Poisson Actions and Scattering Theory for Integrable Systems

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Abstract

Conservation laws, heirarchies, scattering theory and Bäcklund transformations are known to be the building blocks of integrable partial differential equations. We identify these as facets of a theory of Poisson group actions, and apply the theory to the ZS-AKNS nxn heirarchy (which includes the non-linear Schrödinger equation, modified KdV, and the n-wave equation). We first find a simple model Poisson group action that contains flows for systems with a Lax pair whose terms all decay on R. Bäcklund transformations and flows arise from subgroups of this single Poisson group. For the ZS-AKNS nxn heirarchy defined by a constant $a \in u(n)$, the simple model is no longer correct. The a determines two separate Poisson structures. The flows come from the Poisson action of the centralizer H_a of a in the dual Poisson group (due to the behavior of $e^{a\lambda x}$ at infinity). When a has distinct eigenvalues, H_a is abelian and it acts symplectically. The phase space of these flows is the space S_a of left cosets of the centralizer of a in D_-, where D_- is a certain loop group. The group D_- contains both a Poisson subgroup corresponding to the continuous scattering data, and a rational loop group corresponding to the discrete scattering data. The H_a-action is the right dressing action on S_a. Bäcklund transformations arise from the action of the simple rational loops on S_a by right multiplication.

*Research supported in part by NSF Grant DMS 9626130

†Research supported in part by Sid Richardson Regents' Chair Funds, University of Texas system

Various geometric equations arise from appropriate choice of a and restrictions of the phase space and flows. In particuar, we discuss applications to the sine-Gordon equation, harmonic maps, Schrödinger flows on symmetric spaces, Darboux orthogonal coordinates, and isometric immersions of one space-form in another.

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1 Introducton

Soliton theory is an enticingly elegant part of modern mathematics. It has a multitude of interpretations in geometry, analysis and algebra. The main goal of this paper is to relate loop groups actions, scattering theory, and Bäcklund transformations within the same narrative, via Poisson actions. Our work is motivated by Beals and Coifman's rigorous and beautiful treatment of scattering and inverse scattering theory of the first order systems ([BC 1, 2, 3]). An expository version of our main result on scattering theory is contained in lecture notes by Richard Palais [Pa]. In retrospect,, we also find that many of our results in the $su(2)$ case are contained in the book by Faddeev and Takhtajian ([FT]). Throughout the paper, the matrix non-linear Schrödinger equation is used as a motivating example. In the final section we discuss a number of applications in geometry, including Darboux orthogonal coordinates and isometric immerisons of space forms in space forms. We begin the paper with a survey.

- **Finite dimensional mechanics**

To give some perspective, we start with a short review of finite dimensional Hamiltonian systems, complete integrability, symplectic actions, Poisson actions, and moment maps. A more detailed review of Poisson actions is given in section 2. A *symplectic structure* on a $2n$-dimensional manifold M is a closed, non-degenerate two form w on M. Since w is non-degenerate, it induces an isomorphism $J : T^*M \to TM$. A Hamiltonian on M is a smooth funtion $f : M \to R$. The Hamiltonian vector field X_f corresponding to f is the symplectic dual of df, i.e.,

$$X_f = J(df), \qquad \text{or} \qquad i_{X_f} w = df.$$

It follows from this definition that X_f is symplectic, i.e., $L_{X_f} w = 0$, or equivalently the one parameter subgroup generated by X_f preserves w.

A Poisson structure on M is a Lie bracket $\{\,,\,\}$ on $C^\infty(M, R)$, which satisfies the Leibnitz rule

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

A symplectic form w induces a natural Poisson structure on M by

$$\{f, g\} = w(X_f, X_g) = df(X_g).$$

Then the map from $C^\infty(M, R)$ to the Lie algebra of vector fields on M defined by $f \mapsto X_f$ is a Lie algebra homomorphism, i.e.,

$$[X_f, X_g] = X_{\{f,g\}}.$$

Two Hamiltonians f, g are said to be *in involution* if $\{f, g\} = 0$. In this case the corresponding Hamiltonian flows commute, and g is a *conservation law* for the Hamiltonian system

$$\frac{dx}{dt} = X_f(x(t)), \tag{1.1}$$

i.e., g is constant on the integral curves of X_f.

The Hamiltonian system (1.1) on M^{2n} is called *completely integrable* if there exists n conservation laws $f_1 = f, f_2, \ldots, f_n$ that are in involution and $df_1, \ldots, df_n$ are linearly independent. For example, the Hamiltonian systems given by the Kowalevsky top, the Toda system and the geodesic flow on an ellipsoid are completely integrable.

Suppose f is completely integrable and $f = f_1, \ldots, f_n$ are in involution. Then $X_{f_1}, \ldots, X_{f_n}$ generate an action of R^n on M. If the map $\mu = (f_1, \ldots, f_n) : M \to R^n$ is proper, then $X_{f_1}, \ldots, X_{f_n}$ generate an action of the n-torus T^n on each R^n-orbit of M. Let $\theta_1, \ldots, \theta_n$ denote the angular coordinates on the torus orbit. Then $(f_1, \ldots, f_n, \theta_1, \ldots, \theta_n)$ is a coordinate system on M and the system

(1.1) is linearized in these coordinates. These are the *action-angle coordinates* in Liouville's Theorem (for detail see [AbM], [Ar]).

The notion of complete integrability can be extended to the notion of symplectic action of a Lie group. An action of G on (M, w) is *symplectic* if the action preserves w. A symplectic action of a Lie group G on M is *Hamiltonian* if there exists a map $\mu : M \to \mathcal{G}^*$ such that the infinitesimal vector field on M corresponding to $\xi \in \mathcal{G}$ is the Hamiltonian vector field of the function f_ξ defined by $f_\xi(x) = \mu(x)(\xi)$. Such μ is called a *moment map*. When G is abelian, the flows generated by the action commute. In particular, the study of completely integrable systems on M^{2n} is the same as the study of Hamiltonian actions of R^n on M^{2n}. When G is non-abelian, the flows generated by η in the centralizer of ξ $\mathcal{G}_\xi = \{\eta \in \mathcal{G} \mid [\xi, \eta] = 0\}$ commute with the flow generated by ξ. In other words, f_η is a conservation law for the flow generated by ξ. But the flows generated by $\eta_1, \eta_2 \in \mathcal{G}_\xi$ in general do not commute.

- **Poisson groups**

Given two Poisson manifolds $(M_1, \{\,,\,\}_1)$ and $(M_2, \{\,,\,\}_2)$, the *product Poisson structure* on $M_1 \times M_2$ is defined by

$$\{f, g\}(x, y) = \{f(\cdot, y), g(\cdot, y)\}_1(x) + \{f(x, \cdot), g(x, \cdot)\}(y).$$

A map $\phi : M_1 \to M_2$ is Poisson if ϕ preserves the Poisson structure, i.e.,

$$\{f, g\}_2 \circ \phi = \{f \circ \phi, g \circ \phi\}_1.$$

A Poisson group is a Poisson manifold $(G, \{\,,\,\})$ such that G is a Lie group and the multiplication map $m : G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure. The modern study of Poisson groups was initiated by Drinfeld in [Dr] and there are several good articles by Lu and Weinstein [LW] and Semenov-Tian-Shanksy [Se1, 2]. Given a Poisson group G, there is a canonical construction of a dual Poisson group G^* (cf. [LW]). The simplest Poisson group is a Lie group G with the trivial Poisson structure, and its dual Poisson group is the dual $\mathcal{G}^*$ of Lie algebra $\mathcal{G}$ with the standard Lie Poisson structure and viewed as an abelian Lie group. In general, Poisson groups are best understood as part of a Manin triple. A Manin triple is a triple $(\mathcal{G}, \mathcal{G}_+, \mathcal{G}_-)$, where $\mathcal{G}$ is a Lie algebra with a non-degenerate bi-linear form $\langle\,,\rangle$, $\mathcal{G}_+, \mathcal{G}_-$ are Lie subalgebras of $\mathcal{G}$, $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ as direct sum of vector spaces, and $\langle \mathcal{G}_+, \mathcal{G}_+ \rangle = \langle \mathcal{G}_-, \mathcal{G}_- \rangle = 0$. Then the corresponding Lie group G_+ is Poisson and G_- is its Poisson dual. The triple (G, G_+, G_-) is called a *double group* in the literature. In this paper, we will call this triple a *Manin triple group* to avoid confusion with the completely different concept of a double loop group. For example, $(SL(n), SU(n), B_n)$ is a Manin triple group, where B_n is the subgroup of upper triangular matrices in $SL(n, C)$ with real diagonal entries and $\langle x, y \rangle = \mathrm{Im}(\mathrm{tr}(xy))$ is the non-degenerate bi-linear form. For the trivial Poisson structure on G, the Manin triple group is $(G \ltimes_{\mathrm{ad}} \mathcal{G}^*, G, \mathcal{G}^*)$, where $G \ltimes_{\mathrm{ad}} \mathcal{G}^*$ is the semi-direct product.

- **Adler-Kostant-Symes Theorem**

Let $\mathcal{G}$ be a Lie algebra equipped with an ad-invariant, non-degenerate bilinear form $\langle\ ,\rangle$. Suppose $\mathcal{K}$ and $\mathcal{N}$ are Lie subalgebras of $\mathcal{G}$ such that $\mathcal{G}$ is the direct sum of $\mathcal{K}$ and $\mathcal{N}$ as vector space. Then the space $\mathcal{K}^\perp$ perpendicular to $\mathcal{K}$ in $\mathcal{G}$ with respect to $\langle\ ,\rangle$ can be identified as the dual $\mathcal{N}^*$ of $\mathcal{N}$. Let $M \subset \mathcal{K}^\perp$ be a coadjoint N-orbit equipped with the standard co-adjoint orbit symplectic structure. The Adler-Kostant-Symes theorem ([AdM], [Kos]) states that if f and g are Ad-invariant function from $\mathcal{G}$ to R then $f\,|\,M$ and $g\,|\,M$ are commuting Hamiltonians. For example, let $\langle x, y\rangle = \mathrm{tr}(xy)$ and $sl(n,R) = \mathcal{K} + \mathcal{N}$, where $\mathcal{K} = so(n)$ and $\mathcal{N}$ is the subalgebra of real, trace zero, upper triangular matrices. Then $\mathcal{K}^\perp$ is the space of real, symmetric, trace zero matrices, and the coadjoint N-orbit M at $x_0 = \sum_{i=1}^{n-1}(e_{i,i+1} + e_{i+1,i})$ is the set of all tridiagonal matrices $z = \sum_{i=1}^{n} x_i e_{ii} + \sum_{i=1}^{n-1} y_i(e_{i,i+1} + e_{i+1,i})$ such that all $y_i > 0$ and $\sum_i x_i = 0$. Note that $f_k(x) = \mathrm{tr}(x^k)$ is Ad-invariant function on $sl(n,R)$. So the Hamiltonians $H_2 = f_2\,|\,M, \ldots, H_n = f_n\,|\,M$ are commuting, and the Hamiltonian system on M corresponding to H_2 is the Toda lattice.

Adler and van Moerbeke [AdM] have shown that many finite dimensional completely integrable systems can be obtained by applying the Adler-Kostant-Symes theorem to suitable Lie algebras. For more examples, see also the paper by Reyman [R].

- **Poisson actions and dressing actions**

An action of a Poisson group G on a Poisson manifold M is Poisson if the action $G \times M \to M$ is a Poisson map. When G is equipped with the trivial Poisson structure, a G-action on a symplectic manifold is Poisson if and only if it is symplectic. The coadjoint action of G on $\mathcal{G}^*$ is Poisson in this trivial structure. In general, if (G, G_+, G_-) is a Manin triple group such that the multiplication map from $G_- \times G_+$ to G is an isomorphism, then the action of G_+ on G_- defined by $g_+ * g_- = \tilde{g}_+$, where $\tilde{g}_+$ is obtained from the factorization

$$g_+ g_- = \tilde{g}_- \tilde{g}_+ \in G_- \times G_+,$$

is Poisson. This action of G_+ on G_- is called the *dressing action*. To construct a global dressing action, every element in G must be factored as a product $g_+ g_- \in G_+ \times G_-$. For example, in reference to the example in the previous paragraph, the factorization of $g \in GL(n)$ as $g_+ g_- \in U(n) \times B_n$ can be obtained by applying the Gram-Schmidt process to the columns of g. In general, this factorization cannot be carried out in the entire group G.

- **Birkhoff decompositions theorems**

We remark here that all of the definitions and results in symplectic and Poisson geometry mentioned above make sense in infinite dimensions.

Two typical examples of infinite dimensional Manin triple groups are given by:

(1) G = the loop group of smooth maps from S^1 to $GL(n, C)$, G_+ is the subgroup of $g \in G$ such that g is the boundary value of a holomorphic map in the disk $|\lambda| < 1$, and G_- is the subgroup of $g \in G$ such that g is the boundary value of a holomorphic maps in $|\lambda| > 1$ and $g(1) = I$.

(2) G and G_+ are the same as in example (1), and G_- is the subgroup of $g \in G$ such that $g(S^1) \subset U(n)$ and $g(1) = I$.

The two Birkhoff decomposition theorems, which are carefully explained by Pressley and Segal ([PrS]) state that the multiplication map from $G_+ \times G_-$ to G is injective onto an open dense subset of G in example (1) and is a diffeomorphism in example (2). We also need a third analytic theorem on how the decomposition depends on a parameter $x \in (x_0, \infty)$ (Theorem 7.14).

- **Soliton equations and inverse scattering**

Infinite dimensional completely integrable systems are defined in terms of the existence of action-angle coordinates. All interesting infinite dimensional completely integrable Hamiltonian systems seem to be generalizations of the "classical" soliton equations. To set the stage, we give a brief, biased history of some of the work of these equations that is directly related to our paper. It is impossible to give a full history here and we have omitted many major developments.

Solitons were first observed by J. Scott Russel in 1834 while riding on horseback following the bow-wave of a barge along a narrow canal. In 1895, Korteweg and de Vries [KdV] derived the equation

$$q_t = -q_{xxx} - 6qq_x \qquad (KdV)$$

to model the wave propagation in a shallow channel of water, and obtained solitary wave solutions, i.e., $q(x,t) = f(x-ct)$ and f decays at $\pm\infty$. The modern theory of soliton equations started with the famous numerical computation of the interaction of solitary waves of the KdV equation by Zabusky and Kruskal ([ZK]) in 1965. In 1967, Gardner, Green, Kruskal, and Miura [GGKM] used a method called inverse scattering of the one-dimensional linear Schrödinger operator to solve the Cauchy problem for rapidly decaying initial data for the KdV equation. In 1968, Lax ([La]) introduced the concept of Lax-pair for KdV and wrote KdV as the condition for a pair of commuting linear operators. Zakharov and Faddeev ([ZF] 1971) gave a Hamiltonian formulation of KdV, and proved that KdV is completely integrable by finding action-angle variables. Zakharov and Shabat ([ZS1] 1972) found a Lax pair of 2×2 first order differential operators for the non-linear Schrödinger equation:

$$q_t = \frac{i}{2}(q_{xx} + 2|q|^2 q) \qquad (NLS)$$

and solved the Cauchy problem via a similar inverse scattering. Wadati ([Wa] 1973) used the same inverse scattering transform for the Modified KdV equation

$$q_t = -(q_{xxx} + 6q^2 q_x). \qquad (mKdV)$$

Ablowitz, Kaup, Newell and Segar ([AKNS1] 1973) again used this same inverse scattering transform for the sine-Gordon equation

$$q_{xt} = \sin q, \qquad (SGE),$$

and also observed ([AKNS2]) that all these equations have a Lax pair of 2×2 linear operators. In 1973, Zakharov and Manakov ([ZMa1], [ZMa2]) "solved" the 3-wave equation

$$(u_{ij})_t = \frac{b_i - b_j}{a_i - a_j}(u_{ij})_x + \sum_{k \neq i,j} \left(\frac{b_k - b_j}{a_k - a_j} - \frac{b_i - b_k}{a_i - a_k} \right) u_{ik} u_{kj}, \qquad i \neq j.$$

using a Lax pair of 3×3 first order linear operators. In 1976, Gelfand and Dikii [GDi] found an evolution equation on the space of n-th order differential operators on the line with a Lax pair (this generalizes the KdV equation). In 1978, Zakharov and Mikhailov studied harmonic maps from $R^{1,1}$ to Lie group G using a Lax pair of $\mathcal{G}$-valued first order linear operators ([ZMi1], [ZMi2]).

The scattering theory of the $n \times n$ linear system was studied by Shabat [Sh], and Beals and Coifman [BC1], [BC2]. In a series of papers, Beals and Coifman studied the scattering and inverse scattering theory of the first order $n \times n$ linear operator:

$$\begin{cases} \frac{\partial \psi}{\partial x} = (a\lambda + u)\psi, \\ \lim_{x \to -\infty} e^{-a\lambda x} \psi(x, \lambda) = I, \\ e^{-a\lambda x}\psi(x, \lambda) \text{ is bounded in } x. \end{cases} \tag{1.2}$$

Here $a = \operatorname{diag}(a_1, \dots, a_n)$ is a constant diagonal matrix with distinct eigenvalues $a_1, \dots, a_n$, and u lies in the space $\mathcal{S}(R, gl_*(n))$ of Schwartz maps from R to the space $gl_*(n)$ of all $y \in gl(n)$ with zero diagonal entries. The "scattering data", S, of u is defined in terms of the singularity of $e^{-a\lambda x}\psi(x, \lambda)$ in λ. Assume $b \in gl(n)$ is a diagonal matrix. Beals and Coifman defined an evolution equation, the j-th flow associated to a, b on $\mathcal{S}(R, gl_*(n))$, such that if $u(x, t)$ is a solution of this equation then the scattering data $S(\cdot, t)$ of $u(\cdot, t)$ is a solution of the following linear equation:

$$\frac{\partial S}{\partial t} = [S, \lambda^j b],$$

i.e., $S(\lambda, t) = e^{-b\lambda^j t} S(\lambda, 0) e^{b\lambda^j t}$. Then by the inverse scattering transform they solved the Cauchy problem for the j-th flow equation globally. When $n = 2$ and $u \in su(2)$, the second flow defined by $a = b = \operatorname{diag}(i, -i)$ is the non-linear Schrödinger equation, and when $n = 3$ and $u \in su(3)$, the first flow defined by a, b with $b \neq a$ is the 3-wave equation. They prove that

$$w = \operatorname{Re} \int_{-\infty}^{\infty} \operatorname{tr}(-\operatorname{ad}(a)^{-1}(v_1)v_2)\, dx \tag{1.3}$$

is a symplectic structure on $\mathcal{S}(R, gl_*(n))$ and all the j-th flows are commuting Hamiltonian flows. In 1991, Beals and Sattinger ([BS]) proved that the j-th flow equation is completely integrable by finding action-angle variables. A good survey on recent results is contained in the article by Beals, Deift and Zhou [BDZ].

- **Soliton equations and geometry**

Even earlier than they appeared in applied problems, soliton equations occurred in classical differential geometry. It was known in the mid 19th century that solutions of the sine-Gordon equation correspond to surfaces in R^3 with constant Gaussian curvature -1. The Bäcklund transformations (cf. [Ba]) of surfaces in R^3 generate families of new surfaces with -1 curvature from a given one, and hence give a method of generating new solutions of the sine-Gordon equation from a given one by solving two compatible ordinary differential equations. Inspired by this classical result, Bäcklund transformations have been constructed for a large class of the equations already mentioned (cf. [Mi], [SZ1, 2], [GZ], [TU1]). Many more equations in differential geometry possesses Bäcklund transformations. For example, equations for submanifolds with constant curvature in Euclidean space ([TT], [Ten]), Darboux orthogonal coordinate systems ([Da2]), and harmonic maps from $R^{1,1}$ into a Lie group ([U1]).

Another interesting soliton equation made its appearance in differential geometry at the beginning of the twentieth century. Da Rios, a student of Levi-Civita, studied the free motion of a thin vortex tube in a liquid medium in his master degree thesis ([dR]). He modeled this motion using the evolution of curves in R^3 (the *vortex filament equation* or the *smoke ring equation*):

$$\gamma_t = \gamma_x \times \gamma_{xx}, \tag{1.4}$$

i.e., γ evolves along the direction of the binormal with curvature as speed. The corresponding evolution of the geometric quantity $q = k\exp(-i\int \tau\, dx)$ satisfies the non-linear Schrödinger equation, where $k(\cdot, t)$ and $\tau(\cdot, t)$ are the curvature and torsion of the curve $\gamma(\cdot, t)$. This is the Hasimoto transformation of the vortex filament equation to the non-linear Schrödinger equation. For an interesting historical account of the multiple rediscoveries of the non-linear Schrödinger equation for vortex tubes see an article by Ricca [Ri].

Recently, techniques developed in soliton theory have also been used successfully in several geometric problems whose differential equations are elliptic. For example, the studies of harmonic maps from S^2 to a compact Lie group by Uhlenbeck ([U1]), harmonic maps from a torus to S^3 by Hitchen ([Hi1]), into a symmetric space by Burstall, Ferus, Pedit and Pinkall ([BFPP]), constant mean curvature tori in S^3 by Pinkall and Sterling ([PiS]), constant mean curvature tori in 3-dimensional space forms by Bobenko ([Bo]), and minimal tori in S^4 by Ferus, Pedit, Pinkall, and Sterling ([FPPS]). For a detailed and beautiful account of these developments see the survey book by Guest [Gu].

- **Main goal of the paper**

The main point of our paper is to show how Bäcklund transformations, scattering theory, and the hierarchy of flows can be obtained in a uniform and natural way from "dressing actions" of suitable infinite dimensional Manin triple groups.

- **Decay case**

Most of the soliton equations considered in our paper are evolution equations on the space $\mathcal{S}_{1,a}$ of $gl(n)$-valued connections with one complex parameter:

$$\frac{d}{dx} + a\lambda + u.$$

Here $a \in u(n)$ is a fixed diagonal element and $u \in \mathcal{S}(R, \mathcal{U}_a^\perp)$, where $\mathcal{U}_a^\perp$ is the orthogonal complement of the centralizer $\mathcal{U}_a = \{y \in u(n) \mid [a, y] = 0\}$ of a.

To help explain the basic Poisson group action for soliton equations, we study a simpler case first: flows on the space $\mathcal{S}_+$ of $gl(n)$-valued connections of the form

$$\frac{d}{dx} + A(x, \lambda),$$

where $A(x,\lambda) = \sum_{j=0}^{k} \alpha_j(x)\lambda^j$ for some k and $\alpha_j \in \mathcal{S}(R, u(n))$ for all $0 \le j \le k$. Hence $A(x, \lambda)$ is rapidly decaying in x for each $\lambda \in C$. Consider now the infinite dimensional Manin triple group (G, G_+, G_-), where G is the group of holomorphic maps g from $\mathcal{O} \setminus \{\infty\}$ satisfying the condition $g(\bar{\lambda})^* g(\lambda) = I$, G_+ is the subgroup of $g \in G$ that are holomorphic in C and G_- is the subgroup of $g \in G$ that is holomorphic in $\mathcal{O}$, where $\mathcal{O}$ is a small neighborhood of ∞ in the Riemann sphere $C \cup \{\infty\}$. Since $\mathcal{S}_+$ can be identified as a subspace of the dual of the Lie algebra $C(R, \mathcal{G}_-)$, and is invariant under the coadjoint action, $\mathcal{S}_+$ is a Poisson manifold with the standard Lie-Poisson structure. Here we use λ to denote the loop variable for G and x for the variable $x \in R$. The trivialization F of $D = \frac{d}{dx} + A(x, \lambda)$ in $\mathcal{S}_+$ is the solution of

$$\begin{cases} F^{-1}F_x = A(x, \lambda), \\ \lim_{x \to -\infty} F(x, \lambda) = I, \end{cases} \tag{1.5}$$

and the *monodromy* of D is

$$F_\infty(\lambda) = \lim_{x \to \infty} F(x, \lambda).$$

Since $A(x, \lambda)$ is decaying in x, it follows that the linear system (1.5) has a unique global solution. This identifies $\mathcal{S}_+$ as a subset of $C(R, G_+)$. Moreover, the monodromy F_∞ exists and is an element in G_+. The group G_- acts on $C(R, G_+)$ by the pointwise dressing action of G_- on G_+, hence it induces an action $*$ of G_- on $\mathcal{S}_+$. In fact, given $g \in G_-$, factor

$$g(\cdot)F(x, \cdot) = \tilde{F}(x, \cdot)\tilde{g}(x, \cdot) \in G_+ \times G_-,$$

then $g * D = \frac{d}{dx} + \tilde{F}^{-1}\tilde{F}_x$. The fundamental theorem for the decay case appears in section 4. We show that the action of G_- on $\mathcal{S}_+$ is Poisson with the monodromy on the line as the moment map. We call the flows generated by the action of G_- on $\mathcal{S}_+$ the *negative flows*.

- **Rational loop group action**

For a fixed constant $a \in u(n)$, the phase space $\mathcal{S}_{1,a}$ is a coadjoint orbit of $C(R, G_-)$, and the w defined by formula (1.3), is the Kostant-Kirillov symplectic form. Since $a\lambda + u$ does not decay in x, the monodromy of the connection $\frac{d}{dx} + a\lambda + u(x) \in \mathcal{S}_{1,a}$ on the line is not defined. The "action" of G_- on $\mathcal{S}_{1,a}$ can still be defined formally by the dressing action. In fact, first we identify A with its trivialization $E \in C(R, G_+)$ of A normalized at $x = 0$, i.e., E is the solution of

$$\begin{cases} E^{-1}E_x = a\lambda + u, \\ E(0, \lambda) = I. \end{cases}$$

Let $\tilde{E}(x, \cdot)$ denote the dressing action of g on $E(x, \cdot)$ for each $x \in R$. Then

$$\tilde{E}^{-1}\tilde{E}_x = a\lambda + \tilde{u}$$

for some smooth $\tilde{u}$. In general, $\tilde{u}$ does not decay at $\pm\infty$ for $g \in G_-$. So the action of G_- on $\mathcal{S}_{1,a}$ does not exist. But $\tilde{u}$ does belong to the Schwartz class for $g \in G_-^m$, the subgroup of rational maps in G_-. Hence the subgroup G_-^m does act on $\mathcal{S}_{1,a}$. Moreover, if $g \in G_-^m$ is a linear fractional transformation then $\tilde{u}$ can be obtained by solving an ordinary differential equation or by an algebraic formula in terms of g and E. These results are proved in section 6.

- **Homogeneous structure of scattering data**

We are motivated by results from scattering theory to choose the group D_- of meromorphic maps f from $C \setminus R$ to $GL(n, C)$, which satisfy the following conditions:

(i) $f(\bar{\lambda})^* f(\lambda) = I$,

(ii) f has a smooth extension to the closure $\bar{C}_\pm$,

(iii) f has an asymptotic expansion at ∞,

(iv) $f_\pm(r) = \lim_{s \searrow} f(r \pm is)$ such that

$f_+ = h_+ v_+$ factors with v_+ unitary and h_+ upper triangular and $h_+ - I$ in the Schwartz class.

Note that G_-^m is a subgroup of D_-. But D_- is not a subgroup of G_- because we do not assume f is holomorphic at $\lambda = \infty$. Let D_-^c denote the subgroup of $f \in D_-$ such that f is holomorphic in $C \setminus R$. In section 7, we prove that D_- is diffeomorphic to $G_-^m \times D_-^c$ by translating the Birkhoff decomposition theorems for maps from the unit circle to maps from the real line using a linear fractional transformation. We identify the space $\mathcal{S}_{1,a}$ as the homogeneous space D_-/H_- of left cosets of H_- in D_- (scattering cosets), where H_- is the subgroup of $f \in D_-$ that commutes with a. In fact, given $f \in D_-$, we use the Birkhoff decomposition

$$f(\lambda)^{-1}e^{a\lambda x} = E(\lambda, x)M(\lambda, x)^{-1}, \tag{1.6}$$

with $E \in C(R, G_+)$ and $M \in C(R, D_-)$. Then $E^{-1}E_x$ is of the form $a\lambda + u$ for some decay map u, and the map sending the left coset $H_- f$ to $\frac{d}{dx} + E^{-1}E_x$ gives the identification of D_-/H_- and $\mathcal{S}_{1,a}$. Moreover, the right action of D_- on D_-/H_- induces an action of D_- on $\mathcal{S}_{1,a}$, which extends the action $*$ of the G_-^m on $\mathcal{S}_{1,a}$ defined in section 6.

- **Poisson structure of positive flows**

Let H_+ denote the subgroup of G_+ generated by $\{e^p \,|\, p$ is a polynomial in λ which commutes with $a\}$. In section 8, we construct an action of H_+ on D_-/H_- by the "dressing action" of H_+ on D_-. Hence it induces an action of H_+ on $\mathcal{S}_{1,a}$. If a is regular (i.e., a has distinct eigenvalues) then H_+ is abelian, the action of H_+ on $\mathcal{S}_{1,a}$ is Hamiltonian, and the flows generated by H_+ are the commuting hierarchy of the j-th flows. If a is singular, then H_+ is a non-abelian Poisson group and H_+ contains a distinguished infinite dimensional abelian subgroup generated by polynomials in a. Although the action of H_+ is not symplectic, we prove the action of H_+ on $\mathcal{S}_{1,a}$ is Poisson by constructing a moment map. Here we need to prove the difficult result that $M_{\pm\infty} \in H_-$, where $M_{\pm\infty}(\lambda) = \lim_{x\to\pm\infty} M(x, \lambda)$. Then $M_{-\infty}^{-1}M_\infty$ is a moment map for the H_+-action. We also show that the pull back of the symplectic form w to the space of continuous scattering cosets $D_-^c/(H_- \cap D_-^c)$ is non-degenerate. We believe the restriction of w to each algebraic component of the space of discrete scattering cosets $G_-^m/(H_- \cap G_-^m)$ is also non-degenerate, and we prove this in one case.

- **Bäcklund transformations**

Since G_-^m acts on the phase space $\mathcal{S}_{1,a}$, it induces an action of G_-^m on the space of solutions of the j-th flow. In general, if G acts on M, the induced action of G on the space of solutions of a dynamical system is not easy to write down. In section 10, we prove that the induced action of G_-^m on the space of solutions of the j-th flow on $\mathcal{S}_{1,a}$ can be constructed again by dressing action as done in section 6. In fact, if $a\lambda + u$ is a solution of the j-th flow and $g \in G_-^m$ is a linear fractional transformation, then $g \sharp (a\lambda + u)$ can be obtained by solving two compatible ordinary differential equations. The action of such g gives the classical Bäcklund transformation for the sine-Gordon equation. The orbit of the rational negative loop group G_-^m through the vacuum (trivial) solution can be computed explicitly, and is the space of pure solitons. Using the action of G_-^m, we are also able to construct periodic (breather) solutions for the harmonic map equation and the j-th flow equation with $j \geq 2$.

- **Geometric Non-linear Schrödinger equation**

In section 11, we apply soliton theory to the Schrödinger flow on $Gr(k, C^n)$. Suppose (M, g, J) is a complex Hermitian manifold. The geometric non-linear Schrödinger equation (GNLS) is the following evolution equation of curves on M:

$$J\phi_t = \Delta\phi = \nabla_{\phi_x}\phi_x, \qquad (GNLS)$$

where ∇ is the Levi-Civita connection of the metric g. When $M = S^2$, this equation is equivalent via the Hasimoto transformation to the non-linear Schrödinger equation. In fact, if γ evolves according to the vortex filament equation (1.4) and x is the arc length parameter, then γ_x satisfies the geometric non-linear Schrödinger equation on S^2. When M is the complex Grassmannian manifold $Gr(k, C^n)$, the GNLS gives the matrix non-linear Schrödinger equation (MNLS) studied by Fordy and Kulish [FK] for maps q from R^2 to the space $\mathcal{M}_{k\times(n-k)}$ of $k \times (n-k)$ complex matrices:

$$q_t = \frac{i}{2}(q_{xx} + 2qq^*q), \qquad (MNLS)$$

where $q^* = \bar{q}^t$. The MNLS is the second flow on $\mathcal{S}_{1,a}$ defined by

$$a = \begin{pmatrix} iI_k & 0 \\ 0 & -iI_{n-k} \end{pmatrix}.$$

This flow has a Lax pair:

$$\left[\frac{\partial}{\partial x} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\lambda + \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \right.$$
$$\left. \frac{\partial}{\partial t} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}\lambda + \frac{1}{2i}\begin{pmatrix} qq^* & -q_x \\ -q_x & -q^*q \end{pmatrix}\right] = 0.$$

By applying soliton theory to the MNLS, we can solve the Cauchy problem globally with decay initial data, and obtain a Poisson action of H_+ on $\mathcal{S}_{1,a}$ such that the flow generated by $a\lambda^2$ is the MNLS. The flow generated by $b\lambda^j$ with $b \in \mathcal{U}_a$ commutes with MNLS. If $n = 2$ then H_+ is abelian and the action is symplectic. If $n > 2$ then H_+ is non-abelian and the flows generated by $b\lambda^k$ with $b \in \mathcal{U}_a$ commute with the MNLS. But the flows generated by $b_1\lambda^j$ and $b_2\lambda^s$ with $b_1, b_2 \in \mathcal{U}_a$ do not commute if $[b_1, b_2] \neq 0$. Not all these flows are described by differential equations. The flow generated by $b\lambda^k$, $b \neq a$ and $k \neq 1$, are mixed integral-differential flows.

- **Restriction of the phase space by an automorphism**

The phase space of the modified KdV (mKdV) equation is the following subspace of $\mathcal{S}_{1,a}$:

$$\mathcal{S}'_{1,a} = \left\{ \frac{d}{dx} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\lambda + \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \,\middle|\, q \in \mathcal{S}(R, R) \right\}.$$

The third flow defined by $b = a = \mathrm{diag}(i, -i)$ leaves $\mathcal{S}'_{1,a}$ invariant and is the modified KdV flow. While all the even flows vanish on $\mathcal{S}'_{1,a}$, all odd flows leave $\mathcal{S}'_{1,a}$ invariant. This is a special case of restrictions given by finite order automorphisms. To explain this in a more general context, we let $\mathcal{U}$ be a semi-simple Lie algebra (not necessary a subalgebra of $su(n)$), and let $\langle\ ,\rangle$ denote the Killing form. Given $a \in \mathcal{U}$, let $\mathcal{S}_{1,a}$ denote the space of all connections of the form $\frac{d}{dx} + a\lambda + u$, where u is a Schwartz class map from R to the orthogonal complement $\mathcal{U}_a^\perp$ of the centralizer $\mathcal{U}_a$ of a in $\mathcal{U}$. Then $\mathrm{ad}(a)$ maps $\mathcal{U}_a^\perp$ isomorphically onto $\mathcal{U}_a^\perp$. Hence

$$w(v_1, v_2) = \mathrm{Re} \int_{-\infty}^{\infty} \langle -\,\mathrm{ad}(a)^{-1}(v_1), v_2\rangle dx$$

still defines a symplectic structure on $\mathcal{S}_{1,a}$.

Suppose σ is an order k Lie algebra automorphism of $\mathcal{U}$ such that there is an eigendecomposition of σ

$$\mathcal{U} = \mathcal{U}_0 + \ldots + \mathcal{U}_{k-1},$$

where $\mathcal{U}_j$ is the eigenspace with eigenvalues $e^{2(j-1)\pi i/k}$ with $1 \le j \le k$. Assume $a \in \mathcal{U}_1$, and consider the following subspace of $\mathcal{S}_{1,a}$:

$$\mathcal{S}^\sigma_{1,a}(\mathcal{U}) = \left\{ \frac{d}{dx} + a\lambda + u \;\middle|\; u \in \mathcal{U}_0 \cap \mathcal{U}_a^\perp. \right\}$$

Note that when $\mathcal{U} = su(2)$, $\sigma(x) = \bar{x}$, and $a = \mathrm{diag}(i, -i)$, we have $\mathcal{S}^\sigma_{1,a} = \mathcal{S}'_{1,a}$.

It was shown by the first author [Te2] that there exist a sequence of symplectic structures w_r such that $w_{-1} = w$ and all positive flows are Hamiltonian with respect to w_r. In section 9, we study the restriction of the sequence w_r of symplectic forms and the hierarchy of flows to the subspace $\mathcal{S}^\sigma_{1,a}$. We generalize results proved in [Te2] when σ is of order 2 and a result for the generalized modified KdV equation proved by Kupershmidt and Wilson [KW] when $\mathcal{U} = gl(n, C)$, σ is the order n automorphism defined by the conjugation of the operator $c \in GL(n)$ that permutes the standard basis of C^n cyclically, and $a = \mathrm{diag}(1, \alpha, \cdot, \alpha^{n-1})$ with $\alpha = \exp(2\pi i/n)$. In fact, we prove:

(i) If $j \not\equiv 1 \pmod{k}$, then the j-th flow vanishes on $\mathcal{S}^\sigma_{1,a}$, and if $j \equiv 1 \pmod{k}$ then the j-th flow leaves $\mathcal{S}^\sigma_{1,a}$ invariant.

(ii) The restriction of w_r on $\mathcal{S}^\sigma_{1,a}$ is zero if $r \not\equiv 0 \pmod{k}$, and is non-degenerate if $r \equiv 0 \pmod{k}$.

(iii) Let J_{rk} denote the Poisson structure corresponding to w_{rk}, and F_{jk+1} the Hamiltonian for the $(jk+1)$-th flow with respect to J_0. Then the $(k+1)$-th flow satisfies the Lenard relation

$$u_t = J_0(\nabla F_{k+1}) = J_k(\nabla F_1).$$

We should point out that when $\mathcal{U} \subset su(n)$, σ must have order 2. So the order k automorphisms occur in a more general context, in situations for which the scattering theory is considerably more difficult than the case we have discussed. This leads us to the question of other algebraic situations.

- **Other semi-simple Lie algebras**

In this paper we have proved that all rational factorizations can be carried out, and all the formal scattering coset data yield actual geometric flows when $\mathcal{U} = su(n)$. It follows that any problem for a Lie algebra $\mathcal{U} \subset su(n)$ becomes purely an algebraic subproblem. However, many interesting equations in differential geometry arise as flows on a twisted space $\mathcal{S}^{\sigma}_{1,a}(\mathcal{U})$, where $\mathcal{U} \not\subset su(n)$. We believe that some form of the discrete factorization theory and construction of scattering coset can be carried out for many real semi-simple Lie algebras. However, one normally expects a certain number of the factorization theorems to fail off a "big cell". Even more complications arise in trying to handle systems which lie properly in the full complex group. For example, the Gelfand-Dikii hierarchy for a k-th order differential operators is linked to a restriction by an order k automorphism (k-twist) in the full $gl(k, C)$, and the scattering theory is along rays in the directions of k-th roots of unity. Our formal observations about twists apply, and can help understand pure soliton solutions, but do not address the scattering theory difficulties.

- **First flows and flat metrics**

A symmetric space U/K is formed by a splitting of the Lie algebra $\mathcal{U} = \mathcal{K} + \mathcal{P}$, where

$$[\mathcal{K},\mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K},\mathcal{P}] \subset \mathcal{P}, \quad [\mathcal{P},\mathcal{P}] \subset \mathcal{K}.$$

The rank of a symmetric space is the maximal number of linearly independent commuting elements in $\mathcal{P}$, i.e., the dimension of a maximal abelian subalgebra $\mathcal{T}$ in $\mathcal{P}$. Choose a basis $b_1, \ldots, b_k$ of $\mathcal{T}$. Then for each element $[f]$ of the scattering coset, from our point of view (at least formally, rigorously if $\mathcal{U} \subset su(n)$), there are k commuting first flows in variables we call $x_1, \ldots, x_k$. This yields a flat connection

$$\frac{\partial}{\partial x_i} + b_i\lambda + u_i$$

of k variables for each scattering coset $[f]$. For example, Darboux orthogonal coordinates in R^n ([Da2]), isometric immersions of R^n into R^{2n} with flat normal bundle and maximal rank ([Te2]), equations of hydrodynamic types ([DN1], [DN2], [Dub1], [Ts]) and Frobenius manifolds ([Dub2], [Hi2]) are of this type. In the appendix, we apply some of the soliton theory to these examples.

The authors would like to thank Mark Adler, Percy Deift, Gang Tian and Pierre Van Moerbeke for many helpful discussions. We are grateful to Dick Palais and Gudlaugur Thorbergsson for reading a draft of this paper.

2 Review of Poisson Actions

In this section, we review basic definitions and theorems on Poisson Lie groups and Poisson actions. Two good introductions for this material are articles by Lu and Weinstein [LW] and Semenov-Tian-Shansky [Se1].

A *Poisson structure* on a smooth manifold M is a smooth section π of $L(T^*M, TM)$ such that the bilinear map

$$\{\,,\,\} : C^\infty(M,R) \times C^\infty(M,R) \to C^\infty(M,R)$$

defined by $\{f,g\} = dg(\pi(df))$ is a Lie bracket and satisfies the condition

$$\{fg,h\} = f\{g,h\} + g\{f,h\}, \quad \text{for all } f,g,h \in C^\infty(M,R).$$

We will refer to either $\{\,,\,\}$ or π as the Poisson structure on M. The section π can also be viewed as a section of $(T^*M \otimes T^*M)^*$ or a section of $TM \otimes TM$, which will still be denoted by π. Symplectic manifolds are well-known examples of Poisson manifolds.

Let $(M, \{\,,\,\}_M)$ and $(N, \{\,,\,\}_N)$ be two Poisson manifolds. A smooth map $\phi : M \to N$ is called a *Poisson map* if $\{f_1 \circ \phi, f_2 \circ \phi\}_M = \{f_1, f_2\}_N \circ \phi$. *The product Poisson structure* on $M \times N$ is defined by

$$\{f,g\}(x,y) = \{f(*,y), g(*,y)\}_M(x) + \{f(x,*), g(x,*)\}_N(y).$$

A submanifold N of M is a *Poisson submanifold* if there exists a Poisson structure on N such that the inclusion map $i : (N, \{\,,\,\}_N) \to (M, \{\,,\,\}_M)$ is Poisson.

The dual $\mathcal{G}^*$ of a Lie algebra has a natural Lie-Poisson structure by

$$\pi_\ell(x,y) = \ell([x,y]), \qquad \ell \in \mathcal{G}^*, x, y \in \mathcal{G} = (\mathcal{G}^*)^*,$$

with coadjoint orbits as its symplectic leaves. If $\mathcal{G}$ has a non-degenerate ad-invariant form $(\,,\,)$, then by identifying $\mathcal{G}^*$ with $\mathcal{G}$ via $(\,,\,)$, the Lie-Poisson structure on $\mathcal{G}$ is $\pi_x(y,z) = (x,[y,z])$ for all $x,y,z \in \mathcal{G}$.

2.1 Definition. A *Poisson group* is a Lie group G together with a Poisson structure π such that the multiplication map $m : G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

Note that $\pi(e) = 0$ when π is viewed as a map from $G \to TG \times TG$. Moreover, the dual of $d\pi_e$ is a map from $\mathcal{G}^* \times \mathcal{G}^* \to \mathcal{G}^*$, which defines a Lie bracket on $\mathcal{G}^*$. The corresponding simply connected Lie group G^* has a natural Poisson structure π^* such that the dual of $d(\pi^*)_e$ is the Lie bracket on $\mathcal{G}$. We will call (G^*, π^*) the *dual Poisson group* of (G, π). This pair often fits into a larger group and we call the collection of three groups a Manin triple group. We first explain the Manin triple at the level of Lie algebras.

2.2 Definition. A *Manin triple* is a collection of three Lie algebras $(\mathcal{G}, \mathcal{G}_+, \mathcal{G}_-)$ and an ad-invariant non-degenerate bilinear form $\langle\,,\,\rangle$ on $\mathcal{G}$ with the properties:

(1) $\mathcal{G}_+, \mathcal{G}_-$ are subalgebras of $\mathcal{G}$ and $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ as direct sum of vector spaces,

(2) $\mathcal{G}_+, \mathcal{G}_-$ are isotropic, i.e., $\langle \mathcal{G}_+, \mathcal{G}_+ \rangle = \langle \mathcal{G}_-, \mathcal{G}_- \rangle = 0$.

Let $(\mathcal{G}, \mathcal{G}_+, \mathcal{G}_-)$ be a Manin triple with respect to $\langle\ ,\rangle$. Then $\mathcal{G}_+ \simeq \mathcal{G}_-^*$ and the infinitesimal vector field corresponding to $x_- \in \mathcal{G}_-$ for the coadjoint action of G_- on $\mathcal{G}_+$ is

$$v_{x_-}(y_+) = [x_-, y_+]_+.$$

The Lie Poisson structure on $\mathcal{G}_+$ is

$$(\pi_+)_{x_+}(y_-) = [x_+, y_-]_+.$$

If there are corresponding Lie groups (G, G_+, G_-) we call this a *Manin triple group*. If (G, G_+, G_-) is a Manin triple group, then G_+ and G_- have natural Poisson group structures. To describe the Poisson structures on G_+ and G_-, we first set up some notation: Given $x_\pm \in \mathcal{G}_\pm$, let $\ell_{x_\pm}, \tau_{x_\pm}$ denote the 1-forms on $G_\mp$ defined by

$$\begin{aligned} \ell_{x_-}(y_+g_+) &= \langle x_-, y_+\rangle, \qquad & \tau_{x_-}(g_+y_+) &= \langle x_-, y_+\rangle, \\ \ell_{x_+}(y_-g_-) &= \langle x_+, y_-\rangle, \qquad & \tau_{x_+}(g_-y_-) &= \langle x_+, y_-\rangle. \end{aligned}$$

Then the Poisson structures on $G_\pm$ are given explicitly:

$$\begin{aligned} (\pi_+)_{g_+}(\ell_{x_-}, \ell_{y_-}) &= \langle (g_+^{-1}x_-g_+)_+, g_+^{-1}y_-g_+\rangle \\ (\pi_-)_{g_-}(\tau_{x_+}, \tau_{y_+}) &= \langle (g_-x_+g_-^{-1})_-, g_-y_+g_-^{-1}\rangle. \end{aligned}$$

This is equivalent to

$$(\pi_+)_{g_+}(\ell_{x_-}) = g_+(g_+^{-1}x_-g_+)_+, \qquad (\pi_-)_{g_-}(\tau_{x_+}) = (g_-x_+g_-^{-1})_-g_-,$$

where $g_\pm \in G_\pm$, $x_\pm \in \mathcal{G}_\pm$ and $y_\pm$ denotes the projection of $y \in \mathcal{G}$ onto $\mathcal{G}_\pm$ with respect to the decomposition $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$. Here we identify $\mathcal{G}_-$ as $\mathcal{G}_+^*$, $\mathcal{G}_+$ as $\mathcal{G}_-^*$ via $\langle\ ,\rangle$, and use the matrix convention $gx = (\ell_g)_*(x)$, $gxg^{-1} = \mathrm{Ad}(g)(x)$, and so forth. Since

$$\ell_{x_+}(g_-) = \tau_{(g_-^{-1}x_+g_-)_+}(g_-),$$

we have

$$\begin{aligned} &(\pi_-)_{g_-}(\ell_{x_+}, \ell_{y_+}) \\ &\quad = \langle (g_-(g_-^{-1}x_+g_-)_+g_-^{-1})_-,\ g_-(g_-^{-1}y_+g_-)_+g_-^{-1}\rangle \\ &\quad = \langle (g_-(g_-^{-1}x_+g_- - (g_-^{-1}x_+g_-)_-)g_-^{-1})_-,\ g_-(g_-^{-1}y_+g_-)_+g_-^{-1}\rangle \\ &\quad = -\langle g_-(g_-^{-1}x_+g_-)_-g_-^{-1},\ g_-(g_-^{-1}y_+g_-)_+g_-^{-1}\rangle \\ &\quad = -\langle (g_-^{-1}x_+g_-)_-,\ (g_-^{-1}y_+g_-)_+\rangle \\ &\quad = -\langle (g_-^{-1}x_+g_-)_-,\ g_-^{-1}y_+g_-\rangle. \end{aligned}$$

Hence (G_+, π_+) is the dual Poisson group of (G_-, π_-). Conversely, if K is a Poisson group and K^* is its dual Poisson group, then there exist an Ad-invariant form $\langle\ ,\rangle$ and a Lie bracket on $\mathcal{G} = \mathcal{K} + \mathcal{K}^*$ such that $(\mathcal{G}, \mathcal{K}, \mathcal{K}^*)$ is a Manin triple. Hence there is a bijective correspondence between the Manin triples and simply connected Poisson groups. The Manin triple group (G, G_+, G_-) is called a *double group* in the literature. In some cases, multiplication in G can not be globally defined. In this case, we call (G, G_+, G_-) a *local Manin triple group.*

2.3 Example. Let $G = SL(n, C)$, $G_+ = SU(n)$, G_- the subgroup of upper triangular matrices with real diagonal entries, and $\langle x, y\rangle = \mathrm{Im}(\mathrm{tr}(xy))$ the non-degenerate bi-invariant form on $\mathcal{G}$. Then (G, G_+, G_-) is a Manin triple group, and the multiplication map $G_+ \times G_- \to G$ and $G_- \times G_+ \to G$ are isomorphisms. The decomposition of $g \in SL(n, C)$ as $g = g_+ g_- \in G_+ \times G_-$ and $g = h_- h_+ \in G_- \times G_+$ are obtained by applying the Gram-Schmidt process to the columns and rows of g respectively.

2.4 Examples. The type of Poisson groups we need in this paper are generally credited to Cherednik ([Ch]). Let Ω_+ and Ω_- be two domains of $S^2 = C \cup \{\infty\}$ such that $S^2 = \Omega_+ \cup \Omega_-$ and both Ω_+ and Ω_- are invariant under complex conjugation. Let $\mathcal{O} = \Omega_+ \cap \Omega_-$. A map $g : \mathcal{O} \to SL(n, C)$ is called *$su(n)$-holomorphic* if g is holomorphic and satisfies the reality condition $g(\bar{\lambda})^* g(\lambda) = I$ for all $\lambda \in \mathcal{O}$. Let

$$G = \{g : \mathcal{O} \to GL(n, C) \mid g \text{ is } su(n)\text{-holomorphic}\}.$$

Now we fix a normalization point $\lambda_o \in C \cup \{\infty\}$. If $\lambda_o \in \Omega_+$, define

$$\begin{aligned} G_+ &= \{g \in G \mid g \text{ extends } su(n)\text{-holomorphically to } \Omega_+ \ g(\lambda_o) = I\}, \\ G_- &= \{g \in G \mid g \text{ extends } su(n)\text{-holomorphically to } \Omega_-\}. \end{aligned}$$

Similarly, if $\lambda_o \in \Omega_-$, we define

$$\begin{aligned} G_+ &= \{g \in G \mid g \text{ extends } su(n)\text{-holomorphically to } \Omega_+\}, \\ G_- &= \{g \in G \mid g \text{ extends } su(n)\text{-holomorphically to } \Omega_- \ g(\lambda_o) = I\}. \end{aligned}$$

The normalization point λ_o determines an Ad-invariant bilinear form $\langle\ ,\rangle$ on $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ such that $(\mathcal{G}, \mathcal{G}_+, \mathcal{G}_-)$ is a Manin triple. In fact,

$$\langle u, v\rangle = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\oint_\gamma \frac{\mathrm{tr}(u(\lambda)v(\lambda))}{(\lambda - \lambda_o)^2} d\lambda, & \text{if } \lambda_o \in C, \\ \dfrac{1}{2\pi i} \displaystyle\oint_\gamma \mathrm{tr}(u(\lambda)v(\lambda)) d\lambda, & \text{if } \lambda_o = \infty, \end{cases}$$

where $\gamma = \partial\mathcal{O}$. Note that if $u(\lambda) = \sum_k u_k(\lambda - \lambda_o)^k$ and $v = \sum_k v_k(\lambda - \lambda_o)^k$, then

$$\langle u, v\rangle_{\lambda_o} = \begin{cases} \sum_k \operatorname{tr}(u_k v_{-k+1}), & \text{if } \lambda_o \in C, \\ \sum_k \operatorname{tr}(u_k v_{-k-1}), & \text{if } \lambda_o = \infty. \end{cases}$$

The main examples we use in this paper are:

(i) $\Omega_+ = C$, $\Omega_- = \mathcal{O}_\infty$ a neighborhood of ∞,

(ii) $\Omega_+ = C \setminus \{0\}$, $\Omega_- = \mathcal{O}_0 \cup \mathcal{O}_\infty$.

It follows from the Birkhoff Decomposition Theorem (cf. p. 120 Theorem 8.1.2 in the book by Pressley and Segal ([PrS])) that the multiplication map $G_+ \times G_- \to G$ for example (i) is injective and maps onto an open dense subset of G. McIntosh shows that the multiplication map for example (ii) is a diffeomorphism [Mc].

Now suppose (G, G_+, G_-) is a Manin triple group, and the multiplication map $G_+ \times G_- \to G$ is a diffeomorphism. Then given $g_\pm \in G_\pm$, we decompose

$$g_+ g_- = f_- f_+ \in G_- G_+, \qquad g_- g_+ = h_+ h_- \in G_+ G_-.$$

Define

$$g_+ \# g_- = f_-, \qquad g_- \# g_+ = h_+.$$

Then $\#$ defines the dressing action of G_+ on G_- on the left, and the dressing action of G_- on G_+ on the left respectively. Let $x_- \in \mathcal{G}_-$, and $\tilde{x}_-$ denote the infinitesimal vector field of the action of G_- on G_+. Then

$$\tilde{x}_-(g_+) = g_+(g_+^{-1} x_- g_+)_+, \qquad \tilde{x}_+(g_-) = g_-(g_-^{-1} x_+ g_-)_-.$$

There are clearly also corresponding dressing action of G_- on G_+ and G_+ on G_- on the right.

Since the image of the multiplication map is an open dense subset for Example 2.4 (i) and the whole group G for Example 2.4 (ii), the dressing actions for the corresponding Manin triple groups are local and global respectively. However, the Lie algebra actions are defined for all elements in both cases.

2.5 Definition. An action of a Poisson group G on a Poisson manifold P is *Poisson* if the action $G \times P \to P$ is a Poisson map.

It is clear that if the G-action on P is Poisson, M is a Poisson submanifold of P, and M is invariant under G, then the G-action on M is also Poisson. Here one must be careful as the requirement that $M \subset P$ is Poisson is quite restrictive.

A *symplectic structure* on P is a Poisson structure π such that $\pi_x : TP_x^* \to TP_x$ is injective for all $x \in P$. This definition agrees with the standard one when P is finite dimensional, and is the definition of a weak symplectic structure defined in the lecture notes of Chernoff and Marsden [CM] when P is of infinite dimension. For simplicity of notation, we still call such structure a symplectic structure. A G-action on P is called symplectic if $g_*(\pi) = \pi$ for all $g \in G$. If G is equipped with the trivial Poisson structure ($\pi_G = 0$), then an action of G on a symplectic manifold P is Poisson if and only if it is symplectic. However, in general these two notions of actions are different on symplectic manifolds.

A *moment map* of a symplectic action of G on a symplectic manifold P is a G-equivariant map $\mu : P \to \mathcal{G}^*$ such that $\pi_P(df_\xi)$ is the infinitesimal vector field $\tilde{\xi}$ associated to ξ, where f_ξ is the function on P defined by $f_\xi(x) = \mu(x)(\xi)$. When the action is Poisson, we can not expect to define a Poisson map $\mu : P \to \mathcal{G}^*$. The following theorem gives a natural generalization of moment map for Poisson actions.

2.6 Theorem ([Lu]). *Suppose the Poisson group (G, π) acts on the Poisson manifold (P, π_P), and there exists a G-equivariant Poisson map*

$$m : (P, \pi_P) \to (G^*, \pi^*)$$

such that

$$\pi_P(((dm)m^{-1})(\xi)) = \tilde{\xi}, \quad \forall\, \xi \in \mathcal{G},$$

where $\tilde{\xi}$ is the infinitesimal vector field on P associated to ξ and (G^*, π^*) is the dual Poisson group of (G, π). Then the action of (G, π) on (P, π_P) is Poisson.

2.7 Definition. A *moment map* for a Poisson action of a Poisson group G on a Poisson manifold P is a map $m : P \to G^*$ which satisfies the assumptions in the above theorem.

2.8 Example. Suppose (G, G_+, G_-) is a Manin triple group, and the multiplication maps $G_+ \times G_- \to G$ and $G_- \times G_+ \to G$ are diffeomorphisms. Then the dressing action of (G_-, π_-) on (G_+, π_+) is Poisson and the identity map $\mathrm{id} : G_+ \to G_-^* = G_+$ is a moment map. To see this, note first that the identity map is Poisson and equivariant. So by Theorem 2.6 it suffices to check

$$(\pi_+)_{g_+}(dg_+ g_+^{-1}(x_-)) = (\pi_+)_{g_+}(\ell_{x_-}) = g_+(g_+^{-1}x_- g_+)_+ = \tilde{x}_-(g_+).$$

Similarly, the dressing action of (G_+, π_+) on (G_-, π_-) is Poisson.

3 Negative flows in the decay case

Our starting point is the Manin triple $(\mathcal{G}, \mathcal{G}_+, \mathcal{G}_-, \langle\ ,\ \rangle)$ of Cherednik type (Example 2.4 (i)) with $\Omega_+ = C$, $\Omega_- = \mathcal{O}_\infty$ and

$$\langle u, v\rangle = \frac{1}{2\pi i}\oint_{\gamma_\infty} \mathrm{tr}(u(\lambda), v(\lambda))d\lambda,$$

where $\gamma_\infty = \partial\mathcal{O}_\infty$ is a contour around ∞. The basic geometric object is a $\mathcal{G}_+$-valued connection on the real line R of the form

$$D = \frac{d}{dx} + A(x,\lambda) = \frac{d}{dx} + \alpha_k(x)\lambda^k + \alpha_{k-1}(x)\lambda^{k-1} + \ldots + \alpha_0(x).$$

From the analytic point of view there are three distinct theories which have very different algebraic structures:

(1) Asymptotically constant cases—the leading term α_k is a constant $a \in gl(n,C)$ and $\alpha_j(x)$ decays in x for $0 \le j < k$.

(2) Decay case—$\alpha_j(x)$ decays in x for all $0 \le j \le k$.

(3) Periodic case—$\alpha_j(x)$ is periodic in x for all j.

Most of the classical scattering theory deals with the asymptotically constant case, which is the case we discuss in most of the paper. For the periodic case we refer the readers to papers by Krichever [Kr1], [Kr2]. We start with the decay case, as a warm-up for the asymptotically constant case.

Fix an element $\alpha_k \in L^1(R)$. An important example would be $\alpha_k(x) = \rho(x)a$ for $\rho \in L^1(R)$ and $a \in gl(n,C)$. If $\rho = dy/dx$, then we can rewrite the connection in y as

$$\frac{dy}{dx}\left(\frac{d}{dy} + a\lambda^k\right) = \frac{d}{dx} + \rho(x)a\lambda^k.$$

Hence the decay case is in reality the case of a "finite interval". However, we use the parametrization of the infinite interval to demonstrate structural relationships with the asymptotically constant case.

Let $C(R,G_\pm)$ be a linear subspace of maps from R to $G_\pm$, that has a formal Lie group structure with Lie algebra $C(R,\mathcal{G}_\pm)$, where $C(R)$ consists of functions which decay at least as fast as those in $L^1(R)$. Identify a map $A \in C(R,\mathcal{G}_+)$ with an element in $C(R,\mathcal{G}_-)^*$ via the pairing

$$\langle\langle A,T\rangle\rangle = \int_{-\infty}^{\infty} \langle A,T\rangle\, dx.$$

(Note that if $A\,dx$ is thought as a one form, then the above formulation is coordinate invariant). Let $\mathcal{S}$ be a subset of $C(R,\mathcal{G}_+)$ that is invariant under the coadjoint action of $C(R,G_-)$. The infinitesimal vector fields for the coadjoint action of $C(R,G_-)$ on $\mathcal{S}$ are

$$v_T(A)(x) = [A(x), T_-(x)]_+,$$

where $+$ indicates the orthogonal projection from $\mathcal{G}$ onto $\mathcal{G}_+$. The Poisson structure on $\mathcal{S}$ is given by

$$\pi_A(T_-) = -[A,T_-]_+,$$

and gives rise to a symplectic structure on the coadjoint orbits of $C(R, G_-)$ on $\mathcal{S}$.

The coadjoint orbit of $\alpha(x)\lambda^k$ under $C(R, G_-)$ is clearly contained in the set of polynomials of degree k of the form

$$A = \alpha(x)\lambda^k + \alpha_{k-1}(x)\lambda^{k-1} + \ldots + \alpha_1(x)\lambda + \alpha_0(x)$$

with the condition that $\alpha_{k-1}(x)$ is of the form $[\alpha(x), v(x)]$ for some v. For many choice of α, this will be the only constraint. The vector field v_T for $T = \sum_{k=1}^{\infty} T_j(x)\lambda^{-j}$ is

$$v_T(A) = [A, T]_+ = \sum_{j=0}^{k-1}\left(\sum_{i=j+1}^{k} [\alpha_i(x), T_{i-j}(x)]\right)\lambda^j,$$

where $\alpha_k = \alpha$.

The negative flows in the decay case can be easily described. Let $\mathcal{P}(R, \mathcal{G}_+)$ denote the Lie algebra of maps $A : R \to \mathcal{G}_+$ such that $A(x)(\lambda)$ is a polynomial in λ and decay in x. Let $\mathcal{P}_k$ denote the set of all $A \in \mathcal{P}(R, \mathcal{G}_-)$ of degree k, and $\mathcal{P}_{k,\alpha}$ the set of all $A \in \mathcal{P}(R, \mathcal{G}_-)$ whose leading term is $\alpha\lambda^k$. Then $\mathcal{P}(R, \mathcal{G}_+)$, $\mathcal{P}_k$ and $\mathcal{P}_{k,\alpha}$ are invariant under the coadjoint action of $C(R, G_-)$, and

$$\pi_A(T_-) = -[A, T_-]_+$$

gives the Poisson structure.

3.1 Definition. The *trivialization of* $A = \sum_{j=0}^{k} \alpha_j(x)\lambda^j$ *normalized at* $x = -\infty$ is the solution $F(A) \in C(R, G_+)$ of

$$F^{-1}F_x = A, \qquad \lim_{x\to-\infty} F(x, \lambda) = I.$$

Given $b \in su(n)$ and $A = \sum_{j=0}^{k} \alpha_j(x)\lambda^j$, then $F(A)^{-1}(x)bF(A)(x) \in \mathcal{G}_+$. Write the expansion of $F(A)^{-1}bF(A)$ at $\lambda = 0$ to get

$$F(A)^{-1}bF(A) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \ldots. \tag{3.1}$$

The β_j's can be computed explicitly from A. Since

$$(F^{-1}bF)_x + [A, F^{-1}bF] = 0, \tag{3.2}$$

we can compare coefficients of λ^j in equation (3.2) to get

$$\begin{cases} (\beta_0)_x + [\alpha_0, \beta_0] = 0, \\ (\beta_j)_x + [\alpha_0, \beta_j] + \sum_{i=1}^{\min\{j,k\}}[\alpha_i, \beta_{j-i}] = 0. \end{cases}$$

The β_j's can be solved explicitly from $\alpha_0, \ldots, \alpha_k$ as follows: Let $g : R \to GL(n, C)$ be the solution to

$$\begin{cases} g^{-1}g_x = \alpha_0 \\ \lim_{x\to-\infty} g(x) = I. \end{cases}$$

Then

$$\begin{cases}\beta_0 = g^{-1}bg, \\ \beta_j(x) = -g^{-1}(x)\sum_{i=1}^{\min\{j,k\}}\left(\int_{-\infty}^{x} g(y)[\alpha_i(y), \beta_{j-i}(y)]g^{-1}(y)dy\right) g(x).\end{cases} \tag{3.3}$$

Hence we have obtained a family of integral equations to describe the β_j's.

The Lax pair for this system is written

$$\left[\frac{\partial}{\partial x} + A,\ \frac{\partial}{\partial t} + (F^{-1}b\lambda^{-m}F)_-\right] = 0. \tag{3.4}$$

It follows from the definition of β's that the coefficient of λ^j with $j < 0$ in the left hand side of equation (3.4) is automatically zero. Setting the coefficients of λ^j $(j \geq 0)$ in equation (3.4) to zero gives a system of equations describing a flow on $\mathcal{P}_k$:

$$\begin{cases}\dfrac{d\alpha_k}{dt} = 0, \\ \dfrac{d\alpha_j}{dt} = \sum_{i=j+1}^{\min\{k,m+j\}}[\alpha_i, \beta_{m+j-i}].\end{cases} \tag{3.5}$$

We call this flow the *$-m$-flow on $\mathcal{P}_{k,\alpha}$ defined by b.* Equation (3.4) also gives

$$\begin{aligned}A_t &= ((\lambda^{-m}F^{-1}bF)_x)_- + [A, (\lambda^{-m}F^{-1}bF)_-] \\ &= [(\lambda^{-m}F^{-1}bF), A]_- + [A, (\lambda^{-m}F^{-1}bF)_-] \\ &= [A, (F^{-1}b\lambda^{-m}F)_-]_+.\end{aligned}$$

So the $-m$-th flow can also be written as

$$A_t = [A, (F^{-1}b\lambda^{-m}F)_-]_+. \tag{3.6}$$

Since the vector field

$$\xi_{b,j}(A) = [A, (F^{-1}b\lambda^{-m}F)_-]_+$$

is bounded in L^1, it is not difficult to see that the $-m$-flow is global. We will prove these flows generate a natural Poisson group action on $\mathcal{P}_{k,\alpha}$ in the next section.

For our basic model, $k = 1$, we have

$$\begin{aligned}&A = \alpha(x)\lambda + u(x), \\ &v_T(u) = [\alpha(x), T_1(x)], \\ &\{T, V\}_A = \int_{-\infty}^{\infty} \operatorname{tr}(\alpha(x)[T_1(x), V_1(x)])\, dx,\end{aligned}$$

where $T = \sum_{j=1}^{\infty} T_j\lambda^{-j}$ and $V = \sum_{j=1}^{\infty} V_j\lambda^{-j}$. This gives our next proposition.

3.2 Proposition. *The $-m$-th flow on $\mathcal{P}_{1,\alpha}$ defined by b is*

$$u_t = [\alpha, \beta_{m-1}],$$

where β_j is defined inductively by

$$\begin{cases} \beta_0 = g^{-1}bg, \\ \beta_j(x) = -g^{-1}(x)\left(\int_{-\infty}^{x} g[\alpha,\beta_{j-1}]g^{-1}dy\right)g(x), \end{cases}$$

and g is the solution to $g^{-1}g_x = u$ and $\lim_{x\to-\infty} g(x) = I$.

A simple change of gauge (cf. [Te2]) implies that the -1-flow describes the geometric equation for harmonic maps from $R^{1,1}$ into $U(n)$ in characteristic coordinates:

3.3 Proposition. *Fix a smooth L^1-map $\alpha : R \to u(n)$ and $b \in u(n)$. Suppose $u(x,t)$ is a solution of the -1-flow equation on $\mathcal{P}_{1,\alpha}$ defined by b:*

$$u_t = [\alpha, g^{-1}bg], \qquad \text{where } g^{-1}g_x = u, \ \lim_{x\to-\infty} g(x) = I. \tag{3.7}$$

Then there exists a unique solution $E(x,t,\lambda)$ for

$$\begin{cases} E^{-1}E_x = \alpha\lambda + u, \\ E^{-1}E_t = \lambda^{-1}g^{-1}bg, \\ E_\lambda(0,0) = I. \end{cases}$$

Set $s(x,t) = E(x,t,-1)E(x,t,1)^{-1}$. Then $s : R^{1,1} \to U(n)$ is harmonic, $(s^{-1}s_x)(x,t)$ is conjugate to $-2\alpha(x)$, and $(s^{-1}s_t)(x,t)$ is conjugate to $-2b$ for all $t \in R$.

Harmonic maps into a symmetric space are obtained by restriction ([Te2]). This is discussed in section 9. Also, a more elaborate choice of Cherednik splittings allows more complicated examples like the harmonic map equation in space-time (laboratory) coordinates.

4 Poisson structure for negative flows (decay case)

The dressing action defines a local action of G_- on $\mathcal{P}(R, \mathcal{G}_+)$ which is Poisson and generates the negative flows. The notation is the same as in section 3.

4.1 Theorem. *For $A \in \mathcal{P}(R, \mathcal{G}_+)$, let $F(A) : R \to G_+$ denote the trivialization of A normalized at $x = -\infty$. Given $g_- \in G_-$, let $\tilde{F}(x) = g_- \sharp (F(A)(x))$, where $\sharp$ denotes the dressing action of G_- at G_+ for each $x \in R$. Define*

$$g_- * A = \tilde{F}^{-1}\tilde{F}_x.$$

Then $g_- * A$ defines a local action of G_- on $\mathcal{P}(R, \mathcal{G}_+)$. Moreover, the infinitesimal vector field $\tilde{\xi}_-$ associated to $\xi_- \in \mathcal{G}_-$ for this action is

$$\tilde{\xi}_-(A) = -[A, (F^{-1}(A)\xi_- F(A))_-]_+. \tag{4.1}$$

Proof. It is clear that $(g_- * A)$ defines a local action of G_- on $\mathcal{C}(R, \mathcal{G}_+)$. Now we compute the infinitesimal vector field $\tilde{\xi}_-$ on $\mathcal{C}(R, \mathcal{G}_+)$. Write $g_- F = \tilde{F} f_-$, and let δ denote the tangent variation. Then $(\delta g_-)F = \delta\tilde{F} + F\delta f_-$, which implies that

$$F^{-1}(\delta g_-)F = F^{-1}\delta\tilde{F} + \delta f_-.$$

If $\xi_- = \delta g_-$, then we have

$$\delta f_- = (F^{-1}\xi_- F)_-, \qquad F^{-1}\delta\tilde{F} = (F^{-1}\xi_- F)_+. \tag{4.2}$$

Since $g_- * A = \tilde{F}^{-1}\tilde{F}_x$, we obtain

$$\begin{aligned}
\tilde{\xi}_-(A) &= -F^{-1}(\delta\tilde{F})F^{-1}F_x + F^{-1}(\delta\tilde{F})_x \\
&= -(F^{-1}\xi_- F)_+ A + F^{-1}(F(F^{-1}\xi_- F)_+)_x \\
&= -(F^{-1}\xi_- F)_+ A + A(F^{-1}\xi_- F)_+ + ((F^{-1}\xi_- F)_x)_+ \\
&= [A, (F^{-1}\xi_- F)_+] + [F^{-1}\xi_- F, A]_+ \\
&= -[A, (F^{-1}\xi_- F)_-]_+.
\end{aligned}$$

Since $x \mapsto A(x)(\lambda)$ is in $L^1(R) \cap C^\infty(R)$ and $x \mapsto F(x, \lambda)$ is bounded for all λ, we have $\tilde{\xi}_-$ is tangent to $\mathcal{P}(R, \mathcal{G}_+)$. □

4.2 Corollary. *The local action of G_- on $\mathcal{P}(R, \mathcal{G}_+)$ leaves $\mathcal{P}_{k,\alpha}$ invariant, and the flow generated by $\xi_- = -b\lambda^{-m}$ is the $-m$-flow on $\mathcal{P}_{k,\alpha}$ defined by b.*

4.3 Theorem. *The local action of G_- on $\mathcal{P}(R, \mathcal{G}_+)$ is Poisson. The infinitesimal vector field corresponding to ξ_- is $\tilde{\xi}_-(A) = -[A, (F^{-1}\xi_- F)_-]_+$, where F is the trivialization of A normalized at $x = -\infty$. In fact, the map $\phi : \mathcal{P}(R, \mathcal{G}_+) \to G_+ = G_-^*$ defined by $\phi(A) = \lim_{x\to\infty} F(A)(x)$ is a moment map for this action.*

To prove the theorem, we first need a lemma:

4.4 Lemma. $d\phi_A(B) = \left(\int_{-\infty}^{\infty} F(A)BF(A)^{-1}dx\right)\phi(A)$.

Proof. Let F denote $F(A)$, and $\delta F = dF_A(B)$. Taking the variation of the equation $F^{-1}F_x = A$, we get $(F^{-1}\delta F)_x + [A, F^{-1}\delta F] = B$. This implies that

$$F^{-1}\delta F = F(A)^{-1}\left(\int_{-\infty}^{x} F(A)(y,\lambda)B(y,\lambda)F^{-1}(A)(y,\lambda)dy\right)F(A).$$

Then the lemma follows from taking the limit as $x \to \infty$. □

4.5 Proof of Theorem 4.3. It suffices to prove that ϕ satisfies the assumption in Theorem 2.6. First we prove that ϕ is G_--equivariant. Taking the limit of $g_-F = \tilde{F}f_-$ as $x \to -\infty$, we get

$$\lim_{x\to-\infty} \tilde{F}(\lambda, x) = I, \qquad \lim_{x\to-\infty} f_-(\lambda, x) = g_-(\lambda).$$

So $F(g_- * A) = \tilde{F}$ and

$$\phi(g_- * A) = \lim_{x\to\infty} \tilde{F} = g_-\phi(A)(\lim_{x\to\infty} f_-)^{-1} = g_-\#\phi(A).$$

This proves that ϕ is G_--equivariant.

Given $\xi_- \in \mathcal{G}_-$ and $B \in \mathcal{P}(R, \mathcal{G}_+)$, using Lemma 4.4 we get

$$\begin{aligned}\langle\langle d\phi_A(B)(\phi(A))^{-1}, \xi_-\rangle\rangle &= \langle\langle F(A)BF(A)^{-1}, \xi_-\rangle\rangle \\ &= \langle\langle B, F(A)^{-1}\xi_-F(A)\rangle\rangle \\ &= \langle\langle B, (F^{-1}\xi_-F)_-\rangle\rangle.\end{aligned}$$

So $(\Pi_+)_A(d\phi_A\phi(A)^{-1}, \xi_-) = \tilde{\xi}_-(A)$.

It remains to prove that ϕ is a Poisson map. Given $\xi_-, \eta_- \in \mathcal{G}_-$, let $g_+ = \phi(A)$, and ℓ_i the linear functional on $T(G_+)_{g_+}$ defined by

$$\ell_1(z_+g_+) = \langle \xi_-, z_+\rangle, \qquad \ell_2(z_+g_+) = \langle \eta_-, z_+\rangle.$$

It follows form Lemma 4.4 that

$$\ell_1 \circ d\phi_A(B) = \langle B, (F^{-1}\xi_-F)_-\rangle, \qquad \ell_2 \circ d\phi_A(B) = \langle B, (F^{-1}\eta_-F)_-\rangle.$$

But $(F^{-1}\xi_-F)_x + [A, F^{-1}\xi_-F] = 0$. So we get

$$\begin{aligned}&\Pi_+(\ell_1 \circ d\phi_A, \ell_2 \circ d\phi_A) \\ &\quad= -\langle\langle [A, (F^{-1}\xi_-F)_-], (F^{-1}\eta_-F)_-\rangle\rangle \\ &\quad= -\langle\langle [A, (F^{-1}\xi_-F) - (F^{-1}\xi_-F)_+], (F^{-1}\eta_-F)_-\rangle\rangle \\ &\quad= \langle\langle (F^{-1}\xi_-F)_x + [A, (F^{-1}\xi_-F)_+], (F^{-1}\eta_-F)_-\rangle\rangle \\ &\quad= \langle\langle (F^{-1}\xi_-F)_x, (F^{-1}\eta_-F)_-\rangle\rangle + \langle\langle [A, (F^{-1}\xi_-F)_+], (F^{-1}\eta_-F)_-\rangle\rangle \\ &\quad= \langle F^{-1}\xi_-F, (F^{-1}\eta_-F)_-\rangle \,|_{x=-\infty}^{x=\infty} - \langle\langle (F^{-1}\xi_-F), ((F^{-1}\eta_-F)_-)_x\rangle\rangle \\ &\qquad\quad + \langle\langle [A, (F^{-1}\xi_-F)_+], (F^{-1}\eta_-F)_-\rangle\rangle.\end{aligned}$$

The first term is equal to

$$\begin{aligned}\langle g_+^{-1}\xi_-g_+, (g_+^{-1}\eta_-g_+)_-\rangle - \langle \xi_-, \eta_-\rangle &= \langle g_+^{-1}\xi_-g_+, (g_+^{-1}\eta_-g_+)_-\rangle \\ &= (\pi_+)_{g_+}(\xi_-g_+, \eta_-g_+) = (\pi_+)_{g_+}(\ell_1, \ell_2),\end{aligned}$$

where $\langle \xi_-, \eta_- \rangle = 0$ because $\mathcal{G}_-$ is isotropic with respect to $(\, ,\,)$. The second term is

$$\begin{aligned}\langle\langle (F^{-1}\xi_- F), ((F^{-1}\eta_- F)_-)_x\rangle\rangle &= \langle\langle (F^{-1}\xi_- F)_+, (F^{-1}\eta_- F)_x\rangle\rangle \\ &= \langle\langle (F^{-1}\xi_- F)_+, -[A, F^{-1}\eta_- F]\rangle\rangle \\ &= \langle\langle [A, (F^{-1}\xi_- F)_+], F^{-1}\eta_- F\rangle\rangle \\ &= \langle\langle [A, (F^{-1}\xi_- F)_+], (F^{-1}\eta_- F)_-\rangle\rangle,\end{aligned}$$

which cancels the third term. This proves that ϕ is Poisson. Since ϕ satisfies all assumptions of Theorem 4.3, the action of G_- on $\mathcal{L}_+$ is Poisson and ϕ is a moment map. □

5 Positive flows in the asymptotically constant case

In this section, we will use the same Manin triple as in section 3, and describe flows in the asymptotically constant case. We restrict our discussion to the simplest cases.

Fix $a \in su(n)$, and set

$$\begin{aligned} U_a &= \{g \in SU(n) \mid ga = ag\}, \\ \mathcal{U}_a &= \{y \in su(n) \mid [a, y] = 0\}, \\ \mathcal{U}_a^\perp &= \{z \in su(n) \mid \langle z, \mathcal{U}_a\rangle = 0\}.\end{aligned}$$

Given a vector space V, we let $\mathcal{S}(R, V)$ denote the space of all maps from R to V that are in the Schwartz class. Let $\mathcal{S}_{1,a}$ denote the space of all maps $A : R \to \mathcal{G}_+$ such that $A(x)(\lambda) = a\lambda + u(x)$ with $u \in \mathcal{S}(R, \mathcal{U}_a^\perp)$. The basic symplectic structure on $\mathcal{S}_{1,a}$ is similar to what we have described already for the decay case. However, the structure of the natural flows is different because we may not normalize at $x = -\infty$. Integration as described in the negative flows will tend to destroy the decay condition. The -1-flow does in some sense exist: $\mathcal{S}_{1,a}$

$$\begin{cases} u_t = [a, g^{-1}bg], \\ g_x = gu, \\ \lim_{x\to-\infty} g = I. \end{cases} \tag{5.1}$$

However, the right-hand boundary at ∞ will not be under control and the symplectic structure does not make coherent sense.

Rather than identify A with the trivialization F normalized at $x = -\infty$, we use two different trivializations. For the purposes of constructing Bäcklund transformations, we identify A with the trivialization E normalized at $x = 0$, i.e.,

$$E^{-1}E_x = a\lambda + u, \qquad E(0,\lambda) = I.$$

When we describe the Poisson structure of the positive flows we use $M(x,\lambda)$, where

$$(e^{a\lambda x}M)^{-1}(e^{a\lambda x}M)_x = A, \qquad \lim_{x\to-\infty} M(x,\lambda) = I.$$

Since both E and $e^{a\lambda x}M$ solve the same linear equation, there exists $f(\lambda)$ such that

$$f(\lambda)E(x,\lambda) = e^{a\lambda x}M(x,\lambda).$$

Note that $f(\lambda) = M(0,\lambda)$ contains all the spectral information. The general condition is that f is not holomorphic at $\lambda = \infty$, but that both $f(\lambda)$ and $M(x,\lambda)$ have asymptotic expansion at $\lambda = \infty$. This is known to be the case in scattering theory, and we need our theory to mesh with this analysis.

The positive flows for the asymptotically constant case are defined in a similar fashion as the negative flows for the decay case with the restriction that the generators commute with a. The hierarchy of flows is now mixed ordinary differential and integral equations. Let $A = a\lambda + u$ with $u \in \mathcal{S}(R, \mathcal{U}_a^{\perp})$, and M as above. Fix $b \in u(n)$ such that $[a,b] = 0$. Then $M^{-1}bM$ has an asymptotic expansion at $\lambda = \infty$ (cf. [BC1,2]):

$$M^{-1}bM \;\sim\; Q_{b,0} + Q_{b,1}\lambda^{-1} + Q_{b,2}\lambda^{-2} + \dots, \qquad Q_{b,0} = b.$$

Since

$$M^{-1}bM = E^{-1}f^{-1}e^{a\lambda x}be^{-a\lambda x}fE = E^{-1}f^{-1}bfE,$$

we get $(M^{-1}bM)_x + [a\lambda + u, M^{-1}bM] = 0$. So we have

$$(Q_{b,i})_x + [u, Q_{b,i}] + [a, Q_{b,i+1}] = 0. \tag{5.2}$$

This defines $Q_{b,i}$'s recursively.

An element $a \in u(n)$ is *regular* if a has distinct eigenvalues. Otherwise, a is *singular*. If a is regular, then it is known that $Q_{b,i}$'s are polynomial differential operators in u (cf [Sa]). But when a is singular, the $Q_{b,i}$'s are integral-differential operators in u. To be more precise, we decompose

$$Q_{b,j} = P_{b,j} + T_{b,j} \in \mathcal{U}_a + \mathcal{U}_a^{\perp}.$$

Using equation (5.2), P's and T's can be solved recursively. In fact,

$$\begin{cases} P_{b,0} = 0, \\ T_{b,0} = b, \\ P_{b,j+1} = -\operatorname{ad}(a)^{-1}((P_{b,j})_x + [u, Q_{b,j}]^{\perp}), \\ T_{b,j+1} = -\int_{-\infty}^{x}[u, P_{b,j+1}]^{\mathcal{T}}dy, \end{cases} \tag{5.3}$$

where $v^{\perp}$ and $v^{\mathcal{T}}$ denote the projection onto $\mathcal{U}_a^{\perp}$ and $\mathcal{U}_a$ respectively, and $-\operatorname{ad}(a)$ maps $\mathcal{U}_a^{\perp}$ isomorphically to $\mathcal{U}_a^{\perp}$. It follows from induction and formula (5.3) that the $T_{b,j}$ are bounded and the $P_{b,j}$ are in the Schwartz class.

Consider the Lax pair

$$\left[\frac{\partial}{\partial x} + A,\ \frac{\partial}{\partial t} + (M^{-1}b\lambda^j M)_+\right] = 0. \tag{5.4}$$

Set the coefficient of λ^{j-k}, $0 \le k < j$, in equation (5.4) equal to zero to get

$$\begin{cases} [a, Q_{b,0}] = 0, \\ (Q_{b,k})_x + [u, Q_{b,k}] + [a, Q_{b,k+1}] = 0, \quad 1 \le k < j. \end{cases} \tag{5.5}$$

This defines the $Q_{b,j}$'s. The constant term gives

$$u_t = (Q_{b,j})_x + [u, Q_{b,j}] = [Q_{b,j+1}, a] \tag{5.6}$$

which is called the *j-th flow equation on $\mathcal{S}_{1,a}$ defined by b.* Equation (5.4) can also be written as

$$\begin{aligned} A_t &= ((M^{-1}b\lambda^j M)_+)_x + [A, (M^{-1}b\lambda^j M)_+] \\ &= [M^{-1}b\lambda^j M, M^{-1}M_x]_+ + [A, (M^{-1}b\lambda^j M)_+] \\ &= [M^{-1}b\lambda^j M, A - M^{-1}a\lambda M]_+ + [A, (M^{-1}b\lambda^j M)_+] \\ &= [M^{-1}b\lambda^j M, A]_+ + [A, (M^{-1}b\lambda^j M)_+] \\ &= [(M^{-1}b\lambda^j M)_-, A]_+ = [Q_{b,j+1}, a]. \end{aligned}$$

It is clear that the following three statements are equivalent:

(i) $\left[\frac{\partial}{\partial x} + A, \frac{\partial}{\partial t} + B\right] = 0$,

(ii) the connection 1-form $\theta = A\,dx + B\,dt$ is flat for all λ, i.e., $d\theta = -\theta \wedge \theta$,

(iii) $\begin{cases} E^{-1}E_x = A, \\ E^{-1}E_t = B, \end{cases}$ is solvable.

So we have

5.1 Proposition. *$A = a\lambda + u$ is a solution of the j-th flow (5.6) on $\mathcal{S}_{1,a}$ defined by b if and only if*

$$\theta(x,t,\lambda) = (a\lambda + u)dx + (b\lambda^j + Q_{b,1}\lambda^{j-1} + \ldots + Q_{b,j})dt$$

is flat on the (x,t)-plane for each λ.

5.2 Definition. The one parameter family of connection 1-form θ defined in Proposition 5.1 is called the *flat connection associated to the solution* A of the j-th flow. The unique solution $E : R^2 \times C \to GL(n, C)$ of

$$\begin{cases} E^{-1}E_x = a\lambda + u, \\ E^{-1}E_t = b\lambda^j + Q_{b,1}\lambda^{j-1} + \ldots + Q_{b,j}, \\ E(0,0,\lambda) = I \end{cases}$$

is called *the trivialization of the flat connection* θ *normalized at the origin* or *the trivialization of the solution* A *at* $(x, t) = (0, 0)$.

When a is regular, positive flows are the familiar hierarchy of commuting Hamiltonian flows described by differential equations. When a is singular, positive flows generate a non-abelian Poisson group action. This will be described in section 8.

5.3 Example. For $su(2)$ with $a = \mathrm{diag}(i, -i)$, $\mathcal{S}_{1,a}$ is the set of A of the form $a\lambda + u$, where

$$u = \begin{pmatrix} 0 & f \\ -\bar{f} & 0 \end{pmatrix}$$

and $f : R \to C$ is in the Schwartz class. The first flow is the translation $u_t = u_x$, the second flow defined by a is the non-linear Schrödinger equation (NLS)

$$q_t = \frac{i}{2}(q_{xx} + 2|q|^2 q), \tag{5.7}$$

and the positive flows are the hierarchy of commuting flows associated to the non-linear Schrödinger equation.

5.4 Example. For $a = \mathrm{diag}(a_1, \ldots, a_n) \in su(n)$ with $a_1 < \ldots < a_n$, $\mathcal{S}_{1,a}$ is the set of all $A = a\lambda + u$, where $u = (u_{ij}) \in su(n)$ and $u_{ii} = 0$ for all $1 \le i \le n$. The first flow on $\mathcal{S}_{1,a}$ defined by a is the translation

$$u_t = u_x.$$

The first flow on $\mathcal{S}_{1,a}$ defined by $b = \mathrm{diag}(b_1, \ldots, b_n)$ $(b \neq a)$ is the n-wave equation ([ZMa1, 2]) for u:

$$(u_{ij})_t = \frac{b_i - b_j}{a_i - a_j}(u_{ij})_x + \sum_{k \neq i,j} \left(\frac{b_k - b_j}{a_k - a_j} - \frac{b_i - b_k}{a_i - a_k} \right) u_{ik}u_{kj}, \qquad i \neq j.$$

If a is singular and $[b, a] = 0$, then the j-th flow on $\mathcal{S}_{1,a}$ defined by b is in general an integro-differential equation. But the j-th flow on $\mathcal{S}_{1,a}$ defined by a is again a differential operator:

5.5 Proposition. *$Q_{a,j}(u)$ is always a polynomial differential operator in u.*

Proof. It is easy to see that $Q_{a,1} = u$. We will prove this Proposition by induction. Suppose $Q_{a,i}$ is a polynomial differential operator in u for $i \leq j$. Write

$$Q_{a,i} = P_{a,i} + T_{a,i} \in \mathcal{U}_a^\perp + \mathcal{U}_a$$

as before. Using formula (5.3), we see that $P_{a,j+1}$ is a polynomial differential operator in u. But we can not conclude from formula (5.3) that $T_{a,j+1}$ is a polynomial differential operator in u. Suppose a has k distinct eigenvalues $c_1, \ldots, c_k$. Then

$$f(t) = (t - c_1)(t - c_2) \ldots (t - c_k)$$

is the minimal polynomial of a. So $f(M^{-1}aM) = 0$, which implies that the formal power series

$$f(a + Q_{a,1}\lambda^{-1} + Q_{a,2}\lambda^{-2} + \ldots) = 0. \tag{5.8}$$

Notice that $f'(a)$ is invertible and $T_{a,j+1}$ commutes with a. Now compare coefficient of $\lambda^{-(j+1)}$ in equation (5.8) implies that $T_{a,j+1}$ can written in terms of $a, Q_{a,1}, \ldots, Q_{a,j}$. This proves that $Q_{a,j+1}$ is a polynomial differential operator in u. □

5.6 Example. For $u(n)$ with

$$a = \begin{pmatrix} iI_k & 0 \\ 0 & -iI_{n-k} \end{pmatrix},$$

$$\mathcal{S}_{1,a} = \left\{ a\lambda + u \;\middle|\; u = \begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix}, X \in \mathcal{M}_{k\times(n-k)} \right\},$$

where $\mathcal{M}_{k\times(n-k)}$ is the space of $k \times (n-k)$ complex matrices. Identifying $\mathcal{S}_{1,a}$ as $\mathcal{S}(R, \mathcal{M}_{k\times(n-k)})$, then the bi-linear form

$$\langle u, v \rangle = \int_{-\infty}^{\infty} \operatorname{tr}(uv)\, dx$$

on $\mathcal{S}(R, \mathcal{U}_a^\perp)$ induces the following bi-linear form on $\mathcal{S}(R, \mathcal{M}_{k\times(n-k)})$:

$$\langle X, Y \rangle = -\int_{-\infty}^{\infty} \operatorname{tr}(XY^* + X^*Y)\, dx.$$

The orbit symplectic structure on $\mathcal{S}_{1,a}$ induces the following symplectic structure on $\mathcal{S}(R, \mathcal{M}_{k\times(n-k)})$:

$$w(X, Y) = \left\langle \frac{i}{2}X, Y \right\rangle.$$

According to Propositions 5.5, the j-th flow defined by a can be written down explicitly. For

$$u = \begin{pmatrix} 0 & B^* \\ -B^* & 0 \end{pmatrix},$$

we have $Q_{a,0} = a$, $Q_{a,1} = u$,

$$Q_{a,2} = \begin{pmatrix} \frac{1}{2i}BB^* & \frac{i}{2}B_x \\ \frac{i}{2}B_x^* & -\frac{1}{2i}B^*B \end{pmatrix}.$$

The first three flows on $\mathcal{S}(R, \mathcal{M}_{k\times(n-k)})$ are

$$\begin{aligned} B_t &= B_x \\ B_t &= \frac{i}{2}(B_{xx} + 2BB^*B) \\ B_t &= -\frac{1}{4}B_{xxx} - \frac{3}{4}(B_xB^*B + BB^*B_x). \end{aligned}$$

Notice that the second flow is the matrix non-linear Schrödinger equation associated to $Gr(k, C^n)$ by Fordy and Kulish [FK]. By Proposition 5.1, B is a solution of the second flow if and only if

$$(a\lambda + u)\; dx + \left(a\lambda^2 + u\lambda + Q_{a,2}\right) dt$$

is flat for all λ.

6 Action of the rational loop group

The rational loop group is used to construct the soliton data for the positive flows discussed in section 5. We first define a local action $\sharp$ of G_- on $C(R, \mathcal{G}_+)$ via the dressing action. In general the G_--action does not preserve the space $\mathcal{S}_{1,a}$ (because the Schwartz condition on u for $A = a\lambda + u$ is not preserved even locally). However, we prove that the action $\sharp$ of the subgroup G_-^m of rational maps in G_- leaves $\mathcal{S}_{1,a}$ invariant. We also show that the factorization can be done explicitly. In particular, the action $g_- \sharp A$ can be computed explicitly in terms of the trivialization $E(A)$ of A normalized at $x = 0$. In fact, $g_- \sharp A$ is given by an algebraic formula in terms of $E(A)$ and g.

Let $A \in C(R, \mathcal{G}_+)$, and $E(x, \lambda)$ denote the trivialization of A normalized at $x = 0$. Then the map $A \mapsto E$ identifies $C(R, \mathcal{G}_+)$ with a subset of $C(R, G_+)$. (We write $E(x)(\lambda) = E(x, \lambda)$).

Given $f_- \in G_-$ and $A \in C(R, \mathcal{G}_+)$, define

$$f \sharp A = \tilde{E}^{-1}(\tilde{E})_x,$$

where $\tilde{E}(x) = f_- \sharp E(x)$ is the dressing action of G_- on G_+ for each $x \in R$. In other words, we factor

$$f_-E(x) = \tilde{E}(x)\tilde{f}_-(x) \in G_+ \times G_-.$$

Clearly, this defines a local action of G_- on $C(R, \mathcal{G}_+)$. For $\xi_- \in \mathcal{G}_-$, the corresponding infinitesimal vector field on $C(R, \mathcal{G}_+)$ is

$$\tilde{\xi}_-(A) = -[A, (E(A)^{-1}\xi_- E(A))_-]_+.$$

6.1 Proposition. *Let a be a fixed diagonal element in $u(n)$, and $C_{1,a}$ the space of all $A \in C(R, \mathcal{G}_+)$ such that $A(x)(\lambda) = a\lambda + u(x)$ and $u : R \to \mathcal{U}_a^\perp$ is a smooth map. Then $\xi_- \mapsto \tilde{\xi}_-$ defines an action of $\mathcal{G}_-$ on $C_{1,a}$.*

In general, the action of G_- does not preserve the Schwartz condition for $\mathcal{S}_{1,a}$. So it does not define an action on $\mathcal{S}_{1,a}$. But the subgroup G_-^m of rational maps does preserve the decay condition.

6.2 Theorem. *Let G_-^m be the subgroup of rational maps $g \in G_-$. Then the $\sharp$ action of G_-^m on $C(R, \mathcal{G}_+)$ leaves the space $\mathcal{S}_{1,a}$ invariant. Moreover, let $g \in G_-^m$, $A \in \mathcal{S}_{1,a}$, and E the trivialization of A normalized at $x = 0$, then*

(i) we can factor $gE(x) = \tilde{E}(x)\tilde{g}(x) \in G_+ \times G_-^m$ and $g \sharp A = \tilde{E}^{-1}(\tilde{E})_x$,

(ii) $g \sharp A$ can be constructed algebraically from E and g.

To prove this theorem, we first recall the following result of the second author [U1]:

6.3 Proposition ([U1]). *Let $z \in C \setminus R$, V a complex linear subspace of C^n, π the projection of C^n onto V, and $\pi^\perp = I - \pi$. Set*

$$g_{z,\pi}(\lambda) = \pi + \frac{\lambda - z}{\lambda - \bar{z}}\pi^\perp. \tag{6.1}$$

Then

(i) $g_{z,\pi} \in G_-^m$,

(ii) G_-^m is generated by $\{g_{z,\pi} \mid z \in C \setminus R, \pi \text{ is a projection}\}$. ($g_{z,\pi}$ will be called a simple element).

6.4 Proposition.

(i) Let $g(\lambda) = \Pi_{j=1}^r \frac{\lambda - z_j}{\lambda - \bar{z}_j}$, and $A = a\lambda + u$. Then $g \in G_-^m$ and $g \sharp A = A$.

(ii) Let $v_1, \ldots, v_k$ be a unitary basis of the linear subspace V, π_j the projection of C^n onto Cv_j, and π the projection onto V. Then

$$\prod_{j=1}^k g_{z,\pi_j} = \left(\frac{\lambda - z}{\lambda - \bar{z}}\right)^{k-1} g_{z,\pi}.$$

Proof. Statement (i) follows from the fact that g commutes with G_+ and G_-. Statement (ii) follows from a direct computation. □

The above two Propositions imply that to prove Theorem 6.2 it suffices to prove $g_{z,\pi} \sharp A \in \mathcal{S}_{1,a}$, where π is the projections onto a one dimensional subspace. First, we give an explicit construction of $g_{z,\pi} \sharp A$.

6.5 Theorem. *Let $A = a\lambda + u \in \mathcal{S}_{1,a}$, and E the trivialization of A normalized at $x = 0$. Let $z \in C\backslash R$, V a complex linear subspace of C^n, and π the projection onto V. Set*

$$\begin{aligned}
\tilde{V}(x) &= E(x,z)^*(V), \\
\tilde{\pi}(x) &= \text{the projection of } C^n \text{ onto } \tilde{V}(x), \\
\tilde{E}(x,\lambda) &= g_{z,\pi}(\lambda) E(x,\lambda) g_{z,\tilde{\pi}(x)}{}^{-1} \\
&= \left(\pi + \frac{\lambda - z}{\lambda - \bar{z}}\pi^{\perp}\right) E(x,\lambda) \left(\tilde{\pi}(x) + \frac{\lambda - \bar{z}}{\lambda - z}\tilde{\pi}(x)^{\perp}\right).
\end{aligned}$$

Then:

(i) $g_{z,\pi} \sharp E = \tilde{E}$.

(ii) $\tilde{\pi}^{\perp}(\tilde{\pi}_x + (a\bar{z} + u)\tilde{\pi}) = 0$.

(iii) If $v : R \to C^n$ is a smooth map such that $v(x) \in \tilde{V}(x)$ for all $x \in R$, then $v_x(x) + (a\bar{z} + u)v(x) \in \tilde{V}(x)$ for all x.

(iv) $g_{z,\pi} \sharp A = A + (z - \bar{z})[\tilde{\pi}, a]$.

Proof. First we claim that $\tilde{E}(x,\lambda)$ is holomorphic for $\lambda \in C$. By definition, $\tilde{E}$ is holomorphic in $\lambda \in C \setminus \{z, \bar{z}\}$ and has possible poles at $z, \bar{z}$ with order one. The residues of $\tilde{E}$ at these two points can be computed easily:

$$\begin{aligned}
\operatorname{Res}(\tilde{E}, z) &= (z - \bar{z})\pi E(x,z)\tilde{\pi}^{\perp}(x), \\
\operatorname{Res}(\tilde{E}, \bar{z}) &= (\bar{z} - z)\pi^{\perp} E(x,\bar{z})\tilde{\pi}(x).
\end{aligned}$$

Since $A(x,\bar{z})^* + A(x,z) = 0$ and $E(0,\lambda) = I$, $E(x,\bar{z})^* E(x,z) = I$. This implies that

$$\tilde{V}(x) = E(x,z)^*(V) = E(x,\bar{z})^{-1}(V).$$

So both residues are zero, and the claim is proved. In particular, we have $g_{z,\pi}E(x) = \tilde{E}(x) g_{z,\tilde{\pi}(x)} \in G_+ \times G_-$. This implies (i).

By Proposition 6.1, $\tilde{E}^{-1}(\tilde{E})_x = a\lambda + \tilde{u}(x)$ for some smooth $\tilde{u} : R \to \mathcal{U}_a^{\perp}$. We get from the formula for $\tilde{E}$ that

$$\begin{aligned}
a\lambda + \tilde{u} &= g_{z,\tilde{\pi}}(a\lambda + u)g_{z,\tilde{\pi}}^{-1} - (g_{z,\tilde{\pi}})_x g_{z,\tilde{\pi}}^{-1} \\
&= \left(\tilde{\pi} + \frac{\lambda - z}{\lambda - \bar{z}}\tilde{\pi}^{\perp}\right)(a\lambda + u)\left(\tilde{\pi} + \frac{\lambda - \bar{z}}{\lambda - z}\tilde{\pi}^{\perp}\right) \\
&\qquad - \left(\tilde{\pi}_x + \frac{\lambda - z}{\lambda - \bar{z}}\tilde{\pi}_x^{\perp}\right)\left(\tilde{\pi} + \frac{\lambda - \bar{z}}{\lambda - z}\tilde{\pi}^{\perp}\right).
\end{aligned} \tag{6.2}$$

Since the left hand side is holomorphic at $\lambda = z$, the residue of the right hand side at $\lambda = z$ is zero. This gives $(\tilde{\pi}(az+u) - \tilde{\pi}_x)\tilde{\pi}^\perp = 0$, which is equivalent to (ii).

Statement (iii) follows from (ii) since

$$\begin{aligned} v_x + (a\bar{z}+u)v &= (\tilde{\pi}(v))_x + (a\bar{z}+u)v \\ &= \tilde{\pi}_x v + \tilde{\pi} v_x + (a\bar{z}+u)v \\ &= (\tilde{\pi}_x + (a\bar{z}+u)\tilde{\pi})(v) + \tilde{\pi}(v_x) \in \tilde{V}(x). \end{aligned}$$

To prove (iv), we multiply $g_{z,\tilde{\pi}}$ to both sides of equation (6.2) and get

$$\begin{aligned} ((\lambda-\bar{z})\tilde{\pi} + (\lambda - z)\tilde{\pi}^\perp)(a\lambda+u) - ((\lambda-\bar{z})\tilde{\pi}_x + (\lambda-z)\tilde{\pi}^\perp_x) \\ = (a\lambda+\tilde{u})((\lambda-\bar{z})\tilde{\pi} + (\lambda-z)\tilde{\pi}^\perp). \end{aligned}$$

Set $\lambda = z$ and $\lambda = \bar{z}$ in the above equation, we get

$$\begin{cases} \tilde{\pi}(az+u) - \tilde{\pi}_x = (az+\tilde{u})\tilde{\pi}, \\ \tilde{\pi}^\perp(a\bar{z}+u) - \tilde{\pi}^\perp_x = (a\bar{z}+\tilde{u})\tilde{\pi}^\perp. \end{cases} \tag{6.3}$$

Add the two equations in (6.3) to get

$$\tilde{u} = u + (z-\bar{z})[\tilde{\pi}, a]. \tag{6.4}$$

□

6.6 Theorem. *The map $\tilde{\pi}$ in Theorem 6.5 is the solution of the following ordinary differential equation:*

$$\begin{cases} (\tilde{\pi})_x + [az+u, \tilde{\pi}] = (\bar{z}-z)[\tilde{\pi}, a]\tilde{\pi}, \\ \tilde{\pi}^* = \tilde{\pi}, \quad \tilde{\pi}^2 = \tilde{\pi}, \quad \tilde{\pi}(0) = \pi. \end{cases} \tag{6.5}$$

Moreover, if $\tilde{\pi}$ is a solution of this equation then $[\tilde{\pi}, a]$ is in the Schwartz class.

Proof. Substitute equation (6.4) into the first equation of (6.3) to get the equation (6.5).

By Proposition 6.4, to prove $[\tilde{\pi}, a]$ is in the Schwartz class it suffices to prove it for the case when V is of one dimensional. By Theorem 6.5 (iii) there exist smooth maps $v : R \to C^n$ and $\phi : R \to C$, such that $v(x)$ spans the linear subspace $\tilde{V}(x)$ and

$$v_x + (a\bar{z}+u)v = \phi v.$$

Set $w = \exp\left(-\int_{-\infty}^x \phi\right) v$. Then $w(x)$ generates $\tilde{V}(x)$ and

$$w_x + (a\bar{z}+u)w = 0. \tag{6.6}$$

We may assume that

$$a = \mathrm{diag}(ic_1, \dots, ic_n), \qquad c_1 \le \dots \le c_n.$$

Let $\psi_j : R \to C^n$ denote the solution of

$$\begin{cases} (\psi_j)_x + (a\bar{z} + u)\psi_j = 0, \\ \lim_{x\to-\infty} e^{-ic_j\bar{z}x}\psi_j(x) = e_j, \end{cases}$$

where $\{e_1, \dots, e_n\}$ is the standard basis of R^n. The construction of the ψ_j is a standard textbook part of the scattering theory. Then $\psi_1, \dots, \psi_n$ form a basis of the solution for equation (6.6). So there exist constants $b_1, \dots, b_n$ such that $w = \sum_{j=1}^n b_j\psi_j$. Let $\bar{z} = r + is$ with $s > 0$ and choose j to be the smallest integer such that $b_j \neq 0$. Then

$$e^{-ic_j\bar{z}x}w = \sum_{k\le j}^{n} e^{-ic_j\bar{z}x} b_k\psi_k = \sum_{k\le j}^{n} e^{i(-c_j+c_k)\bar{z}x} b_k(e^{-ic_k\bar{z}x}\psi_k)$$

Since $\lim_{x\to-\infty} e^{i(-c_j+c_k)\bar{z}x} = 0$ if $c_k < c_j$, we get

$$\lim_{x\to-\infty} e^{-ic_j\bar{z}x}w(x) = \sum_{c_k=c_j} b_k e_k,$$

which is an eigenvector for a. So $\lim_{x\to-\infty}[\tilde{\pi}(x), a] = 0$. Moreover,

$$e^{-ic_j\bar{z}x}w(x) = \sum_{c_k=c_j} b_k e_k + \sum_{c_j<c_k} e^{i(-c_j+c_k)\bar{z}x} b_k e^{-ic_k\bar{z}x}\psi_k.$$

Since $\lim_{x\to-\infty} e^{-ic_k\bar{z}x}\psi_k(x) = e_k$,

$$\left| e^{-ic_j\bar{z}x}w(x) - \sum_{c_k=c_j} b_k e_k \right| = e^{(-c_j+c_m)x\,\mathrm{Im}(z)}\mathcal{O}(1),$$

where c_m is the next non-zero term. Hence $\tilde{\pi}(x) - \lim_{x\to-\infty}\tilde{\pi}(x)$ decays exponentially, so $[\tilde{\pi}, a]$ also decays exponentially as $x \to -\infty$. Similarly, we can prove that $[\tilde{\pi}(x), a] = 0$ decays exponentially when $x \to \infty$.

From equation (6.5)

$$|\tilde{\pi}_x| \le 2|z|\,|[a, \tilde{\pi}]| + 4|u|.$$

So $\tilde{\pi}_x$ decays like u. Repeated differentiation of equation (6.5) gives the desired result. In fact, $\tilde{\pi}_x \in \mathcal{S}(R)$ as well. □

If U is a matrix whose columns form a basis of V, then the projection of C^n onto V is $\pi = U(U^*U)^{-1}U^*$. This follows from elementary linear algebra. So we have:

6.7 Corollary. *Let V be a k-dimensional linear subspace of C^n, and U a matrix whose columns form a basis of V. Then $g_{z,\pi} \sharp A = A + (\bar{z} - z)[\tilde{\pi}, a]$, where*

$$\tilde{\pi}(x) = E^*(x,z)U(U^*E(x,z)E^*(x,z)U)^{-1}U^*E(x,z). \tag{6.7}$$

6.8 Proof of Theorem 6.2.

Given $g \in G_-^m$, write g as product of simple elements $\prod_{j=1}^k g_{z_j,\pi_j}$ (note that the factorization of g into simple elements is not unique, for example see Proposition 6.4 (ii)). Use Theorem 6.5 to see that $g \sharp A \in \mathcal{S}_{1,a}$, and $g \sharp E$ and $g \sharp A$ are obtained by algebraic formulae from E and g. $\square$

7 Scattering data and Birkhoff decomposition

The asymptotically constant case is the case standardly treated in the soliton literature. We obtain many hints of how to describe the theory, since most of what we need is already contained in scattering theory literature. The main purpose of this section is to give a homogeneous structure for the space of scattering data, to obtain the Inverse Scattering Transform using the standard Birkhoff decompositions, and to relate the action of the rational loop group described in section 6 to the scattering data in a natural and simple way.

We first review results of Beals-Coifman ([BC1, 2]) and Zhou ([Zh1, 2]) on scattering theory for $n \times n$ first order linear system. Let

$$a = \mathrm{diag}(ia_1, \dots, ia_n) \in u(n), \qquad a_1 \leq a_2 \leq \dots \leq a_n,$$

and $A = a\lambda + u \in \mathcal{S}_{1,a}$. Consider the linear system

$$\begin{cases} \psi_x = \psi(a\lambda + u), \\ \lim_{x\to-\infty} e^{-a\lambda x}\psi(x,\lambda) = I, \\ m(x,\lambda) = e^{-a\lambda x}\psi(x,\lambda) \text{ is bounded in } x. \end{cases} \tag{7.1}$$

(m will be called the *normalized (matrix) eigenfunction* of A).

7.1 Theorem ([BC1, 2],[Zh2]). *Given $A = a\lambda + u \in \mathcal{S}_{1,a}$ there exists a bounded discrete subset D of $C \setminus R$ such that the normalized eigenfunction $m(x,\lambda) = e^{-a\lambda x}\psi(x,\lambda)$ is holomorphic in $\lambda \in C \setminus (R \cup D)$ and has poles at $z \in D$. Moreover, there exists a dense open subset $\mathcal{S}_{1,a}'$ of $\mathcal{S}_{1,a}$ such that for $A = a\lambda + u \in \mathcal{S}_{1,a}'$, the normalized eigenfunction $m(x,\lambda)$ satisfies the following conditions:*

(i) The subset D is finite, and m has only simple poles at $z \in D$,

(ii) The matrix function m can be extended smoothly to the real axis from the upper and lower half λ-plane,

(iii) As a function of λ, m has an asymptotic expansion at $\lambda = \infty$.

The open dense subset $\mathcal{S}'_{1,a}$ contains all $\{u \in \mathcal{S}_{1,a}$ such that the L^1-norm of u is less than 1 and all u with compact support.

7.2 Theorem ([BC1,2]). *Let m be the normalized eigenfunction of $A = a\lambda + u \in \mathcal{S}_{1,a}$, and $b \in u(n)$ such that $[a,b] = 0$. Set $Q_b = m^{-1}bm$. Then Q_b has an asymptotic expansion at $\lambda = \infty$:*

$$Q_{b,\lambda} \sim b + Q_{b,1}\lambda^{-1} + Q_{b,2}\lambda^{-2} + \dots .$$

Moreover,

(i) $(Q_{b,j})_x + [u, Q_{b,j}] = [Q_{b,j+1}, a]$.

(ii) The j-th flow $u_t = [Q_{b,j+1}, a]$ is symplectic with respect to the symplectic structure $w(v_1, v_2) = \langle -\operatorname{ad}(a)^{-1}(v_1), v_2\rangle$.

Recall

$$\begin{aligned} G_+ &= \{g : C \to GL(n,C) \mid g \text{ is holomorphic, } g(\bar\lambda)^* g(\lambda) = I\}, \\ G_- &= \{g : \mathcal{O}_\infty \to GL(n,C) \mid g \text{ is holomorphic, } g(\bar\lambda)^* g(\lambda) = I, g(\infty) = I\}. \end{aligned}$$

Since $m(x,\lambda)$ is not holomorphic at $\lambda = \infty$, we must change G_-, and restrict G_+ to have a singularity at $\lambda = \infty$ of the type exp(polynomial).

We are motivated by Theorem 7.1 to choose a different negative group D_-:

7.3 Definition. Let D_- denote the group of meromorphic maps f from $C \setminus R$ to $GL(n,C)$ satisfying the following conditions:

(i) $f(\bar\lambda)^* f(\lambda) = I$.

(ii) f has a smooth extension to the closure $\bar C_\pm$, i.e., $f_\pm(r) = \lim_{s \searrow 0} f(r \pm is)$ exists and is smooth for $r \in R$, (since $f(\bar\lambda)^* f(\lambda) = I$, we have $f_-(r) = (f_+(r)^*)^{-1}$).

(iii) f has an asymptotic expansion at ∞.

(iv) $f_+ - I$ lies in the Schwartz class modulo unitary maps. In other words, if we factor $f_+ = h_+ v_+$ with v_+ unitary and h_+ upper triangular then $h_+ - I$ is in the Schwartz class.

Let $m(x,\lambda)$ be the normalized eigenfunction for $A \in \mathcal{S}'_{1,a}$, and E the trivialization of A normalized at $x = 0$. Since both $e^{a\lambda x} m(x,\lambda)$ and $E(x,\lambda)$ satisfy the ordinary differential equation in x:

$$E^{-1}E_x = (e^{a\lambda x} m)^{-1} (e^{a\lambda x} m)_x = A,$$

there exists f such that

$$e^{a\lambda x}m(x,\lambda) = f(\lambda)E(x,\lambda).$$

In fact, $f(\lambda) = m(0,\lambda)$. By Theorem 7.1, $f \in D_-$.

Beals and Coifman [BC1,2] defined the scattering data of $A = a\lambda + u \in \mathcal{S}'_{1,a}$ to be the map $S : R \cup D \to GL(n)$: for $z \in D$, $S(z)$ is the element in $GL(n,C)$ such that

$$(I - (\lambda - z)^{-1}e^{azx}S(z)e^{-azx})m(x,\lambda)$$

has a removable singularity at $\lambda = z$, and for $r \in R$, $S(r) = v_-^{-1}(r)v_+(r)$, where

$$\lim_{s\searrow 0} m(x, r + is) = e^{arx}v_\pm(r)e^{-arx}m(x,r).$$

They prove:

(i) The map sending A to S is injective.

(ii) If $u(x,t)$ is a solution of the j-th flow on $\mathcal{S}_{1,a}$ defined by b, $u_t = [Q_{b,j+1}, a]$, and $S(\lambda,t)$ is the corresponding scattering data, then

$$S_t(\lambda,t) = [S(\lambda,t), \lambda^j b].$$

In particular, $S(\lambda,t) = e^{-b\lambda^j t}S(\lambda,0)e^{b\lambda^j t}$.

We note that scattering data S for A is determined by $f(\lambda) = m(0,\lambda)$. In fact, $S(r) = f_-(r)^{-1}f_+(r)$ for $r \in R$ and $S(z)$ can be obtained from the residue of $f(\lambda)$ at $z \in D$.

7.4 Remark. The rational group G_-^m defined in section 6 is a subgroup of D_-.

Instead of using S as the scattering data, we use the left coset H_-f in D_-/H_- as the scattering data of A, where H_- is the subgroup of $h \in D_-$ that commutes with a. We will call $[f] = H_-f$ the *scattering coset of* A. One advantage of using the scattering cosets is that the inverse scattering transform can be obtained from the standard Birkhoff Decomposition Theorems. Another advantage is that the natural action of the subgroup G_-^m of rational maps on D_-/H_- on the right by multiplication induces the action of G_-^m on $\mathcal{S}_{1,a}$ defined in section 6. To explain this, we first prove a decomposition theorem.

7.5 Theorem. *Let D_-^c denote the subgroup of $v \in D_-$ such that v is holomorphic in $C \setminus R$. Then any $f \in D_-$ can be uniquely factored into*

$$f = gh = \tilde{h}\tilde{g},$$

where $g, \tilde{g} \in G_-^m$ and $h, \tilde{h} \in D_-^c$. Moreover, the multiplication map

$$G_-^m \times D_-^c \to D_-$$

is a diffeomorphism.

This theorem is the real line version of the Birkhoff decomposition theorem, which can be seen by transforming the domain C_+ to the unit disk and real axis to the unit circle S^1 by a linear fractional transformation. To be more precise, let $LGL(n,C)$ denote the loop group of smooth maps from S^1 to $GL(n,C)$, and $L^+GL(n,C)$ the group of maps $g \in LGL(n,C)$ such that g is the boundary value of a holomorphic map

$$g : \{z \,|\, |z| < 1\} \to GL(n,C)\}.$$

Let $\Omega U(n)$ denote the based loop group of maps $g : S^1 \to U(n)$ such that $g(-1) = I$, Recall that the standard Birkhoff Decomposition Theorem (cf. [PrS] p. 120, Theorem 8.1.1) is:

7.6 Birkhoff Decomposition Theorem. *Any $g \in LGL(n,C)$ can be factored uniquely as*

$$g = g_+ g_- = h_- h_+,$$

where $g_+, h_+ \in L^+GL(n,C)$ and $g_-, h_- \in \Omega U(n)$. In other words, the multiplication map

$$L^+GL(n,C) \times \Omega U(n) \to LGL(n,C)$$

is a diffeomorphism.

A direct computation shows:

7.7 Proposition. *Given $g : S^1 \to GL(n,C)$, define $\Phi(g) : R \to GL(n,C)$ by $\Phi(g)(r) = g\left(\frac{1+ir}{1-ir}\right)$. Then*

(i) g is smooth if and only if $\Phi(g)$ is smooth and has the same asymptotic expansions at $-\infty$ and ∞,

(ii) $g - I$ is infinitely flat at $z = -1$ if and only if $\Phi(g) - I$ is in the Schwartz class,

(iii) $g : C \to GL(n,C)$ satisfies the reality condition $g(1/\bar{z})^* g(z) = I$ if and only if $\tilde{g}(\lambda) = g(\frac{1+i\lambda}{1-i\lambda})$ satisfies the reality condition $\tilde{g}(\bar{\lambda})^* \tilde{g}(\lambda) = I$.

7.8 Corollary. *The group D_- is isomorphic to the group of smooth loops $g : S^1 \to GL(n,C)$ that are boundary values of meromorphic maps with finitely many poles in $|\,z\,| < 1$ and $g^* g - I$ is infinitely flat at $z = -1$.*

As a consequence of Theorem 7.6 and Proposition 7.7, we have

7.9 Corollary. *If $f : R \to GL(n,C)$ is smooth and has an asymptotic expansion at $\lambda = \infty$, then f can be factored*

$$f = vg,$$

where g is unitary and v is the boundary value of a holomorphic map on C_+.

7.10 Proof of Theorem 7.5.

It follows from Corollary 7.9 that given $f \in D_-$, we can factor $f_\pm$

$$f_\pm(r) = h_\pm(r) g_\pm(r), \qquad r \in R$$

where $h_\pm$ is the boundary value of a holomorphic map h on $C_\pm$ and $g_\pm$ is a smooth map from R to $U(n)$. It follows from $f_- = (f_+^*)^{-1}$ that we have $g_+ = g_-$ and $h(\bar\lambda)^* h(\lambda) = I$. Write $f = hg$. Since f is meromorphic and h is holomorphic in C_+, g is meromorphic in C_+. However, $g(r)^* g(r) = I$ for $r \in R$ implies that g extends holomorphically across the real axis. So g is meromorphic in C and bounded near infinity. This implies that g is rational, i.e., $g \in G_-^m$. □

Recall that $a = \mathrm{diag}(ia_1, \ldots, ia_n) \in u(n)$ is a fixed diagonal matrix, and G_+ is the group of holomorphic maps $g : C \to GL(n, C)$.

7.11 Theorem. *Let $f \in D_-$, k a positive integer, and $b \in u(n)$ such that $[a, b] = 0$. Let $e_{b,k}(x)(\lambda) = e^{b\lambda^k x}$. Then there exists a unique $E(x, \lambda)$ and $M(x, \lambda)$ such that*

$$f^{-1} e_{b,k}(x) = E(x, \cdot) M^{-1}(x, \cdot) \in G_+ \times D_-.$$

Proof. Write $f = hg$ as in Theorem 7.5 with $h \in D_-^c$ and $g \in G_-^m$. Write $h = pv$, where p is upper triangular and v is unitary. By definition of D_-, $p - I$ is in the Schwartz class when restricted to the real axis in the λ-plane. So $e_{b,k}^{-1}(x) p^{-1} e_{b,k}(x)$ has an asymptotic expansion at $r = \pm\infty$ for each x. Write $e_{b,k}^{-1}(x) p^{-1} e_{b,k}(x) = \tilde v(x) \tilde h(x)$, where $\tilde v$ is unitary and $\tilde h$ is the boundary value of a holomorphic map on C_+. Notice f, p, v and h do not depend on x, whereas the rest of the matrix functions do depend on x. So

$$h^{-1} e_{b,k}(x) = v^{-1} e_{b,k}(x) \tilde v(x) \tilde h(x) = B(x) \tilde h(x),$$

where $B(x) = v^{-1} e_{b,k}(x) \tilde v(x)$ is unitary. Both $\tilde h(x)$ and h are holomorphic in $\lambda \in C_+$, and $e_{b,k}(x)$ is holomorphic in C_+. Hence $B(x)$ is holomorphic in $\lambda \in C_+$. However, $B(x)$ is unitary hence it is holomorphic in $\lambda \in C$.

Next we claim that we can factor $g^{-1} B(x) = E(x) g_1^{-1}(x)$ with E holomorphic in C and $g_1 \in G_-^m$. This can be proved exactly the same way as Theorem 6.2. Then

$$\begin{aligned} f^{-1} e_{b,k}(x) &= (hg)^{-1} e_{b,k}(x) = g^{-1} h^{-1} e_{b,k}(x) \\ &= g^{-1} B(x) \tilde h(x) = E(x) g_1^{-1}(x) \tilde h(x) = E(x) M^{-1}(x), \end{aligned}$$

which finishes the proof. □

7.12 Definition. A matrix q is called a-diagonal if $q_{jk} = 0$ whenever $a_j \neq a_k$, q is (strictly) upper a-triangular if $q_{jk} = 0$ whenever $a_j > a_k$ (and $q_{jk} = 0$ or I if $a_j = a_k$), and q is (strictly) lower a-triangular if $q_{jk} = 0$ whenever $a_j < a_k$ (and $q_{jk} = 0$ or I if $a_j = a_k$). Let q_d denote the *a-diagonal projection of* q, i.e.,

$$(q_d)_{ij} = \begin{cases} q_{ij}, & \text{if } a_i = a_j, \\ 0, & \text{if } a_j \neq a_k. \end{cases}$$

7.13 Proposition. *Any $f \in D_-$ can be factored uniquely as*

$$f = pv = q\tilde{v},$$

where p is upper a-triangular, q is lower a-triangular, v and $\tilde{v}$ are unitary, and the a-diagonal projections p_d, q_d are holomorphic in $C_\pm$.

Proof. Write $g = p_0 v_0$, where p_0 is upper a-triangular and v_0 is unitary. Such p_0, v_0 are not unique because an element in $U_a = \{y \in U(n) \mid ay = ya\}$ is both a-triangular and unitary. Write $p_0 = p_1 p_2$, where p_1 is strictly upper a-triangular and p_2 is in a-diagonal. Factor $p_2 = p_3 h$, where p_3 is holomorphic in C_+ and h is unitary. Then $g = p_1 p_3 h v_0 = pv$, where $p = p_1 p_3$ is upper a-triangular and $v = hv_0$ is unitary. Since p_1 is strictly upper a-triangular, $p_d = p_3$ is holomorphic. □

To study how the Birkhoff factorization of Theorem 7.11 depends on parameter x, we introduce the class of Schwartz maps from $[r_0, \infty)$ to a Hilbert space. Let H be a Hilbert space, a map $\phi : [r_0, \infty) \to H$ is in $\mathcal{S}([r_0, \infty), H)$ if for each pair of integers (m, s) there exists a constant $c_{m,s}$ such that

$$\left\| \left(\frac{d}{dx}\right)^m \phi(x) \right\| \leq \frac{c_{m,s}}{(1 + |x|)^s}.$$

Let H_1 denote the Sobolev space for maps from $R^+ = [0, \infty)$ to $\mathcal{U}_a^\perp$. In other words, $u \in H_1$ if

$$\begin{aligned} ||u||_1^2 &= \int_0^\infty \left(\left\| \frac{du}{dr} \right\|^2 + ||u||^2 \right) dr \\ &= \int_0^\infty (y^2 + 1) ||\hat{u}||^2 dy \quad < \infty, \end{aligned}$$

where $\hat{u}$ is the Fourier transform of u.

The following is a functional analytic extension of Birkhoff decomposition.

7.14 Theorem. *Let $I + D(x, \cdot) = (I + h(x, \cdot))V(x, \cdot)$ be the Birkhoff decomposition, where $h(x)(r)$ is the boundary value of a holomorphic map in the upper half plane and $V(x)(r)$ is unitary. If $D \in \mathcal{S}(R^+, H_1)$, then h and $V - I$ are in $\mathcal{S}(R^+, H_1)$.*

Proof. This should be regarded as an implicit function theorem. It is based on the two facts about the Sobolev space H_1. The first is that H_1 is an algebra under multiplication and $\exp : H_1 \to H_1$ is smooth. The second is that the linear Birkhoff decomposition can be defined using the Fourier transform $\mathcal{F}$. Let $\Psi : L^2(R) \to L^2(R)$ denote the linear operator defined by

$$\Psi(f)(y) = \begin{cases} f(y) + f(-y)^*, & \text{if } y \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and let $\pi_+ : H_1 \to H_1$ be the bounded linear map

$$\pi_+ = \mathcal{F}^{-1}\Psi\mathcal{F}, \qquad \text{i.e.,}$$
$$\pi_+(f)(r) = \int_0^\infty (\hat{f}(y) + \hat{f}(-y)^*)e^{iry}dy.$$

We claim that $f = \pi_+(f) + (I - \pi_+)(f)$ is the linear Birkhoff Decomposition, or equivalently, $\pi_+(f)$ is the boundary value of a holomorphic map on C_+ and $(I - \pi_+)(f)$ is is in $u(n)$. To see this, we note that $\pi_+(f)$ is the boundary value of the holomorphic map

$$\lambda \in C_+ \mapsto \int_0^\infty (\hat{f}(y) + \hat{f}(-y)^*)e^{i\lambda y}dy.$$

Then

$$\begin{aligned}(I - \pi_+)(f) &= f(r) - \int_0^\infty (\hat{f}(y) + \hat{f}(-y)^*)e^{iry}dy \\ &= \int_{-\infty}^\infty \hat{f}(y)e^{iry}dy - \int_0^\infty (\hat{f}(y) + \hat{f}(-y)^*)e^{iry}dy \\ &= \int_{-\infty}^0 \hat{f}(y)e^{iry}dy - \int_0^\infty \hat{f}(-y)e^{iry}dy.\end{aligned}$$

It follows that $(I - \pi_+)(f)^* = -(I - \pi_+)(f)$.

Due to the linearity of π_+, it is easy to see that this extends to the parameter version in x. We write this as

$$\pi_+ : \mathcal{S}(R^+, H_1) \to \mathcal{S}(R^+, H_1).$$

Now the Birkhoff decomposition is a non-linear operator. However we are near the identity, so it can be regarded as a perturbation of the linear operation because the exponential map is smooth on $H_1(R)$.

Let $Y : \mathcal{S}(R^+, H_1) \to \mathcal{S}(R^+, H_1)$ be the map defined by

$$Y(f) = e^{\pi_+(f)}e^{(I-\pi_+)(f)}.$$

Given D, we wish to find $\tilde{D}$ such that

$$(I + D) = \exp(\pi_+(\tilde{D}))\exp((I - \pi_+)\tilde{D}) = Y(\tilde{D}).$$

Since $dY_0 = I$, for x sufficiently large

$$\|\pi_+(\tilde{D})\|_1 \leq \|\tilde{D}\|_1 \leq O_s,$$

where $O_s(x) = c\|D\|_1 \leq cc_s/(1 + |\,x\,|)^s$. The estimate on derivatives in x is more difficult. Let $I + h = \exp(\pi_+(\tilde{D}))$. Then

$$(I + h)^{-1}D_x V^{-1} = (I + h)^{-1}h_x + V_x V^{-1}.$$

On the right, the first term is holomorphic in the upper half plane, the second term is unitary. Hence

$$(I + h)^{-1}h_x = \pi_+((I + h)^{-1}D_x V^{-1}) = \pi_+((I + h)^{-1}D_x(I + D)^{-1}(I + h)).$$

Or

$$h_x = (I + h)\pi_+((I + h)^{-1}D_x(I + D)^{-1}(I + h)).$$

Certainly, $D_x(I + D)^{-1} \in \mathcal{S}(R^+, H_1)$, π_+ is linear, and H_1 is an algebra. Using the Leibniz rule repeatedly, we can obtain

$$\left\| \left(\frac{\partial}{\partial x}\right)^m h(x) \right\|_1 \leq C_m(h) \max_{j \leq l} \left\| \left(\frac{\partial^m D}{\partial x^m}(x)\right) \right\|_1,$$

where $C_m(h) = C(\|h\|, \ldots, \|(\partial/\partial x)^{m-1}h\|)$. Estimates in the Schwartz topology follows by induction on m. $\square$

7.15 Remark. The awkwardness of this proof reminds one that the classical use of the Schwartz space is probably not as natural for the analysis as various choices of Hilbert or Banach spaces in x would be. The above proof would then be a straight forward use of the usual implicit function theorem (rather than a reproof). Notice that in fact $H_1(R)$ could be replaced by any $H_k(R)$, $k > \frac{1}{2}$.

Recall that $e_{a,1}(x)(\lambda) = e^{a\lambda x}$.

7.16 Theorem. *In the Birkhoff factorization of Theorem 7.11*

$$f^{-1}e_{a,1}(x) = E(x)M^{-1}(x) \in G_+ \times D_-.$$

We have in addition the following properties

(i) $E^{-1}E_x = A$, *where* $A(x, \lambda) = a\lambda + u(x)$ *for some* $u \in \mathcal{S}(R, \mathcal{U}_a^{\perp})$,

(ii) $M_{\pm\infty} \in H_-$, *where* $M_{\pm\infty}(\lambda) = \lim_{x \to \pm\infty} M(x, \lambda)$,

(iii) if λ *is not a pole of* f *then* $\tilde{M}_\pm(\cdot, \lambda) = M(\cdot, \lambda) - M_{\pm\infty}(\lambda)$ *is in the Schwartz class.*

To prove this theorem, we need the following Lemma:

7.17 Lemma. *Given $q \in \mathcal{S}(R)$ and $\beta < 0$ a constant, and set*

$$\Delta(x)(r) = \int_{-\infty}^{0} q(y+\beta x)e^{iry}dy,$$
$$\xi(x)(r) = \int_{0}^{\infty} q(y+\beta x)e^{iry}dy.$$

Then

(i) $\Delta \in \mathcal{S}(R^+, H_1)$,

(ii) for any integer $m \geq 0$ there exists a constant c_m such that

$$\left|\frac{\partial^m \xi}{\partial x^m}(x,r)\right| \leq c_m.$$

Proof. Write $\Delta(x,r) = \Delta(x)(r)$ and $\xi(x,r) = \xi(x)(r)$. Then

$$\left(\frac{\partial}{\partial x}\right)^m \Delta(x,r) = \beta^m \int_{-\infty}^{0} \frac{\partial^m q}{\partial y^m}(y+\beta x)e^{iry}dy.$$

Since $q \in \mathcal{S}(R^-)$,

$$\left|\left(\frac{\partial}{\partial y}\right)^m q(y)\right| \leq \frac{c_{m,s}}{(1+|y|)^s}.$$

But by the Plancherel Theorem

$$\begin{aligned}\left\|\left(\frac{\partial}{\partial x}\right)^m \Delta(x,\cdot)\right\|_1^2 &= \beta^{2m}\int_{-\infty}^{0}(1+y^2)\left(\frac{\partial^m q}{\partial y^m}(y+\beta x)\right)^2 dy\\ &\leq \beta^{2m}\int_{-\infty}^{0}\frac{(1+|y|^2)c_{m,s}^2}{(1+|y|+|\beta x|)^{2s}}dy\\ &\leq \frac{\beta^{2m}c_{m,s}^2}{(1+|\beta x|)^{2s-3}}\end{aligned}$$

.

An adjustment of the constants completes the proof of (i). A straight forward computation gives (ii). □

7.18 Proof of Theorem 7.16.
Take the variation with respect to $f \in D_-$ in the formulas

$$f^{-1}(\lambda)e^{a\lambda x} = E(x,\lambda)M^{-1}(x,\lambda),$$
$$A(x,\lambda) = E^{-1}(x,\lambda)E_x(x,\lambda).$$

We get

$$-E^{-1}f^{-1}\delta f E = E^{-1}\delta E - M^{-1}\delta M,$$
$$\delta A = [A, (E^{-1}f^{-1}\delta f E)_-]_+.$$

For $f = I$, we have $A = a\lambda$. So $\delta A(x)$ is independent of λ and lies in $\mathcal{U}_a^\perp$ for all x. This implies that $E^{-1}E_x = a\lambda + u$ for some $u : R \to \mathcal{U}_a^\perp$. The fact that $u \in \mathcal{S}(R, \mathcal{U}_a^\perp)$ follows directly from (ii) and (iii). Write

$$u = M^{-1}M_x + M^{-1}[a\lambda, M].$$

Now $M(\cdot, \lambda) - M_\infty \in \mathcal{S}(R^+)$ and $[M_\infty, a] = 0$ imply that $u \,|\, R^+ \in \mathcal{S}(R^+)$. The corresponding argument gives $u \,|\, R^- \in \mathcal{S}(R^-)$.

We first prove the theorem for $f \in D_-^c$. Use Proposition 7.13 to write

$$f = p_d(I + p)v,$$

where p_d is a-diagonal and holomorphic in C_+, p is strictly upper a-triangular, and v is unitary. We will be looking at $x \to \infty$. Examine the formula for $\lambda \in C_+$:

$$(e^{-a\lambda x}p(\lambda)e^{a\lambda x})_{jk} = \begin{cases} 0, & \text{if } a_j \geq a_k, \\ p_{jk}(\lambda)e^{-i(a_j - a_k)\lambda x}, & \text{if } a_j < a_k. \end{cases} \tag{7.2}$$

Here $p_{jk} \,|\, R$ lies in the Schwartz space if $a_j < a_k$.

Use inverse Fourier transform to write

$$p_{jk}(r) = \int_{-\infty}^{\infty} \hat{p}_{jk}(y)e^{iry}dy.$$

So

$$p_{jk}(r)e^{-i(a_j - a_k)rx} = \int_{-\infty}^{\infty} \hat{p}_{jk}(y + (a_j - a_k)x)e^{iry}dy.$$

The piece $\int_0^\infty \hat{p}_{jk}(y + (a_j - a_k)x)e^{iry}dy$ is the boundary value of a holomorphic map in C_+, which can be written

$$\xi_{jk}(x, \lambda) = \int_0^\infty \hat{p}_{jk}(y + (a_j - a_k)x)e^{i\lambda y}dy.$$

So

$$p_{jk}(r)e^{-i(a_j - a_k)rx} = \xi_{jk}(r, x) + \triangle_{jk}(x, r), \qquad \text{where}$$
$$\triangle_{jk}(x, r) = \int_{-\infty}^{0} \hat{p}_{jk}(y + (a_j - a_k)x)e^{iry}dy.$$

It follows from Lemma 7.17 that $\triangle \in \mathcal{S}(R^+, H_1)$ and $\left\| \frac{\partial^m \xi}{\partial x^m(x,r)} \right\| \leq c_m$.

Now write

$$\begin{aligned} e_a^{-1} p_d (I+p) e_a &= p_d e_a^{-1} (I+p) e_a = p_d (I + \xi + \triangle) \\ &= p_d ((I+\xi)(I+\triangle) - \xi\triangle) \\ &= p_d (I+\xi)(I + \triangle - (I+\xi)^{-1}\xi\triangle). \end{aligned}$$

We claim that $D = \triangle - (I+\xi)^{-1}\xi\triangle \in \mathcal{S}(R^+, H_1)$. Note that

$$(I+\xi)^{-1} = I - \xi + \xi^2 - \xi^3 + \ldots + \xi^n$$

is a finite series since ξ is strictly upper a-triangular. The rules of multiplication of $\mathcal{S}(R^+, H_1)$ by a smooth bounded function give the result that $D \in \mathcal{S}(R^+, H_1)$.

Let $(I + D) = (I + h)V$ be the Birkhoff decomposition. By Theorem 7.14, h and $V - I$ are in $\mathcal{S}(R^+, H_1)$. So

$$\begin{aligned} e_a^{-1} f = e_a^{-1} p_d (I+p) v &= e_a^{-1} p_d (I+p) e_a e_a^{-1} v \\ &= p_d (I+\xi)(I+D) e_a^{-1} v \\ &= p_d (I+\xi)(I+h) V e_a^{-1} v. \end{aligned}$$

By definition $M = p_d(I+\xi)(I+h)$, and

$$M - p_d(I+\xi) = p_d(I+\xi)h.$$

Since $h \in \mathcal{S}(R^+, H_1)$, and we have uniform estimates on all derivatives of $p_d(I+p)$, $M - p_d(I+\xi) \in \mathcal{S}(R^+, H_1)$. The same argument, in which a factorization $f = q\tilde{v}$ for q lower a-triangular and $\tilde{v}$ unitary, proves Schwartz space decay as $x \to -\infty$.

To complete the proof, given $f \in D_-$, write $f = hg \in D_-^c \times G_-^m$. Write

$$h^{-1} e_{a,1}(x) = E_0(x) M_0^{-1}(x) \in G_+ \times D_-.$$

By Theorem 6.5, we factor $g^{-1}E_0(x) = \tilde{E(x)} g(x) \in G_+ \times G_-^m$. Then

$$\begin{aligned} f^{-1} e_{a,1}(x) = g^{-1} h^{-1} e_{a,1}(x) &= g^{-1} E_0(x) M_0^{-1}(x) = E(x) \tilde{g}(x) M_0^{-1}(x) \\ &= E(x) M(x). \end{aligned}$$

By Theorem 6.6 $\tilde{g}$ satisfies condition (ii) and (iii). But we just proved that M_0 satisfies (ii) and (iii), so is $M = \tilde{g} M_0^{-1}$. □

Note the convergence is actually uniform in the argument in Theorem 7.16. So we have

7.19 Theorem. *As in Theorem 7.16 let $f \in D_-$, $f^{-1}e_{a,1}(x) = E(x)M(x)^{-1} \in G_+ \times D_-$, and $f_+ = \lim_{s \searrow 0} f(r+is)$. Factor $f_+ = Pv = Q\tilde{v}$, where $v, \tilde{v}$ are unitary, P is upper a-triangular, Q is lower a-triangular, and P_d, Q_d is holomorphic in C_+. Then*

$$\lim_{x\to\infty} e^{arx} M_+(x,r) e^{-arx} = P(r),$$
$$\lim_{x\to-\infty} e^{arx} M_+(x,r) e^{-arx} = Q(r).$$

7.20 Theorem. *Let $\Psi : D_- \to \mathcal{S}_{1,a}$ be the map defined by $\Psi(f) = E^{-1}E_x$, where E is obtained from f as in Theorem 7.16. Let H_- denote the subgroup of $f \in D_-$ such that $fa = af$. Then*

(i) $\mathcal{S}^0_{1,a} = \Psi(D_-)$ *is an open and dense subset of* $\mathcal{S}_{1,a}$,

(ii) $\Psi(f) = \Psi(g)$ *if and only if there exist* $h \in H_-$ *such that* $g = hf$,

(iii) $\mathcal{S}^0_{1,a}$ *is isomorphic to the homogeneous space* D_-/H_- *of left cosets of* H_- *in* G_-,

(iv) if $A = \Psi(f)$ *and* M *is as in Theorem 7.16, then the normalized eigenfunction* m *in Theorem 7.1 of* A *is* $M_{-\infty}^{-1}M$.

Proof. The first part (i) is a consequence of Theorem 7.1. Both (iii) and (iv) follow from (ii). To prove (ii), recall if

$$f^{-1}e_{a,1}(x) = E(x)M^{-1}(x) \in G_+ \times D_-,$$
$$g^{-1}e_{a,1}(x) = E(x)N^{-1}(x) \in G_+ \times D_-.$$

Then

$$M(x)N^{-1}(x) = e_{a,1}(x)^{-1} f g^{-1} e_{a,1}(x).$$

Suppose $\mathrm{Im}\,\lambda > 0$. Then the limit of the right hand side is upper a-triangular when $x \to \infty$, and the limit is lower a-triangular when $x \to -\infty$. So MN^{-1} is both upper and lower a-triangular. Hence it is a-diagonal, i.e., $MN^{-1} \in H_-$. So $fg^{-1} \in H_-$.

Conversely, if $g = hf$ for some $h \in H_-$ and $f^{-1}e_{a,1}(x) = E(x)M(x)^{-1} \in G_+ \times D_-$, then

$$\begin{aligned} g^{-1}e_{a,1}(x) &= f^{-1}h^{-1}e_{a,1}(x) = f^{-1}e_{a,1}(x)h^{-1} \\ &= E(x)(M(x)^{-1}h^{-1}) \in G_+ \times D_-. \end{aligned}$$

So $\Psi(f) = \Psi(g)$. □

In summary, we have shown that given $f \in D_-$, we can construct an $A \in \mathcal{S}_{1,a}$ such that $A = \Psi(f)$ by using various Birkhoff decomposition theorems repeatedly.

7.21 Theorem. *The natural right action of D_- on the space D_-/H_- of left cosets induces a natural action $*$ of D_- on $\mathcal{S}^0_{1,a}$ via the isomorphism $\bar{\Psi}$ from D_-/H_- to $\mathcal{S}^0_{1,a}$. Equivalently, if $A = \Psi(f)$ and $g \in D_-$ then $g * A = \Psi(fg^{-1})$. Moreover:*

(i) Let $g \in D_-$, $A \in \mathcal{S}^0_{1,a}$, and E the trivialization of A normalized at $x = 0$, then we can factor

$$gE(x) = \tilde{E}(x)\tilde{g}(x) \in G_+ \times D_-,$$

and $g * A = \tilde{E}^{-1}\tilde{E}_x$.

(ii) If $g \in G^m_-$, then $g * A = g \sharp A$, where $\sharp$ is the action of G^m_- on $\mathcal{S}_{1,a}$ defined in Theorem 6.2. In other words, if $A = \Psi(f)$ and $g \in G^m_-$, then $g \sharp A = \Psi(fg)$. Or equivalently, if H_-f is the scattering coset of A then H_-fg is the scattering coset of $g \sharp f$.

Proof. Given $f, g \in D_-$, we factor

$$f^{-1}e_{a,1}(x) = E(x)M^{-1}(x), \qquad (fg)^{-1}e_{a,1}(x) = \tilde{E}(x)\tilde{M}^{-1}(x).$$

Then $g^{-1}E(x) = \tilde{E}(x)(\tilde{M}^{-1}(x)M(x)) \in G_+ \times D_-$. This defines the action of D_- on $\mathcal{S}^0_{1,a}$, and it extends the action of G^m_- on $\mathcal{S}_{1,a}$ defined in Theorem 6.2. □

7.22 Remark. If the scattering data of A has k poles counted with multiplicity, then $g_{z,\pi} \sharp A$ typically has $k+1$ poles, but it may have k or $k-1$ poles for special choices of z and π. To see this, let $z \in C \setminus R$, and π a projection such that $\pi a \neq a\pi$. If $\bar{z}$ is not a pole of the scattering data of A then $g_{z,\pi} \sharp A$ add one pole $\bar{z}$ to the scattering data. Let $A = g_{\bar{z},\pi} \sharp A_0$, where A_0 is the vacuum solution. Then the scattering data of $g_{z,\pi_1} \sharp A$

(i) has no poles if $\pi_1 = \pi$,

(ii) has only one pole z if π_1 and π commute and $\pi + \pi_1 \neq I$.

8 Poisson structure for the positive flows

Let H_+ denote the subgroup of G_+ generated by

$$\{e^{p(\lambda)} \mid p(\lambda) \text{ is a polynomial } p(\lambda)a = ap(\lambda)\}.$$

In this section, we prove that the right dressing action of H_+ on D_- induces a Poisson group action of H_+ on $\mathcal{S}^0_{1,a}$ and show that it generates the positive flows defined in section 5. We also study the induced symplectic structure on the space of discrete scattering data $G^m_-/(G^m_- \cap H_-)$, and the space of the continuous scattering data $D^c_-/(D^c_- \cap H_-)$.

Set $e_{a,j,b}(x,t)(\lambda) = e^{a\lambda x + b\lambda^j t}$ and recall $e_{b,j}(x)(\lambda) = e^{b\lambda^j x}$.

8.1 Theorem. *Let $a, b \in u(n)$ such that $[a,b]=0$. Then we can factor*

$$f^{-1}e_{a,j,b}(x,t) = E(x,t)M^{-1}(x,t) \in G_+ \times D_-.$$

Moreover, E and M satisfy the following conditions:

(i) $E^{-1}E_x = A$ is a solution of the j-th flow defined by b, where $A(x,\lambda) = a\lambda + u(x)$.

(ii) $E^{-1}E_t = B$, where

$$B(\cdot,\lambda) = b\lambda^j + Q_{b,1}(u)\lambda^{j-1} + \ldots + Q_{b,j}(u) = (M^{-1}b\lambda^j M)_+.$$

Proof. Since $[a,b]=0$, $\exp(a\lambda x + b\lambda^j t) = e^{a\lambda x}e^{b\lambda^j t}$. Use Theorem 7.11 to factor

$$f^{-1}e_{a,j,b}(x,t) = f^{-1}e_{a,1}(x)e_{b,j}(t) = E_0(x)M_0^{-1}(x)e_{b,j}(t).$$

Use Theorem 7.11 again to factor

$$M_0^{-1}(x)e_{b,j}(t) = E_1(x,t)M(x,t) \in G_+ \times D_-.$$

The variational form of $f^{-1}e_{a,j,b} = EM^{-1}$ implies

$$\begin{cases} E^{-1}E_x = a\lambda + u, \\ E^{-1}E_t = b\lambda^j + q_1\lambda^{j-1} + \ldots + q_j. \end{cases}$$

So

$$\left[\frac{\partial}{\partial x} + a\lambda + u,\ \frac{\partial}{\partial t} + b\lambda^j + q_1\lambda^{j-1} + \ldots + q_j\right] = 0. \tag{8.1}$$

Compare coefficient of λ^i in equation (8.1) to get

$$\begin{cases} (q_i)_x + [u, q_i] = [q_{i+1}, a], & \text{if } 0 \le i < j, \\ u_t = (q_j)_x + [u, q_{j+1}]. \end{cases}$$

This is the same system as (5.2) defining the $Q'_{b,i}s$. Hence $q_i = Q_{b,i}$. □

8.2 Corollary. *The dressing action $\natural$ of H_+ on D_- on the right is well-defined and H_- is fixed under this action. Hence an action $\natural$ of H_+ on D_-/H_- is defined, which leads to an action on $\mathcal{S}^0_{1,a}$. In fact, this action is defined as follows: Write $A = \Psi(f)$, $f^{-1}e_{a,1}(x) = E(x)M(x)^{-1}$. For $h \in H_+$, we factor*

$$hM(x) = \tilde{M}(x)\tilde{h}(x) \in D_- \times G_+$$

to get

$$h \natural A = (e^{a\lambda x}\tilde{M})^{-1}(e^{a\lambda x}\tilde{M})_x.$$

8.3 Corollary. *If $A_0 = \Psi(f_0)$, then $A(t) = \Psi(e_{b,j}(t) \natural f_0)$ is the solution of the j-th flow on $\mathcal{S}_{1,a}$ defined by b with $A(0) = A_0$.*

8.4 Corollary. *Let $a_1, \dots, a_n$ be a basis of the space $\mathcal{T}$ of diagonal matrices in $u(n)$, $f \in D_-$, and $e_{a_1,\dots,a_n}(x_1, \dots, x_n)(\lambda) = \exp(\sum_{j=1}^n a_j x_j \lambda)$. Factor*

$$f^{-1} e_{a_1,\dots,a_n}(x) = E(x)M(x) \in G_+ \times D_-.$$

Then there exists $v : R^n \to \mathcal{T}^\perp$ such that

(i) $E^{-1}E_{x_j} = a_j \lambda + [a_j, v]$ for all $1 \le j \le n$,

(ii) v is a solution of equation

$$\left[a_i, \frac{\partial v}{\partial x_i}\right] - \left[a_j, \frac{\partial v}{\partial x_j}\right] = [[a_i, v], [a_j, v]]. \tag{8.2}$$

8.5 Remark. Equation (8.2) is the n-dimensional system associated to $U(n)$ constructed in the paper of the first author [Te2].

8.6 Theorem. *The action $\natural$ of H_+ on $\mathcal{S}_{1,a}^0$ is Poisson. Moreover, the map $\mu : \mathcal{S}_{1,a}^0 \to H_- = H_+^*$ defined by $\mu(A)(\lambda) = M_{-\infty}^{-1} M_\infty$ is a moment map, where $A = \Psi(f)$, $f^{-1}e_{a,1}(x) = E(x)M(x)^{-1} \in G_+ \times D_-$ and $M_{\pm\infty}(\lambda) = \lim_{x\to\pm\infty} M(x, \lambda)$.*

Proof. Suppose $A = \Psi(f)$, i.e.,

$$\begin{cases} f^{-1}e_{a,1}(x) = E(x)M^{-1}(x) \in G_+ \times D_-, \\ A = E^{-1}E_x = (e^{a\lambda x}M)^{-1}(e^{a\lambda x}M)_x. \end{cases}$$

The second equation implies

$$M^{-1}M_x + \lambda M^{-1}aM = A. \tag{8.3}$$

Set $\eta = M^{-1}\delta M$, $B = \delta A$ and $\psi = e^{a\lambda x}M$. Compute the variation directly from equation (8.3) to derive

$$\eta_x + [A, \eta] = B, \qquad \lim_{x\to-\infty} \eta = 0,$$
$$\eta(x) = \psi(x)^{-1} \int_{-\infty}^x (\psi B \psi^{-1}) dy\, \psi(x).$$

For $\xi_+ \in \mathcal{H}_+$, since $[\xi_+, a] = 0$, we have $M^{-1}\xi_+ M = \psi^{-1}\xi_+\psi$.

$$\begin{aligned}
&\langle d\mu_A(B)\mu(A)^{-1}\,,\,\xi_+\rangle \\
&= \lim_{x\to\infty}\left\langle M(x)\psi^{-1}(x)\int_{-\infty}^{x}(\psi B\psi^{-1})dy\psi(x)M^{-1}(x)\,,\,\xi_+\right\rangle, \\
&= \lim_{x\to\infty}\left\langle e^{-a\lambda x}\int_{-\infty}^{x}(\psi B\psi^{-1})dy e^{a\lambda x}\,,\,\xi_+\right\rangle \\
&= \lim_{x\to\infty}\left\langle \int_{-\infty}^{x}(\psi B\psi^{-1})dy\,,\,e^{a\lambda x}\xi_+ e^{-a\lambda x}\right\rangle \\
&= \lim_{x\to\infty}\left\langle \int_{-\infty}^{x}(\psi B\psi^{-1})dy\,,\,\xi_+\right\rangle = \lim_{x\to\infty}\int_{-\infty}^{x}\langle \psi B\psi^{-1}\,,\,\xi_+\rangle dy \\
&= \lim_{x\to\infty}\int_{-\infty}^{x}\langle B\,,\,\psi^{-1}e^{-a\lambda y}\xi_+ e^{a\lambda y}\psi\rangle dy \\
&= \langle\langle B\,,\,(M^{-1}\xi_+ M)_-\rangle\rangle.
\end{aligned}$$

The rest of the proof goes exactly the same as for Theorem 4.3. □

8.7 Remark. Let $a = \mathrm{diag}(ia_1, \ldots, ia_n)$, and $a_1 < \ldots < a_n$. Then $\mathcal{U}_a$ is the set of all diagonal matrices in $u(n)$, $\mathcal{U}_a^{\perp}$ is the set of all matrices $u \in u(n)$ such that $u_{ii} = 0$ for all $1 \leq i \leq n$. So H_+ is abelian and the action of H_+ on $\mathcal{S}_{1,a}$ is in fact symplectic.

The following theorem was proved by Flaschke, Newell and Ratiu [FNR1, 2] for $n = 2$ and by one of us [Te2] for general n:

8.8 Theorem ([Te2]). *The Hamiltonian function on $\mathcal{S}_{1,a}$ corresponding to the j-th flow defined by a is:*

$$F_{a,j}(u) = -\frac{1}{j+1}\int_{-\infty}^{\infty}(Q_{a,j+2}, a)\,dx, \tag{8.4}$$

i.e., $\nabla F_{b,j}(u) = Q_{b,j+1}^{\perp}(u)$.

8.9 Remark. Let $b, c \in \mathcal{U}_a$, and $\xi_{b,j}$ and $\xi_{c,k}$ denote infinitesimal vector field for the H_+-action on $\mathcal{S}_{1,a}^0$ corresponding to $b\lambda^j$ and $c\lambda^k$ respectively. Then the bracket $[\xi_{b,j}, \xi_{c,k}]$ is equal to the infinitesimal vector field corresponding to $[b, c]\lambda^{k+j}$. Unless $[b, c] = 0$, these two flows do not commute.

8.10 Remark. If we replace the group $SU(n)$ by a simple compact Lie group, then what we have discussed still holds if appropriate algebraic conditions are prescribed.

In the end of this section, we will study the pull back of the symplectic structure w on $\mathcal{S}_{1,a}$ to D_-/H_- via the isomorphism Ψ. Note that $\Psi(G_-^m)$ ($\Psi(D_-^c)$ resp.) is the space of A's with only discrete (continuous resp.) scattering data.

We have been using the base point $x = 0$, i.e., $f(\lambda) = M(0, \lambda)$. But there is nothing special about $x = 0$. In the following, we choose a base point y and let $y \to -\infty$. The expression with base point 0, and with y, differ by a term which cancels out when we evaluate integrals at the end points.

We only deal with the symplectic structure on $\Psi(D^c_-)$. However, this set is pretty large. For example, Beals and Coifman and later Zhou show the following:

8.11 Theorem ([BDZ]). *Let B_1 denote the unit ball in $\mathcal{S}_{1,a}$ with respect to the L^1-norm, i.e., B_1 is set of all $A = a\lambda + u \in \mathcal{S}_{1,a}$ such that $\int_{-\infty}^{\infty} \|u\| dr < 1$. Then $B_1 \subset \Psi(D^c_-)$.*

Let S denote the scattering transform that maps $A \in \mathcal{S}'_{1,a}$ to its scattering data S (defined in section 7). The restriction of the symplectic form w on $\mathcal{S}_{1,a}$ to $S(B_1)$ was computed by Beals and Sattinger [BS]. We will compute the restriction of w to $\Psi(D^c_-)$ in terms of variations in D^c_- below. Let

$$\langle f(r), g(r) \rangle = \int_{-\infty}^{\infty} \mathrm{Im}(\mathrm{tr}(f(r)g(r))dr.$$

By the same computation as in Theorem 4.3, the Poisson bracket on $B_1 \subset \mathcal{S}_{1,a}$ is

$$\begin{aligned}
\{\delta_1 A, \delta_2 A\} &= \lim_{x\to\infty} \langle (E^{-1}(x) f^{-1}\delta_1 f E(x))_-, E(x)^{-1} f^{-1}\delta_2 f E(x)\rangle \\
&\quad - \lim_{y\to-\infty} \langle (E^{-1}(y) f^{-1}\delta_1 f E(y))_-, E(y)^{-1} f^{-1}\delta_2 f E(y)\rangle \\
&= \lim_{x\to\infty} \langle E^{-1}(x)\delta_1 E(x), M^{-1}\delta_2 M(x)\rangle \\
&\quad - \lim_{y\to-\infty} \langle E^{-1}(y)\delta_1 E(y), M^{-1}\delta_2 M(y)\rangle.
\end{aligned}$$

Now, let the vacuum be based at y, i.e., factor

$$f^{-1}(\lambda) e^{a\lambda(x-y)} = E(x, y, \lambda) M^{-1}(x, y, \lambda).$$

Hence $f(\lambda) = M(y, y, \lambda)$ and $\delta E(y, y, \lambda) = 0$. The y-term in the above description is now zero, and we have

$$\begin{aligned}
&= \lim_{x\to\infty, y\to-\infty} -\langle M(x)^{-1} e^{-a\lambda(x-y)} \delta_1 f f^{-1} e^{a\lambda(x-y)} M(x),\ M^{-1}(x)\delta M(x)\rangle \\
&= \lim_{x\to\infty, y\to-\infty} -\langle e^{-a\lambda(x-y)} \delta_1 f f^{-1} e^{a\lambda(x-y)},\ \delta_2 M(x) M^{-1}(x)\rangle \\
&= \lim_{x\to\infty, y\to-\infty} -\langle e^{a\lambda y} \delta_1 M(y) M^{-1}(y) e^{-a\lambda y},\ e^{a\lambda x}\delta_2 M(x) M^{-1}(x) e^{-a\lambda x}\rangle.
\end{aligned}$$

Now by Theorem 7.19, we get

8.12 Theorem. *The Poisson structure on the unit ball B_1 in $\mathcal{S}_{1,a}$ with respect to the L^1-norm, written in terms of variations in D_-^c, is*

$$\{\delta_1 A, \delta_2 A\} = \int_{-\infty}^{\infty} \langle \delta_1 P P^{-1}, \delta_2 Q Q^{-1}\rangle,$$

where $A = \Psi(f)$, $f_+ = Pv = Q\tilde{v}$ is the factorization of f_+ into upper a-triangular and lower a-triangular times unitary and P_d, Q_d are holomorphic as given in Proposition 7.13.

Next we study the pull back the symplectic form

$$w(q_1, q_2) = \int_{-\infty}^{\infty} \operatorname{tr}(\operatorname{ad}(a)^{-1}(q_1)(q_2))\, dx$$

on $\mathcal{S}_{1,a}$ to the space $\Psi(G_-^m)$. This space has many complicated algebraic components. For example the space of all $A \in \mathcal{S}_{1,a}$ whose scattering data have only one pole (or equivalently, $A = \Psi(g)$, where g is a simple element) can be parametrized by

$$\bigcup_{k=1}^{n} C_+ \times \{V \in Gr(k, C) \mid a(V) \not\subset V\}.$$

However, the space of A whose scattering data has only two poles immediately becomes complicated as the factorization of $g \in G_-^m$ as product of simple elements is not unique. The following Proposition gives the restriction of w to the simplest component of $\Psi(G_-^m)$. We believe that the restriction of w to each algebraic component should be symplectic, but we have not yet found an efficient way to compute the general case.

8.13 Proposition. *Let $a = \operatorname{diag}(-i, i, \ldots, i)$. Then:*

(i) *The space of all $A = \Psi(g_{z,\pi})$, where π is the projection onto a one dimensional subspace Cv, is isomorphic to*

$$N = C_+ \times (C^{n-1} \setminus 0) = \{(z, v) \mid z \in C \setminus R, v = (v_2, \ldots, v_n) \neq 0\}.$$

(ii) *The pull back of the symplectic form w to N is*

$$2\operatorname{Re}\left(d\bar{z} \wedge \partial \log(|v|^2) + (z - \bar{z})\partial\bar{\partial}\log(|v|^2)\right),$$

where $|v| = \sum_{j=2}^{n} |v_j|^2$.

Proof. Let π denote the projection of C^n onto the one dimensional subspace spanned by $(1, v)$, where $v = (v_2, \ldots, v_n) \in C^{n-1}$. By Theorem 6.5 (vi) and formula (6.7), $g_{z,\pi} \sharp 0 = (u_{ij})$, where $(u_{ij}) \in u(n)$, $u_{ij} = 0$ if $2 \leq i, j \leq n$ and

$$u_{1j}(x) = \frac{2i(z - \bar{z})\bar{v}_j e^{i(z+\bar{z})x}}{e^{-i(z-\bar{z})x} + e^{i(z-\bar{z})}(\mid v_2\mid^2 + \ldots + \mid v_n\mid^2)}.$$

Then the proposition follows from at least two separate computations, neither of which is very illuminating. We hope to provide the more general results in a future paper. □

8.14 Remark. Fix $z \in C_+$. Then the restriction of the symplectic form to the subset $\{(z,v) \mid v \in C^{n-1} \setminus \{0\}, \|v\| = 1\}$ of M in the theorem above gives the standard symplectic structure of CP^{n-2}.

9 Symplectic structures for the restricted case

Most of the interesting applications in geometry come from restrictions of the full flow equation to a smaller phase space satisfying additional algebraic conditions. This leads to a serious problem, not with the flows and the scattering cosets, but with the symplectic structure. Generally the original symplectic structure we have used to this point vanishes on the restricted submanifolds. In this section, we describe a typical restrictions and the construction of the hierarchy of symplectic structures. We give an outline of the theory, and explain how it can be applied. The details of this construction for involutions appear in [Te2].

Let U be a simple Lie group, $\langle\ ,\rangle$ a non-degenerate, ad-invariant bilinear form on the Lie algebra $\mathcal{U}$, and σ an order m automorphism of U. For simplicity, we denote the Lie algebra automorphism $d\sigma_e$ on $\mathcal{U}$ again by σ. Fix a primitive m-th root of unity α. Suppose σ has an eigendecomposition on $\mathcal{U}$:

$$\mathcal{U} = \sum_{j=0}^{m} \mathcal{U}_j,$$

where $\mathcal{U}_j$ is the eigenspace of σ on $\mathcal{U}$ with eigenvalue α^j. Then

$$\begin{cases} [\mathcal{U}_j, \mathcal{U}_k] \subset \mathcal{U}_{j+k}, & \text{for all } j,k, \\ \langle \mathcal{U}_j, \mathcal{U}_k \rangle = 0, & \text{if } j \neq k. \end{cases}$$

Here we use the convention $\mathcal{U}_j = \mathcal{U}_k$ if $j \equiv k \pmod m$.

Fix an element $a \in \mathcal{U}_1$. Let $\mathcal{U}_a$ denote the centralizer of a, and $\mathcal{U}_a^\perp$ the orthogonal complement of $\mathcal{U}_a$ in $\mathcal{U}$. Consider a subspace of $\mathcal{S}_{1,a}$:

$$\mathcal{S}_{1,a}^\sigma = \{A = a\lambda + u \in \mathcal{S}_{1,a} \mid u \in \mathcal{U}_0 \cap \mathcal{U}_a^\perp\} \quad \subset \mathcal{S}_{1,a}.$$

Then $A \in \mathcal{S}_{1,a}^\sigma$ satisfies the reality condition

$$\sigma(A(\alpha^{-1}\lambda)) = A(\lambda). \tag{9.1}$$

Hence the trivialization E of $A \in \mathcal{S}_{1,a}$ normalized at the origin satisfies the condition $\sigma(E(\alpha^{-1}\lambda)) = E(\lambda)$. Then

9.1 Proposition. *$\mathcal{S}_{1,a}^\sigma$ is invariant under the action $\sharp$ of $G_-^{m,\sigma}$ on $\mathcal{S}_{1,a}$ defined in section 6, where $G_-^{m,\sigma}$ is the subgroup of $g \in G_-^m$ such that $\sigma(g(\alpha^{-1}\lambda)) = g(\lambda)$.*

The trivialization M of $A \in \mathcal{S}^{\sigma}_{1,a}$ normalized at infinity also satisfies the same reality condition as E. So for $b \in \mathcal{U}_1 \cap \mathcal{U}_a$, we have

$$\sigma(M^{-1}(\alpha^{-1}\lambda)bM(\alpha^{-1}\lambda)) = \alpha M^{-1}(\lambda)bM(\lambda).$$

Since

$$M^{-1}bM \sim \sum_{j=0}^{\infty} Q_{b,j}\lambda^{-j},$$

we get

$$Q_{b,j} \in \mathcal{U}_{-j+1}, \qquad \text{for all } j \geq 0.$$

In particular, $[Q_{j+1}, a] \in \mathcal{U}_{-j+1}$. So $[Q_{b,j+1}(u), a]$ is normal to $\mathcal{S}^{\sigma}_{1,a}$ if $j \not\equiv 1 \pmod{m}$, and is tangent to $\mathcal{S}^{\sigma}_{1,a}$ if $j \equiv 1 \pmod{m}$. And we have

9.2 Proposition. *The j-th flow preserves $\mathcal{S}^{\sigma}_{1,a}$ for all j and any order m automorphism σ. If $j \not\equiv 1 \pmod{m}$, then the flows are identically constant.*

However, the symplectic form

$$w(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty} \operatorname{tr}(-\operatorname{ad}(a)^{-1}(\delta_1 u), \delta_2 u)\, dx$$

vanishes on $\mathcal{S}^{\sigma}_{1,a}$.

The sequence of symplectic structures constructed by Terng can be described using a sequence of coadjoint orbits, which arise using a shift in the bi-linear form $\langle\ ,\ \rangle$ on the loop algebra $\mathcal{G}$.

For $k \leq -1$, let M_k denote the coadjoint $C(R, G_-)$-orbit at $\left(\frac{d}{dx} + a\lambda\right)\nu_k^{-1}$ in $C(R, \mathcal{G}_+)$, where $\nu_k(\lambda) = \lambda^{k+1}$. Set

$$\mathcal{S}_{1,a,k} = (M_k \nu_k) \cap \mathcal{S}_{1,a}.$$

Then δu lies in the tangent space of $\mathcal{S}_{1,a,k}$ at $\frac{d}{dx} + a\lambda + u$ if and only if

$$\delta u(x) = \left(\left[\xi_-(x), \frac{d}{dx} + A(x)\right]\nu_k^{-1}\right)_+ \nu_k, \tag{9.2}$$

where formally $\xi_-(x) \in \mathcal{G}_-$. Here $(\cdot)_+$ is the projection into $\mathcal{G}_+$, and the construction is entirely algebraic. For $A = a\lambda + u$, write

$$\xi_-(\delta u) = \xi_{-1}(\delta u)\lambda^{-1} + \xi_{-2}(\delta u)\lambda^{-2} + \dots .$$

Then equation (9.2) gives

$$\begin{aligned} [\xi_{-1}, a] &= \delta u, \\ [\xi_j, a] &= \left[\frac{d}{dx} + u, \xi_{j+1}\right], \qquad k \leq j \leq -1. \end{aligned}$$

This gives a recipe to compute $\xi_-(\delta u)$ explicitly via a mixed integro-differential operation:

$$\begin{aligned} \xi_{-1}^{\perp}(\delta u) &= J_a^{-1}(\delta u), \\ \xi_j^{\perp}(\delta u) &= (J_a^{-1}P_u)^{-j-1}J_a^{-1}(\delta u), \end{aligned}$$

where

$$\begin{aligned} J_a(v) &= [v,a], \\ P_u(v) &= v_x + [u,v]^{\perp} - [u,\eta_u(v)], \\ \eta_u(v) &= \int_{-\infty}^{x} [u(y),v(y)]^d dy. \end{aligned}$$

Here $v^{\perp}$ and v^d denote the projection onto a-off diagonal ($\mathcal{U}_a^{\perp}$) and a-diagonal ($\mathcal{U}_a$) respectively. Set

$$J_k = J_a(J_a^{-1}P_u)^{k+1}.$$

The natural shifted symplectic structure is given by

$$\begin{aligned} w_k(\delta_1 u, \delta_2 u) &= \int_{-\infty}^{\infty} \left\langle \frac{d}{dx} + A, \nu_k^{-1}[\xi_-(\delta_1 u), \xi_-(\delta_2 u)] \right\rangle dx \\ &= \int_{-\infty}^{\infty} \mathrm{tr}\left(\left(\frac{d}{dx} + a\lambda + u \right) ([\xi_-(\delta_1 u), \xi_-(\delta_2 u)]) \right)_k dx, \\ &= \int_{-\infty}^{\infty} \mathrm{tr}\left((\delta_1 u)\, \xi_k^{\perp}\, (\delta_2 u) \right) dx, \\ &= \int_{-\infty}^{\infty} \mathrm{tr}\left((\delta_1 u)\, J_k^{-1}\, (\delta_2 u) \right) dx, \end{aligned}$$

where $(\cdot)_k$ denote the coefficient of λ^k in $(\cdot)$. In particular, the first two in the series are:

$$\begin{aligned} w_{-1}(\delta_1 u, \delta_2 u) = w(\delta_1 u, \delta_2 u) &= \int_{-\infty}^{\infty} \mathrm{tr}((-\,\mathrm{ad}(a)^{-1}(\delta_1 u))\delta_2 u)\, dx, \\ w_{-2}(\delta_1 u, \delta_2 u) &= \int_{-\infty}^{\infty} \mathrm{tr}((\delta_1 u)(J_{-2})_u^{-1}(\delta_2 u))\, dx \\ &= \int_{-\infty}^{\infty} \mathrm{tr}((\delta_1 u)J_a^{-1}P_u J_a^{-1}(\delta_2 u))\, dx. \end{aligned}$$

The natural coadjoint orbits require the relevant terms of ξ_- to lie in the Schwartz class. So the tangent space of the smaller submanifold $\mathcal{S}_{1,a,k} =$

$(M_k\nu_k)\cap \mathcal{S}_{1,a}$ at u is

$$\{\delta u \mid \xi_{-j}(\delta u)(\infty)=0, 1\le j\le -k\}.$$

Hence $\mathcal{S}_{1,a,k}$ is a finite codimension submanifold of $\mathcal{S}_{1,a}$ and the formulas we write down for w_k are skew symmetric.

For $k\ge 0$, let M_k denote the coadjoint $C(R,G_+)$-orbit at $\left(\frac{d}{dx}+a\lambda\right)\nu_k^{-1}$ in $C(R,\mathcal{G}_-)$, and $\mathcal{S}_{1,a,k}=(M_k\nu_k)\cap\mathcal{S}_{1,a}$, where $\nu_k(\lambda)=\lambda^{k+1}$. Then δu lies in the tangent space of $\mathcal{S}_{1,a,k}$ at $\frac{d}{dx}+A$ if and only if

$$\delta u(x)=\left(\left[\xi_+(x),\frac{d}{dx}+A(x)\right]\nu_k^{-1}\right)_-\nu_k, \tag{9.3}$$

where formally $\xi_+(x)\in\mathcal{G}_+$. Here $()_-$ is the projection into $\mathcal{G}_-$. For $A=a\lambda+u$, write

$$\xi_+(\delta u)=\xi_0(\delta u)+\xi_1(\delta u)\lambda+\dots.$$

Then equation (9.3) gives

$$\begin{aligned}\left[\frac{d}{dx}+u,\xi_0\right]&=\delta u,\\ \left[\frac{d}{dx}+u,\xi_j\right]&=[\xi_{j-1},a], \qquad 1\le j\le k.\end{aligned}$$

Hence

$$\xi_j^\perp(\delta u)=(P_u^{-1}J_a)^jP_u^{-1}(\delta u)=J_j^{-1}(\delta u).$$

The natural shifted symplectic structure is given by

$$w_k(\delta_1u,\delta_2u)=\int_{-\infty}^{\infty}\operatorname{tr}((\delta_1u)J_k^{-1}(\delta_2u))\,dx.$$

In particular,

$$w_0(\delta_1u,\delta_2u)=\int_{-\infty}^{\infty}\operatorname{tr}((\delta_1u)P_u^{-1}(\delta_2u))\,dx.$$

If $a\in\mathcal{U}_1$, then $J_a=-\operatorname{ad}(a)$ maps $\mathcal{U}_j$ to $\mathcal{U}_{j+1}$. This implies that J_k maps $\mathcal{U}_j$ to $\mathcal{U}_{j-k}$. Thus we obtain:

9.3 Proposition ([Te2]). *w_k is a symplectic structure on $\mathcal{S}_{1,a,k}$. Moreover,*

(i) $w_k=0$ on $\mathcal{S}_{1,a,k}\cap\mathcal{S}_{1,a}^\sigma$ if $k\not\equiv 0\pmod m$,

(ii) w_k is non-degenerate on $\mathcal{S}_{1,a,k}\cap\mathcal{S}_{1,a}^\sigma$ if $k\equiv 0\pmod m$.

Recall that $F_{b,j}$ defined by formula (8.4) is the Hamiltonian for the j-th flow on $\mathcal{S}_{1,a}$ defined by b with respect to the symplectic form w_{-1}, and $\nabla F_{b,j}=Q_{b,j+1}^\perp$. Since $P_u(Q_{b,j}^\perp)=[Q_{b,j+1},a]$, we get

9.4 Theorem ([Te2]). *If a is regular, then*

(i) $J_r(\nabla F_{b,j}) = [Q_{b,j+r+2}, a]$,

(ii) the Hamiltonian flow corresponding to $F_{b,j}$ on $(\mathcal{S}_{1,a}, w_r)$ is the $(j+r+1)$-th flow defined by b.

9.5 Examples.

Example 1. Let σ denote the involution $\sigma(y) = -y^t$ of $SU(n)$, and $a = \operatorname{diag}(i, -i, \ldots, -i)$. Then $\mathcal{S}^{\sigma}_{1,a,0}$ is the set of all $A = a\lambda + u$ with

$$u = \begin{pmatrix} 0 & v \\ -v^t & 0 \end{pmatrix},$$

where $v : R \to \mathcal{M}_{1\times(n-1)}$ is a decay map from R to the space $\mathcal{M}_{1\times(n-1)}$ of real $1 \times (n-1)$ matrices. The even flows vanishes on $\mathcal{S}^{\sigma}_{1,a,0}$, and the odd flows are extensions of the usual hierarchy of flows for the modified KdV. The third flow written in terms of $v : R \to \mathcal{M}_{1\times(n-1)}$ is the matrix modified KdV equation:

$$v_t = -\frac{1}{4}(v_{xxx} + 3(v_x v^t v + v v^t v_x).$$

(When $n = 2$, v is a scalar function and the above equation is the classic modified KdV equation.) The 2-form w_0 gives the appropriate non-degenerate symplectic structure for the matrix modified KdV equation and the hierarchy of odd flows.

Example 2. It seems appropriate to mention the relation of the restriction to the sine-Gordon equation. The sine-Gordon equation is written in space time coordinates (τ, y) as

$$\frac{\partial^2 q}{\partial \tau^2} - \frac{\partial^2 q}{\partial y^2} + \sin q = 0,$$

or

$$q_{xt} = \sin q$$

in characteristic coordinates. This is the -1-flow on $\mathcal{S}^{\sigma}_{1,a}$ defined by b, where $\sigma(y) = -y^t$ is the involution on $su(2)$, $a = \operatorname{diag}(i, -i)$ and $b = -a/4$. The Lax pair is best written in characteristic coordinates:

$$\left[\frac{\partial}{\partial x} + a\lambda + u, \frac{\partial}{\partial t} + \lambda^{-1} B\right] = 0,$$

where

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad u = \begin{pmatrix} 0 & \frac{q_x}{2} \\ -\frac{q_x}{2} & 0 \end{pmatrix}, \quad B = i \begin{pmatrix} \cos q & \sin q \\ \sin q & -\cos q \end{pmatrix}.$$

The restriction is the same as for the modified KdV. The natural Cauchy problem is in space time coordinates (τ, y), but the scattering theory has been developed for characteristic coordinates. However, the classical Bäcklund transformations work well with either choice of coordinates , and preserve whatever decay conditions have been described in either coordinate systems.

Example 3. We obtain the Kupershmidt and Wilson equation ([KW]) in terms of a restriction by an order n automorphism of $sl(n)$. Let $\alpha = e^{2\pi i/m}$, and $p \in SL(n)$ the matrix representing the cyclic permutation $(12\ldots n)$, i.e., $p(e_i) = e_{i+1}$ (here we use the convention that $e_i = e_j$ if $i \equiv j \pmod n$). Let $\sigma : sl(n) \to sl(n)$ be the order n automorphism defined by $\sigma(y) = p^{-1}yp$. Then $X \in \mathcal{U}_j$ if and only if $\sigma(X) = \alpha^j X$. Let

$$a = \operatorname{diag}(1, \alpha, \ldots, \alpha^{n-1}) \in \mathcal{U}_1.$$

Note that $\mathcal{U}_a^\perp$ is the space of all matrices $X \in sl(n)$ such that $X_{ii} = 0$ for all $i = 1, \ldots, n$. So

$$\mathcal{S}_{1,a}^\sigma = \begin{cases} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda + \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} \,\middle|\, v \in \mathcal{S}(R, C) \right\}, & \text{if } n = 2, \\ \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \lambda + \begin{pmatrix} 0 & v_1 & v_2 \\ v_2 & 0 & v_1 \\ v_1 & v_2 & 0 \end{pmatrix} \,\middle|\, v_1, v_2 \in \mathcal{S}(R, C) \right\}, & \text{if } n = 3. \end{cases}$$

In general, $A = a\lambda + u \in \mathcal{S}_{1,a}^\sigma$ is determined by $(n-1)$ functions (the first row of u). By Propositions 9.3, $\{w_{rn} \mid r \in Z\}$ is a sequence of symplectic forms on $\mathcal{S}_{1,a}^\sigma$. The $(n+1)$-th flow is the Kupershmidt-Wilson equation. By Theorem 9.4 it satisfies the Lenard relation:

$$u_t = [Q_{a,n+2}, a] = J_0(\nabla F_{b,n}) = J_n(\nabla F_{b,0}).$$

When $n = 2$, the third flow on $\mathcal{S}_{1,a}^\sigma$ defined by a gives the modified KdV equation:

$$v_t = \frac{1}{4}(v_{xxx} - 6v^2 v_x), \tag{9.4}$$

and all the odd flows are the hierarchy of commuting flows of the modified KdV equation. For $n > 2$, this gives another generalization of modified KdV equation.

10 Bäcklund transformations for j-th flows

This section contains a brief outline of ideas and results in [TU1]. The classical Bäcklund transformations are originally geometric constructions by which a two parameters family of constant Gaussian curvature -1 surfaces is obtained from a single surface of Gaussian curvature -1. This is accomplished by solving two ordinary differential equations with a parameter s. The second parameter is the initial data. Since surfaces of Gaussian curvature -1 are classically known to be equivalent to local solutions of the sine-Gordon equation ([Dal], [Ei])

$$q_{xt} = \sin q$$

this provides a method of deriving new solutions of a partial differential equation from a given solution via the solution of ordinary differential equations. Most of the known "integrable systems" possess transformations of this type, which are sometimes called *Darboux transformations*. Ribaucour and Lie transformations are other classical transformations that generate new solutions from a given one.

The action of the rational loop group we constructed in section 6 can be extended to an action which transforms solutions of the j-th flow equation. In this section we describe very briefly the results in [TU1], which will construct an action of the semi-direct product of $R^* \ltimes G_-^m$ on the solution space of the j-th flow. The construction of this loop group action is motivated by the construction given by the second author in [U1] for harmonic maps. We will see

(1) the action of a simple element $g_{z,\pi}$ corresponds to a Bäcklund transformation,

(2) the action of R^* corresponds to the Lie transformations,

(3) the Bianchi permutability formula arises from the various ways of factoring quadratic elements in the rational loop group into simple elements,

(4) the Bäcklund transformations arise from ordinary differential equations if one solution is known,

(5) once given the trivialization of the Lax pair corresponding to a given solution, the Bäcklund transformations become algebraic.

Since the sine-Gordon equation arises as part of the algebraic structure (the -1-flow for $su(2)$ with an involution constraint), we can check that we are generalizing the classical theory. The choice of group structure depends on the choice of the base point (just as the scattering theory depends on the choice of a vacuum, or the choice of $0 \in R$). Hence the group structure was not apparent to the classical geometers.

One of the most interesting observations is that appropriate choices of poles for the rational loop yield time periodic solutions. This yields an interesting insight into the construction of time-periodic solutions (or the classical breathers) to the sine-Gordon equation as explained in Darboux ([Da1]). For recent developments concerning breathers of the sine-Gordon equation see [BMW], [De], [SS]. There are no simple factors in the rational loop group corresponding to the placement of poles for time periodic solutions. However, there are quadratic elements, whose simple factors do not satisfy the algebraic constraints to preserve sine-Gordon, but which nevertheless generate the well-known breathers (one way to think of them is as the product of two complex conjugate Bäcklund transformations). The product of these quadratic factors generate arbitrarily complicated time periodic solutions.

The classical theory of Bäcklund transformations is based on ordinary differential equations.

10.1 Theorem ([Ei]). *Suppose q is a solution of the sine-Gordon equation, and $s \neq 0$ is a real number. Then the following first order system is solvable for q^*:*

$$\begin{cases} (q^* - q)_x = 4s \sin\left(\dfrac{q^* + q}{2}\right) \\ (q^* + q)_t = \dfrac{1}{s} \sin\left(\dfrac{q^* - q}{2}\right). \end{cases} \tag{10.1}$$

Moreover, q^* is again a solution of the sine-Gordon equation.

10.2 Definition. If q is a solution of the sine-Gordon equation, then given any $c_o \in R$ there is a unique solution q^* for equation (10.1) such that $q^*(0,0) = c_o$. Then $B_{s,c_o}(q) = q^*$ is a transformation on the space of solutions of the sine-Gordon equation, which will be called a *Bäcklund transformation* for the sine-Gordon equation.

10.3 Proposition ([Ei]). *Define $L_s(q)(x,t) = q(sx, s^{-1}t)$. Then q is a solution of the sine-Gordon equation if and only if $L_s(q)$ is a solution of the sine-Gordon equation. (L_s is called a Lie transformation).*

10.4 Proposition ([Ei]). *Bäcklund transformations and Lie transformations of the sine-Gordon equation are related by the following formula:*

$$B_{s,c_o} = L_s^{-1} B_{1,c_o} L_s.$$

There is also a Bianchi permutability theorem for surfaces with Gaussian curvature -1 in R^3, which gives the following analytical formula for the sine-Gordon equation:

10.5 Theorem ([Ei]). *Suppose q_0 is a solution of the sine-Gordon equation, $s_1^2 \neq s_2^2$, and $s_1 s_2 \neq 0$. Let $q_i = B_{s_i,c_i}(q_0)$ for $i = 1,2$. Then there exist $d_1, d_2 \in R$, which can be constructed algebraically, such that*

(1) $B_{s_1,d_1} B_{s_2,c_2} = B_{s_2,d_2} B_{s_1,c_1}$,

(2) let $q_3 = B_{s_1,d_1} B_{s_2,c_2}(q_0)$, *then*

$$\tan \frac{q_3 - q_0}{4} = \frac{s_1 + s_2}{s_1 - s_2} \tan \frac{q_1 - q_2}{4}. \tag{10.2}$$

This is called the Bianchi permutability formula for the sine-Gordon equation.

Next we describe the action of G_-^m on the spaces of solutions of the j-th flow ($j \geq -1$). This construction is again using dressing action as in section 6 for the action of G_-^m on $\mathbb{S}_{1,a}$. First we make some definitions:

10.6 Definition. Let $\mathcal{M}(j,a,b)$ denote the space of all solutions of the j-th flow (equation (5.6)) on $\mathcal{S}_{1,a}$ defined by b with $[a,b]=0$ for $j=-1$ and $j\geq 1$ respectively.

Assume $j\geq 1$. Let $A=a\lambda+u\in\mathcal{M}(j,a,b)$, and $E(x,t,\lambda)$ the trivialization of A normalized at $(x,t)=0$, i.e.,

$$\begin{cases} E^{-1}E_x = a\lambda+u, \\ E^{-1}E_t = b\lambda^j + Q_{b,1}\lambda^{j-1}+\ldots+Q_{b,j}, \\ E(0,0,\lambda)=I. \end{cases}$$

Given $g\in G_-^m$, by exactly the same method as in section 6, we can factor

$$g(\lambda)E(x,t,\lambda)=\tilde{E}(x,t,\lambda)\tilde{g}(x,t,\lambda),$$

such that $E(x,t,\cdot)\in G_+$ and $\tilde{g}(x,t,\cdot)\in G_-^m$. Define

$$\begin{cases} g\bullet E=\tilde{E}, \\ g\bullet A=\tilde{E}^{-1}\tilde{E}_x \\ . \end{cases}$$

Then $g\bullet A\in\mathcal{M}(j,a,b)$, and $\bullet$ defines an action of G_-^m on $\mathcal{M}(j,a,b)$.

Recall that the $g\in G_-^m$ can be generated by simple elements $g_{z,\pi}\in G_-^m$, which are rational functions of degree 1. Choose a pole $z\in C\setminus R$, and a subspace $V\subset C^n$, which is identified with the Hermitian projection

$$\pi: C^n\to V.$$

Write

$$g_{z,\pi}(\lambda)=\pi+\frac{\lambda-z}{\lambda-\bar{z}}\pi^{\perp}$$

as in Proposition 6.3.

10.7 Definition. $A\mapsto g_{z,\pi}\bullet A$ is a Bäcklund transformation for the j-th flow.

Compute the action of $g_{z,\pi}$ explicitly as in section 6 to get:

10.8 Theorem. *Let $g_{z,\pi}$ be a generator in G_-^m, where π is the projection of C^n onto a k-dimensional complex linear subspace V. Let $A=a\lambda+u\in\mathcal{M}(j,a,b)$, and $E(x,t,\lambda)$ the trivialization of A normalized at $(x,t)=0$. Set $\tilde{V}(x,t)=E(x,t,z)^*(V)$, and let $\tilde{\pi}(x,t)$ denote the projection of C^n onto $\tilde{V}(x,t)$. Set*

$$\begin{aligned} \tilde{\pi}(x,t) &= E^*(x,t,z)U(U^*E(x,t,z)E^*(x,t,z)U)^{-1}U^*E(x,t,z), \\ g_{z,\tilde{\pi}(x,t)(\lambda)} &= \tilde{\pi}(x,t)+\frac{\lambda-z}{\lambda-\bar{z}}\tilde{\pi}(x,t)^{\perp} \end{aligned} \tag{10.3}$$

where U is a $n\times k$ matrix whose columns form a basis for V. Then

(i) $g_{z,\pi} \bullet E = g_{z,\pi} E g_{z,\tilde{\pi}}{}^{-1}$,

(ii) $g_{z,\pi} \bullet A = A + (z - \bar{z})[\tilde{\pi}, a]$.

10.9 Theorem. *The $\tilde{\pi}$ constructed in Theorem 10.8 is the solution of the following compatible first order system:*

$$\begin{cases} (\tilde{\pi})_x + [az + u, \tilde{\pi}] = (\bar{z} - z)[\tilde{\pi}, a]\tilde{\pi}, \\ (\tilde{\pi})_t = \sum_{k=0}^{j} [\tilde{\pi}, Q_{b,j-k}(u)](z + (\bar{z} - z)\tilde{\pi})^k, \\ \tilde{\pi}^* = \tilde{\pi}, \quad \tilde{\pi}^2 = \tilde{\pi}, \quad \tilde{\pi}(0,0) = \pi. \end{cases} \tag{10.4}$$

Moreover,

(i) *equation (10.4) is solvable for $\tilde{\pi}$ if and only if $A = a\lambda + u$ is a solution of the j-th flow on $\mathcal{S}_{1,a}$ defined by b,*

(ii) *if $A = a\lambda + u$ is a solution of the j-th flow and $\tilde{\pi}$ is a solution of equation (10.4), then $\tilde{A} = A + (z - \bar{z})[\tilde{\pi}, a]$ is again a solution of the j-th flow.*

10.10 Definition. Let $R^* = \{r \in R \,|\, r \neq 0\}$ denote the multiplicative group, and $R^* \ltimes G_-^m$ the semi-direct product of R^* and G_-^m defined by the homomorphism

$$\rho : R^* \to \mathrm{Aut}(G_-^m), \qquad \rho(r)(g)(\lambda) = g(r\lambda),$$

i.e., the multiplication in $R^* \ltimes \Omega(G)$ is defined by

$$(r_1, h_1) \cdot (r_2, h_2) = (r_1 r_2, h_1(\rho(r_1)(h_2))).$$

10.11 Theorem. *The action $\bullet$ of G_-^m extends to an action of $R^* \ltimes G_-^*$ on the space $\mathcal{M}(j, a, b)$ of solutions of the j-th flow on $\mathcal{S}_{1,a}$ defined by b. In fact, if $A = a\lambda + u \in \mathcal{M}(j, a, b)$ and E is the trivialization of A normalized at $(x, t) = 0$, then*

$$\begin{cases} (r \bullet E)(x, t, \lambda) = E(r^{-1}x, r^{-j}t, r\lambda), \\ (r \bullet A)(x, t, \lambda) = a\lambda + r^{-1}u(r^{-1}x, r^{-j}t). \end{cases}$$

Since $(r^{-1}, 1)(1, g_{e^{i\alpha},\pi})(r, 1) = (1, g_{re^{i\alpha},\pi})$, we have

10.12 Corollary. *If $A \in \mathcal{M}(j, a, b)$, then*

$$r^{-1} \bullet (g_{e^{i\alpha},\pi} \bullet (r \bullet A)) = g_{re^{i\alpha},\pi} \bullet A.$$

Next we state an analogue of the Bianchi Permutability Theorem for the positive flows:

10.13 Theorem. *Let $z_1 = r_1 + is_1, z_2 = r_2 + is_2 \in C \setminus R$ such that $r_1 \neq r_2$ or $s_1^2 \neq s_2^2$, and π_1, π_2 projections of C^n. Let $A_0 \in \mathcal{M}(j, a, b)$, and $A_i = g_{z_i,\pi_i} \bullet A_0$ for $i = 1, 2$. Set*

$$\begin{aligned} \xi_i &= \left(-(z_1 - z_2)I + 2i(s_1\pi_1 - s_2\pi_2)\right)\pi_i\left((-(z_1 - z_2)I + 2i(s_1\pi_1 - s_2\pi_2)\right)^{-1}, \\ \tilde{\xi}_i &= (-(z_1 - z_2)I + 2i(s_1\tilde{\pi}_1 - s_2\tilde{\pi}_2))\tilde{\pi}_i(-(z_1 - z_2)I + 2i(s_1\tilde{\pi}_1 - s_2\tilde{\pi}_2))^{-1}, \end{aligned} \tag{10.5}$$

for $i = 1, 2$, where $\tilde{\pi}_i$ is as in Theorem 10.8 and $A_i = A_0 + 2is[\tilde{\pi}_i, a]$. Then

(i) $g_{z_2,\xi_2} g_{z_1,\pi_1} = g_{z_1,\xi_1} g_{z_2,\pi_2}$,

(ii)

$$\begin{aligned} A_3 &= (g_{z_2,\xi_2} g_{z_1,\pi_1}) \bullet A_0 = A_0 + 2i[s_1\tilde{\pi}_1 + s_2\tilde{\xi}_2, a] \\ &= (g_{z_1,\xi_1} g_{z_2,\pi_2}) \bullet A_0 = A_0 + 2i[s_1\tilde{\xi}_1 + s_2\tilde{\pi}_2, a]. \end{aligned} \tag{10.6}$$

Note that if A lies in the orbit of $R^* \ltimes G_-^m$ through the vacuum solution $A_0 = a\lambda$ of the j-th flow then A has no continuous scattering data.

10.14 Definition. An element A in the orbit of $R^* \ltimes G_-^m$ through the vacuum solution $A_0 = a\lambda$ of the j-th flow is called a *k-soliton* if the normalized eigenfunction has k poles counted with multiplicities.

10.15 Remark. The trivialization of the vacuum solution $A_0 = a\lambda$ of the j-th flow on $\mathcal{S}_{1,a}$ defined by b is $E = \exp(a\lambda x + b\lambda^j t)$. So 1-soliton is

$$g_{z,\pi} \bullet A_0 = a\lambda + (z - \bar{z}) \left[e^{-a\bar{z}x - b\bar{z}^j t} U (U^* e^{a(z-\bar{z})x + b(z^j - \bar{z}^j)t} U)^{-1} U^* e^{azx + bz^j t},\ a \right],$$

where U is a matrix whose columns form a basis of $\mathrm{Im}(\pi)$. If $g = \prod_{i=1}^k g_{z_i,\pi_i}$, then $g \sharp A_0$ can be written in terms of 1-solitons $g_{z_1,\pi_1} \sharp A, \dots, g_{z_k,\pi_k} \sharp A$ algebraically by applying the permutability formula (10.6) repeatedly.

10.16 Remark. If A is a soliton solution of the j-th flow on $\mathcal{S}_{1,a}$ defined by b, then by Corollary 8.3 there exists $g \in G_-^m$ such that

$$A(t) = \Psi(e_{b,j}(t) \natural g).$$

As noted in Remark 7.22, given $g_{z,\pi} \notin H_-$, $g_{z,\pi} \bullet A$ can be a $m+1, m$ or $m-1$-soliton if A is a m-soliton solution.

10.17 Proposition. *The set of all soliton solutions of the j-th flow on $\mathcal{S}_{1,a}$ defined by b is isomorphic to the space G_-^m / H_-^m of left cosets, where $H_-^m = G_-^m \cap H_-$.*

10.18 Remark. Although the space G^m_-/H^m_- of all multi-solitons does not have a manifold structure, the set $\mathcal{B}_m$ all m-solitons is the union of complex algebraic varieties. For example,

$$\begin{aligned}\mathcal{B}_1 &= \bigcup_{k=1}^{n-1} (C_+ \times X_k), \text{ where}\\ X_k &= Gr(k, C^n) \setminus \{V \in Gr(k, C^n) \mid a(V) = V\}.\end{aligned}$$

But $\mathcal{B}_m$ with $m \geq 2$ is much more complicated because of the non-uniqueness of the factorization and the fact that generators of G^m_- have complicated relations such as the permutability formula given in Theorem 10.13 (i).

Next we apply the action of G^m_- on the first flows to get actions of G^m_- on the space $\mathcal{M}$ of solutions of the n-dimension system (8.2) associated to $U(n)$. Given $v \in \mathcal{M}$, the trivialization E of v normalized at the origin is the solution

$$\begin{cases} E^{-1}E_{x_j} = a_j\lambda + [a_j, v], \quad 1 \leq j \leq n \\ E(0, \lambda) = I. \end{cases}$$

10.19 Theorem. *The group $R^* \times G^m_-$ acts on $\mathcal{M}$, and the action $\bullet$ is constructed in the same manner as on the spaces of solutions of the first flow. In fact, given $g_{z,\pi} \in G^m_-$, the following initial value problem is solvable for $\tilde{\pi}$ and has a unique solution:*

$$\begin{cases} (\tilde{\pi})_{x_j} + [a_j z + [a_j, v], \tilde{\pi}] = (\bar{z} - z)[\tilde{\pi}, a_j]\tilde{\pi}, \\ \tilde{\pi}^* = \tilde{\pi}, \quad \tilde{\pi}^2 = \tilde{\pi}, \quad \tilde{\pi}(0) = \pi. \end{cases}$$

Moreover,

(i) $g_{z,\pi} \bullet v = v - ((z - \bar{z})\tilde{\pi})^\perp$, *where $y^\perp$ denote the projection of $y \in \mathcal{U}$ onto $\mathcal{T}^\perp$,*

(ii) the trivialization of $g_{z,\pi} \bullet v$ is $g_{z,\pi} E g^{-1}_{z,\tilde{\pi}}$,

(iii) $\tilde{\pi}$ is the projection onto the linear subspace $E^*_z(V)$, where V is the image of the projection π,

(iv) $(r \bullet v)(x) = r^{-1}v(r^{-1}x)$ for $r \in R^*$.

10.20 Remark. The permutability Theorem 10.13 holds for system (8.2) with the same formula.

There are also analogous results for the -1-flow:

10.21 Theorem. *The group $R^* \ltimes G^m_-$ acts on the space $\mathcal{M}(-1, a, b)$ of solutions of the -1-flow on $\mathcal{S}_{1,a}$ defined by b:*

$$\begin{cases} u_t = [a, g^{-1}bg], \\ g_t = gu, \qquad \lim_{x\to-\infty} g(x) = I. \end{cases} \tag{10.7}$$

Moreover, let $A \in \mathcal{M}(-1, a, b)$, E the trivialization of A normalized at $(x, t) = 0$, and $g_{z,\pi}$ a simple element of G_-^m, then

(i) Theorems 10.8, 10.13 and Corollary 10.12 hold with the same formulas,

(ii) $\tilde{\pi}$ is the solution to

$$\begin{cases} (\tilde{\pi})_x + [az + u, \tilde{\pi}] = (\bar{z} - z)[\tilde{\pi}, a]\tilde{\pi}, \\ (\tilde{\pi})_t = \dfrac{1}{|\,z\,|^2}\left((z - \bar{z})\tilde{\pi}g^{-1}bg\tilde{\pi} - zg^{-1}bg\tilde{\pi} + \bar{z}\tilde{\pi}g^{-1}bg\right), \\ \tilde{\pi}^* = \tilde{\pi}, \quad \tilde{\pi}^2 = \tilde{\pi}, \quad \tilde{\pi}(0,0) = \pi, \end{cases} \tag{10.8}$$

(iii) for $r \in R^*$ we have

$$\begin{cases} (r \bullet E)(x, t, \lambda) = E(r^{-1}x, rt, r\lambda), \\ (r \bullet A)(x, t, \lambda) = a\lambda + r^{-1}u(r^{-1}x, rt). \end{cases}$$

10.22 Remark. Let $A \in \mathcal{M}(-1, a, b)$, and E its trivialization normalized at the origin. Then $s(x, t) = E(x, t, -1)E(x, t, 1)^{-1}$ is a harmonic map from $R^{1,1}$ with the metric $2dx\,dt$ to $U(n)$ and $s^{-1}s_x$ is conjugate to a and $s^{-1}s_t$ is conjugate to b. In particular, this says that $\mathcal{M}(-1, a, b)$ is a subset of the space $\mathcal{F}$ of harmonic maps from $R^{1,1}$ to $U(n)$. The action of G_-^m on $\mathcal{F}$ constructed in [U1] leaves $\mathcal{M}(-1, a, b)$ invariant, and agrees with the action $\bullet$ we constructed here.

Recall that given an involution σ of $su(n)$, $G_-^{m,\sigma}$ is the subgroup of $g \in G_-^m$ such that $\sigma(g(-\lambda)) = g(\lambda)$ and $\mathcal{S}_{1,a}^\sigma = \{A \in \mathcal{S}_{1,a} \mid \sigma(A(-\lambda)) = A(\lambda)\}$. Since the action of $G_-^{m,\sigma}$ leaves $\mathcal{S}_{1,a}^\sigma$ invariant, we have

10.23 Theorem. *The space $\mathcal{M}^\sigma(2j-1, a, b)$ of solutions of the $(2j-1)$-th flow associated to U/K defined by b is a subset of $\mathcal{M}(2j-1, a, b)$ for $j \geq 0$. Moreover, the action $\bullet$ of $R^* \ltimes G_-^{m,\sigma}$ leaves $\mathcal{M}^\sigma(2j-1, a, b)$ invariant, and Theorems 10.8, 10.9, 10.11 and Corollary 10.12 hold for $\mathcal{M}^\sigma(2j-1, a, b)$ and $R^* \ltimes G_-^{m,\sigma}$.*

10.24 Theorem. *The space $\mathcal{M}^\sigma(-1, a, b)$ of solutions of the -1-th flow equation on $\mathcal{S}_{1,a}^\sigma$ defined by b is a subset of $\mathcal{M}(-1, a, b)$. Moreover, the action of $R^* \ltimes G_-^{m,\sigma}$ leaves $\mathcal{M}^\sigma(-1, a, b)$ invariant and Theorem 10.21 holds for $\mathcal{M}_{1,a}^\sigma$ and $R^* \ltimes G_-^{m,\sigma}$.*

10.25 Proposition. *Let σ denote the involution on $SU(n)$ defined by $\sigma(y) = \bar{y}$. Then*

(i) $g_{z,\pi} \in G_-^{m,\sigma}$ if and only if $z = -\bar{z}$ and $\bar{\pi} = \pi$,

(ii) if $\bar{\pi} = \pi$, then $g_{z,\pi}g_{-\bar{z},\pi} \in G_-^{m,\sigma}$.

Next we explain the relation between the classical Bäcklund transformations and the action of $G_-^{m,\sigma}$ on the space of solutions of the sine-Gordon equation (or the space $\mathcal{M}^\sigma(-1,a,b)$ with σ, a,b defined as in Example 9.5). If $s \in R$, $\tilde{\pi}^* = \tilde{\pi} = (\tilde{\pi})^t$ and $(\tilde{\pi})^* = \tilde{\pi}$, then by Proposition 10.25, $g_{is,\tilde{\pi}} \in G_-^{m,\sigma}$. Hence

$$\tilde{\pi} = \begin{pmatrix} \cos^2 \frac{f}{2} & \sin\frac{f}{2}\cos\frac{f}{2} \\ \sin\frac{f}{2}\cos\frac{f}{2} & \sin^2\frac{f}{2} \end{pmatrix}$$

for some function f, i.e., $\tilde{\pi}$ is the projection onto $\begin{pmatrix} \cos\frac{f}{2} \\ \sin\frac{f}{2} \end{pmatrix}$. So the first order system (10.8) for $\tilde{\pi}$ becomes

$$\begin{cases} f_x = \frac{q_x}{2} + 2s\sin f, \\ f_t = \frac{1}{2s}\sin(f-q). \end{cases} \tag{10.9}$$

Write

$$\tilde{u} = g_{is,\beta} \bullet u = \begin{pmatrix} 0 & \tilde{q}_x/2 \\ -\tilde{q}_x/2 & 0 \end{pmatrix}.$$

But $\tilde{u} = u + 2is[\tilde{\pi}, a]$, hence we have $\tilde{q} = 2f - q$. Writing equation (10.9) in terms of $\tilde{q}$, we get

$$\begin{cases} (q^* - q)_x = 4s\sin(\frac{q^*+q}{2}) \\ (q^* + q)_t = \frac{1}{s}\sin(\frac{q^*-q}{2}), \end{cases}$$

which is the classical Bäcklund transformation for the sine-Gordon equation. So we have:

10.26 Proposition. *Let q be a solution of the sine-Gordon equation, and $0 < c_0 < \pi$. Set*

$$\begin{cases} A = a\lambda + \begin{pmatrix} 0 & \frac{q_x}{2} \\ -\frac{q_x}{2} & 0 \end{pmatrix}, \\ f_o = 1/2(q(0,0) + c_o) \\ \pi = \begin{pmatrix} \cos^2\frac{f_0}{2} & \sin\frac{f_0}{2}\cos\frac{f_0}{2} \\ \sin\frac{f_0}{2}\cos\frac{f_0}{2} & \sin^2\frac{f_0}{2} \end{pmatrix}. \end{cases}$$

Then

$$B_{s,c_o}(q) = g_{is,\pi} \bullet A.$$

(We will still use $\bullet$ to denote the induced action of $G_-^{m,\sigma}$ on the set of solutions of the sine-Gordon equation).

10.27 Proposition. *Let q is a solution of the sine-Gordon equation. Then:*

(i) $s \bullet q$ is the Lie transformation $L_s(q)$.

(ii) Proposition 10.4 is a consequence of the following equality in the group $R^* \ltimes G_-^m$:

$$(s^{-1}, I)(1, g_{e^{i\alpha},\beta})(s, I) = g_{se^{i\alpha},\beta}.$$

(iii) The Permutability Formula (10.2) in Theorem 10.5 is the same formula (10.6) given in Theorem 10.13, which follows from the following relation of the generators of G_-^m:

$$g_{z_1,\xi_1} g_{z_2,\pi_2} = g_{z_2,\xi_2} g_{z_1,\pi_1},$$

where π_i and ξ_i are related by formula (10.5).

10.28 Remark. It is noted by Xi Du [Du] that a classical Ribaucour transformations as defined in Darboux ([Da1]) for surfaces of $K = -1$ in R^3 correspond to the action of an element $g \in G_-^{m,\sigma}$, which is the product of two simple elements as in Proposition 10.25 (ii).

Using the action of G_-^m, we obtain many solutions of the j-th flow that are periodic in time. This is an algebraic calculation, which shows that when the poles are properly placed, the solutions are periodic. Multi-solitons will be time periodic if the periods of the component solitons are rationally related.

10.29 Theorem. *Let $j > 1$ be an integer, $a = \mathrm{diag}(ia_1, \dots, ia_n)$, and $b = \mathrm{diag}(ib_1, \dots, ib_n)$. If $b_1, \dots, b_n$ are rational numbers. Then the j-th flow equation defined by a, b:*

$$u_t = [Q_{b,j+1}(u), a]$$

has infinitely many m-soliton solutions that are periodic in t.

The trivialization of the vacuum solution for the -1-flow defined by $a = \mathrm{diag}(i, \dots, i, -i, \dots, -i)$ is $E(\lambda, x, t) = \exp(a(\lambda x + \lambda^{-1} t))$. By Theorem 10.21, the 1-soliton $g_{e^{i\theta},\pi} \bullet 0$ is a function of

$$\exp(i(\cos\theta(x+t) - i\sin(x-t))) = \exp(i\tau\cos\theta + y\sin\theta),$$

where $y = x - t$ and $\tau = x + t$ are the space-time coordinates. This gives

10.30 Theorem. *If $z = e^{i\theta}$ and $a = \mathrm{diag}(i, \dots, i, -i, \dots, -i)$, then the 1-soliton $g_{z,\pi} \bullet 0$ for the -1-flow (harmonic maps from $R^{1,1}$ to $SU(n)$) is periodic in time with period $\frac{2\pi}{\cos\theta}$. A multiple soliton generated by a rational loop with poles at $z_1 = e^{i\theta_1}, \dots, z_r = e^{i\theta_r}$ will be periodic with period τ if there exists integers $k_1, \dots, k_r$ such that*

$$\tau = \frac{2\pi k_j}{\cos\theta_j} \qquad \forall\, 1 \le j \le r.$$

The multi-solitons above satisfy the sine-Gordon equation if the rational loop satisfies $\overline{f(-\bar\lambda)} = f(\lambda)$. This means the poles occur in pairs $(e^{i\theta_j}, -e^{-i\theta_j})$ and the projection matrices π_j must be real.

10.31 Corollary. *Multiple-breather solutions exists for the sine-Gordon equation.*

10.32 Example. If π is real, then

$$(g_{e^{i\theta},\pi} g_{-e^{-i\theta},\pi}) \bullet 0 = 4\tan^{-1}\left(\frac{\sin\theta\sin((x+t)\cos\theta)}{\cos\theta\cosh((x-t)\sin\theta)}\right)$$

is the classical breather solution for the sine-Gordon equation. Theorems 10.24 and 10.30 give m-breather solutions explicitly, although the computations are quite long.

10.33 Corollary. *There are infinitely many harmonic maps from $R^{1,1}$ to a symmetric space that are periodic in time.*

11 Geometric non-linear Schrödinger equation

Consider the evolution of curves in R^3

$$\gamma_t = \gamma_x \times \gamma_{xx}, \tag{11.1}$$

where $\times$ denote the cross-product in R^3. This equation is known as the *vortex filament equation* , and has a long and interesting history (cf. [Ri]). It is easy to see that $||\gamma_x||^2$ is preserved under the evolution. It follows that if $\gamma(\cdot, 0)$ is parametrized by its arc length, then so are all $\gamma(\cdot, t)$ for all t. So equation (11.1) can also be viewed as the evolution of a curve that moves along the direction of binormal with the curvature as its speed. Let $k(\cdot, t)$ and $\tau(\cdot, t)$ be the curvature and torsion of the curve $\gamma(\cdot, t)$. Then there exists a unique $\theta(x, t)$ such that $\theta_x = \tau$ and

$$q(x,t) = k(x,t)e^{-i\int^x \tau(s,t)ds}$$

is a solution of the non-linear Schrödinger equation:

$$q_t = \frac{i}{2}\left(q_{xx} + 2|q|^2 q\right).$$

There is another interesting evolution of curves in S^2 that is also associated to the non-linear Schrödinger equation:

11.1 Proposition. *$\gamma(x, t)$ is a solution of equation (11.1) with x as the arc length parameter if and only if $\phi(x, t) = \gamma_x(x, t) : R^2 \to S^2$ satisfies the equation*

$$J(\phi_t) = \nabla_{\phi_x}\phi_x, \tag{11.2}$$

where ∇ is the Levi-Civita connection and J is the complex structure of the standard two sphere S^2.

Equation (11.2) is the geometric non-linear Schrödinger equation (GNLS) on S^2. Such equation can be defined on any complex Hermitian manifold (M, g, J). Consider the Schrödinger flow on the space $\mathcal{S}(R, M)$ of Schwartz maps, i.e., the equation for maps $\phi : R \times R \to M$:

$$J\left(\frac{\partial \phi}{\partial t}\right) = \triangle\phi,$$

where $\triangle\phi = \nabla_{\phi_x}\phi_x$ is the gradient of the energy functional on $\mathcal{S}(R, M)$, or the accelleration.

In this section, we give a brief outline of ideas and results in a forthcoming paper [TU3]. There is a Hasimoto type transformation that transforms the GNLS equation associated to $Gr(k, C^n)$ to the the second flow on $\mathcal{S}_{1,a}$ defined by

$$a = \begin{pmatrix} iI_k & 0 \\ 0 & -iI_{n-k} \end{pmatrix} \in u(k) \times u(n-k). \tag{11.3}$$

We have seen in Example 5.6 that identifying $\mathcal{S}_{1,a}$ as the space $\mathcal{M}_{k\times(n-k)}$ of $k \times (n-k)$ matrices, the second flow defined by a is the matrix non-linear Schrödinger equation:

$$B_t = \frac{i}{2}(B_{xx} + 2BB^*B). \tag{11.4}$$

Applying our theory to equation (11.4), we obtain many beautiful properties for the GNLS associated to $Gr(k, C^n)$. For example, we have

(i) a Hamiltonian formulation,

(ii) long time existence for the Cauchy problem,

(iii) a sequence of commuting Hamiltonian flows,

(iv) explicit soliton solutions,

(v) a non-abelian Poisson group action on the space of solutions of the GNLS,

(vi) a sequence of compatible symplectic structures on the space $\mathcal{S}(R, Gr(k, C^n))$ in which the GNLS is Hamiltonian and has a Lenard relation.

Let $U(n)$ be equipped with a bi-invariant metric. It is well-known that $Gr(k, C^n)$ can be naturally embedded as a totally geodesic submanifold M of $U(n)$. In fact, M is the set of all $X \in U(n)$ such that X is conjugate to a as described by formula (11.3). The invariant complex structure on M is given by

$$J_h(v) = [v, h].$$

Consider the following equation for maps $\phi : R^2 \to M$:

$$J_\phi(\phi_t) = \nabla_{\phi_x}(\phi_x), \qquad \phi : R^2 \to M \tag{11.5}$$

where ∇ is the Levi-Civita connection of the standard Kahler metric on M. A direct computation gives

$$\nabla_{\phi_x}(\phi_x) = \phi(\phi^{-1}\phi_x)_x.$$

So equation (11.5) becomes

$$\phi_t = -\frac{1}{2}(\phi^{-1}\phi_x)_x. \tag{GNLS}$$

Next we want to associate to each solution of equation (11.4) a solution of the GNLS. This is a generalization of the Hasimoto transformation of the vortex filament equation to non-linear Schrödinger equation. As noted in Example 5.6, $A = a\lambda + u \in$ is a solution of (11.4) if and only if

$$\theta_\lambda = (a\lambda + u)dx + (a\lambda^2 + u\lambda + Q_{a,2}(u))dt$$

is flat for all λ, where

$$u = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \qquad Q_{a,2} = \begin{pmatrix} \frac{1}{2i}BB^* & \frac{i}{2}B_x \\ \frac{i}{2}B_x^* & -\frac{1}{2i}B^*B \end{pmatrix}.$$

In particular, $\theta_0 = u\,dx + Q_{a,2}(u)dt$ is flat. Let g be the trivialization of θ_0, i.e.,

$$\begin{cases} g^{-1}g_x = u, \\ g^{-1}g_t = Q_{a,2}(u). \end{cases}$$

Set $\phi = gag^{-1}$. Changing the gauge of θ_λ by g gives

$$\begin{aligned} \tau_\lambda = g\theta_\lambda g^{-1} - dgg^{-1} &= (gag^{-1}\lambda)dx + (gag^{-1}\lambda^2 + g_xg^{-1}\lambda)dt \\ &= \phi\lambda\,dx + (\phi\lambda^2 + gug^{-1}\lambda)dt. \end{aligned}$$

Since τ_λ is flat for all λ, we get

$$\begin{cases} \phi_t = (gug^{-1})_x, \\ \phi_x = -[\phi, gug^{-1}]. \end{cases} \tag{11.6}$$

But for $u \in \mathcal{U}_a^\perp$, we have $a^{-1}ua = -u$. Hence

$$\begin{aligned} \phi^{-1}\phi_x = ga^{-1}g^{-1}(g_xag^{-1} - gag^{-1}g_xg^{-1}) &= ga^{-1}uag^{-1} - g_xg^{-1} \\ &= -gug^{-1} - g_xg^{-1} = -2g_xg^{-1} = -2gug^{-1}. \end{aligned} \tag{11.7}$$

So the first equation of (11.6) implies that ϕ is a solution of the GNLS.

Conversely, suppose $\phi : R^2 \to M$ is a solution of the GNLS. Then there exists $g : R^2 \to U(n)$ such that $\phi = gag^{-1}$ and $g^{-1}g_x(x,t) \in \mathcal{U}_a^\perp$ for all (x,t). Set

$$u = g^{-1}g_x, \qquad f = -\frac{1}{2}(\phi^{-1}\phi_x).$$

Then the equation (11.7) implies that $\phi^{-1}\phi_x = g_x g^{-1} = f$. Differentiate $\phi = gag^{-1}$ with respect to x to get

$$\phi_x = [g_x g^{-1}, \phi] = [f, \phi].$$

But the GNLS gives $\phi_t = f_x$. So

$$(\phi\lambda)dx + (\phi\lambda^2 + f\lambda)dt$$

is flat for all λ. Changing the gauge by g^{-1}, we find that

$$\beta_\lambda = (a\lambda + u)dx + (a\lambda^2 + u\lambda + h)dt$$

is flat, where $h = g^{-1}g_t$. Flatness of β_λ on the (x,t)-plane for all λ implies that $h = Q_{a,2}(u)$. So this proves that u is a solution of the second flow equation (11.4). To summarize,

11.2 Proposition. *If $B : R^2 \to \mathcal{M}(k \times (n-k))$ is a solution of the matrix NLS (11.4), then there is $g : R^2 \to U(n)$ such that*

$$g^{-1}g_x = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}, \qquad g^{-1}g_t = \begin{pmatrix} \frac{1}{2i}BB^* & \frac{i}{2}B_x \\ \frac{i}{2}B_x^* & -\frac{1}{2i}B^*B \end{pmatrix}$$

and $\phi = gag^{-1}$ is a solution of the GNLS. Conversely, if ϕ is a solution of the GNLS, then there is $g : R^2 \to U(n)$ such that

$$\phi = g\begin{pmatrix} iI_k & 0 \\ 0 & -I_{n-k} \end{pmatrix} g^{-1}, \qquad g^{-1}g_x = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix},$$

and B is a solution of the matrix NLS equation (11.4).

To end this section, we will translate properties for the second flow (11.4) to properties of the GNLS equation.

When a is singular, formula (8.4) implies that the corresponding Hamiltonians for the first three flows on $\mathcal{S}_{1,a}$ defined by a are

$$\begin{aligned}
F_1(B) &= \frac{1}{4}\int_{-\infty}^{\infty} \operatorname{tr}(iB_xB^* - iBB_x)dx = \frac{1}{4}\langle iB_x, B\rangle, \\
F_2(B) &= \frac{1}{4}\int_{-\infty}^{\infty} \operatorname{tr}(-B_xB_x^* + B^*BB^*B)\,dx \\
&= \frac{1}{8}(-\langle B_x, B_x\rangle + \langle B^*B, B^*B\rangle), \\
F_3(B) &= \frac{i}{16}\int_{-\infty}^{\infty} \operatorname{tr}(-BB_{xxx}^* + B_{xxx}B^*) + 3\operatorname{tr}(-BB^*BB_x^* + B_xB^*BB^*)dt \\
&= -\frac{1}{16}(\langle iB_{xxx}, B\rangle + 3\langle iB_x, BB^*B\rangle).
\end{aligned}$$

11.3 Remark. If $b \in \mathcal{U}_a$, i.e., $[a,b] = 0$, and $b \neq a$, then

(i) $Q_{b,j}(u)$ in general is not a local operator in u,

(ii) the flow generated by $b\lambda^n \in \mathcal{H}_+$ commutes with the flow (11.4),

(iii) given $b_1, b_2 \in \mathcal{U}_a$, the flow generated by $b_1\lambda^k$ and $b_2\lambda^j$ need not commute and

$$[\xi_{b_1,k}, \xi_{b_2,j}] = \xi_{[b_1,b_2],k+j},$$

where $\xi_{b,m}$ denotes the infinitesimal vector fields corresponding to $b\lambda^m$,

(iv) the action of $H_+ = \{g \in G_+ \mid ga = ag\}$ on $\mathcal{S}_{1,a}$ is Poisson.

It follows from the discussion in section 7 that the Cauchy problem for equation (11.4) with initial condition $A_0 = a\lambda + u_0 \in \mathcal{S}'_{1,a}$ can be solved by using factorizations. First we use the direct scattering of $A_0 = a\lambda + u_0$ on the line, i.e., solve equation (7.1). Set $f(\lambda) = \psi(0, \lambda)$. Then $f \in D_-$. Decompose

$$f(\lambda)e^{a\lambda x + b\lambda^j t} = E(x,t,\lambda)M(x,t,\lambda)^{-1}$$

as in section 7 by applying Birkhoff decompositions repeatedly. Then $A = E^{-1}E_x$ is the solution of the Cauchy problem.

The rational group G_-^m acts on the space of solutions of the GNLS, and soliton solutions can be calculated explicitly using the formulas in Theorem 10.8.

11.4 Proposition. *Let $a = \operatorname{diag}(-i, i, \dots, i)$, and choose a pole $z = r + is \in C \setminus R$. Let π be the projection on the subspace spanned by $(1, v)^t = (1, v_2, \dots, v_n)^t$. Then the one-solitons for the j-th flow defined by a generated by Bäcklund transformations from the vacuum solution $A_0(x,t,\lambda) = a\lambda$ are of the form*

$$A(x,t,\lambda) = a\lambda + u(x,t),$$
$$u(x,t) = \begin{pmatrix} 0 & B(x,t) \\ -B^*(x,t) & 0 \end{pmatrix},$$

where

$$B(x,t) = \frac{4se^{-2i(rx + \mathrm{Re}(z^j)t)}}{e^{-2(sx+\mathrm{Im}(z^j)t)} + e^{2(sx+\mathrm{Im}(z^j)t)}\|v\|^2}\,\bar{v}.$$

Proof. We use Theorem 10.8 to make our computations. We start with $A_0 = a\lambda$. According to Theorem 10.8,

$$A_0 \mapsto a\lambda + (z - \bar{z})[\tilde{\pi}, a],$$

where $a = \operatorname{diag}(-i, i, \dots, i)$. Here $\tilde{\pi}$ is the projection on $(1, \tilde{v})^t$, where

$$\hat{v} = (1, \tilde{v}) = (1, e^{2i(\bar{z}x + \bar{z}^j t)}v).$$

Let $z = r + is$. Then

$$\tilde{\pi}(x,t) = \frac{1}{e^{-2(sx+\mathrm{Im}(z^j)t)} + e^{2(sx+\mathrm{Im}(z^j)t)}\|v\|^2}\,\hat{v}^*\hat{v}.$$

The formula for B follows. □

12 First flows and flat metrics

The integrable equations of evolution we have been describing up to this point have at most two independent variables. The flow of the first variable, regarded as a spacial variable, is used to construct the initial Cauchy data from the scattering coset (hence the "first flow" terminology). The second variable is considered to be the time variable, and the flow in this variable is the evolution. Many authors consider a commuting heirarchy of flows to generate functions of an infinite sequence of time variables. However, the physical and geometric applications do not require this consideration.

We turn our attention to a family of geometric problems in n spacial variables, which we shall call *n-dimensional systems* or *n-dimensional flows*. In the applications, the n variables are on an equal footing, and the flows in each variable is a first flow. The flows commute, and hence the resulting geometric object is always a flat connection on a region of R^n with special properties. From our viewpoint, the natural parameter (moduli) space of solutions is a coset space of the sort we have just described. In many cases, we have obtained global results on connections in R^n via the decay theorems in section 7.

The n commuting first flows associated to a rank n symmetric space have been discussed in a paper by the first author ([Te2]). We outline the general theory and give some of the basic examples. The results on coset spaces and Bäcklund transformations apply naturally to these systems.

12.1 Definition ([Te2]). Let U be a rank n Lie group, $\mathcal{T}$ a maximal abelian subalgebra of the Lie algebra $\mathcal{U}$, and $a_1, \ldots, a_n$ a basis of $\mathcal{T}$. *The n-dimensional system associated to U* is the following first order system:

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \qquad v : R^n \to \mathcal{T}^\perp. \tag{12.1}$$

12.2 Definition ([Te2]). Let U/K be a rank n symmetric space, $\sigma : \mathcal{U} \to \mathcal{U}$ the corresponding involution, $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the Cartan decomposition, $\mathcal{A}$ a maximal abelian subalgebra in $\mathcal{P}$, and $a_1, \ldots, a_n$ a basis of $\mathcal{A}$. *The n-dimensional system associated to U/K* is the first order system:

$$[a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], \qquad v : R^n \to \mathcal{P} \cap \mathcal{A}^\perp. \tag{12.2}$$

12.3 Theorem ([Te2]). *The following conditions are equivalent:*

(i) v is a solution of equation (12.1) (or (12.2))

(ii) *the connection* 1*-form* $\theta = \sum_{j=1}^{n}(a_j\lambda + [a_j, v])\, dx_j$ *is left flat, i.e.,* $d\theta = -\theta \wedge \theta$,

(iii) $\left[\frac{\partial}{\partial x_i} + (a_i\lambda + [a_i, v]), \frac{\partial}{\partial x_j} + (a_j\lambda + [a_j, v])\right] = 0$ *for all* $i \neq j$.

12.4 Theorem ([Te2]). *Let* $\mathcal{T}$ *be a maximal abelian subalgebra of* $\mathcal{U}$, $\mathcal{T}^{\perp}$ *the orthogonal complement of* $\mathcal{T}$ *in* $\mathcal{U}$, *and let* $\mathcal{S}(R^n, \mathcal{T}^{\perp})$ *denote the space of Schwartz maps from* R^n *to* $\mathcal{T}^{\perp}$. *Let* $a_1, \dots, a_n$ *be regular elements and form a basis of* $\mathcal{T}$. *Then there exists a dense open subset* $\mathcal{S}_0$ *of* $\mathcal{S}(R^n, \mathcal{T}^{\perp})$ *such that given* $v_0 \in \mathcal{S}_0$, *the following Cauchy problem for equation (12.1) has a unique solution:*

$$\begin{cases} [a_i, v_{x_j}] - [a_j, v_{x_i}] = [[a_i, v], [a_j, v]], & \text{if } i \neq j, \\ v(t, 0, \dots, 0) = v_0(t). \end{cases}$$

At this point, it is important to give some explaination and application. Because these flows all commute, a change of basis in the abelian subalgebra $\mathcal{T}$ or $\mathcal{A}$ can be represented by composition with an element of $GL(n, R)$. So we might as well assume that $a_1, \dots, a_n$ are generic or regular (have distinct eigenvalues). Starting at $0 \in R^n$, given an element in the coset space D_-/H_-, we can solve for $E(x_1, 0, \dots, 0, \lambda)$ and find $\frac{d}{dx} + \lambda a_1 + u(x_1, 0, \dots, 0)$ as if we were solving for initial Cauchy data. Instead of going to one of the heirarchy of flows, we solve for the entire family of first flows in variables $(x_1, \dots, x_n)$. This gives us a map

$$D_-/H_- \mapsto \{\text{flat connections on a region of } R^n\}.$$

In the case that our flows can be embedded in the unitary flows,

$$D_-/H_- \mapsto \{\text{flat connections on } R^n \text{ decaying at } \infty\}$$

(by Theorem 7.16). This gives a proof of Theorem 12.4.

12.5 Remark. Note that we are constructing a more rigid structure then a flat connection. We are actually constructing special connection one-forms, and we do not allow arbitray coordinate change or gauge changes in the theory.

By expressing the parameter space in this form, we have made a beginning towards thinking about the natural symplectic structure on this solution space. In the case of one-dimensional Cauchy data, the symplectic structures were averaged out over the one-parameter flow. Here we need to average them out over an n-dimensional flow to obtain a natural structure.

The canonical examples of these flat connections are quite easy to describe.

12.6 Examples.

Example (i) Let $U = GL(n, R)$, $\mathcal{T}$ the maximal abelian subalgebra of diagonal matrices, and $\{e_{11}, \dots, e_{nn}\}$ a basis of $\mathcal{T}$, where e_{ij} denote the matrix in $gl(n)$ all whose entries are zero except that the ij-th entry is equal to 1. Then the n-dimensional system associated to $GL(n,R)$ is the system for

$$f = (f_{ij}) : R^n \to gl(n,R), \quad f_{ii} = 0,\ 1 \le i \le n$$

$$\begin{cases} (f_{ij})_{x_i} + (f_{ij})_{x_j} + \sum_k f_{ik} f_{kj} = 0, & \text{if } i \ne j, \\ (f_{ij})_{x_k} = f_{ik} f_{kj}, & \text{if } i,\ j,\ k \text{ are distinct.} \end{cases} \tag{12.3}$$

Example (ii) Let $U/K = GL(n,R)/SO(n)$, and $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the corresponding Cartan decomposition. Then $\mathcal{P}$ is the set of all real symmetric $n \times n$ matrices, the space $\mathcal{A}$ of all diagonal matrices is a maximal abelian subalgebra in $\mathcal{P}$, $e_{11}, \ldots, e_{nn}$ form a basis of $\mathcal{A}$, and $\mathcal{P} \cap \mathcal{A}^\perp$ is the space $gl_s(n)$ of all symmetric $n \times n$ matrices whose diagonal entries are zero. The n dimensional system (12.2) associated to $GL(n,R)/SO(n)$ is the system for

$$F = (f_{ij}) : R^n \to gl(n,R), \quad f_{ij} = f_{ji}, \quad f_{ii} = 0 \text{ if } 1 \le i \le n$$

$$\begin{cases} (f_{ij})_{x_i} + (f_{ij})_{x_j} + \sum_k f_{ik} f_{kj} = 0, & \text{if } i \ne j, \\ (f_{ij})_{x_k} = f_{ik} f_{kj}, & \text{if } i,\ j,\ k \text{ are distinct.} \end{cases} \tag{12.4}$$

Note that system (12.4) is the system (12.3) restricted to maps $f = (f_{ij})$ that are symmetric.

Example (iii) Let $U/K = U(n)/SO(n)$, and $u(n) = so(n) + \mathcal{P}$ a Cartan decomposition. Then $\mathcal{P}$ is the set of all symmetric pure imaginary $n \times n$ matrices and the space $\mathcal{A}$ of all diagonal matrices in $\mathcal{P}$ is a maximal abelian algebra. Let $ia_1, \ldots, ia_n$ be a basis of $\mathcal{A}$. Write $v : R^n \to \mathcal{P} \cap \mathcal{A}^\perp$ as $v = -iF$, where F is a real $n \times n$ symmetric matrix. Then equation (12.2) for v is the equation (12.4) for F. This is a special case of the general fact that the n-dimensional system associated to a compact symmetric space is the same as that associated to its non-compact dual.

Example (iv) Let $U/K = SO(2n)/S(O(n) \times O(n))$, and $\mathcal{U} = \mathcal{K} + \mathcal{P}$ the corresponding Cartan decomposition. Then

$$\mathcal{K} = so(n) \times so(n) = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \;\middle|\; B, C \in so(n) \right\},$$

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & F \\ -F^t & 0 \end{pmatrix} \;\middle|\; F \in gl(n) \right\},$$

and

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \;\middle|\; D \text{ is diagonal} \right\}$$

is a maximal abelian subalgebra of $\mathcal{P}$. Let $a_i = \begin{pmatrix} 0 & -e_{ii} \\ e_{ii} & 0 \end{pmatrix}$. Then $a_1, \ldots, a_n$ form a basis of $\mathcal{A}$, and $\mathcal{P} \cap \mathcal{A}^\perp$ is the set of matrices of the form $\begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix}$

such that $X = (x_{ij})$ is $n \times n$ matrix with $x_{ii} = 0$ for all i. Then equation (12.2) for $v = \begin{pmatrix} 0 & F \\ -F^t & 0 \end{pmatrix}$, with $F = (f_{ij}) : R^n \to gl(n,R)$ and $f_{ii} = 0$ for all $1 \le i \le n$, is

$$\begin{cases} (f_{ij})_{x_i} + (f_{ji})_{x_j} + \sum_k f_{ki} f_{kj} = 0, & \text{if } i \neq j \\ (f_{ij})_{x_j} + (f_{ji})_{x_i} + \sum_k f_{ik} f_{jk} = 0, & \text{if } i \neq j \\ (f_{ij})_{x_k} = f_{ik} f_{kj}, & \text{if } i,\ j,\ k \text{ are distinct.} \end{cases} \tag{12.5}$$

Up to now, the flat connections were not constructed to relate to Riemannian geometry. To explain the relation of these flat connections to geometry, we need to set up some notations. A *diagonal metric* is a metric of the form

$$ds^2 = \sum_j b_j(x)^2 dx_j^2.$$

If this diagonal metric is flat, then $(x_1, \dots, x_n)$ is an orthogonal coordinate system on R^n in the sense of Darboux ([Da2]). These examples arise in the study of isometric immersions of constant sectional curvature n-manifolds into Euclidean space.

On the other hand, to study Lagrangian flat submanifolds in C^n or Frobenius manifolds (used in quantum cohomology), we consider *Egoroff metrics*. These are metrics of the form

$$ds^2 = \sum_j \phi_{x_j} dx_j^2$$

for some function ϕ.

The Levi-Civita connection 1-form for the diagonal metric $\sum_{i=1}^n b_i(x)^2 dx_i^2$ is

$$w = (w_{ij}) = (-f_{ij} dx_i + f_{ji} dx_j), \qquad f_{ij} = \frac{(b_i)_{x_j}}{b_j}.$$

Or equivalently,

$$w = -\delta F + F^t \delta, \qquad \text{where } \delta = \operatorname{diag}(dx_1, \dots, dx_n).$$

Hence we are looking for flat connections o this special form.

The Levi-Civita connection of a Egoroff metric is $w = [F, \delta]$ with $F = F^t$. It is easy to see that a diagonal metric is *Egoroff* if and only if $f_{ij} = f_{ji}$.

12.7 Definition. A *Darboux connection* is a connection of the form $-\delta F + F^t \delta$, and an *Egoroff connection* is a connection of the form $[F, \delta]$ with F symmetric, where $\delta = \operatorname{diag}(dx_1, \dots, dx_n)$.

By definition of flatness, we get

12.8 Proposition. *A Darboux connection $-\delta F + F^t\delta$ is flat if and only if $F = (f_{ij})$ satisfies*

$$\begin{cases} (f_{ij})_{x_j} + (f_{ji})_{x_i} + \sum_k f_{ik}f_{jk} = 0, & \text{if } i \neq j \\ (f_{ij})_{x_k} = f_{ik}f_{kj}, & \text{if } i,\ j,\ k \text{ are distinct.} \end{cases} \tag{12.6}$$

An Egoroff connection $[F,\delta]$ (with $F = F^t$) is flat if and only if $F = (f_{ij})$ is a solution of equation (12.4), the n-dimensional system associated to the symmetric space $GL(n,R)/SO(n)$.

Let $w = -\delta F + F^t\delta$ be a flat Darboux connection. Then a metric $ds^2 = \sum_j b_j^2(x)dx_j^2$ has w as its Levi-Civita connection if and only if $(b_1, \dots, b_n)$ is a solution of

$$\frac{(b_i)_{x_j}}{b_j} = f_{ij}, \quad i \neq j. \tag{12.7}$$

In general, given a solution $F = (f_{ij})$ of equation (12.6), equation (12.7) has infinitely many local solutions parametrized by n functions b_i defined on the line $x_j = 0$ for $j \neq i$. These are used as the initial conditions for the ordinary differential equations (12.7).

Next we will explain the relation between the space of solutions of equation (12.6) and the set of flat n-submanifolds in R^{2n} with flat normal bundle and maximal rank. First we need the following definition:

12.9 Definition. The rank of a submanifold M^n of R^m at $x \in M$ is the dimension of the space of shape operators at $x \in M$. M is said to have constant rank k if the rank of M at x is equal to k for all $x \in M$. In general, the rank k of M at x is less than or equal to the codimension of M in R^m.

Using the local theory of submanifolds, it is easy to see that (cf. [Te2]) if M^n is a flat submanifold of R^{2n} with flat normal bundle and constant rank n, then locally there exist a coordinate system $x : \mathcal{O} \to M \subset R^{2n}$, $A = (a_{ij}) : \mathcal{O} \to O(n)$, $b_i : \mathcal{O} \to R$ and parallel normal frame $\{e_{n+1}, \dots, e_{2n}\}$ such that the two fundamental forms are:

$$\begin{cases} I = \sum_i b_i(x)^2 dx_i^2, \\ II = \sum_{i,j=1}^n a_{ji}b_i dx_i^2 e_{n+j}. \end{cases}$$

The coordinate system x is unique up to permutation and changing x_i to $-x_i$ (i.e., the action of the Weyl group B_n). Such coordinates will be called *principal curvature coordinates*, and (b, A) will be called the *fundamental data* of M.

12.10 Theorem ([Te2]). *Suppose M^n is flat submanifold of R^{2n} with flat normal bundle and constant rank n, x is a principal curvature coordinate system, and (b, A) is the fundamental data of M. Set*

$$f_{ij} = \begin{cases} (b_i)_{x_j}/b_j, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Then $F = (f_{ij})$ is a solution of equation (12.5), the system associated to the rank n symmetric space $SO(2n)/S(O(n) \times O(n))$. Conversely, if $F = (f_{ij})$ is a solution of equation (12.5), then there exist an open subset $\mathcal{O}$ of R^n, $b : \mathcal{O} \to R^n$, $A : \mathcal{O} \to O(n)$ and an immersion $X : \mathcal{O} \to R^{2n}$ such that

$$\begin{cases} dA = A(-F\delta + \delta F^t), & \textit{where } \delta = \mathrm{diag}(dx_1, \dots, dx_n) \\ (b_i)_{x_j} = f_{ij} b_j, & \textit{if } i \neq j, \end{cases} \tag{12.8}$$

and

(i) the immersion X is flat, has flat normal bundle and constant rank n,

(ii) x is a principal curvature coordinate system for $X(\mathcal{O})$ and (b, A) is its fundamental data,

(iii) given any constants $c_1, \dots, c_n$ and set $b_i = \sum_j c_j a_{ji}$ for $1 \leq i \leq n$. Then $(b_1, \dots, b_n)$ is a solution of the second equation of system (12.8),

(iv) let $b = (a_{11}, \dots, a_{1n})$, then $X(\mathcal{O}) \subset S^{2n-1}$.

12.11 Remark. If $F = F^t$ is a solution of equation (12.4), then F is a solution of equation (12.5). Let A be as in Theorem 12.10 and $b = (a_{11}, \dots, a_{1n})$, and $X : \mathcal{O} \to R^{2n}$ the corresponding immersion. It was observed by Dajczer and Tojeiro [DaR2] that the condition $F = F^t$ is equivalent to the condition that $X(\mathcal{O})$ is a Lagrangian flat submanifold of $R^{2n} = C^n$.

12.12 Remark. Let $N^n(c)$ denote the n-dimensional space form of sectional curvature c. It was proved by Cartan ([Ca]) that if $c < c'$ then $N^n(c)$ can not be locally isometrically embedded in $N^m(c')$ when $m < 2n - 1$, but can if $m \geq 2n - 1$. An analogue of Bäcklund's theorem for immersions of $N^n(c)$ into $N^{2n-1}(c')$ was constructed by Tenenblat and the first author [TT] for $c = -1$ and by Tenenblat [Ten] for $c = 0$. The corresponding Gauss-Codazzi equations for these immersions are called the *generalized sine-Gordon equation (GSGE)* and *generalized wave equation GWE* respectively. GSGE and GWE arise as the n-dimensional system associated to the symmetric spaces $SO(2n, 1)/SO(n) \times SO(n, 1)$ and $SO(2n)/S(O(n) \times O(n))$ respectively (cf. [Te2]). Du ([Du]) noted that the equation for isometric immersions of $N^k(c)$ in $N^m(c')$ is the k-dimensional system associated to a suitable rank k symmetric space. For example, the equation for immersions of R^k into S^n, $n \geq 2k - 1$, is the k-dimensional system associated to $Gr(k, R^{n+1})$. Du also proved that the Bäcklund transformations constructed in [TT], [Ten], and Ribaucour transformations constructed in [DaR1] are given by actions of certain order two elements in G_-^m.

Darboux' orthogonal coordinate systems arises naturally in the work of Dubrovin and Novikov ([DN1, 2]) and Tsarev ([Ts]) on Hamiltonian system of hydrodynamic type. A brief review of their results follows. Given a smooth section P of $L(TR^n, TR^n)$, the following first order quasi-linear system for $\gamma : R^2 \to R^n$

$$\frac{\partial \gamma}{\partial t} = P(\gamma)\left(\frac{\partial \gamma}{\partial x}\right) \tag{12.9}$$

is called a *hydrodynamic system*. If $(u_1, \ldots, u_n)$ is a local coordinate system on R^n, then $P(u)\left(\frac{\partial}{\partial u_i}\right) = \sum_{j=1}^n v_{ij}(u)\frac{\partial}{\partial u_j}$ for some smooth map $v = (v_{ij}) : R^n \to gl(n)$. System (12.9) is said to be *diagonalizable* if given any point $x \in R^n$ there is a local coordinate u around x such that the corresponding matrix map v for the smooth section P is diagonal. Let ds_0^2 denote the standard flat metric on R^n, and ∇ its Levi-Civita connection. Given two functionals F and G on $\mathcal{S}(R, R^n)$,

$$\{F, G\}(\gamma) = \int_{-\infty}^{\infty} (\delta F(\gamma), \nabla_{\gamma_x}(\delta G(\gamma))\, dx$$

defines a Poisson structure on $\mathcal{S}(R, R^n)$. Dubrovin and Novikov ([DN1], [DN2]) proved that this is the only Poisson structure on $\mathcal{S}(R, R^n)$ that is given by a first order differential operator. Given a zero order Lagrangian F with density $f : R^n \to R$, i.e.,

$$F(\gamma) = \int_{-\infty}^{\infty} f(\gamma(x))\, dx,$$

the Hamiltonian equation with respect to the Poisson structure defined above is

$$\frac{\partial \gamma}{\partial t} = \nabla_{\gamma_x}(\nabla f(\gamma)). \tag{12.10}$$

Such system is called *Hamiltonian system of hydrodynamic type*. Novikov conjectured that if system (12.10) is diagonalizable then it is completely integrable. This conjecture is proved by Tsarev in [Ts]. In these results, the boundary conditions for the Poisson bracket are ignored and equation (12.10) is defined on an open subset of R^n. In other words, this is the local theory of Hamiltonian hydrodynamic systems. Below we state some of Tsarev's results:

12.13 Theorem ([Ts]). *Suppose the Hamiltonian system for*

$$F(\gamma) = \int_{-\infty}^{\infty} f(\gamma(x))\, dx$$

is diagonalizable with respect to a local coordinate system $(u_1, \ldots, u_n)$. Then $\nabla^2 f = \sum_i v_i(u) du_i \otimes \frac{\partial}{\partial u_i}$, and the Hamiltonian system (12.10) is

$$\frac{\partial u_i}{\partial t} = v_i(u)\frac{\partial u_i}{\partial x}. \tag{12.11}$$

Moreover:

(i) $(u_1, \ldots, u_n)$ *is a local orthogonal coordinate system on* R^n*, i.e., the standard metric* ds_0^2 *on* R^n *is* $\sum_{i=1}^n b_i(u)^2 du_i^2$ *for some smooth functions* $b_1, \ldots, b_n$*. Moreover,*

$$\frac{1}{v_i - v_j}\frac{\partial v_i}{\partial u_j} = -\frac{1}{b_j}\frac{\partial b_i}{\partial u_j}. \tag{12.12}$$

(ii) If $(\tilde{v}_1, \ldots, \tilde{v}_n)$ *is a solution of system (12.12), then* $\frac{\partial u_i}{\partial t} = \tilde{v}_i(u)\frac{\partial u_i}{\partial x}$ *is also a Hamiltonian system of hydrodynamic type.*

(iii) Suppose $\sum_{i=1}^n \tilde{b}_i^2(u)du_i^2$ *and* $\sum_{i=1}^n b_i(u)^2 du_i^2$ *have the same connection 1-form, i.e.,* $(b_i)_{u_j}/b_j = (\tilde{b}_i)_{u_j}/\tilde{b}_j$ *for all* $i \neq j$*. Set* $h_i = \tilde{b}_i/b_i$*. Then*

$$\frac{\partial u_i}{\partial t} = h_i(u)\frac{\partial u_i}{\partial x} \tag{12.13}$$

is a Hamiltonian system of hydrodynamic type and commutes with system (12.11).

(iv) If $v_i, \ldots, v_n$ *are distinct, then system (12.11) is completely integrable.*

To end this section, we review some of the elementary relations between Dubrovin's Frobenius manifolds ([Dub2]) and the n-dimensional system (12.4) associated to the symmetric space $GL(n)/SO(n)$. For more deep and detailed results of Frobenius manifolds, we refer the reader to Dubrovin's article [Dub2].

12.14 Definition ([Dub2], [Hi2]). A *Frobenius manifold* of degree m (not necessarily an integer) is a quintuple (R^n, x, g, θ, ξ), where x is a coordinate system on R^n, $g = \sum_j \phi_{x_j} dx_j^2$ a flat Egoroff metric, $\theta = \sum_j \phi_{x_j} dx_j$ and $\xi = \sum_j \phi_{x_j} dx_j^3$ for some function ϕ such that ϕ, g, θ, ξ satisfy the following conditions:

(i) $\nabla\theta = 0$, where ∇ is the Levi-Civita connection of g,

(ii) $\nabla\xi$ is a symmetric 4 tensor,

(iii) ϕ_{x_j} is homogeneous of degree m for all j, i.e., $\phi_{x_j}(rx) = r^m\phi_{x_j}(x)$ for all $r \in R^*$ and $x \in R^n$.

The coordinate system x is called a *canonical coordinate system.*

Each tangent plane of the Frobenius manifold has a natural multiplication defined as follows: Set $v_i = \partial/\partial x_i$. Then

$$v_i v_j = v_j v_i = \delta_{ij} v_i, \quad \forall\ 1 \leq i, j \leq n$$

defines a multiplication on the tangent plane of R^n. Moreover, $T(R^n)_x$ is a commutative algebra and

$$\theta(uv) = g(u, v), \qquad \xi(u, v, w) = g(uvw). \tag{12.14}$$

The dual of the 1-form θ is $e = \sum_j \partial/\partial x_j$, which is the identity, i.e., $ve = ev = v$ for all $v \in T(R^n)_x$.

The following Proposition gives the relation between Frobenius manifolds and solutions of the n-dimensional system associated to $GL(n)/SU(n)$.

12.15 Proposition ([Dub2], [Hi2]). *Let (R^n, x, g, θ, ξ) be a Frobenius manifold of degree m, $g = \sum_j b_j^2 dx_j^2$, and $w_{ij} = f_{ij}(-dx_i + dx_j)$ the Levi-Civita connection of g, i.e., $b_i^2 = \phi_{x_i}$ and $f_{ij} = (b_i)_{x_j}/b_j$. Set*

$$F = (f_{ij}), \qquad S(x) = (S_{ij}(x)) = (f_{ij}(x)(x_i - x_j)).$$

Then

(i) $F = (f_{ij})$ is a solution of equation (12.4), the n-dimensional system associated to $GL(n)/SO(n)$,

(ii) F is invariant under the action of R^* defined in section 9, i.e.,

$$r \cdot F(x) = r^{-1} F(r^{-1}x), \text{for all } r \neq 0.$$

(iii) $\frac{\partial S}{\partial x_i} = [[F, e_{ii}], S]$, where e_{ii} is the diagonal matrix with all entries zero except the ii-th entry is 1,

(iv) $(b_1, \dots, b_n)^t$ is an eigenvector of the matrix (S_{ij}) with eigenvalue $m/2$.

Since Frobenius manifolds are flat, there are also coordinate systems such that all the coordinate vector fields are covariant constant. A coordinate system $(t_1, \dots, t_n)$ on a Frobenius manifold (R^n, x, g, θ, ξ) is called a *flat coordinate system* if g has constant coefficients with respect to the t-coordinates. Since $\nabla\theta = 0$ and e is the dual of θ, we have $\nabla e = 0$. So there exists a flat coordinates $(t_1, \dots, t_n)$ such that $e = \partial/\partial t_1$. It follows from the condition that $\nabla\xi$ is symmetric, there exists a function $h(t)$ such that

$$\begin{cases} g = \sum_{jk} h_{t_1 t_j t_k} dt_j dt_k, \\ \theta = dt_1, \\ \xi = \sum_{ijk} h_{t_i t_j t_k} dt_i dt_j dt_k. \end{cases}$$

Using condition (12.14) and the fact that $\partial/\partial t_1$ is the identity, the multiplication can be written down explicitly in terms of h. In order for this multiplication to be associative, h has to satisfy a complicated non-linear equation, which is the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation. The WDVV equation arises in the study of Gromov-Witten invariants, and we refer the readers to work of Dubrovin ([Dub2]) and Ruan and Tian ([RT]).

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