathrm{R}_{AL\underline{L}L},\\
&\rho = \frac{1}{4\Omega^4}\mathrm{R}_{L\underline{L}\underline{L}L}, && \sigma \not\in_{AB} = \frac{1}{2\Omega^2}\mathrm{R}_{AB\underline{L}L}.
\end{aligned}$$

Then the metric components $\Omega, b, \slashed{g}$, the structure coefficients $\underline{\chi}, \chi, \eta, \underline{\omega}, \omega$ and the curvature components $\underline{\alpha}, \alpha, \underline{\beta}, \beta, \rho, \sigma$ encode the geometric information of the vacuum perturbed Schwarzschild spacetime. A good reference for the above constructions can be found in [C4]. By comparing these quantities, we can obtain the quantitative description of the perturbation of the Schwarzschild metric.

The class of vacuum perturbed Schwarzschild metrics considered in the proof of the null Penrose inequality satisfies the following assumptions:

i. $\Sigma_{0,0}$ is marginally trapped;
ii. the differences of the geometric quantities between the perturbed metric and the Schwarzschild metric are small and have certain decaying rate in r at past null infinity. For examples, we list the assumptions on $\slashed{g}$, η, α: let $\mathring{g}$ denote the standard round metric $\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ on the sphere,

$$\left|\slashed{g} - \slashed{g}_S\right|_{\mathring{g}} < \epsilon r^2, \qquad \left|\partial_s^l \partial_{\underline{s}}^m \left(\slashed{g} - \slashed{g}_S\right)\right|_{\mathring{g}} < \frac{\epsilon}{r^{l-1} r_0^{m-1}},$$

$$|\eta|_{\overset{\circ}{g}} < \frac{\epsilon r_0}{r}, \qquad |\partial_s^l \partial_{\underline{s}}^m \eta|_{\overset{\circ}{g}} < \frac{\epsilon}{r^{1+l} r_0^{m-1}},$$
$$|\alpha|_{\overset{\circ}{g}} < \frac{\epsilon r}{r_0}, \qquad |\partial_s^l \partial_{\underline{s}}^m \alpha|_{\overset{\circ}{g}} < \frac{\epsilon}{r^{l-1} r_0^{m+1}}.$$

The assumptions on the decaying rate in r are motived from the results in the global nonlinear stability of Minkowski spacetime [CK]. In fact, these decaying assumptions are exactly the ones proved in [CK].

3.2. Main results on the null Penrose inequality

In 2018 [L2], we carry out the strategy proposed by Christodoulou and Sauter that, by varying the null hypersurface in a vacuum perturbed Schwarzschild spacetime, we found not only one single null hypersurface but a 4-parameter family of null hypersurfaces, on which the constant mass aspect function foliation starting from the marginally trapped surface is asymptotically round at past null infinity.

Theorem 3.1 ([L2])**.** *There is a 4-parameter family of marginally trapped surfaces near $\Sigma_{0,0}$, such that the constant mass aspect function foliation starting from any surface in this family is asymptotically round at past null infinity, hence defining a reference frame at past null infinity.*

As the corollary of the above theorem, combining the result of Sauter on the monotonicity of Hawking mass along the constant mass aspect function foliation, we proved the following weak null Penrose inequality, where the Bondi mass is replaced by the Bondi energy.

Theorem 3.2 ([L2])**.** *Let $\bar{\Sigma}$ be an arbitrary one in the 4-parameter family of marginally trapped surfaces, γ^∞ be the asymptotic reference frame defined by the constant mass aspect function foliation starting from $\bar{\Sigma}$, and $E_{Bondi}^{\gamma^\infty}\left(\underline{\bar{\mathcal{H}}}\right)$ be the Bondi energy measured w.r.t. the asymptotic reference frame γ^∞ on $\underline{\bar{\mathcal{H}}}$ at past null infinity, then we have*

$$m_H(\bar{\Sigma}) = \sqrt{\frac{|\bar{\Sigma}|}{16\pi}} \leq E_{Bondi}^{\gamma^\infty}\left(\underline{\bar{\mathcal{H}}}\right).$$

The above result is close to the null Penrose inequality, however it suffers a severe drawback: as long as the Bondi mass is non-zero, the Bondi energy could be made arbitrarily large by choosing suitable reference frames. However there is still hope that the asymptotic reference frame in theorem 3.2 could be an asymptotic center-of-mass frame, as the degree of freedom of marginally trapped surfaces found in theorem 3.2 is 4, which coincides with the dimension of energy-momentum vector. Thus it is possible to eliminating the 3-dimensional linear momentum vector to pick up an asymptotic center-of-mass reference frame by making use of the additional 4 parameters. We confirmed this intuition and proved the following result, improving the inequality from the Bondi energy to the Bondi mass.

Theorem 3.3 ([L2])**.** *There is a* 1*-parameter family of marginally trapped surfaces inside the 4-parameter family in theorem 3.2, such that the constant mass aspect function foliation starting from an arbitrary* $\bar{\Sigma}$ *in this* 1*-parameter family defines an asymptotic center-of-mass frame on* $\underline{\bar{\mathcal{H}}}$ *at past null infinity, hence the null Penrose inequality on* $\underline{\bar{\mathcal{H}}}$ *holds*

$$m_H(\bar{\Sigma}) = \sqrt{\frac{|\bar{\Sigma}|}{16\pi}} \leq m_{Bondi}\left(\underline{\bar{\mathcal{H}}}\right).$$

The above theorem almost concludes the goal of the strategy of Christodoulou and Sauter, with one question left: within the 1-parameter family of marginally trapped surfaces, does there exist one $\bar{\Sigma}$ having the same area as the original marginally trapped surface $\Sigma_{0,0}$? As the climax of [L2], we confirmed it positively and obtained a spacetime Penrose inequality for $\Sigma_{0,0}$.

Theorem 3.4 ([L2])**.** *There exists at least one* $\bar{\Sigma}$ *in the* 1*-parameter family of marginally trapped surfaces, such that* $|\bar{\Sigma}| = |\Sigma_{0,0}|$*. Then we have*

$$m_H(\Sigma_{0,0}) = \sqrt{\frac{|\Sigma_{0,0}|}{16\pi}} = \sqrt{\frac{|\bar{\Sigma}|}{16\pi}} \leq m_{Bondi}\left(\underline{\bar{\mathcal{H}}}\right).$$

Suppose that the vacuum perturbed Schwarzschild spacetime is embedded in an asymptotically flat vacuum spacetime with a complete past null infinity, then

$$m_H(\Sigma_{0,0}) \leq m_{Bondi}\left(\underline{\bar{\mathcal{H}}}\right) \leq m_{ADM}.$$

The last inequality follows from the Bondi mass loss formula at past null infinity. In theorem 3.4, it isnot answered whether that the null Penrose inequality holds on the original null hypersurface $\underline{C}$ where $\Sigma_{0,0}$ is located. However we can give a heuristic argument of the null Penrose inequality on the original null hypersurface $\underline{C}$ as follows: if there is no incoming gravitational radiation after $\underline{C}$ at past null infinity, then the Bondi mass loss formula implies that the Bondi mass is constant after $\underline{C}$, thus it implies the null Penrose inequality on the original null hypersurface $\underline{C}$, i.e.

$$m_H(\Sigma_{0,0}) \le m_{ADM} = m(\underline{C}).$$

Thus if assuming the existence of such a spacetime with no incoming gravitational radiation after $\underline{C}$, the null Penrose inequality on the original null hypersurface $\underline{C}$ follows form theorem 3.4.

4. Some discussions about the proof

4.1. Basic system of equations for constant mass aspect function foliation

A useful method to handle the asymptotic geometry of constant mass aspect function foliation at past null infinity is to investigate the equations of the geometric quantities along the foliation. For the constant mass aspect function foliation, we have a system of such equations, which is very useful for the study of the foliation's geometry.

The system consists of two groups of equations: one group of propagation equations, and the other group of elliptic equations. Suppose $\{\Sigma_v\}$ is a constant mass aspect function foliation. Firstly, since the mass aspect function is constant on each Σ_s, then

$$\mu = \overline{\mu}^{g} = \frac{\int_{\Sigma_v} \mu \mathrm{dvol}_{g}}{\int_{\Sigma_v} \mathrm{dvol}_{g}}.$$

Secondly, we choose the foliation being parameterised by area radius, i.e.

$$r_{\Sigma_v} = r_{\Sigma_0} + v \quad \Leftrightarrow \quad \overline{\mathrm{tr}\underline{\chi}}^{g} = \frac{2}{r_{\Sigma_v}}.$$

The group of propagation equations consists of the following equations:

$$\begin{aligned}
\underline{L}\mathrm{tr}\underline{\chi} &= 2\underline{\omega}\mathrm{tr}\underline{\chi} - |\hat{\underline{\chi}}|^2_{g} - \frac{1}{2}(\mathrm{tr}\underline{\chi})^2, \\
\underline{L}\mathrm{tr}\chi &= -2\underline{\omega}\mathrm{tr}\chi - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi - \left(\hat{\chi}, \hat{\underline{\chi}}\right)_{g} - 2\left(\mathrm{d}\!\!/\mathrm{iv}\,\eta + |\eta|^2_{g}\right) - 2\rho \\
&= -2\underline{\omega}\mathrm{tr}\chi - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi - 2|\eta|^2_{g} + 2\mu, \\
\underline{L}\overline{\mu}^{g} &= -\frac{3}{2}\overline{\mathrm{tr}\underline{\chi}}^{g}\overline{\mu}^{g} + \frac{1}{4}\overline{\mathrm{tr}\chi|\hat{\underline{\chi}}|^2_{g}}^{g} + \frac{1}{2}\overline{\mathrm{tr}\underline{\chi}|\eta|^2_{g}}^{g}.
\end{aligned}$$

The group of elliptic equations is the following:

$$
\begin{aligned}
2\not\Delta\underline{\omega} = &-\frac{3}{2}\mu(\mathrm{tr}\underline{\chi} - \overline{\mathrm{tr}\underline{\chi}}^{\not g}) + \frac{1}{2}(\mathrm{tr}\underline{\chi}|\eta|^2 - \overline{\mathrm{tr}\underline{\chi}|\eta|^2}^{\not g}) + \frac{1}{4}(\mathrm{tr}\chi|\hat{\underline{\chi}}|^2 - \overline{\mathrm{tr}\chi|\hat{\underline{\chi}}|^2}^{\not g}) \\
&+ 4\left(\not{\mathrm{div}}\,\hat{\underline{\chi}}, \eta\right)_{\not g} + 4\left(\hat{\underline{\chi}}, \not\nabla\eta\right)_{\not g} - 2\not{\mathrm{div}}\,\underline{\beta},
\end{aligned}
$$

$$
\not{\mathrm{div}}\,\hat{\underline{\chi}} - \frac{1}{2}\not d\,\mathrm{tr}\underline{\chi} - \hat{\underline{\chi}}\cdot\eta + \frac{1}{2}\mathrm{tr}\underline{\chi}\eta = -\underline{\beta},
$$

$$
\not{\mathrm{div}}\,\hat{\chi} - \frac{1}{2}\not d\,\mathrm{tr}\chi + \hat{\chi}\cdot\eta - \frac{1}{2}\mathrm{tr}\chi\eta = -\beta,
$$

$$
\begin{cases}
\not{\mathrm{div}}\,\eta = K - \frac{1}{4}\mathrm{tr}\underline{\chi}\,\mathrm{tr}\chi - \mu = -\rho - \frac{1}{2}\left(\hat{\chi}, \hat{\underline{\chi}}\right)_{\not g} - \mu, \\
\not{\mathrm{curl}}\,\eta = \frac{1}{2}\hat{\chi}\wedge\hat{\underline{\chi}} + \sigma.
\end{cases}
$$

Note that if the curvature components $\beta, \underline{\beta}, \rho, \sigma$ are known functions, then the above equations form a closed system for the structure coefficients $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}, \underline{\omega}, \hat{\chi}, \hat{\underline{\chi}}, \eta$ and the mass aspect function μ: if their values were known on the starting surface Σ_0, then in principle, we could obtain their values on each Σ_s by solving the system. Thus we call the above system the basic system of equations for constant mass aspect function foliation.

4.2. Perturbation of null hypersurfaces

In our proof of the null Penrose inequality, we have to perturb null hypersurfaces, therefore we must find a method to describe the perturbation of null hypersurfaces. The simplest way to achieve such a description is to use the double null coordinate system to parameterise null hypersurfaces and study the perturbation of the parameterisations.

In [L2], and later in [L5] with more relaxed assumptions of the spacetime, we describe the following method to parameterise a null hypersurface as a graph of $\underline{s}$ over the (s, ϑ) domain. As illustrated in figure 14, the null hypersurface $\underline{\mathcal{H}}$ is a graph $s = \underline{h}(s, \vartheta)$ over the (s, ϑ) domain. $\underline{h}$ is the parameterisation of the null hypersurface $\underline{\mathcal{H}}$. Σ_s is the intersection surface of $\underline{\mathcal{H}}$ with the outgoing null hypersurface C_s. Then introduce the function ${}^s\underline{f}$ by

$$
{}^s\underline{f}(\vartheta) = \underline{h}(s, \vartheta).
$$

Similar to the parameterisation of $\underline{\mathcal{H}}$, Σ_s can be parameterised as a graph of $(s, \underline{s} = {}^s\underline{f}(\vartheta))$ over the ϑ domain.

Since $\underline{\mathcal{H}}$ is null, geometrically it is easy to observe that $\underline{\mathcal{H}}$ is determined by $\Sigma_{s=0}$, the intersection of $\underline{\mathcal{H}}$ with $C_{s=0}$. This means that the parameterisation

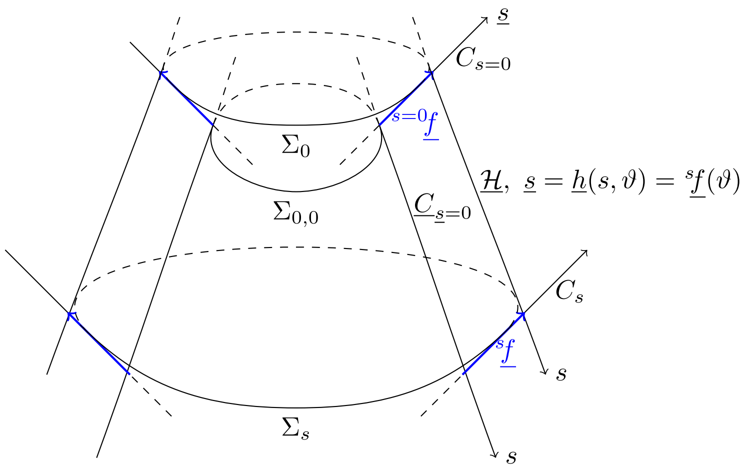


Figure 14: Null hypersurface $\underline{\mathcal{H}}$.

$\underline{h}$ of $\underline{\mathcal{H}}$ is solely determined by $^{s=0}\underline{f} = \underline{h}(s=0,\cdot)$. In fact, we can derive the following first order nonlinear partial differential equation satisfied by $\underline{h}$

$$\partial_s \underline{h} = -b^i \cdot \underline{h}_i + \Omega^2 \left(\not{g}^{-1}\right)^{ij} \cdot \underline{h}_i \cdot \underline{h}_j, \text{ where } \underline{h}_i = \partial \underline{h}/\partial \theta^i. \tag{4.1}$$

The converse of the above is also true that if $\underline{h}$ solves equation (4.1) then $\underline{\mathcal{H}}$ is null. Thus as long as the parameterisation $^{s=0}\underline{f}$ of $\Sigma_{s=0}$ is known, by solving equation (4.1), we can obtain the parameterisation $\underline{h}$ of the null hypersurface $\underline{\mathcal{H}}$.

With the above method to parameterise a null hypersurface, it is conceptually easy to describe the perturbation of null hypersurfaces. As showed in the following figure, suppose that $\underline{\mathcal{H}}_a, a=1,2$ are two null hypersurfaces with the parameterisation functions $^{a}\underline{h}$, then the perturbation from $\underline{\mathcal{H}}_1$ to $\underline{\mathcal{H}}_2$ can be described quantitatively by the difference between their parameterisation functions, i.e. if define

$$\mathfrak{d}\{\underline{h}\} = {}^{a=2}\underline{h} - {}^{a=1}\underline{h}, \quad \mathfrak{d}\{{}^{s}\underline{f}\} = {}^{a=2,s}\underline{f} - {}^{a=1,s}\underline{f},$$

then these functions tell us how much the null hypersurface $\underline{\mathcal{H}}_2$ deforms away from $\underline{\mathcal{H}}_1$.

4.3. Space of closed marginally trapped surfaces near $\Sigma_{0,0}$

Let $\underline{\mathcal{H}}$ be a null hypersurface near the original null hypersurface $\underline{C}_{\underline{s}=0}$. When considering the null Penrose inequality on $\underline{\mathcal{H}}$, we have to locate the marginally

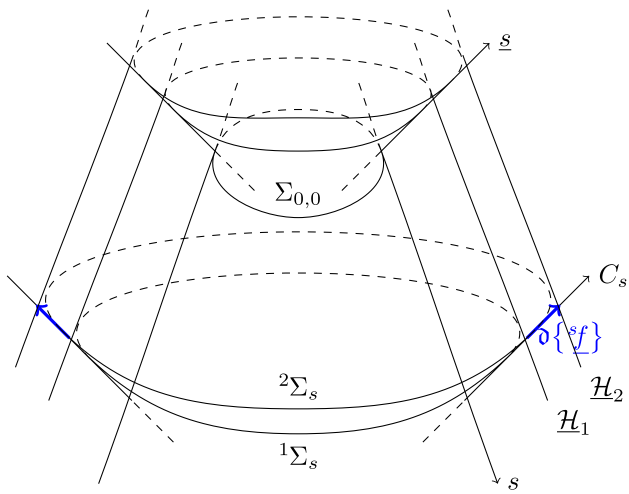


Figure 15: A perturbation of null hypersurfaces: $\underline{\mathcal{H}}_1$ and $\underline{\mathcal{H}}_2$.

trapped surface embedded in it. This naturally leads us to consider the space of all marginally trapped surfaces near $\Sigma_{0,0}$.

The first question that needs to be answered is that how we describe an arbitrary spacelike surface near $\Sigma_{0,0}$. In [L2] and also [L3], we adopt a two-step procedure to parameterise a spacelike surface, which is visualised in figure 16.

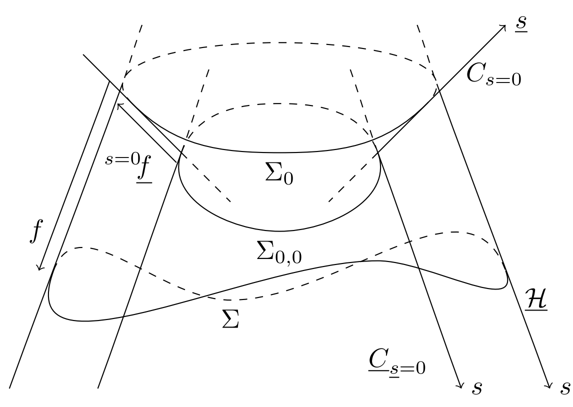


Figure 16: Parameterisation of a spacelike surface Σ.

Geometrically the position of a spacelike surface Σ can be determined by two sets of data:

i. the null hypersurface $\underline{\mathcal{H}}$ where Σ is embedded in;
ii. its position inside the null hypersurface $\underline{\mathcal{H}}$.

As figured out in previous section, $\underline{\mathcal{H}}$ is solely determined by its intersection surface Σ_0 with $C_{s=0}$, thus we can apply the parameterisation function ${}^{s=0}\underline{f}$ of Σ_0 to locate $\underline{\mathcal{H}}$. For the position of Σ in $\underline{\mathcal{H}}$, we use the restriction of (s, ϑ) on $\underline{\mathcal{H}}$ as a coordinate system and represent Σ as a graph of s over the ϑ domain. Therefore the above gives a way to parameterise Σ by a pair of functions $({}^{s=0}\underline{f}, f)$ over the ϑ domain:

i. ${}^{s=0}\underline{f}$: the parameterisation function of Σ_0, the intersection surface of $\underline{\mathcal{H}}$ with $C_{s=0}$;
ii. f: Σ is a graph of $s = f(\vartheta)$ over the ϑ domain in $\underline{\mathcal{H}}$.

With the above parameterisation of a spacelike surface Σ, we can transform the question of finding marginally trapped surfaces to a problem of analysis as follows: consider the map from the parameterisation functions $({}^{s=0}\underline{f}, f)$ to the outgoing null expansion $(\mathrm{tr}\chi)_\Sigma$,

$$({}^{s=0}\underline{f}, f) \quad \to \quad \Sigma \quad \to \quad (\mathrm{tr}\chi)_\Sigma,$$

we denote this map by $\mathrm{tr}\chi({}^{s=0}\underline{f}, f)$. Therefore finding a marginally trapped surface is equivalent to solving the equation

$$\mathrm{tr}\chi({}^{s=0}\underline{f}, f) = 0.$$

In order solve the above equation, one need to obtain the formula of the outgoing null expansion in terms of ${}^{s=0}\underline{f}$ and f. The key feature in the formula is that $\mathrm{tr}\chi({}^{s=0}\underline{f}, f)$ is an elliptic equation in f. This can be demonstrated clearly by the following case in the Schwarzschild spacetime.

Suppose that Σ is embedded in $\underline{C}_{\underline{s}=0}$ in the Schwarzschild spacetime, then the parameterisation of Σ is $(0, f)$. The outgoing null expansion of Σ is given by the formula below

$$\begin{aligned}\mathrm{tr}\chi(0, f) =& \mathrm{tr}\chi_{Sch} - 2r^{-2}\mathring{\Delta} f + \mathrm{tr}\underline{\chi}_{Sch} \cdot r^{-2} |\not{d} f|^2_{\mathring{g}} \\ =& 2(r_0 + f)^{-2}\left[f - \mathring{\Delta} f + (r_0 + f)^{-1} |\not{d} f|^2_{\mathring{g}}\right].\end{aligned}$$

In [L4], the formula of the outgoing null expansion is applied to prove the following interesting result: if a hypersurface satisfies the property that each spacelike section inside the hypersurface is marginally trapped, then the hypersurface must be null. Another interesting application of the formula of the

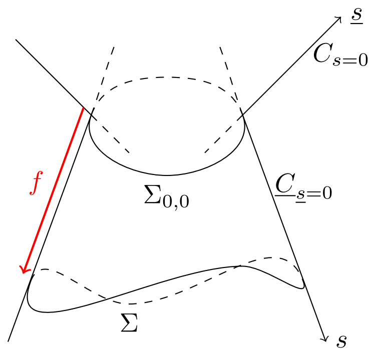


Figure 17: A spacelike surface Σ embedded in $\underline{C}_{\underline{s}=0}$ in the Schwarzschild spacetime.

outgoing null expansion is the anisotropic criteria for formation of trapped surfaces in vacuum, see [KLR], [L1].

In [L2], and also [L3], we proved that, in a perturbed Schwarzschild spacetime, for each ${}^{s=0}\underline{f}$ which is small in a certain Sobolev norm, the equation $\mathrm{tr}\chi({}^{s=0}\underline{f}, f) = 0$ has a unique solution f which is also small in the same Sobolev norm. Geometrically the result means that in any null hypersurface $\underline{\mathcal{H}}$ which is a perturbation of $\underline{C}_{\underline{s}=0}$, there exists a unique marginally trapped surface Σ close to Σ_0, the intersection of $\underline{\mathcal{H}}$ with $C_{s=0}$.

4.4. Linearised perturbation of asymptotic geometry at past null infinity

The main problem in the proof of the null Penrose inequality in a perturbed Schwarzschild spacetime can be stated as follows: considering the map from the space of marginally trapped surfaces to the asymptotic geometry at past null infinity of constant mass aspect function foliations starting from marginally trapped surfaces, does the image of this map contain the round metric on the sphere?

For the space of marginally trapped surfaces, we have a desired description from the point of view of analysis. For the asymptotic geometry at past null infinity, it is natural to consider the Gauss curvature as the characterisation. Suppose that $\{\Sigma_v\}$ is a foliation on a null hypersurface $\underline{\mathcal{H}}$, then define the renormalised Gauss curvature ${}^{r}K(v, \vartheta)$ to be

$${}^{r}K(v, \vartheta) = r_v^2 K_{\Sigma_v}(\vartheta),$$

where r_v is the area radius of Σ_v: $|\Sigma_v| = 4\pi r_v^2$. We define the limit of ${}^r\!K(v,\cdot)$ as $v \to +\infty$ as the limit renormalised Gauss curvature of the foliation at past null infinity, which is denoted by ${}^{\infty,r}\!K$.

Therefore we have the following description of the problem from the point of view of analysis: consider the following map from the space of marginally trapped surfaces to the asymptotic geometry at past null infinity

$$ {}^{s=0}\underline{f} \xrightarrow{\text{a.}} ({}^{s=0}\underline{f}, f) \xrightarrow{\text{b.}} \{\Sigma_v\} \xrightarrow{\text{c.}} {}^r\!K(v,\cdot) \xrightarrow{\text{d.}} {}^{\infty,r}\!K $$

a. $\mathrm{tr}\chi({}^{s=0}\underline{f}, f) = 0$, i.e. the surface Σ_0 parameterised by $({}^{s=0}\underline{f}, f)$ is marginally trapped;
b. $\{\Sigma_v\}$ is the constant mass aspect function of foliation starting from Σ_0;
c. ${}^r\!K(v,\cdot)$ is the renormalised Gauss curvature of Σ_v;
d. ${}^{\infty,r}\!K(\vartheta) = \lim_{v\to+\infty} {}^r\!K(v,\vartheta)$.

Denote the above map by $\mathbf{k}$:

$$ \mathbf{k} : {}^{s=0}\underline{f} \to {}^{\infty,r}\!K. $$

The problem is to solve the equation

$$ \mathbf{k}({}^{s=0}\underline{f}) = 1. $$

In [L2], this equation is solved by the idea of the inverse function theorem. We study the linearisation of the map $\mathbf{k}$, use the linearisation to construct approximating solutions of equation $\mathbf{k}({}^{s=0}\underline{f}) = 1$, and eventually prove that the approximating solutions converge to an exact solution of the equation.

One key step is to construct the linearisation for the map $\mathbf{k}$ in a perturbed Schwarzschild spacetime. In general, the construction is involved and the linearisation is complicated, however in the Schwarzschild spacetime, the linearisation of $\mathbf{k}$ at ${}^{s=0}\underline{f} = 0$ can be calculated explicitly. We demonstrate the result in this special case in [L2] and [L6]. Denote the linearisation of $\mathbf{k}$ by $\delta\mathbf{k}$. Let Y_l be a spherical harmonic of degree l, i.e.

$$ \mathring{\Delta} Y_l = -\lambda_l Y_l, \quad \lambda_l = l(l+1), \quad l = 0, 1, 2, \cdots $$

Then the function $Y_l r_0$ is an eigenfunction of $\delta\mathbf{k}$,

$$ \begin{aligned} \delta\mathbf{k}(Y_l r_0) &= \left\{ \left(\frac{\lambda_l}{3} + \frac{2}{3} \right) - \frac{1}{3}(\lambda_l - 1) \exp\left[\frac{3}{\lambda_l} \right] \right\} (\lambda_l - 2)\lambda_l Y_l \\ &\sim \left[-\frac{\lambda_l}{2} + 1 + \frac{3}{8\lambda_l} + O\left(\frac{1}{\lambda_l^2} \right) \right] Y_l. \end{aligned} $$

The factor $(\lambda_l - 2)\lambda_l$ implies that the spherical harmonics of degree 0 or 1 are the kernel of the linearisation $\delta\mathbf{k}$. In fact, $\delta\mathbf{k}$ is a bounded linear map from the Hilbert space $H^2(\mathbb{S}^2, \mathring{g})$ to $L^2(\mathbb{S}^2, \mathring{g})$. Moreover $\delta\mathbf{k}$ is diagonal in the basis of spherical harmonics. Let $V = \mathrm{span}\{1, Y_{1,1}, Y_{1,2}, Y_{1,3}\}$ be 4-dimensional space spanned by the spherical harmonics of degree 0 or 1 and $V^\perp$ be the L^2 orthogonal complement of V, then

$$\ker \delta\mathbf{k} = V, \quad \mathrm{im}\delta\mathbf{k} = V^\perp,$$

and $\delta\mathbf{k} : H^2(\mathbb{S}^2, \mathring{g}) \cap V^\perp \to V^\perp$ is a bounded self-adjoint bijection.

The above special case is enlightening, where the 4-dimensional kernel of $\delta\mathbf{k}$ corresponds to the 4-parameter family of marginally trapped surfaces in theorem 3.3. From this special case, we also observe a difficulty when applying the idea of the inverse function theorem, that $\delta\mathbf{k}$ is not surjective. To resolve this difficulty, we apply the well-known Kazdan-Warner identity on the conformal deformation of the standard round metric on the sphere, which says that its Gauss curvature must lie in $V^\perp$.

In [L2] and [L6], we also calculated the linearised perturbation of the energy-momentum vector with respect to the linearised perturbation of the marginally trapped surface in $\ker(\delta\mathbf{k}) = V$ in the Schwarzschild spacetime. We use γ_∞ to denote the asymptotic reference frame defined by the asymptotically round constant mass aspect function foliation. Let δE^{γ_∞} and $\delta P^{\gamma_\infty, i}$ be the linearised perturbations of the Bondi energy and the linear momentum respectively. For a linearised perturbation $\delta({}^{s=0}\underline{f}) = c_0 + c_1 Y^1_{l=1} r_0 + c_2 Y^2_{l=1} r_0 + c_3 Y^3_{l=1} r_0$ in V, the corresponding linearised perturbations δE^{γ_∞} and $\delta P^{\gamma_\infty, i}$ are

$$\delta E^{\gamma_\infty} = 0, \quad \delta P^{\gamma_\infty, i} = c_i \left(\frac{1}{3} e^{\frac{3}{2}} - \frac{4}{3}\right) r_0.$$

The above result implies that the linear map from V to the 3-dimensional space of linearised perturbation of the linear momentum is surjective, and the kernel V_0 of this map is the 1-dimensional space of constant functions. This partially indicates that one can eliminate the linear momentum to achieve the asymptotic center-of-mass reference frame, thus to obtain the 1-parameter family of marginally trapped surfaces in theorem 3.3 from the 4-parameter family of marginally trapped surfaces in theorem 3.2.

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