Metric and comparison geometry

Jeff Cheeger and Karsten Grove

The present volume surveys some of the important recent developments in metric geometry and comparison geometry. These areas represent a vital and expanding part of modern geometry. We begin with an indication of their scope and perspective.

While *metric geometry* could be taken as refering simply to the geometry of metric spaces with no additional structure, our interest here is in those metric spaces which are smooth riemannian manifolds, or more generally, in metric spaces which are either constructed from smooth riemannian manifolds via natural geometric operations such as the process of taking weak geometric limits, or which share properties of riemannian manifolds when these properties are formulated in a suitably weak sense.

Also characteristic of the subject is a certain *synthetic* mode of argument driven by an associated sequence of mental pictures. In establishing the geometric tools on which such synthetic arguments are based, analysis (calculus in some form) intervenes to an extent which varies from instance to instance.

By *comparison geometry* we mean the study of manifolds (or more general metric spaces) whose curvature satisfies definite bounds, for instance, the classification problem for manifolds of positive curvature, which is still far from solved.

Comparison theorems are a key tool in comparison geometry. These are theorems which assert that if a particular inequality on curvature is satisfied, then some associated geometric property holds at least to the extent that it does in a corresponding model case, often one in which the curvature is constant. Classical comparison theorems pertain to the behavior of geodesics, and related objects such as the Jacobi equation, the formula for the second variation of arc length and the index form in Morse theory.

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Increasingly however, the solution of a problem in comparison geometry is likely to call for a combination of synthetic, analytic and topological arguments.

Scalar curvature, for example, is too weak an invariant to control the behavior of geodesics. At the infinitesimal level scalar curvature controls the volume of balls, but it fails to do so for balls of any definite positive radius. Nonetheless, although manifolds with positive scalar curvature are much less constrained than those with positive sectional curvature, there are analytically based results for positive scalar curvature which do *not* (at present) have synthetic proofs, even under the assumption that the sectional curvature is positive e.g. the vanishing of the $\hat{A}$-genus for spin manifolds with positive scalar curvature. The $\hat{A}$-genus is actually a concept from topology, and topological techniques, such as surgery theory, also play an important role in the subject.

In the study of manifolds with a lower bound on Ricci curvature, estimates on geodesics can be used to control volume. They are also important in situations which are highly constrained (in fact almost rigid). However, some control over geodesics is lost when one passes to weak geometric limits; for example, one can have infinitely many distinct geodesics with the same tangent vector. So in addition to comparison theorems like the Bishop-Gromov inquality, ideas from analysis such as Bochner's formula, the maximum principle, gradient estimates for harmonic functions are required. The theories of Einstein manifolds and Ricci flow involve geometry as well as analysis.

Even in the world of more general metric spaces, the connection with analysis is seen in results where the existence of a *measure* satisfying a suitable compatibility condition with the metric, say a doubling condition, or Poincaré inequality, leads to metric or topological conclusions, in whose statement the measure does not appear. In metric riemannian geometry, this is particularly relevant in the context of lower (and two-sided) Ricci curvature bounds.

Thus, it seems that distinctions such as "metric geometry" versus "geometric analysis" are to some extent artificial and if pressed too far, are genuinely destructive. To reiterate, increasingly, the solution of specific geometric problems requires a mixture of synthetic, analytic and topological arguments — the work of Perelman (on the program originated by Hamilton) being just one, albeit spectacular, example. This circumstance can only make the subject more interesting.

This having been said, our purpose here is to focus primarily, but not exclusively, on techniques from metric geometry and their use in the study of comparison geometry.

What then does the metric point of view have to offer? Here are few of the possible answers.

The metric space structure does not presuppose any assumption of smoothness. Attempting to extend notions of curvature to objects with minimal smoothness can provide fundamental insight. For instance, one can try to understand which notions of curvature are defined for piecewise flat spaces, or for convex hypersurfaces, or what Ricci curvature bounds might mean for a space equipped with a metric and a measure.

At a more practical level, certain results which have arisen in metric riemannian geometry have subsequently been realized to have natural extensions to much more general contexts and these have turned out to be of fundamental importance. A prime example is furnished by the work of Gromov in geometric group theory.

Finally, even if one were only interested in riemannian manifolds *per se*, the study of more general metric measure spaces provides useful information, for example, via the following route. Compactness theorems, such as that of Gromov, (whose hypothesis requires a definite lower bound on Ricci curvature) enable one to take weak geometric limits of sequences of riemannian manifolds M_i^n. The resulting limit spaces, Y, may be viewed as playing a role in riemannian geometry which is analogous to that played by distributions or Sobolev functions in analysis. Information on the regularity and singularity structure of such Y (the analog of Sobolev embedding theorems) provides information on the sequence, M_i^n, in some cases enough information to show that in actuality, it could not have existed in the first place, and in other cases, information on structure of the M_i^n, for i large.

On the face of it, this sort of reasoning might appear circular, since it would seem that the only possibility for obtaining nontrivial information on the limit space would have to be *via* uniform estimates on the approximating sequence. While initially this is so, once some preliminary properties of the limit objects have been established, additional properties can be deduced purely synthetically i.e. without further reference to the approximating sequence M_i^n. This in turn, provides new information on the M_i^n themselves.

We now turn to the articles in the present volume.

As we have indicated, lower bounds on Ricci curvature and in particular lower bounds on sectional curvature provide a natural setting in which convergence methods play a significant role. Their utility in applications is governed by the extent to which the limit objects and their relation to elements of a limiting sequence is well understood. Such understanding is most complete in the case of bounded sectional curvature; see the article by Rong.

When only a lower sectional curvature bound is imposed, the stability theorem of Perelman provides good information in the noncollapsing case. A (long awaited) complete and detailed exposition of Perelman's stability theorem is presented in the article by Kapovitch.

Although there have been important recent advances in the collapsing case, the crucially important class of almost nonnegatively curved manifolds is still poorly understood. Since all limit objects are Alexandrov spaces, the theory of these spaces will be indispensible in future developements; see the article by Petrunin.

Convergence theory in the case in which only a lower Ricci curvature bound is imposed, is surveyed in the article by Wei.

A class of metric measure spaces, for which a sythetic definitition of lower Ricci curvature bounds is possible has long been sought and has recently emerged. This class, includes in particular, weak limits riemannian manifolds with lower Ricci curvature bounds. The general theory will surely undergo considerable further developement; see the article by Lott.

Progress in the classical areas of manifolds with nonnegative or positive sectional curvature is discussed in the articles of Wilking and of Ziller. The latter provides a self contained account on all examples known to date.

After this volume was completed a milestone was reached in the classical pinching problem: A (pointwise) weakly 1/4 pinched manifold is diffeomorphic to either a space form or it is isometric to a rank one symmetric space. The proof due to Brendle and Schoen is an amazing application of the Ricci flow. An account provided by wilking of this exciting development can be found as an "added in proof" section of his article in this volume.

In cases where uniform curvature comparisons are available, convergence methods have played a role when only upper curvature bounds are present e.g. for simply connected manifolds with nonpositive sectional curvature and in parts of geometric group theory. A survey on the current state of affairs for general spaces with an upper curvature bound is provided in the article by Schröder and Buyalo.

The article by Farrell, Jones and Ontaneda is concerned with geometric and topological rigidity and flexibility issues for negatively curved manifolds.

An update on the status of the classification problem for manifolds with positive and nonnegative scalar curvature is given in the article by Rosenberg.

The selection of topics treated in this volume has been influenced by several factors, including space, existence of other sources and our success (or failure) in attracting contributors. So we will conclude by mentioning some topics that might have well been included, but for whatever reason, ended up being omitted, or almost so.

In the first place, lower curvature bounds are very much emphasized over upper bounds. While this was not by design, there do exist several excellent surveys which treat nonpositively curved manifolds. Recent progress on Einstein manifolds, in whose proofs metric geometry plays a role, might have been discussed. Nonsmooth calculus on metric measure spaces is another topic which has close relations with material considered here. (For a very informative overview, see Heinonen's recent survey in the Bulletin of the AMS.) Several topics from Perelman's papers on geometrization, such as comparison theorems in generalized and possibly infinite dimensional settings, would have been natural to include had they not been exposed at great length elsewhere. Other natural topics which wound up being left out are geometric group theory, isoperimetric inequalities and "curvature free" metric geometry (including results on systols).

The excluded topics could easily fill a second volume. Perhaps, at some future time, they will.