Global methods of solving equations on manifolds

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Abstract. We survey our recent works on certain global methods of solving equations on complex manifolds and present several geometric applications.

Contents

1. Introduction

In [**21**], we developed a global method to deal with various problems related to variations of complex structures. These problems can be reduced to solving certain $\overline{\partial}$-equations over complex manifolds. With the help of Hodge theory, we change the $\overline{\partial}$-equation into an associated integral equation, the novelty is that we use the Banach fixed point theorem or the quasi-isometry formula to obtain a global solution of the associated integral equation. These global solutions have many geometric applications. In this survey, we present several results in [**21**] and describe some geometric applications of our methods. In most cases we will only outline the main ideas of proofs and refer the reader to [**21**] for details.

1.1. A simple approach to classical deformation theory. The first $\overline{\partial}$-equation which we deal with is the famous Maurer-Cartan equation arising from the deformation theory of complex structures. In the classical deformation theory developed by Kodaira et al. [**22, 14**], in order to construct a complex analytic family of compact complex manifold M, the starting point is to solve an integrable equation, i.e. Maurer-Cartan equation. When $H^2(M, T^{1,0}M) = 0$, Kodaira et al. showed that the deformation is unobstructed and constructed the solution of Maurer-Cartan equation by formal power series. Then they showed the convergence of this series through comparing with an artificial majorant series.

In our new approach [**21**], the existence and convergence of the solution is directly derived from the classical Banach fixed point theorem. We think that this is the elementary proof which Kodaira looked for (cf. the open problem asked in page 55 of [**22**]). As a consequence, we also present a simple proof of the unobstructed theorem for Calabi-Yau manifold due to [**29, 30**]. The method is global in nature and can be applied to more general deformation problems.

1.2. Variations of pluricanonical forms and deformation cohomology. The second $\overline{\partial}$-equation is from the variation problem of holomorphic pluricanonical form which is closely related to Siu's conjecture on the invariance of plurigenera [**26**].

Let (M, ω) be a compact Kähler manifold of complex dimension n with Kähler form ω, and $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential. Let m be a positive integer. We consider a pluricanonical form σ_0 which is a holomorphic section of $K_M^{\otimes m}$ over M, where K_M denotes the canonical line bundle of M. An important question is how to construct the pluricanonical forms $\sigma(\varphi)$ on M_φ, such that $\sigma(0) = \sigma_0$. In fact, for projective manifolds, the existence of this variation was proved by Y.-T. Siu. In general there is a famous conjecture due to Siu [26], about the invariance of plurigenera for compact Kähler manifolds.

In our approach, Siu's conjecture is reduced to solving the variation equation (4.5). By using Hodge theory, we can provide a closed formula for the solution of this variation equation under certain conditions. A related new deformation cohomology theory is also introduced.

More precisely, let $(\mathcal{L}, h)$ be an Hermitian holomorphic line bundle over (M, ω). Let $\nabla = \nabla' + \overline{\partial}$ be the Chern connection of $(\mathcal{L}, h)$ with curvature Θ. We introduce an operator

$$T^{\nabla'} = \overline{\partial}^* G \nabla'$$

where G is the Green operator associated to the Laplacian $\square_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. We consider the holomorphic line bundle $\mathcal{L}_M = K_M^{\otimes(m-1)}$ over (M, ω) with the induced Hermitian metric $h_\omega = \det(g)^{-(m-1)}$, where g denotes the Kähler metric associated to the Kähler form ω.

Let φ act on differential forms by contraction $i_\varphi = \varphi \lrcorner$ and the denote the exponential operator $\rho_\varphi = e^{i_\varphi}$. We will establish the following result in Section 4 to which we refer the reader for precise definitions of the notations in the following theorem.

THEOREM 1.1. *Suppose $(\mathcal{L}_M, h_\omega)$ is a positive line bundle over a compact Kähler manifold (M, ω) with curvature $\sqrt{-1}\Theta = \rho\omega$ for a constant $\rho > 0$, let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential satisfying two conditions $div\varphi = 0$ and L_∞-norm $\|\varphi\|_\infty < 1$. Then, for any holomorphic pluricanonical form $\sigma_0 \in A^{n,0}(M, \mathcal{L}_M)$,*

$$\sigma(\varphi) = \rho_\varphi((I + T^{\nabla'}\varphi)^{-1}\sigma_0)$$

is a holomorphic pluricanonical form in $A^{n,0}(M_\varphi, \mathcal{L}_{M_\varphi})$.

Note that Theorem 1.1 is global in the sense that it does not depend on the local deformation family. Theorem 1.1 can be used to construct the closed variation formula for pluricanonical forms of Kähler-Einstein manifold of general type, see Definition 4.12.

Indeed, let $\pi : \mathcal{M} \to B_\epsilon$ be a local Kuranishi family of compact Kähler-Einstein manifolds of general type over a small disc. For $t \in B_\epsilon$, we assume $M_t = \pi^{-1}(t) = M_{\varphi(t)}$, where $\varphi(t) \in A^{0,1}(M_0, T^{1,0}M_0)$ denotes an integrable Beltrami differential satisfying the Kuranishi gauge $\overline{\partial}^*\varphi(t) = 0$. As an application of Theorem 1.1, we obtain

COROLLARY 1.2. *Given a local Kuranishi family of compact Kähler-Einstein manifolds of general type, and any holomorphic pluricanonical form*

$$\sigma_0 \in A^{n,0}(M_0, \mathcal{L}_{M_0}),$$

we have that

$$\sigma(t) = \rho_t((I + T^{\nabla'}\varphi(t))^{-1}\sigma_0) \tag{1.1}$$

is a holomorphic pluricanonical form in $A^{n,0}(M_t, \mathcal{L}_{M_t})$ *with* $\sigma(0) = \sigma_0$*, where* $\rho_t = \rho_{\varphi(t)}$*.*

Corollary 1.2 implies the invariance of plurigenera for Kähler-Einstein manifolds of general type, which has been obtained in [**27**]. The new feature of formula (1.1) is that it provides a simple closed explicit formula for the variation which will have interesting geometric applications, such as curvature computations for L^2-metric on the vector bundle of pluricanonical sections.

Next, motivated by our approach to Siu's conjecture of invariance of plurigenera, we introduce a new cohomology theory on compact Kähler manifold M. Let us define an operator

$$D_\varphi \sigma = \overline{\partial}\sigma + \nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma$$

for any $\sigma \in A^{n,*}(M, \mathcal{L}_M)$. Then we show that $D_\varphi^2 = 0$. We therefore obtain the following D_φ-complex on M

$$0 \to A^{n,0}(M, \mathcal{L}_M) \xrightarrow{D_\varphi} A^{n,1}(M, \mathcal{L}_M) \xrightarrow{D_\varphi} \cdots \xrightarrow{D_\varphi} A^{n,n}(M, \mathcal{L}_M) \to 0.$$

We define the p-th deformation cohomology $H^{n,p}_{D_\varphi}(M, \mathcal{L}_M)$ as the p-th cohomology group of the complex $(A^{n,*}(M, \mathcal{L}_M), D_\varphi)$. Then we can reformulate Siu's conjecture [**26**] to terms of the deformation cohomology in the following equivalent form.

Conjecture 1.3 (Invariance of plurigenera). Let M be a compact Kähler manifold, and $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential comes from the local Kuranishi family of M, then there exists an isomorphism

$$H^{n,0}_{D_\varphi}(M, \mathcal{L}_M) \xrightarrow{\simeq} H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M). \tag{1.2}$$

By using Hodge theory on the compact Kähler manifold M, we will show that the harmonic projection map actually gives an explicit injection map from $H^{n,0}_{D_\varphi}(M, \mathcal{L}_M)$ to $H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M)$, see Lemma 4.19. Therefore, the proof of Conjecture 1.3 is reduced to construct an injective map from $H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M)$ to $H^{n,0}_{D_\varphi}(M, \mathcal{L}_M)$.

1.3. Canonical families for compact Kähler manifolds. Then we specialize the above discussions to the special case of $m = 1$. Namely, we study the variations of canonical forms, such as the holomorphic $(n, 0)$-forms over compact Kähler manifolds. In this case, we have the following global variation formula.

THEOREM 1.4. *Given any integrable Beltrami differential φ with $\|\varphi\|_\infty < 1$, and any holomorphic $(n,0)$-form Ω_0 on M, we have that*

$$\Omega(\varphi) = \rho_\varphi((I + T\varphi)^{-1}\Omega_0) \tag{1.3}$$

is a holomorphic $(n,0)$-form on M_φ with $\Omega(0) = \Omega_0$.

Next, we apply the above variation formula (1.3) to construct canonical families of holomorphic $(n,0)$-forms on Calabi-Yau and hyperkähler manifolds. Assume that M is a Calabi-Yau manifold, and let Ω_0 be the nonwhere vanishing $(n,0)$-form on M.

For the local Kuranishi family $\{M_t\}_{t\in\Delta_\epsilon}$ of the Calabi-Yau manifold M as constructed in Theorem 3.8, the global variation formula (1.3) implies that

$$\Omega^c(t) := \rho_{\varphi(t)}(\Omega_0) \tag{1.4}$$

is a canonical holomorphic $(n,0)$-form on M_t for $t \in \Delta_\epsilon$. Then, we compute the cohomology class $[\Omega^c(t)]$, and show that it has the following expansion:

$$[\Omega^c(t)] = [\Omega_0] + \sum_{i=1}^{N}[\eta_i \lrcorner \Omega_0]t_i + \frac{1}{2}\sum_{i,j=1}^{N}[\mathbb{H}(\eta_i \lrcorner \eta_j \lrcorner \Omega_0)]t_i t_j + O(|t|^3) \tag{1.5}$$

where $O(|t|^3)$ denotes the terms in $\bigoplus_{j=2}^{n} H^{n-j,j}(M)$ of orders at least 3 in t. Here $\{\eta_i\}$ is an orthonormal basis of the space of harmonic Beltrami differentials $\mathbb{H}^{0,1}(M, T^{1,0}M)$. By using formula (1.5), we immediately get that the above expansion gives a local normal coordinate for the Weil-Petersson metric of the Teichmüller space $\mathcal{T}$ of polarized Calabi-Yau manifolds.

Then we consider hyperkähler manifolds. By definition, a hyperkähler manifold M carries a trivial canonical bundle. We still denote by $\mathcal{T}$ the Teichmüller space of polarized hyperkähler manifolds, then we have the following polynomial expansions for the canonical families of holomorphic forms.

THEOREM 1.5. *Fix $p \in \mathcal{T}$, let (M, L) be the corresponding polarized hyperkähler manifold and $\Omega^{2,0}$ be a nonzero holomorphic nondegenerate $(2,0)$-form over M and $\{\eta_i\}_{i=1}^N$ be an orthonormal basis $\mathbb{H}^1_L(M, T^{1,0}M)$ with respect to the Kähler Ricci-flat metric. Then in a neighborhood U of p, there exists a local canonical family of nondegenerate holomorphic $(2,0)$-forms,*

$$\Omega^{c;2,0}(t) = \rho_{\varphi(t)}(\Omega^{2,0}),$$

which defines a canonical family of $(2,0)$-classes

$$[\Omega^{c;2,0}(t)] = [\Omega^{2,0}] + \sum_{i=1}^{N}[\eta_i \lrcorner \Omega^{2,0}]t_i + \frac{1}{2}\sum_{i,j=1}^{N}[\eta_i \lrcorner \eta_j \lrcorner \Omega^{2,0}]t_i t_j$$

Moreover, let $\Omega = \wedge^n \Omega^{2,0}$ be the canonical $(2n,0)$-form on M, then in a neighborhood U of p, the canonical family of holomorphic $(2n,0)$-forms

$\Omega^c(t) = \rho_{\varphi(t)}(\Omega)$ *defines a canonical family of* $(2n,0)$*-classes*

$$[\Omega^c(t)] = [\Omega] + \sum_{i=1}^{N} [\eta_i \lrcorner \Omega] t_i + \cdots + \frac{1}{(2n)!} \sum_{i_1,..,i_{2n}=1}^{N} [\eta_{i_1} \lrcorner \cdots \lrcorner \eta_{i_{2n}} \lrcorner \Omega] t_{i_1} t_{i_2} \cdots t_{i_{2n}}.$$

We also present a criterion about when $\mathcal{T}$ is locally Hermitian symmetric under the Weil-Petersson metric. See Definition 6.3 and Proposition 6.4 for the definition of locally Hermitian symmetric manifolds and precise results there.

1.4. Solving the Beltrami equations. Finally, we present a global method to solve the classical Beltrami equation which is very important in the development of complex analysis and moduli theory of Riemann surfaces and also has many important applications in other subjects. There is a huge literature on the Beltrami equation. See, for examples [**1**], [**3**] and [**9**]. In particular the construction by Ahlfors in [**1**] depends on rather deep analysis and estimate of Calderón-Zygmund. The method of [**3**] is by using local integral operators and their regularity theory. Our method is global in the sense that we use L^2-Hodge theory.

Given a measurable function μ_0 on the unit disc $D \subset \mathbb{C}$, suppose $\sup|\mu_0| < 1$, let $\mu = \mu_0 \frac{\partial}{\partial z} \otimes d\bar{z}$ be a Beltrami differential on D with coordinate z. Recall that solving the Beltrami equation is equivalent to finding a function f on the unit disc D, such that

$$\overline{\partial} f = \mu \partial f.$$

Our observation is that the Beltrami equation can be easily solved by using the L^2-Hodge theory on D. We will see that the L^2-Hodge theory holds on disk D with the Poincáre metric ω_P. So we also have the operator $T = \overline{\partial}^* G \partial$ with norm $\|T\| \le 1$.

Note that the L_∞-norm of μ is independent of the Hermitian metric on D and is equal to $\sup|\mu_0|$, i.e. $\|\mu\|_\infty < 1$. Similarly, we show that for a holomorphic one form h_0 on D, the equation

$$\overline{\partial} h = -\partial \mu h$$

has a solution

$$h = (I + T\mu)^{-1} h_0.$$

As a corollary we can directly get a solution of the Beltrami equation for any measurable μ_0. In particular, we have,

THEOREM 1.6. *Assume that* $||\mu||_\infty = \sup|\mu_0| < 1$*, if* μ_0 *is of regularity* C^k*, then the Beltrami equation*

$$\overline{\partial} f = \mu \partial f$$

has a solution of regularity C^{k+1}*.*

The rest of this paper is organized as follows. In Section 2, we briefly review the Hodge theory on compact complex manifolds and the quasi-isometry formula for compact Kähler manifolds. Then in Section 3, we present a very simple and global method to solve the obstruction equation for variation of complex structure by using the Banach fixed point theorem. As a consequence, this gives a much simpler treatment of the local deformation theory of Kodaira-Spencer-Kuranishi. Next, in Section 4, we construct a closed formula for the variations of holomorphic pluricanonical forms under certain conditions. A new deformation cohomology theory is also introduced.

We discuss the variations of canonical forms in Section 5, and apply this variation formula to the cases of Calabi-Yau and hyperkähler manifolds in Section 6, and obtain the corresponding canonical families which are simply given by polynomial expansions in certain canonical local coordinates. In Section 7, we present our result in [**21**] about solving the famous Beltrami equations with measurable coefficients by using L^2-Hodge theory on Poincaré disk. Section 8 is devoted to discussing various further applications and extensions of our method.

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2. Hodge theory on compact complex manifolds

In this section, we briefly review the Hodge theory on compact complex manifolds and fix the notations used in this article.

Let (E, h) be a Hermitian holomorphic vector bundle over a compact complex manifold M with Hermitian metric g. Let $\nabla = \nabla' + \overline{\partial}$ be the Chern connections of (E, h). The Hermitian metrics on E and M induce an L^2 inner produce on the space $A^{p,q}(M, E)$ of E-valued (p, q)-forms on M. We set the Laplacians

$$\square_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \quad \text{and} \quad \square' = \nabla'\nabla'^* + \nabla'^*\nabla'.$$

Hodge theory implies that there are Green operator G (resp. G') and harmonic projection $\mathbb{H}$ (resp. $\mathbb{H}'$) in the Hodge decomposition corresponding to $\square_{\overline{\partial}}$ (resp. $\square'$).

PROPOSITION 2.1. *We have the following identities:*

$$\square_{\overline{\partial}} G = G\square_{\overline{\partial}} = I - \mathbb{H},\ \overline{\partial}G = G\overline{\partial},\ \overline{\partial}^* G = G\overline{\partial}^*,\ \mathbb{H}G = G\mathbb{H} = 0.$$

Moreover, $\overline{\partial}\mathbb{H} = \mathbb{H}\overline{\partial} = 0$, $\overline{\partial}^*\mathbb{H} = \mathbb{H}\overline{\partial}^* = 0$. *The similar identities holds among the operators* G', $\mathbb{H}'$, ∇' *and* ∇'^*.

Then we suppose (M, ω) is an n-dimensional compact Kähler manifold with Kähler metric ω, and $\|\cdot\|_{L^2}$ be the L^2-norm on the space $A^{p,q}(M)$ of smooth differential forms induced by the metric ω. We set

$$\Box_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, \quad \Box_\partial = \partial\partial^* + \partial^*\partial \text{ and } \Delta_d = dd^* + d^*d,$$

where $d = \partial + \overline{\partial}$. On $A^{p,q}(M)$, we have the equality of the Laplacians $\Box_{\overline{\partial}} = \Box_\partial = \frac{1}{2}\Delta_d$. We also let $\mathbb{H}$ to be the orthogonal projection from $A^{p,q}(M)$ to the harmonic space $\mathbb{H}^{p,q}(M) = \ker \Box_{\overline{\partial}}$. Then the corresponding identities in Proposition 2.1 hold. Furthermore, since $\Box_{\overline{\partial}} = \Box_\partial$ on compact Kähler manifold, we can derive the following quasi-isometry formula. For any $g \in A^{p,q}(M)$, we have

$$\begin{aligned}
\|\overline{\partial}^* G\partial g\|^2_{L^2} &= \langle \overline{\partial}^* G\partial g, \overline{\partial}^* G\partial g\rangle = \langle \overline{\partial}\overline{\partial}^* G\partial g, G\partial g\rangle \\
&= \langle \Box_{\overline{\partial}} G\partial g, G\partial g\rangle - \langle \overline{\partial}^*\overline{\partial} G\partial g, G\partial g\rangle \\
&= \langle \partial g, G\partial g\rangle - \langle \overline{\partial} G\partial g, \overline{\partial} G\partial g\rangle \\
&= \langle g, \Box_\partial G g\rangle - \langle g, \partial\partial^* G g\rangle - \|\overline{\partial} G\partial g\|^2_{L^2} \\
&= \langle g, g - \mathbb{H} g\rangle - \langle \partial^* g, G\partial^* g\rangle - \|\overline{\partial} G\partial g\|^2_{L^2} \\
&= \|g\|^2_{L^2} - \|\mathbb{H} g\|^2_{L^2} - \langle \partial^* g, G\partial^* g\rangle - \|\overline{\partial} G\partial g\|^2_{L^2} \\
&\leq \|g\|^2_{L^2}.
\end{aligned}$$

The last inequality holds since the Green operator G is a non-negative operator. We introduce the operator $T = \overline{\partial}^* G\partial$. Therefore,

$$\|Tg\|_{L^2} = \|\overline{\partial}^* G\partial g\|_{L^2} \leq \|g\|_{L^2} \tag{2.1}$$

which is referred to as being a *quasi-isometry* in [**18**].

3. A simple approach to classical deformation theory

The classical deformation theory of complex structures was developed by Kodaira-Spencer and Kuranishi in 1960s [**14, 22, 10, 11**]. In this section we present a simple and global method to solve the obstruction equation for variation of complex structures by using the Banach fixed point theorem, the details is given in [**21**]. As a consequence, this method also gives a much simpler treatment of the general local deformation theory of Kodaira-Spencer-Kuranishi, for example, two classical unobstructed deformation theorems due to [**14**] and [**29, 30**] will follows easily from our method.

3.1. Beltrami differentials. Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$, and we denote by X the underlying real manifold of M of real dimension $2n$. The associated almost complex structure of the complex manifold M gives a direct sum decomposition of the complexified tangent bundle,

$$T_{\mathbb{C}}X = T^{1,0}M \oplus T^{0,1}M.$$

Let J be another almost complex structure on X. Then, J gives another direct sum decomposition,

$$T_{\mathbb{C}}X = T^{1,0}M_J \oplus T^{0,1}M_J.$$

Denote by

$$\iota_1 : T_{\mathbb{C}}X \to T^{1,0}M,\ \iota_2 : T_{\mathbb{C}}X \to T^{0,1}M,$$

the two projection maps.

Definition 3.1 (cf. Definition 4.2 [12]). Let J be an almost complex structure on X, we say that J is of finite distance from the given complex structure M on X, if the restriction map

$$\iota_1|_{T^{1,0}M_J} : T^{1,0}M_J \to T^{1,0}M$$

is an isomorphism.

Therefore, if J is of finite distance from M, one can define a map

$$\overline{\varphi} : T^{1,0}M \to T^{0,1}M$$

by setting

$$\overline{\varphi}(v) = -\iota_2 \circ \left(\iota_1|_{T^{1,0}M_J}\right)^{-1}(v).$$

This map is well-defined since $\iota_1|_{T^{1,0}M_J}$ is an isomorphism. It is clear that

$$T^{1,0}M_J = \{v - \overline{\varphi}(v)|v \in T^{1,0}M\},\ T^{0,1}M_J = \{v - \varphi(v)|v \in T^{0,1}M\},$$

and their corresponding dual spaces are

(3.1)
$$\Lambda^{1,0}M_J = \{w + \varphi(w)|w \in \Lambda^{1,0}M\},\ \Lambda^{0,1}M_J = \{w + \overline{\varphi}(w)|w \in \Lambda^{0,1}M\}.$$

In this way, the complex conjugate

$$\varphi :\ T^{0.1}M \to T^{1,0}M,$$

defined by $\overline{\varphi(v)} = \overline{\varphi}(\overline{v})$ determines a $T^{1,0}M$-valued $(0,1)$-form which is also denoted by $\varphi \in A^{0,1}(M, T^{1,0}M)$ for convenience. By the condition

$$T^{1,0}M \oplus T^{0,1}M = T_{\mathbb{C}}X = T^{1,0}M_J \oplus T^{0,1}M_J,$$

the transformation matrix

$$\begin{pmatrix} I_n & -\overline{\varphi} \\ -\varphi & I_n \end{pmatrix}$$

from a basis of $T^{1,0}M \oplus T^{0,1}M$ to a basis of $T^{1,0}M_J \oplus T^{0,1}M_J$ must be nondegenerate. Therefore $\det(I_n - \varphi\overline{\varphi}) \neq 0$. In fact, we have

Proposition 3.2 (cf. Proposition 4.3 [12]). *There is a bijective correspondence between the set of almost complex structures of finite distance from M and the set of all $\varphi \in A^{0,1}(M, T^{1,0}M)$ such that, at each point $p \in X$, the map*

$$\varphi\overline{\varphi} :\ T^{1,0}M \to T^{1,0}(M)$$

does not have eigenvalue 1.

Definition 3.3. If $\varphi \in A^{0,1}(M, T^{1,0}M)$ satisfies the condition in Proposition 3.2, we say that φ is a *Beltrami differential*. If φ satisfies the integrability condition

$$\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi], \tag{3.2}$$

we call φ an *integrable Beltrami differential*. The equation (3.2) is usually referred as *Maurer-Cartan equation*.

In conclusion, a Beltrami differential φ determines an almost complex structure of finite distance from M. We denote the corresponding almost complex structure (i.e. almost complex manifold) by M_φ. An integrable Beltrami differential φ gives a new complex structure on X by the Newlander-Nirenberg theorem [**23**], the corresponding complex manifold is denoted by M_φ.

3.2. Deformation theorems. In this section we first present a simple method to solve the Kuranishi equations and the Maurer-Cartan equations in deformation theory, by using the Banach fixed point theorem and Hodge theory on compact complex manifold. As corollaries, we give simple and global treatments of the two unobstructedness theorems, due to Kodaira-Nirenberg-Spencer [**14**] and Bogomorov-Tian-Todorov [**29, 30**]. We will only describe the proofs briefly, and refer the reader to [**21**] for details.

First note that in order to prove the existence of deformations, we only need to show that for any $\eta \in \mathbb{H}^1(M, T^{1,0}M)$, there exists a family of Beltrami differentials $\varphi(t) \in A^{0,1}(M, T^{1,0}M)$ satisfying the Maurer-Cartan equation (3.2). We consider the following integral equation

$$\varphi = \eta + \frac{1}{2}\overline{\partial}^* G[\varphi, \varphi] \tag{3.3}$$

which is usually referred as the *Kuranishi equation*.

By using the Banach fixed point theorem and Hodge theory on holomorphic tangent bundle $T^{1,0}M$, we provide a simple proof for the following results in the classical papers [**14, 22**]. We think that this is the elementary proof which Kodaira looked for (cf. the open problem asked in page 55 of [**22**]).

Proposition 3.4. *Given an orthonormal basis $\eta_1, ..., \eta_N \in \mathbb{H}^1(M, T^{1,0}M)$. Let $\eta(t) = \sum_{i=1}^N \eta_i t_i$, then there is a positive constant ϵ_k which depends on a positive integer k, such that the Kuranishi equation (3.3) with initial value $\eta(t)$ has a unique solution $\varphi(t)$ which analytically depends on t for $|t| < \epsilon_k$.*

Proposition 3.5. *For the above solution $\varphi(t)$, if it satisfies $\mathbb{H}[\varphi(t), \varphi(t)] = 0$, then*

(i) *there exists $\epsilon > 0$ such that $\varphi(t)$ is smooth in (z, t) and analytic in t for $|t| < \epsilon$,*
(ii) *the solution $\varphi(t)$ satisfies the Maurer-Cartan equation (3.2).*

In conclusion, in order to prove the existence of deformations, we only need to prove that for the $\varphi(t)$ constructed in Proposition 3.4, satisfies the equation

$$\mathbb{H}([\varphi(t), \varphi(t)]) = 0. \tag{3.4}$$

Therefore, it immediately implies that

THEOREM 3.6 (Kodaria-Spencer-Nirenberg, 1958). *For a compact complex manifold M with $H^2(M, T^{1,0}M) = 0$, its deformation is unobstructed.*

Furthermore, if M is a Calabi-Yau manifold, then as a easy consequence, we can also prove that $\mathbb{H}[\varphi(t), \varphi(t)] = 0$. Therefore, we obtain

THEOREM 3.7 (Bogolomov-Tian-Todorov). *For a Calabi-Yau manifold, its deformation is unobstructed.*

Indeed, let Ω_0 be a nonzero holomorphic section of K_M. The key property of Calabi-Yau is that the contraction operator

$$\lrcorner\Omega_0 : A^{0,k}(M, T^{1,0}M) \to A^{n-1,k}(M) \quad \text{is an isomorphism.}$$

Furthermore, if we choose the Ricci-flat Kähler metric (i.e. Calabi-Yau metric) g on M, $\lrcorner\Omega_0$ induces an isomorphism between the two Hodge theories on $A^{0,k}(M, T^{1,0}M)$ and $A^{n-1,k}(M)$. Hence $\mathbb{H}[\varphi(t), \varphi(t)] = 0$ if and only if $\mathbb{H}([\varphi(t), \varphi(t)]\lrcorner\Omega_0) = 0$.

By the so-called Tian-Todorov Lemma which is a direct consequence of the Cartan formula for Lie derivatives [16], we have

$$[\varphi(t), \varphi(t)]\lrcorner\Omega_0 = -\partial(\varphi(t)\lrcorner\varphi(t)\lrcorner\Omega_0) + 2\varphi(t)\lrcorner\partial(\varphi(t)\lrcorner\Omega_0).$$

Since $\mathbb{H}\partial = 0$, in order to prove $\mathbb{H}([\varphi(t), \varphi(t)]\lrcorner\Omega_0) = 0$, we only need to show

$$\partial(\varphi(t)\lrcorner\Omega_0) = 0.$$

Applying $\lrcorner\Omega_0$ to equation $\varphi(t) = \eta(t) + \frac{1}{2}\overline{\partial}^* G[\varphi(t), \varphi(t)]$, we obtain

$$\begin{aligned}
(\varphi(t) - \eta(t))\lrcorner\Omega_0 &= \left(\frac{1}{2}\overline{\partial}^* G[\varphi(t), \varphi(t)]\right)\lrcorner\Omega_0 \\
&= \frac{1}{2}\overline{\partial}^* G\left([\varphi(t), \varphi(t)]\lrcorner\Omega_0\right) \\
&= \overline{\partial}^* G\left(\varphi(t)\lrcorner\partial(\varphi(t)\lrcorner\Omega_0)\right) - \frac{1}{2}\overline{\partial}^* G\partial\left(\varphi(t)\lrcorner\varphi(t)\lrcorner\Omega_0\right).
\end{aligned}$$

Note that the harmonicity of $\eta(t)$ implies the harmonicity of $\eta(t)\lrcorner\Omega_0$, hence $\partial(\eta(t)\lrcorner\Omega_0) = 0$ by Kähler condition. Let $\Psi = \varphi(t)\lrcorner\Omega_0$, we have

$$\Psi = \overline{\partial}^* G(\varphi(t)\lrcorner\partial\Psi) + \partial\left(\frac{1}{2}\overline{\partial}^* G\left(\varphi(t)\lrcorner\varphi(t)\lrcorner\Omega_0\right)\right). \tag{3.5}$$

Applying the ∂-operator to equation (3.5), we obtain $\partial\Psi = \partial\overline{\partial}^* G(\varphi(t)\lrcorner\partial\Psi)$. Hence

$$\|\partial\Psi\|_k = \|\partial\overline{\partial}^* G(\varphi(t)\lrcorner\partial\Psi)\|_k \leq C_k\|\varphi(t)\|_k \cdot \|\partial\Psi\|_k \tag{3.6}$$

Therefore, $\partial\Psi = 0$ when t is small enough.

Furthermore, the above proof immediately implies the following result of Todorov.

THEOREM 3.8. *There exists $\epsilon > 0$, for $t \in \Delta_\epsilon$, there is a unique $\varphi(t) \in A^{0,1}(M, T^{1,0}M)$ satisfies $\varphi(t) = \eta(t) + \frac{1}{2}\overline{\partial}^* G[\varphi(t), \varphi(t)]$, and $\varphi(t)$ has the following properties:*

(1) $\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$;
(2) $\overline{\partial}^*\varphi(t) = 0$;
(3) $\left(\varphi(t) - \sum_{i=1}^N \eta_i t_i\right) \lrcorner \Omega_0$ *is ∂-exact, and* $\partial(\varphi(t) \lrcorner \Omega_0) = 0$.

4. Variations of pluricanonical forms

In this section, we will introduce a global method to construct a closed variation formula for the pluricanonical form. A new deformation cohomology theory will also be introduced, which is closely related to Siu's conjecture on the invariance of plurigenera for compact Kähler manifolds [**26**].

In Section 4.1, we derive the variation equation for variation of pluricanonical forms over compact Kähler manifold. In Section 4.2, we introduce the quasi-isometry formula for Hodge theory of positive line bundle. Then we solve the variation equation in Section 4.3. In Section 4.4, we present a closed formula for the variation of pluricanonical forms over the Kähler-Einstein manifolds of general type. The new deformation cohomology is introduced in Section 4.5 which sheds some new light on Siu's conjecture on invariance of plurigenera for compact Kähler manifolds.

4.1. Variation equations. Let (M, ω) be a compact Kähler manifold of dimension n, let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential. Then φ determines a new complex manifold denoted by M_φ. Given a positive integer m, we introduce the line bundles $\mathcal{L}_M = K_M^{\otimes(m-1)}$ and $\mathcal{L}_{M_\varphi} = K_{M_\varphi}^{\otimes(m-1)}$. For a $\mathcal{L}_M$-valued $(n,0)$-form σ on M, we can deform it via φ. We define the map

$$\rho_\varphi : A^{n,0}(M, \mathcal{L}_M) \to A^{n,0}(M_\varphi, \mathcal{L}_{M_\varphi})$$

as follows. For any $x \in M$, one can pick a local holomorphic coordinate system $\{z^1, ..., z^n\}$ near x, we write the integrable Beltrami differential φ as

$$\varphi = \varphi^i_{\overline{k}} d\overline{z}^k \otimes \partial_i.$$

Let $\sigma = f(z)dz^1 \wedge \cdots \wedge dz^n \otimes e$, where $e = (dz^1 \wedge \cdots \wedge dz^n)^{m-1}$, we define

$$\rho_\varphi(\sigma) = f(z)((dz^1 + \varphi(dz^1)) \wedge \cdots \wedge (dz^n + \varphi(dz^n)))^{\otimes m}. \tag{4.1}$$

Equivalently we denote the contraction map by φ as $i_\varphi = \varphi \lrcorner$, then one can write

$$\rho_\varphi = e^{i\varphi}$$

as the exponential operator of i_φ.

Let $\{w^1, ..., w^n\}$ be a local holomorphic coordinate system of M_φ. Then

$$dw^i = \partial_j w^i dz^j + \partial_{\bar{j}} w^i d\bar{z}^j = \partial_j w^i (dz^j + \varphi(dz^j))$$

by the definition of the Beltrami differential φ. Indeed, if we let $a = (a_{ij}) = (\partial_j w^i)$ and $a^{-1} = (a^{ij})$, then

$$\varphi^i_{\bar{k}} = a^{ij} \partial_{\bar{k}} w^j \text{ and } \varphi = \varphi^i_{\bar{k}} d\bar{z}^k \otimes \partial_i.$$

Hence, the formula (4.1) can be rewritten as

$$\rho_\varphi(\sigma) = \frac{f(z)}{\det(a)^m} (dw^1 \wedge \cdots \wedge dw^n)^{\otimes m}.$$

LEMMA 4.1. *Given $\sigma = f(z) dz^1 \wedge \cdots \wedge dz^n \otimes e \in A^{n,0}(M, \mathcal{L}_M)$, then $\rho_\varphi(\sigma)$ is holomorphic in $A^{n,0}(M_\varphi, \mathcal{L}_{M_\varphi})$ if and only if for $j = 1, ..., n$,*

$$\overline{\partial}_j f = \varphi^i_{\bar{j}} \partial_i f + m f \partial_i \varphi^i_{\bar{j}}. \tag{4.2}$$

PROOF. Since a local smooth function h is holomorphic on M_φ if and only if

$$\overline{\partial} h = \varphi \lrcorner (\partial h),$$

i.e. for $j = 1, ..., n$,

$$\overline{\partial}_j h = \varphi^i_{\bar{j}} \partial_i h.$$

Therefore, $\rho_\varphi(\sigma)$ is holomorphic i.e. $\frac{f(z)}{\det(a)^m}$ is holomorphic on M_φ, if and only if

$$\overline{\partial}_j \left(\frac{f(z)}{\det(a)^m} \right) = \varphi^i_{\bar{j}} \partial_i \left(\frac{f(z)}{\det(a)^m} \right)$$

which is equivalent to

$$\left(\overline{\partial}_j f - m f a^{ik} \overline{\partial}_j a_{ki} \right) = \left(\varphi^i_{\bar{j}} \partial_i f - m f \varphi^i_{\bar{j}} a^{pl} \partial_i a_{lp} \right) \tag{4.3}$$

through a straightforward computation.

We claim that

$$a^{ik} \overline{\partial}_j a_{ki} - \varphi^i_{\bar{j}} a^{pl} \partial_i a_{lp} = \partial_i \varphi^i_{\bar{j}}. \tag{4.4}$$

In fact, we have

$$\begin{aligned} \partial_i \varphi^i_{\bar{j}} &= \partial_i (a^{ik} \overline{\partial}_j w^k) = \partial_i a^{ik} \overline{\partial}_j w^k + a^{ik} \partial_i \overline{\partial}_j w^k \\ &= -a^{ip} \partial_i a_{pl} a^{lk} \overline{\partial}_j w^k + a^{ik} \overline{\partial}_j a_{ki} \\ &= -a^{ip} \partial_l a_{pi} \varphi^l_{\bar{j}} + a^{ik} \overline{\partial}_j a_{ki} \end{aligned}$$

which gives (4.4). Therefore, substituting (4.4) into (4.3), we obtain (4.2). ∎

Let $D = D' + \overline{\partial}$ be the Chern connection of the holomorphic bundle $T^{1,0}M$ over M. The connection matrix is given by $\theta = (\partial g g^{-1})$, where $g = (g_{i\bar{j}})$ denotes the Kähler metric matrix associated to the Kähler form ω. We define the divergence operator div as $tr \circ D'$. For $\varphi = \varphi^i_{\bar{j}} d\bar{z}^j \otimes \partial_i \in A^{0,1}(M, T^{1,0}M)$, we have

$$\begin{aligned} D'\varphi &= \partial(\varphi^i_{\bar{j}} d\bar{z}^j)\partial_i - \varphi^i_{\bar{j}} d\bar{z}^j (\partial g g^{-1})^p_i \partial_p \\ &= \partial_k \varphi^i_{\bar{j}} dz^k \wedge d\bar{z}^j \partial_i - \varphi^i_{\bar{j}} d\bar{z}^j \partial_k g_{i\bar{l}} g^{\bar{l}p} dz^k \partial_p. \end{aligned}$$

Therefore

$$\begin{aligned} div\, \varphi = tr \circ D'(\varphi) &= \partial_i \varphi^i_{\bar{j}} d\bar{z}^j + \varphi^i_{\bar{j}} \partial_k g_{i\bar{l}} g^{\bar{l}k} d\bar{z}^j \\ &= (\partial_i \varphi^i_{\bar{j}} + \varphi^i_{\bar{j}} \partial_i g_{k\bar{l}} g^{\bar{l}k}) d\bar{z}^j \\ &= (\partial_i \varphi^i_{\bar{j}} + \varphi^i_{\bar{j}} \partial_i \log\det(g)) d\bar{z}^j \end{aligned}$$

where we have used the Kähler condition $\partial_k g_{i\bar{l}} = \partial_i g_{k\bar{l}}$.

Let ∇' be the $(1,0)$-component of the naturally induced Chern connection on the holomorphic line bundle $\mathcal{L}_M = K_M^{\otimes(m-1)}$. The induced Hermitian metric on $\mathcal{L}_M$ is given by $(\det g)^{-(m-1)}$. For a holomorphic section e of $\mathcal{L}_M$, we have

$$\begin{aligned} \nabla' e &= \partial\left((\det g)^{-(m-1)}\right)(\det g)^{(m-1)} e \\ &= -(m-1)\partial_i \log(\det g) dz^i \otimes e. \end{aligned}$$

Proposition 4.2. *Given $\sigma \in A^{n,0}(M, \mathcal{L}_M)$, then $\rho_\varphi(\sigma)$ is holomorphic in $A^{n,0}(M_\varphi, \mathcal{L}_{M_\varphi})$ if and only if*

$$\overline{\partial}\sigma = -\nabla'(\varphi \lrcorner \sigma) + (m-1)\, div\, \varphi \wedge \sigma. \tag{4.5}$$

Proof. Let $\sigma = f dz^1 \wedge \cdots \wedge dz^n \otimes e \in A^{n,0}(M_0, \mathcal{L}_{M_0})$, then

$$\varphi \lrcorner \sigma = (-1)^{n+i} f \varphi^i_{\bar{j}} dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^n \wedge d\bar{z}^j \otimes e,$$

by a straightforward computation. We have

$$\begin{aligned} \nabla'(\varphi \lrcorner \sigma) &= (-1)^{n+i} \partial(f \varphi^i_{\bar{j}}) dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^n \wedge d\bar{z}^j \otimes e \\ &\quad + (-1)^i f \varphi^i_{\bar{j}} dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^n \wedge d\bar{z}^j \wedge \nabla' e \\ &= -\partial_i (f \varphi^i_{\bar{j}}) d\bar{z}^j \wedge dz^1 \wedge \cdots \wedge dz^n \otimes e \\ &\quad + (m-1) f \varphi^i_{\bar{j}} \partial_i \log(\det g) d\bar{z}^j \wedge dz^1 \wedge \cdots \wedge dz^n \otimes e. \end{aligned}$$

We also have

$$(m-1)\, div\, \varphi \wedge \sigma = (m-1) f \left(\partial_i \varphi^i_{\bar{j}} + \varphi^i_{\bar{j}} \partial_i \log(\det g)\right) d\bar{z}^j \wedge dz^1 \wedge \cdots \wedge dz^n \otimes e,$$

$$\overline{\partial}\sigma = (\overline{\partial}_j f) d\bar{z}^j \wedge dz^1 \wedge \cdots \wedge dz^n \otimes e.$$

Therefore, identity (4.5) follows from Lemma 4.1. ∎

REMARK 4.3. Note that $\sigma \in A^{n,0}(M, \mathcal{L}_M)$ can also be regarded as a smooth section of the holomorphic line bundle $K_M^{\otimes m}$ since $A^{n,0}(M, \mathcal{L}_M) = A^{0,0}(M, K_M^{\otimes m})$. The equation (4.5) is called the *variation equation* of the pluricanonical form, which gives the criterion when the variation pluricanonical form is holomorphic under the new complex structure.

On the other hand, if we let $\widehat{\nabla}'$ be the $(1,0)$-part of the Chern connection on $K_M^{\otimes m}$, it is easy to see that the variation equation (4.5) is equivalent to the equation

$$\overline{\partial}\sigma = \varphi \lrcorner \widehat{\nabla}'\sigma + m\, div\, \varphi \wedge \sigma.$$

4.2. Bundle-valued quasi-isometry over compact Kähler manifold. In this section, we first refine the bundle-valued quasi-isometry formula obtained in [18]. Let (E, h) be a Hermitian holomorphic vector bundle over the compact Kähler manifold (M, ω) and $\nabla = \nabla' + \overline{\partial}$ be the Chern connection of (E, h). With respect to the metrics on E and M, we set

$$\square_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \text{ and } \square' = \nabla'\nabla'^* + \nabla'^*\nabla'.$$

Accordingly, we have the Green operator G (resp. G') and harmonic projection $\mathbb{H}$ (resp. $\mathbb{H}'$) in the Hodge decomposition corresponding to $\square_{\overline{\partial}}$ (resp. $\square'$). Then we have the Proposition 2.1.

Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on M and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. Let $h = (h_{\alpha\overline{\beta}})$ where $h_{\alpha\overline{\beta}} = h(e_\alpha, e_\beta)$, and the inverse matrix $h^{-1} = (h^{\overline{\alpha}\beta})$. By the curvature formula of Chern connection $\Theta = \overline{\partial}(\partial h h^{-1})$, we obtain

$$\Theta^{\delta}_{i\overline{j}\alpha} = -\left(\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j}\right) h^{\overline{\beta}\delta} - \frac{\partial h_{\alpha\overline{\beta}}}{\partial z^i}\frac{\partial h^{\overline{\beta}\delta}}{\partial \overline{z}^j}.$$

Let $R_{i\overline{j}} = \sum_{\alpha=1}^r \Theta^{\alpha}_{i\overline{j}\alpha}$, we define the Chern-Ricci form of (E, h) by

$$Ric(E, h) = \frac{\sqrt{-1}}{2} R_{i\overline{j}} dz^i \wedge d\overline{z}^j.$$

In particular, when $E = T^{1,0}M$, the corresponding Chern-Ricci form is given by

$$Ric(\omega) = \frac{\sqrt{-1}}{2}\overline{\partial}\partial \log(\det g).$$

Let $\Theta_{i\overline{j}\alpha\overline{\beta}} = \Theta^{\gamma}_{i\overline{j}\alpha} h_{\gamma\overline{\beta}}$, we obtain

$$\Theta_{i\overline{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\overline{\delta}\gamma}\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}.$$

DEFINITION 4.4. An Hermitian vector bundle (E, h) is said to be semi-Nakano positive (resp. Nakano-positive), if for any non-zero vector $u =$

$u^{i\alpha}\partial_i \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} \Theta_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha}\overline{u}^{j\beta} \geq 0, \text{ (resp. } > 0).$$

In particular, for a line bundle, we say that it is positive, if it is Nakano-positive.

Proposition 4.5 (cf. Theorem 1.1(2) in [18]). *If $(\mathcal{L}, h)$ is a positive line bundle over a compact Kähler manifold (M, ω) and $\sqrt{-1}\Theta = \rho\omega$ for a constant $\rho > 0$, then for any $f \in A^{n-1,\bullet}(M, \mathcal{L})$, we have*

$$\|\overline{\partial}^* G\nabla' f\| \leq \|f\|. \tag{4.6}$$

For reader's convenience, we provide the proof of Proposition 4.5 here.

Proof. By the well-known Bochner-Kodaira-Nakano identity

$$\Box_{\overline{\partial}} = \Box' + [\sqrt{-1}\Theta, \Lambda_\omega],$$

and $[\omega, \Lambda_\omega] = (k-n)I$ on $A^k(M)$, we have

$$\Box_{\overline{\partial}}(\nabla' f) = \Box'(\nabla' f) + \rho q(\nabla' f) = (\Box' + \rho q)(\nabla' f),$$

for any $f \in A^{n-1,q}(M, \mathcal{L})$. Hence

$$\mathrm{Ker}\, \Box_{\overline{\partial}} \subseteq \mathrm{Ker}\, \Box'$$

which implies that $\mathbb{H}\nabla' f = 0$. Thus

$$\Box_{\overline{\partial}} G(\nabla' f) = \nabla' f = \Box' G'(\nabla' f)$$

by $\mathbb{H}'(\nabla' f) = 0$ and the Hodge decomposition for $\nabla' f$. Then

$$\begin{aligned}
\langle \nabla' f, G(\nabla' f)\rangle &= \langle \nabla' f, \Box_{\overline{\partial}}^{-1}(\nabla' f)\rangle \\
&= \langle \nabla' f, (\Box' + \rho q)^{-1}(\nabla' f)\rangle \\
&\leq \langle \nabla' f, \Box'^{-1}(\nabla' f)\rangle \\
&= \langle \nabla' f, G'(\nabla' f)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\overline{\partial}^* G\nabla' f\|^2 &= \langle \overline{\partial}^* G\nabla' f, \overline{\partial}^* G(\nabla' f)\rangle \\
&= \langle G\nabla' f, \overline{\partial}\overline{\partial}^* G(\nabla' f)\rangle \\
&= \langle G\nabla' f, (\Box_{\overline{\partial}} - \overline{\partial}^*\overline{\partial}) G(\nabla' f)\rangle \\
&= \langle G\nabla' f, \nabla' f\rangle - \langle \overline{\partial} G\nabla' f, \overline{\partial} G\nabla' f\rangle \\
&\leq \langle \nabla' f, G(\nabla' f)\rangle \\
&\leq \langle \nabla' f, G'(\nabla' f)\rangle \\
&= \langle f, \nabla'^*\nabla' G' f\rangle\rangle \\
&= \langle f, f - \mathbb{H}'(f) - \nabla'\nabla'^* G' f\rangle \\
&= \|f\|^2 - \|\mathbb{H}'(f)\|^2 - \langle \nabla'^* f, G'\nabla'^* f\rangle \\
&\leq \|f\|^2.
\end{aligned}$$

□

We introduce the operator

$$T^{\nabla'} = \overline{\partial}^* G\nabla'.$$

The quasi-isometry formula (4.6) implies that $T^{\nabla'}$ is an operator of norm less than or equal to 1 in the L^2 Hilbert space of the $\mathcal{L}$-valued forms. So we have

COROLLARY 4.6. *Let $(\mathcal{L}, h)$ be a positive line bundle over a compact Kähler manifold (M, ω), with $\sqrt{-1}\Theta = \rho\omega$ for a constant $\rho > 0$. Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be a Beltrami differential acting on the Hilbert space $L_2^{n,\bullet}(X, \mathcal{L})$ by contraction such that its L_∞-norm $\|\varphi\|_\infty < 1$. Then the operator $I + T^{\nabla'}\varphi$ is invertible.*

EXAMPLE 4.7. We will consider the holomorphic line bundle $\mathcal{L}_M = K_M^{\otimes(m-1)}$ over the compact Kähler manifold (M, ω), the corresponding Hermitian metric is given by $h_\omega = (\det g)^{-(m-1)}$. In this case, the curvature of the Chern connection of $\mathcal{L}_M$ is given by

$$\Theta = -(m-1)\overline{\partial}\partial \log(\det g).$$

Therefore

$$\sqrt{-1}\Theta = -2(m-1)Ric(\omega). \tag{4.7}$$

In particular, if (M, ω) is a Kähler-Einstein manifold of general type as defined in Definition 4.12, i.e. $Ric(\omega) = -\omega$, then we have

$$\sqrt{-1}\Theta = 2(m-1)\omega.$$

4.3. Solving the variation equations. As discussed in Section 4.1, in order to construct the variation of a pluricanonical form over M_φ, we need to solve the variation equation (4.5).

Before going further, we need the following lemma.

LEMMA 4.8. *Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential and let $\sigma \in A^{n,0}(M, \mathcal{L}_M)$, we set*

$$\Psi = \overline{\partial}\sigma + \nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma, \tag{4.8}$$

then we have the identity:

$$\overline{\partial}(\nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma) = -\left(\nabla'(\varphi \lrcorner \Psi) - (m-1)\, div\, \varphi \wedge \Psi\right). \tag{4.9}$$

PROOF. Locally, we write $\varphi = \varphi^i_{\bar{j}} d\bar{z}^j \otimes \partial_i \in A^{0,1}(M, T^{1,0}M)$, $\sigma = f dz^1 \wedge \cdots \wedge dz^n \otimes e \in A^{n,0}(M, \mathcal{L}_M)$ where $e = (dz^1 \wedge \cdots \wedge dz^n)^{\otimes(m-1)}$. Then

$$div\, \varphi = (\partial_i \varphi^i_{\bar{j}} + \varphi^i_{\bar{j}} \partial_i \log\det(g)) d\bar{z}^j.$$

For brevity, we introduce the notations $dZ = dz^1 \wedge \cdots \wedge dz^n$ and $dZ^{[k]} = dz^1 \wedge \cdots \wedge \widehat{dz^k} \wedge \cdots \wedge dz^n$, where the hat indicates that the corresponding term is to be dropped.

By the computations in the proof of Proposition 4.2, we have

$$\nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma = -\left((\partial_i f)\varphi^i_{\bar{j}} + mf\partial_i \varphi^i_{\bar{j}}\right) d\bar{z}^j \wedge dZ \otimes e.$$

Therefore

$$\begin{aligned}
&\overline{\partial}(\nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma) \\
&= -\overline{\partial}_l \left((\partial_i f)\varphi^i_{\bar{j}} + mf\partial_i \varphi^i_{\bar{j}}\right) d\bar{z}^l \wedge d\bar{z}^j \wedge dZ \otimes e \\
&= \sum_{1 \le l < j \le n} \Big((\overline{\partial}_j \partial_i f \varphi^i_{\bar{l}} - \overline{\partial}_l \partial_i f \varphi^i_{\bar{j}}) + \partial_i f(\overline{\partial}_j \varphi^i_{\bar{l}} - \overline{\partial}_l \varphi^i_{\bar{j}}) \\
&+ m(\overline{\partial}_j f \partial_i \varphi^i_{\bar{l}} - \overline{\partial}_l f \partial_i \varphi^i_{\bar{j}}) + mf(\overline{\partial}_j \partial_i \varphi^i_{\bar{l}} - \overline{\partial}_l \partial_i \varphi^i_{\bar{j}})\Big) d\bar{z}^l \wedge d\bar{z}^j \wedge dZ \otimes e.
\end{aligned} \tag{4.10}$$

On the other hand side, since

$$\begin{aligned}
\Psi &= \overline{\partial}\sigma + \nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma \\
&= \left(\overline{\partial}_j f - \varphi^i_{\bar{j}} \partial_i f - mf\partial_i \varphi^i_{\bar{j}}\right) d\bar{z}^j \wedge dZ \otimes e,
\end{aligned}$$

we have

$$\begin{aligned}
\varphi \lrcorner \Psi &= (\varphi^k_{\bar{l}} d\bar{z}^l \otimes \partial_k) \lrcorner \left(\overline{\partial}_j f - \varphi^i_{\bar{j}} \partial_i f - mf\partial_i \varphi^i_{\bar{j}}\right) d\bar{z}^j \wedge dZ \otimes e \\
&= \sum_{k=1}^n (-1)^k \varphi^k_{\bar{l}} (\overline{\partial}_j f - \varphi^i_{\bar{j}} \partial_i f - mf\partial_i \varphi^i_{\bar{j}}) d\bar{z}^l \wedge d\bar{z}^j \wedge dZ^{[k]} \otimes e.
\end{aligned}$$

Thus

$$
\begin{aligned}
&\nabla'(\varphi\lrcorner\Psi)\\
&= \partial\left(\sum_{k=1}^{n}(-1)^k\varphi_{\bar l}^k(\overline{\partial}_j f-\varphi_{\bar j}^i\partial_i f-mf\partial_i\varphi_{\bar j}^i)\right)d\bar z^l\wedge d\bar z^j\wedge dZ^{[k]}\otimes e\\
&+(-1)^{n+1}\sum_{k=1}^{n}(-1)^k\varphi_{\bar l}^k(\overline{\partial}_j f-\varphi_{\bar j}^i\partial_i f-mf\partial_i\varphi_{\bar j}^i)d\bar z^l\wedge d\bar z^j\wedge dZ^{[k]}\wedge\nabla' e\\
&= -\partial_k\left(\varphi_{\bar l}^k(\overline{\partial}_j f-\varphi_{\bar j}^i\partial_i f-mf\partial_i\varphi_{\bar j}^i)\right)d\bar z^l\wedge d\bar z^j\wedge dZ\otimes e\\
&+\varphi_{\bar l}^k\left(\overline{\partial}_j f-\varphi_{\bar j}^i\partial_i f-mf\partial_i\varphi_{\bar j}^i\right)(m-1)\partial_k\log(\det g)d\bar z^l\wedge d\bar z^j\wedge dZ\otimes e.
\end{aligned}
$$

We also have

$$
\begin{aligned}
-(m-1)\,div\,\varphi\wedge\Psi = -(m-1)&\left(\partial_i\varphi_{\bar l}^i+\varphi_{\bar l}^i\partial_i\log\det(g)\right)\\
&\cdot\left(\overline{\partial}_j f-\varphi_{\bar j}^i\partial_i f-mf\partial_i\varphi_{\bar j}^i\right)d\bar z^l\wedge d\bar z^j\wedge dZ\otimes e.
\end{aligned}
$$

Therefore

(4.11)

$$
\begin{aligned}
&-\left(\nabla'(\varphi\lrcorner\Psi)-(m-1)\,div\,\varphi\wedge\Psi\right)\\
&= m(\partial_k\varphi_{\bar l}^k)\left(\overline{\partial}_j f-\varphi_{\bar j}^i\partial_i f-mf\partial_i\varphi_{\bar j}^i\right)+\varphi_{\bar l}^k\partial_k\left(\overline{\partial}_j f-\varphi_{\bar j}^i\partial_i f-mf\partial_i\varphi_{\bar j}^i\right)\\
&\cdot d\bar z^l\wedge d\bar z^j\wedge dZ\otimes e\\
&= \sum_{0\le l\le j\le n}\left(m\left(\partial_k\varphi_{\bar l}^k\overline{\partial}_j f-\partial_k\varphi_{\bar j}^k\overline{\partial}_l f\right)+\left(\varphi_{\bar l}^k\partial_k\overline{\partial}_j f-\varphi_{\bar j}^k\partial_k\overline{\partial}_l f\right)\right.\\
&+\left.\left(\varphi_{\bar j}^k\partial_k\varphi_{\bar l}^i\partial_i f-\varphi_{\bar l}^k\partial_k\varphi_{\bar j}^i\partial_i f\right)+mf\left(\varphi_{\bar j}^k\partial_k\partial_i\varphi_{\bar l}^i-\varphi_{\bar l}^k\partial_k\partial_i\varphi_{\bar j}^i\right)\right)\\
&\cdot d\bar z^l\wedge d\bar z^j\wedge dZ\otimes e.
\end{aligned}
$$

Since φ is integrable, i.e. $\overline{\partial}\varphi=\frac{1}{2}[\varphi,\varphi]$, we obtain

$$
\varphi_{\bar l}^k\partial_k\varphi_{\bar j}^i-\varphi_{\bar j}^k\partial_k\varphi_{\bar l}^i=\overline{\partial}_l\varphi_{\bar j}^i-\overline{\partial}_j\varphi_{\bar l}^i,
$$

and

$$
\begin{aligned}
\partial_i\left(\overline{\partial}_l\varphi_{\bar j}^i-\overline{\partial}_j\varphi_{\bar l}^i\right) &= \partial_i\left(\varphi_{\bar l}^k\partial_k\varphi_{\bar j}^i-\varphi_{\bar j}^k\partial_k\varphi_{\bar l}^i\right)\\
&= \varphi_{\bar l}^k\partial_i\partial_k\varphi_{\bar j}^i-\varphi_{\bar j}^k\partial_i\partial_k\varphi_{\bar l}^i.
\end{aligned}
$$

Comparing the two expressions in formulas (4.10) and (4.11), we finally obtain the identity (4.9). $\square$

PROPOSITION 4.9. *Suppose that $(\mathcal{L}_M,h_\omega)$ is a positive line bundle over a compact Kähler manifold (M,ω) with $\sqrt{-1}\Theta=\rho\omega$ for a constant $\rho>0$, let $\varphi\in A^{0,1}(M,T^{1,0}M)$ be an integrable Beltrami differential which satisfies the conditions that $div\,\varphi=0$ and L_∞-norm $\|\varphi\|_\infty<1$. Then for any* *holomorphic* $\sigma_0 \in A^{n,0}(M, \mathcal{L}_M)$*, a solution of the following integral equation*

$$\sigma - \sigma_0 = -\overline{\partial}^* G\nabla'(\varphi \lrcorner \sigma) \tag{4.12}$$

is a solution of the equation

$$\overline{\partial}\sigma = -\nabla'(\varphi \lrcorner \sigma). \tag{4.13}$$

Proof. Suppose that $\sigma \in A^{n,0}(M, \mathcal{L}_M)$ satisfies the equation (4.12). First, by using the positivity condition for $\mathcal{L}_M$, we have

$$\begin{aligned}
\overline{\partial}\sigma &= -\overline{\partial}\overline{\partial}^* G\nabla'(\varphi \lrcorner \sigma) \\
&= (\overline{\partial}^*\overline{\partial} - \Box_{\overline{\partial}})G\nabla'(\varphi \lrcorner \sigma) \\
&= (\overline{\partial}^*\overline{\partial}G - I + \mathbb{H})\nabla'(\varphi \lrcorner \sigma) \\
&= -\nabla'(\varphi \lrcorner \sigma) + \overline{\partial}^*\overline{\partial}G\nabla'(\varphi \lrcorner \sigma).
\end{aligned}$$

Let $\Phi = \overline{\partial}\sigma + \nabla'(\varphi \lrcorner \sigma)$, then under the condition $div\, \varphi = 0$, we obtain

$$\overline{\partial}\nabla'(\varphi \lrcorner \sigma) = -\nabla'(\varphi \lrcorner \Phi)$$

by Lemma 4.8. Therefore

$$\begin{aligned}
\Phi &= \overline{\partial}\sigma + \nabla'(\varphi \lrcorner \sigma) \\
&= \overline{\partial}^*\overline{\partial}G\nabla'(\varphi \lrcorner \sigma) \\
&= \overline{\partial}^* G\overline{\partial}\nabla'(\varphi \lrcorner \sigma) \\
&= -\overline{\partial}^* G\nabla'\left(\varphi \lrcorner \Phi\right).
\end{aligned}$$

By quasi-isometry formula (4.6) and the condition $||\varphi||_\infty < 1$, we have

$$||\Phi||^2 \leq ||\varphi \lrcorner \Phi||^2 \leq ||\varphi||_\infty ||\Phi||^2 < ||\Phi||^2$$

and we get the contradiction $\|\Phi\|^2 < \|\Phi\|^2$ unless $\Phi = 0$. Hence,

$$\overline{\partial}\sigma = -\nabla'(\varphi \lrcorner \sigma).$$

□

Remark 4.10. When $(\mathcal{L}_M, h_\omega)$ is semi-positive, we can obtain the same conclusion as in Proposition 4.9, if we substitute the global condition $\|\varphi\|_\infty < 1$ by requiring that φ (with the Hölder norm as in [**22**]) is small enough. We leave further discussion of the variation equation (4.5) to another paper.

By Corollary 4.6, it is easy to see that the integral equation

$$\sigma - \sigma_0 = -\overline{\partial}^* G\nabla'(\varphi \lrcorner \sigma) = T^{\nabla'}\varphi \lrcorner \sigma$$

has a unique solution given by

$$\sigma = (I + T^{\nabla'}\varphi)^{-1}\sigma_0.$$

In conclusion, we obtain

THEOREM 4.11. *Suppose that* $(\mathcal{L}_M, h_\omega)$ *is a positive line bundle over a compact Kähler manifold* (M,ω) *with curvature* $\sqrt{-1}\Theta = \rho\omega$ *for a constant* $\rho > 0$, *let* $\varphi \in A^{0,1}(M, T^{1,0}M)$ *be an integrable Beltrami differential satisfying the two conditions* $div\,\varphi = 0$ *and* L_∞*-norm* $\|\varphi\|_\infty < 1$. *Then, given any holomorphic pluricanonical form* $\sigma_0 \in A^{n,0}(M, \mathcal{L}_M)$,

$$\sigma(\varphi) = \rho_\varphi((I + T^{\nabla'}\varphi)^{-1}\sigma_0)$$

gives a holomorphic pluricanonical form in $A^{n,0}(M_\varphi, \mathcal{L}_{M_\varphi})$.

Theorem 4.11 gives a closed formula for the variation of pluricanonical forms. Note that the above construction is global in the sense that it does not depend on the local deformation theory of Kodaira-Spencer and Kuranishi [**22**]. Some application of Theorem 4.11 will be discussed in the following section.

4.4. Applications to Kähler-Einstein manifold of general type. The invariance of plurigenera for Kähler-Einstein manifold of general type has already been known, see for example [**27**]. Here we derive an explicit and closed formula as a direct application of Theorem 4.11.

Let (M,ω) be a Kähler manifold. Denote the associated Kähler form by $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. The corresponding Chern-Ricci form $Ric(\omega)$ is given by

$$Ric(\omega) = \frac{\sqrt{-1}}{2}\bar{\partial}\partial \log(\det g).$$

DEFINITION 4.12. We say (M,ω) is a Kähler-Einstein manifold of general type if $Ric(\omega) = -\omega$.

In the following discussion, we assume that (M,ω) is a Kähler-Einstein manifold of general type.

PROPOSITION 4.13 (cf. Theorem 1.1 in [**27**]). *Let* $\varphi \in A^{0,1}(M, T^{1,0}M)$ *be an integrable Beltrami differential, then* $\bar{\partial}^*\varphi = 0$ *if and only if* $div\,\varphi = 0$.

Now we assume that

$$\pi : \mathcal{M} \to B_\epsilon$$

is a Kuranishi family of Kähler-Einstein manifolds of general type. Let t be the holomorphic coordinate on B_ϵ. For $t \in B_\epsilon$, we let $M_t = \pi^{-1}(t)$ be the fiber with the complex structure induced by the integrable Beltrami differential $\varphi(t) \in A^{0,1}(M_0, T^{1,0}M_0)$, which satisfies

$$\begin{cases} \bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)] \\ \bar{\partial}^*\varphi(t) = 0, \end{cases}$$

where $\partial, \bar{\partial}^*$ are the operators on M_0 and $\bar{\partial}^*$ is defined with respect to the Kähler-Einstein metric g_0. We can choose ϵ small enough, such that $\|\varphi(t)\|_\infty < 1$.

Let $\mathcal{L}_{M_0} = K_{M_0}^{\otimes(m-1)}$ be the holomorphic line bundle over M_0, and the corresponding Hermitian metric be given by $h_0 = (\det g_0)^{-(m-1)}$. Let $\nabla = \nabla' + \overline{\partial}$ be the Chern connection of $(\mathcal{L}_{M_0}, h_0)$. We have $\sqrt{-1}\Theta = -2(m-1)Ric(\omega_0)$ by formula (4.7). Recall the operator $T^{\nabla'} = \overline{\partial}^* G\nabla'$ we introduced, we have

Corollary 4.14. *Given any holomorphic pluricanonical form $\sigma_0 \in A^{n,0}(M_0, \mathcal{L}_{M_0})$, we have that*

$$\sigma(t) = \rho_t((I + T^{\nabla'}\varphi(t))^{-1}\sigma_0)$$

is a holomorphic pluricanonical form in $A^{n,0}(M_t, \mathcal{L}_{M_t})$ with $\sigma(0) = \sigma_0$, where $\rho_t = \rho_{\varphi(t)}$.

Proof. Since (M_0, ω_0) is Kähler-Einstein of general type, i.e. $Ric(\omega_0) = -\omega_0$. Hence $\sqrt{-1}\Theta = 2(m-1)\omega_0$. Thus $(\mathcal{L}_{M_0}, h_0)$ is a line bundle which satisfies the conditions in Theorem 4.11. Then Corollary 4.14 followed by Theorem 4.11 and Proposition 4.13. □

By using Corollary 4.14, one can write down the curvature formula of the induced L^2 metric on the generalized Hodge bundle over the base B_ϵ with fiber $H^0(M_t, K_{M_t}^{\otimes m})$ as shown in [**27**].

4.5. Deformation cohomology. Let us fix the same notations as in Section 4.1, and let (M, ω) be a compact Kähler manifold of dimension n. For a positive integer $m \geq 1$, we let $\mathcal{L}_M := K_M^{\otimes(m-1)}$ and $\nabla := \nabla' + \overline{\partial}$ be the Chern connection of $\mathcal{L}_M$ with the induced Hermitian metric. In particular, when $m = 1$, $\nabla = d$ is the ordinary de Rham differential operator.

Motivated by the variation equation (4.5) for varying a pluricanonical section under the deformation of complex structure. We introduce the deformed differential operator D_φ as follows:

Definition 4.15. For any $\sigma \in A^{n,*}(M, \mathcal{L}_M)$, we define

$$D_\varphi \sigma = \overline{\partial}\sigma + \nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma.$$

According to the proof of Lemma 4.8, we have $D_\varphi^2 = 0$ on $\sigma \in A^{n,*}(M, \mathcal{L}_M)$. Therefore, we obtain the following D_φ-complex on M

$$0 \to A^{n,0}(M, \mathcal{L}_M) \xrightarrow{D_\varphi} A^{n,1}(M, \mathcal{L}_M) \xrightarrow{D_\varphi} \cdots \xrightarrow{D_\varphi} A^{n,n}(M, \mathcal{L}_M) \to 0,$$

which induces a new cohomology on M.

Definition 4.16. We define the p-th deformation cohomology on M as follows

$$H^{n,p}_{D_\varphi}(M, \mathcal{L}_M) := \frac{Ker(D_\varphi : A^{n,p}(M, \mathcal{L}_M) \to A^{n,p+1}(M, \mathcal{L}_M))}{Im(D_\varphi : A^{n,p-1}(M, \mathcal{L}_M) \to A^{n,p}(M, \mathcal{L}_M)}.$$

Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential. On the complex manifold M_φ, we have the corresponding $\overline{\partial}_\varphi$-complex

$$0 \to A^{n,0}(M_\varphi, \mathcal{L}_{M_\varphi}) \xrightarrow{\overline{\partial}_\varphi} A^{n,1}(M_\varphi, \mathcal{L}_{M_\varphi}) \xrightarrow{\overline{\partial}_\varphi} \cdots \xrightarrow{\overline{\partial}_\varphi} A^{n,n}(M_\varphi, \mathcal{L}_{M_\varphi}) \to 0$$

which gives the Dolbeault cohomology $H^{n,p}_{\overline{\partial}_\varphi}(M_\varphi, \mathcal{L}_{M_\varphi})$ on M_φ.

By the definition of the map ρ_φ given in (4.1) and the variation equation (4.5), we have

Lemma 4.17. *The map ρ_φ gives an isomorphism*

$$H^{n,0}_{D_\varphi}(M, \mathcal{L}_M) \xrightarrow{\simeq} H^{n,0}_{\overline{\partial}_\varphi}(M_\varphi, \mathcal{L}_{M_\varphi}).$$

Now, we can reformulate Siu's conjecture [**26**] of invariance of plurigenera as follows:

Conjecture 4.18 (Invariance of plurigenera). Let M be a compact Kähler manifold, and $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential comes from the local Kuranishi family of M, then there exists an isomorphism

$$H^{n,0}_{D_\varphi}(M, \mathcal{L}_M) \xrightarrow{\simeq} H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M).$$

In order to prove Conjecture 4.18, we need to find an isomorphic map from $H^{n,0}_{D_\varphi}(M, \mathcal{L}_M)$ to $H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M)$.

Let

$$\mathbb{H} : A^{n,*}(M, \mathcal{L}_M) \to \mathbb{H}^{n,*}(M, \mathcal{L}_M)$$

be the harmonic projection map. By using Hodge theory, we have

Lemma 4.19. *Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential coming from a local Kuranishi family, then $\mathbb{H}$ gives an injective map*

$$\mathbb{H} : H^{n,0}_{D_\varphi}(M, \mathcal{L}_M) \to H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M).$$

Proof. Given any $\sigma \in H^{n,0}_{D_\varphi}(M, \mathcal{L}_M)$, then

$$D_\varphi \sigma = \overline{\partial}\sigma + \nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma = 0. \tag{4.14}$$

It is clear that $\mathbb{H}\sigma \in H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M)$. In order to show that $\mathbb{H}$ is injective, we only need to show that if $\mathbb{H}\sigma = 0$, then $\sigma = 0$.

We assume $\mathbb{H}\sigma = 0$ in the following. Applying the operator $\overline{\partial}^* G$ to formula (4.14), we obtain

$$\overline{\partial}^* G \overline{\partial}\sigma = -\overline{\partial}^* G(\nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma).$$

Since

$$\overline{\partial}^* G \overline{\partial}\sigma = \overline{\partial}^* \overline{\partial} G\sigma = (\Box_{\overline{\partial}} - \overline{\partial}\overline{\partial}^*) G\sigma = \Box_{\overline{\partial}} G\sigma - \overline{\partial} G \overline{\partial}^* \sigma = \sigma - \mathbb{H}\sigma = \sigma,$$

We get

$$\sigma = -\overline{\partial}^* G(\nabla'(\varphi \lrcorner \sigma) - (m-1)\, div\, \varphi \wedge \sigma).$$

Consider the Hölder norm $\|\cdot\|_k$ as in [**22**], by the standard estimates of the operator $\overline{\partial}^* G\nabla'$, there is a constant C_k, such that

$$\|\sigma\|_k \leq C_k \|\varphi\|_k \|\sigma\|_k.$$

Therefore, if $\|\varphi\|_k < \frac{1}{C_k}$, we must have $\sigma = 0$. $\square$

Therefore, the proof of Conjecture 4.18 is reduced to find an injective map from $H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M)$ to $H^{n,0}_{D_\varphi}(M, \mathcal{L}_M)$.

Lemma 4.19 is clearly stronger than the semi-continuity relation

$$\dim H^{n,0}_{\overline{\partial}_\varphi}(M_\varphi, \mathcal{L}_{M_\varphi}) = \dim H^{n,0}_{D_\varphi}(M, \mathcal{L}_M) \leq \dim H^{n,0}_{\overline{\partial}}(M, \mathcal{L}_M)$$

for small φ. We refer to [**33**] for a more general discussion of the deformation cohomologies and applications.

Example 4.20. If $m = 1$, for any $\sigma_0 \in H^{n,0}_{\overline{\partial}}(M)$, we construct the map

$$\Phi(\sigma_0) = (I + T\varphi)^{-1}(\sigma_0).$$

Then Proposition 5.2 shows that $\Phi(\sigma_0) \in H^{n,0}_{D_\varphi}(M)$. It is clear that Φ is an injective map form $H^{n,0}_{\overline{\partial}}(M)$ to $H^{n,0}_{D_\varphi}(M)$.

5. Variations of holomorphic canonical forms

In this section, we consider the special case $m = 1$. Namely, we study the variations of holomorphic canonical forms over compact complex manifold. We emphasize that the results in this section do not need the Kähler condition.

Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential. Recall the definition of the map

$$\rho_\varphi : A^0(M, K_M) \to A^0(M_\varphi, K_{M_\varphi}) \tag{5.1}$$

in (4.1). Then $\rho_\varphi(\Omega) = f(z)(dz^1 + \varphi(dz^1)) \wedge \cdots \wedge (dz^n + \varphi(dz^n))$ if we write $\Omega = f(z)dz^1 \wedge \cdots \wedge dz^n$ in local coordinate (U, z).

5.1. Variation equations for holomorphic canonical forms.

Proposition 5.1. *Given an integrable Beltrami differential $\varphi \in A^{0,1}(M, T^{1,0}M)$, for any $(n,0)$-form Ω on M, the corresponding $(n,0)$-form $\rho_\varphi(\Omega)$ on M_φ is holomorphic, if and only if*

$$\overline{\partial}\Omega = -\partial(\varphi \lrcorner \Omega). \tag{5.2}$$

Proof. By Lemma 4.1, for $\Omega = f(z)dz^1 \wedge \cdots \wedge dz^n \in A^{n,0}(M, K_M)$, then $\rho_\varphi(\Omega)$ is holomorphic in $A^{n,0}(M_\varphi, K_{M_\varphi})$ if and only if for $j = 1, ..., n$,

$$\overline{\partial}_j f = \varphi^i_{\bar{j}} \partial_i f + f \partial_i \varphi^i_{\bar{j}} = \partial_i (f \varphi^i_{\bar{j}})$$

which is equivalent to equation (5.2) obviously. $\square$

One can also derive the variation equation (5.2) by using more direct method, see [**18, 21**]. The solution of the equation (5.2) can be used to construct the variation of a holomorphic canonical form from M to M_φ. See [**21**] for more details.

5.2. Closed formulas for variations of canonical forms. From the discussions in Section 5.1, we know that, given an integrable Beltrami differential φ on M, in order to find an $(n,0)$-form Ω on M such that the corresponding $(n,0)$-form $\rho_\varphi(\Omega) = e^{i_\varphi}\Omega$ is holomorphic on M_φ, we only need to find an $(n,0)$-form Ω on M such that Ω satisfies the variation equation

$$\overline{\partial}\Omega = -\partial(\varphi \lrcorner \Omega). \tag{5.3}$$

In [**21**], we will show that the equation (5.3) can be solved by using the Hodge theory on M and the quasi-isometric formula (2.1) reviewed in Section 2. The method is the same as discussed in previous section, and we have

PROPOSITION 5.2. *Let φ be an integrable Beltrami differential of M with L_∞-norm $||\varphi||_\infty < 1$. Given a holomorphic $(n,0)$-form Ω_0 on M, if Ω is a solution of the equation*

$$\Omega = \Omega_0 - \overline{\partial}^* G\partial\,(\varphi \lrcorner \Omega) = \Omega_0 - T\varphi\Omega, \tag{5.4}$$

then Ω is the solution of the equation (5.3).

Conversely, we have

PROPOSITION 5.3. *If the $(n,0)$-form Ω satisfies the equation (5.3), then there exists a unique holomorphic $(n,0)$-form Ω_0, such that Ω satisfies the equation (5.4).*

Furthermore, it is easy to show that the equation (5.4) has a unique solution. Indeed, if we assume that the equation (5.4) has two different solutions Ω and Ω', i.e. $\Omega - \Omega' \neq 0$. Then

$$\Omega - \Omega' = -T\varphi(\Omega - \Omega').$$

By quasi-isometry (2.1), we have

$$\|\Omega - \Omega'\| = \|T\varphi(\Omega - \Omega')\| \leq \|\varphi(\Omega - \Omega')\| \leq \|\varphi\|_\infty \|\Omega - \Omega'\| < \|\Omega - \Omega'\|,$$

which contradicts to $\Omega - \Omega' \neq 0$.

Moreover, this unique solution of the equation (5.4) is given by

$$\Omega = (I + T\varphi)^{-1}\Omega_0,$$

which is a smooth $(n,0)$-form, since Ω_0 is holomorphic.

In conclusion, we have

THEOREM 5.4. *Given any integrable Beltrami differential φ with $\|\varphi\|_\infty < 1$, and any holomorphic $(n,0)$-form Ω_0 on M, we have that*

$$\Omega(\varphi) = \rho_\varphi((I + T\varphi)^{-1}\Omega_0)$$

is a holomorphic $(n,0)$-form on M_φ with $\Omega(0) = \Omega_0$.

Applying Theorem 5.4 to the integrable Beltrami differential $\varphi(t)$ from the local Kodaira-Spencer-Kuranishi deformation theory, one can choose t small enough such that $\|\varphi(t)\|_\infty < 1$, we immediately obtain

Corollary 5.5. *For any holomorphic $(n,0)$-form $\Omega_0 \in A^{n,0}(M)$, and the Beltrami differential $\varphi = \varphi(t)$ with $|t| < \epsilon$ small, there is a holomorphic $(n,0)$-form $\Omega(t)$ on M_t,*

$$\Omega(t) = \rho_t((I + T\varphi)^{-1}\Omega_0),$$

where $\rho_t = \rho_{\varphi(t)}$, with $\Omega(0) = \Omega_0$.

6. Applications to Calabi-Yau and hyperkähler manifolds

In this section, we apply the above closed formula in Theorem 5.4 to construct the canonical families of holomorphic forms over Calabi-Yau and hyperkähler manifolds, and give several geometric applications. As application we give a characterizition when the Teichmüller space of Calabi-Yau manifold with Weil-Petersson metric is locally symmetric under the Weil-Petersson metric. In Section 6.3, we construct the canonical families over hyperkähler manifolds, as an application, we deduce that the Teichmüller space of hyperkähler manifolds is locally Hermitian symmetric with the Weil-Petersson metric.

6.1. Canonical family of classes on Calabi-Yau manifolds. Fixing a nowhere vanishing holomorphic $(n,0)$-form Ω_0 on M, we consider the Kuranishi family $\{M_t\}_{t\in\Delta_\epsilon}$ constructed by Theorem 3.8. Then $T\varphi(t)\Omega_0 = \overline{\partial}^* G\partial(\varphi(t)\lrcorner\Omega_0) = 0$. As a direct application of the Theorem 5.4, we obtain that the holomorphic canonical form on M_t is given by

$$\rho_{\varphi(t)}\left((I + T\varphi(t))^{-1}\,\Omega_0\right) = \rho_{\varphi(t)}(\Omega_0). \tag{6.1}$$

Let $\Omega^c(t) := \rho_{\varphi(t)}(\Omega_0)$ denote this holomorphic family of canonical forms on M_t for $t \in \Delta_\epsilon$, we have the following result which was first observed in a joint project of the first author with X. Sun, A. Todorov and S.-T. Yau by using the Kodaira-Spencer-Kuranishi local deformation theory.

Theorem 6.1. *The cohomology class $[\Omega^c(t)]$ has the following expansion:*

$$[\Omega^c(t)] = [\Omega_0] + \sum_{i=1}^{N}[\eta_i\lrcorner\Omega_0]t_i + \frac{1}{2}\sum_{i,j=1}^{N}[\mathbb{H}(\eta_i\lrcorner\eta_j\lrcorner\Omega_0)]t_it_j + O(|t|^3) \tag{6.2}$$

where $O(|t|^3)$ denotes the terms in $\bigoplus_{j=2}^{n} H^{n-j,j}(M)$ of orders at least 3 in t.

Proof. Let us consider the Taylor expansion of $\varphi(t)$, by Hodge theory, we have

$$[\Omega^c(t)] = [\Omega_0] + \sum_{i=1}^{N}[\mathbb{H}(\eta_i\lrcorner\Omega_0)]t_i + \sum_{|I|\geq 2}[\mathbb{H}(\varphi_I\lrcorner\Omega_0)]t_I + \sum_{k\geq 2}\frac{1}{k!}\left[\mathbb{H}(\varphi(t)^k\lrcorner\Omega_0)\right].$$

Since $\eta_i \lrcorner \Omega_0$ is harmonic and that $\varphi_I \lrcorner \Omega_0$ is ∂-exact for $|I| \geq 2$, we have

$$[\Omega^c(t)] = [\Omega_0] + \sum_{i=1}^{N} [\eta_i \lrcorner \Omega_0] t_i + \sum_{k \geq 2} \frac{1}{k!} \left[\mathbb{H}(\varphi(t)^k \lrcorner \Omega_0) \right].$$

It is obvious that the degree two terms in $\sum_{k\geq 2} \frac{1}{k!} \left[\mathbb{H}(\varphi(t)^k \lrcorner \Omega_0) \right]$ is given by

$$\frac{1}{2} \sum_{i,j=1}^{N} [\mathbb{H}(\eta_i \lrcorner \eta_j \lrcorner \Omega_0)] t_i t_j.$$

□

6.2. Weil-Petersson geometry of the Teichmüller space of Calabi-Yau. Let (M, L) be a polarized Calabi-Yau manifold. Recall that a basis of the quotient space $(H_n(M, \mathbb{Z})/Tor)/m(H_n(M, \mathbb{Z})/Tor)$ is called a level m structure on the polarized Calabi-Yau manifold with $m \geq 3$. It is a well-known fact that there is a quasi-projective space $\mathcal{Z}_m$ parameterizing the polarized Calabi-Yau manifold with level m structure. We define the Teichmüller space $\mathcal{T}$ to be the universal cover of the base space $\mathcal{Z}_m$. One can look at [**28, 19**] for more details about the construction of the Teichmüller space $\mathcal{T}$.

Proposition 6.2. *The Teichmüller space $\mathcal{T}$ is a simply connected smooth complex manifold, and there is a versal family $\mathcal{U} \to \mathcal{T}$ containing M as a fiber, is local Kuranishi at each point of the Teichmüller space $\mathcal{T}$.*

Let $p \in \mathcal{T}$, we denote the corresponding polarized Calabi-Yau manifold by (M, L). Then the holomorphic tangent space of $\mathcal{T}$ at p is given by

$$\mathbb{H}^1_L(M, T^{1,0}M) = \{\varphi \in \mathbb{H}^1(M, T^{1,0}M), [\varphi \lrcorner \omega] = 0\},$$

where ω is any Kähler form in the polarization L. Clearly, we have

$$\mathbb{H}^1_L(M, T^{1,0}M) = \mathbb{H}^1(M, T^{1,0}M)$$

by the condition $H^2(M, \mathcal{O}_M) = 0$ in the definition of Calabi-Yau manifold.

By Theorem 3.8, there is a normal coordinate t in the neighborhood U of p, such that $t(p) = 0$. The Weil-Petersson metric of the Teimüller space $\mathcal{T}$ in local coordinate (U, t) is given by

$$g^{WP}_{i\bar{j}} = -\frac{\partial^2}{\partial t_i \partial \bar{t}_j} \log \widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)}),$$

where $\widetilde{Q}(\cdot, \cdot) = (\sqrt{-1})^n Q(\cdot, \cdot)$ and Q is the Poincaré bilinear form

$$Q(\sigma, \tau) = (-1)^{\frac{n(n-1)}{2}} \int_M \sigma \wedge \tau.$$

for any d-closed forms σ, τ on M.

By using the definition of $\Omega^c(t)$ and the expansion formula (6.2), one can show that

$$g^{WP}_{i\bar{j}} = \delta_{ij} + \delta_{ij}\sum_{k=1}^{N} t_k\bar{t}_k + t_i\bar{t}_j - \sum_{r,s=1}^{N} q_{ir,\bar{j}\bar{s}} t_r\bar{t}_s + O(|t|^3), \tag{6.3}$$

where $q_{ir,\bar{j}\bar{s}} = \widetilde{Q}(\mathbb{H}(\eta_i \lrcorner \eta_r \lrcorner \Omega_0), \overline{\mathbb{H}(\eta_j \lrcorner \eta_s \lrcorner \Omega_0)})$. Formula (6.3) immediately implies that the Weil-Petersson metric is Kähler and the coordinate t is a normal coordinate at $t = 0$.

Therefore, the Christoffel symbols at point $t = 0$ is zero, i.e. $\Gamma^k_{ij}(0) = 0$. So the full curvature tensor at $t = 0$ is given by

$$R^{WP}_{i\bar{j}k\bar{l}}(0) = \frac{\partial^2 g^{WP}_{k\bar{l}}}{\partial t_i \partial \bar{t}_j}(0) = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - q_{ik,\bar{j}\bar{l}}. \tag{6.4}$$

Note that the above expression of the Weil-Petersson curvature formula (6.4) first appeared in [**31**].

Let ∇ be the Levi-Civita connection associated to the underlying Riemannian metric g, and J be the complex structure of M.

Definition 6.3. An Hermitian manifold (M, g) is called locally Hermitian symmetric if

$$\nabla R = 0 = \nabla J.$$

If the metric is complete, then (M, g) is an Hermitian or locally Hermitian symmetric space.

Proposition 6.4. *Let $\mathcal{T}$ be the Teichmüller space of polarized and marked Calabi-Yau manifolds and $\Omega^c(t)$ the canonical form given by (6.1). If the Weil-Petersson potential $\widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)})$ is a polynomial in terms of the normal coordinate t, then $\mathcal{T}$ is locally Hermitian symmetric with respect to the Weil-Petersson metric.*

Proof. By the theorem of Nomizu and Ozeki [**24**], if $\nabla^k R = 0$ for some positive integer k, then $\nabla R = 0$. Therefore, if g is a Kähler metric on M, in order to prove (M, g) is locally Hermitian symmetric, we only need to show that $\nabla^k R = 0$ for some positive integer k.

If the Weil-Petersson potential $\widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)})$ only has finite terms, i.e. it is a polynomial of the normal coordinate $t = (t_1, .., t_N)$, then the coefficients of the Weil-Petersson metric and its curvature tensor

$$g^{WP}_{k\bar{l}} = -\frac{\partial^2}{\partial t_k \partial \bar{t}_l} \log \widetilde{Q}(\Omega^c(t), \overline{\Omega^c(t)})$$
$$R^{WP}_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial t_k \overline{\partial} t_l} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial t_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{t}_l}$$

is a polynomial in the variables $t = (t_1, .., t_N, \bar{t}_1, ..., \bar{t}_N)$. On the other hand, from formula (6.3), the coordinate t is a normal coordinate at the point $t = 0$. So we have that the Christoffel symbols at the point $t = 0$ vanish, i.e. $\Gamma_{ij}^k(0) = 0$. Thus at the point $t = 0$, the covariant derivative $\nabla_j T = \partial_j T$ for any $(0, m)$-tensor T. Therefore, for a large enough integer k, we have $\nabla^m R(0) = 0$. Thus the Teichmüller space $\mathcal{T}$ is locally Hermitian symmetric with respect to the Weil-Petersson metric. $\square$

If the Weil-Petersson metric is complete, then as a consequence, we know that the Teichmüller space $\mathcal{T}$ is an Hermitian symmetric space under the assumption in the above theorem.

6.3. Canonical families on hyperkähler manifolds. First let us recall the definition of hyperkähler manifold. Let M be a compact and simply-connected Kähler manifold of complex dimension $2n \geq 4$, if there exists a non-zero holomorphic nondegenerate $(2,0)$-form $\Omega^{2,0}$ on M, unique up to a constant such that $\det(\Omega^{2,0}) \neq 0$ at each point $x \in M$ and $H^1(M, \mathcal{O}_M) = 0$, then M is called a hyperkähler manifold.

By its definition, the $(2n, 0)$-form $\wedge^n \Omega^{2,0}$ gives a nonzero holomorphic section of canonical line bundle K_M, hence K_M is trivial. Similarly, we also denote by $\mathcal{T}$ the Teichmüller space of the polarized hyperkähler manifolds.

First we review the following well-known result,

Proposition 6.5 (Bochner's principle). *On a compact Kähler Ricci-flat manifold, any holomorphic tensor field (covariant or contravariant) is parallel.*

The proof rests on the following formula, which follows from a straightforward computation [4]: if τ is any holomorphic tensor field,

$$\Delta(\|\tau\|^2) = \|\nabla\tau\|^2. \tag{6.5}$$

From this it follows immediately that τ is parallel.

For $\Omega^{2,0}$, by using the following formulas

$$\begin{aligned}(\overline{\partial}\psi)_{A_p,\overline{\alpha B}_q} &= (-1)^p \sum_{\alpha} \nabla_{\overline{\alpha}} \psi_{A_p,\overline{B}_q}, \\ (\overline{\partial}^* \psi)_{A_p,\overline{B}_q} &= (-1)^{p+1} \sum_{\alpha,\beta} g^{\overline{\beta}\alpha} \nabla_{\alpha} \psi_{A_p,\overline{\beta B}_q}\end{aligned} \tag{6.6}$$

and their conjugates, which can be found in [22], we obtain

$$\overline{\partial}(\eta \lrcorner \Omega^{2,0}) = \overline{\partial}\eta \lrcorner \Omega^{2,0},\ \overline{\partial}^*(\eta \lrcorner \Omega^{2,0}) = \overline{\partial}^* \eta \lrcorner \Omega^{2,0}, \tag{6.7}$$

for $\eta \in A^{0,k}(M, T^{1,0}M)$. Therefore, the map ι given by

$$\iota(\varphi) = \varphi \lrcorner \Omega_0$$

is an isometry with respect to the Hermitian metrics on both spaces induced by g. Moreover, ι preserves the Hodge decomposition.

Let us assume that $\{\eta_i\}_{i=1}^N$ is an orthonormal basis $\mathbb{H}^1_L(M, T^{1,0}M)$ with respect to the Kähler Ricci-flat metric. Then we have the following result.

THEOREM 6.6. *Fix $p \in \mathcal{T}$, let (M, L) be the corresponding polarized hyperkähler manifold and $\Omega^{2,0}$ be a nonzero holomorphic nondegenerate $(2,0)$-form over M. Then in a neighborhood U of p, there exists a local canonical family of nondegenerate holomorphic $(2,0)$-forms $\Omega^{c;2,0}(t) = \rho_{\varphi(t)}(\Omega^{2,0})$ which defines a canonical family of $(2,0)$-classes*

$$[\Omega^{c;2,0}(t)] = [\Omega^{2,0}] + \sum_{i=1}^{N}[\eta_i \lrcorner \Omega^{2,0}]t_i + \frac{1}{2}\sum_{i,j=1}^{N}[\eta_i \lrcorner \eta_j \lrcorner \Omega^{2,0}]t_i t_j. \tag{6.8}$$

PROOF. By using the Beltrami differentials $\varphi(t)$ constructed in Theorem 3.8, we have

$$\begin{aligned}\rho_{\varphi(t)}(\Omega^{2,0}) &= \Omega^{2,0} + \sum_{i=1}^{N}\eta_i \lrcorner \Omega^{2,0} t_i + \frac{1}{2}(\eta_i \lrcorner \eta_j \lrcorner \Omega^{2,0} + \varphi_{(ij)} \lrcorner \Omega^{2,0})t_i t_j \\ &\quad + \sum_{|I|\geq 3}\left(\varphi_I \lrcorner \Omega^{2,0} + \frac{1}{2}\sum_{J+K=I}\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}\right)t_I\end{aligned}$$

In order to prove the expansion (6.8), we only need to show the following:

(a) $\eta_i \lrcorner \Omega^{2,0}$ is harmonic for $1 \leq i \leq N$;
(b) For any multi-index I with $|I| \geq 2$, $\varphi_I \lrcorner \Omega^{2,0}$ is ∂-exact, which implies that $\mathbb{H}(\varphi_I \lrcorner \Omega^{2,0}) = 0$;
(c) For any multiple-index J, K with $|J| \geq 2$, $\mathbb{H}(\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}) = 0$.

Indeed, (a) follows directly from the isomorphism (6.7) of two corresponding Hodge theories, and that η_i is harmonic. As to (b), since $\Omega^{2,0}$ is a nowhere vanishing holomorphic $(2,0)$-form, we can define $\Omega^{*2,0} \in \Gamma(M, \wedge^2 T^{1,0}M)$ by requiring $\Omega^{*2,0} \lrcorner \Omega^{2,0} = 1$ pointwise on M. Actually, in a coordinate chart $\{z_1, z_2, .., z_{2n}\}$, we can assume

$$\Omega^{2,0} = \sum_{i,j=1}^{2n} a_{ij} dz^i \wedge dz^j, \; \Omega^{*2,0} = \sum_{i,j=1}^{2n} b_{ij}\frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j}$$

with $a_{ij} = -a_{ji}$ and $b_{ij} = -b_{ji}$. Then, if we define matrices $A = (a_{ij})$ and $B = (b_{ij})$, then $\det(A) \neq 0$ and

$$\langle \Omega^{2,0}, \Omega^{*2,0} \rangle = \sum_{i,j}^{2n} a_{ij} b_{ij} = tr(AB^T) = 1.$$

Therefore, locally, the matrix B can be defined by $B = \frac{1}{2n}(A^{-1})^T$. It is easy to check that this definition is independent of the local coordinates and $\nabla \Omega^{*2,0} = 0$ by the Bochner's principle. Then, by Theorem 3.8, we have

$$\varphi_I \lrcorner \wedge^n \Omega^{2,0} = \partial \psi_I, \; |I| \geq 2,$$

which implies that

$$\varphi_I \lrcorner \Omega^{2,0} = \wedge^{n-1}\Omega^{*2,0} \lrcorner (\varphi_I \lrcorner \wedge^n \Omega^{2,0}) = \wedge^{n-1}\Omega^{*2,0} \lrcorner \partial \psi_I = \partial(\wedge^{n-1}\Omega^{*2,0} \lrcorner \psi_I).$$

As to (c), we first assume that $\mathbb{H}(\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}) = c \cdot \overline{\Omega^{2,0}}$ for some constant c, then we need to show $c = 0$.

From Hodge decomposition, we have

$$\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0} = c \cdot \overline{\Omega^{2,0}} + d\alpha_1 + d^*\alpha_2$$

for $\alpha_1 \in A^1(M)$ and $\alpha_2 \in A^3(M)$. By Bochner principle 6.5, formulae (6.6) and their conjugates, which can be found in [**22**], we obtain

$$\begin{aligned} d\alpha_1 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}} &= d(\alpha_1 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}) \\ d^*\alpha_2 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}} &= d^*(\alpha_2 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}) = 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} &(\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}} \\ &\quad = c \wedge^n \Omega^{2,0} \wedge \wedge^n\overline{\Omega^{2,0}} + d(\alpha_1 \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}}). \end{aligned} \tag{6.9}$$

On the other hand, since

$$\begin{aligned} 0 &= \varphi_J \lrcorner \left((\varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^n\Omega^{2,0}\right) \\ &= (\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^n\Omega^{2,0} + (\varphi_K \lrcorner \Omega^{2,0}) \wedge (\varphi_J \lrcorner \wedge^n \Omega^{2,0}) \end{aligned}$$

we have

$$\begin{aligned} &\int_M (\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}} \\ &= -\int_M (\varphi_J \lrcorner \wedge^n \Omega^{2,0}) \wedge (\varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^{n-1}\overline{\Omega^{2,0}} \\ &= -\int_M \partial\psi_J \wedge (\varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^{n-1}\overline{\Omega^{2,0}} \quad (\text{since } \varphi_K \lrcorner \Omega^{2,0} \text{ is } \partial\text{-exact}) \\ &= -\int_M \partial(\psi_J \wedge (\varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^{n-1}\overline{\Omega^{2,0}}) \\ &= -\int_M d(\psi_J \wedge (\varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^{n-1}\overline{\Omega^{2,0}}) \end{aligned}$$

where in the last "=", we have used that

$$\psi_J \wedge (\varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^{n-1}\overline{\Omega^{2,0}}$$

is a $(2n-1, 2n)$-form which is $\overline{\partial}$-closed.

Therefore, by using Stokes formula and formula (6.9), we obtain

$$0 = \int_M (\varphi_J \lrcorner \varphi_K \lrcorner \Omega^{2,0}) \wedge \wedge^n\Omega^{2,0} \wedge \wedge^{n-1}\overline{\Omega^{2,0}} = c \cdot \int_M \wedge^n\Omega^{2,0} \wedge \wedge^n\overline{\Omega^{2,0}}.$$

So we have $c = 0$. The proof is completed. □

Again assume that $\{\eta_i\}_{i=1}^N$ is an orthonormal basis $\mathbb{H}^1_L(M, T^{1,0}M)$ with respect to the Kähler Ricci-flat metric, and let $\Omega = \wedge^n\Omega^{2,0}$. The we deduce the following corollary.

Corollary 6.7. *Fix $p \in \mathcal{T}$, let (M, L) be the corresponding polarized hyperkähler manifold and $\Omega^{2,0}$ the nondegenerate holomorphic two form over M. then in a neighborhood U of p, then there exists a canonical family of holomorphic $(2n, 0)$-forms $\Omega^c(t) = \rho_{\varphi(t)}(\Omega)$ which defines a canonical family of $(2n, 0)$-classes*

$$[\Omega^c(t)] = [\Omega] + \sum_{i=1}^{N} [\eta_i \lrcorner \Omega] t_i + \cdots + \frac{1}{(2n)!} \sum_{i_1,..,i_{2n}=1}^{N} [\eta_{i_1} \lrcorner \cdots \lrcorner \eta_{i_{2n}} \lrcorner \Omega] t_{i_1} t_{i_2} \cdots t_{i_{2n}}.$$

Proof. First, we know that the harmonic projection on M, $\mathbb{H}(\rho_{\varphi(t)}(\Omega)) \in \mathbb{H}^{2n,0}(M_t)$ and $\mathbb{H}(\rho_{\varphi(t)}(\Omega^{2,0})) \in \mathbb{H}^{2,0}(M_t)$. Then, we have

$$\mathbb{H}(\wedge^n \mathbb{H}(\rho_{\varphi(t)}(\Omega^{2,0}))) \in \mathbb{H}^{2n,0}(M_t).$$

Since $\dim \mathbb{H}^{2n,0}(M_t) = 1$, there exists $\lambda \in \mathbb{C}$ such that

$$\mathbb{H}(\rho_{\varphi(t)}(\Omega)) = \lambda \mathbb{H}(\wedge^n \mathbb{H}(\rho_{\varphi(t)}(\Omega^{2,0}))).$$

On the other hand, by Theorem 6.6, we have

$$Pr_{\mathbb{H}^{2n,0}(M)}(\mathbb{H}(\rho_{\varphi(t)}(\Omega))) = Pr_{\mathbb{H}^{2n,0}(M)}(\mathbb{H}(\wedge^n \mathbb{H}(\rho_{\varphi(t)}(\Omega^{2,0}))) = \Omega.$$

Hence $\lambda = 1$, and we obtain

$$\begin{aligned}
[\Omega^c(t)] = [\rho_{\varphi(t)}(\Omega)] &= \left[\wedge^n \left(\Omega^{2,0} + \sum_{i=1}^{N} \eta_i \lrcorner \Omega^{2,0} t_i + \frac{1}{2} \sum_{i,j=1}^{2N} (\eta_i \lrcorner \eta_j \lrcorner \Omega^{2,0}) t_i t_j\right)\right] \\
&= [\Omega] + \sum_{i=1}^{N} [\eta_i \lrcorner \Omega] t_i + \cdots + \frac{1}{(2n)!} \sum_{i_1,..,i_{2n}=1}^{N} [\eta_{i_1} \lrcorner \cdots \lrcorner \eta_{i_{2n}} \lrcorner \Omega] t_{i_1} t_{i_2} \cdots t_{i_{2n}}
\end{aligned}$$

which is a polynomial in terms of the coordinate $t = (t_1, t_2, ..., t_N)$. $\square$

As a direct consequence of the Proposition 6.4, we obtain

Corollary 6.8. *The Teichmüller space $\mathcal{T}$ of polarized and marked hyperkähler manifold is locally Hermitian symmetric with the Weil-Petersson metric.*

7. Solving the Beltrami equations

In this section, we briefly review a global method given in [**21**] to solve the Beltrami equation by using the L^2-Hodge theory on complete manifolds.

7.1. L^2-Hodge theory on Poincaré disk. First, it is not hard to show that the L^2-Hodge theory [**13**] holds on the disk D with Poincarè metric g_P. So there exists a bounded operator G on $L_2^{p,q}(D, g_P)$, called the Green operator such that

$$\Box_{\overline{\partial}} G = G \Box_{\overline{\partial}} = I - \mathbb{H}, \mathbb{H}G = G\mathbb{H} = 0, \tag{7.1}$$

Moreover, the Poincaré metric is Kähler, we have the identity

$$\square_{\overline{\partial}} = \square_{\partial} = \frac{1}{2}\Delta_d. \tag{7.2}$$

We consider the operator

$$T = \overline{\partial}^* G \partial$$

in L^2-Hodge theory. Therefore we have the following quasi-isometry formula in L^2-Hodge theory. Its proof is completely the same as in the case for compact Kähler manifold as given in Section 2.

PROPOSITION 7.1. *For $g \in Dom(\partial) \subset L_2^{p,q}(D, g_P)$, we have that*

$$\|Tg\|^2 \leq \|g\|^2.$$

Proposition 7.1 tells us that T is an operator of norm less than or equal to 1.

7.2. Beltrami equations. Beltrami equations are very important in the development of complex analysis and moduli theory of Riemann surfaces. It also has many important applications in other subjects. There is a huge literature on the topic. See, for examples [**1**], [**3**] and [**9**]. In particular the construction by Ahlfors in [**1**] depends on rather deep analysis and estimate of Calderón-Zygmund. The method of [**3**] is by using local integral operators and their regularity theory. Our method is global in the sense that we use L^2-Hodge theory.

Given a measurable function μ_0 on the unit disc $D \subset \mathbb{C}$, suppose $\sup|\mu_0| < 1$, let $\mu = \mu_0 \frac{\partial}{\partial z} \otimes d\bar{z}$ be a Beltrami differential on D with coordinate z. Recall that solving the Beltrami equation is equivalent to finding a function f on the unit disc D, such that

$$\overline{\partial} f = \mu \partial f.$$

Our observation is that the Beltrami equation can be solved by using the L^2-Hodge theory and quasi-isometry Proposition 7.1 with the same method as in Section 5.

Note that the L_∞-norm of μ is independent of the Hermitian metric on D and is equal to $\sup|\mu_0|$, i.e. $\|\mu\|_\infty < 1$. Similarly, we show that for a holomorphic one form h_0 on D, the equation

$$\overline{\partial} h = -\partial \mu h$$

has a solution

$$h = (I + T\mu)^{-1} h_0. \tag{7.3}$$

As a corollary we can directly get a solution of the Beltrami equation for any measurable μ_0. In particular, we have, THEOREM 7.2. *Assume that* $||\mu||_\infty = \sup|\mu_0| < 1$*, if* μ_0 *is of regularity* C^k*, then the Beltrami equation*

$$\overline{\partial} f = \mu \partial f$$

has a solution of regularity C^{k+1}.

PROOF. First note that the solution h is a $(1,0)$-form of regularity C^k on D. Recall the definition of the map ρ in formula (5.1), it follows that

$$d(\rho_\mu(h)) = \overline{\partial} h + \partial \mu h = 0.$$

According to Poincaré lemma, there is a function f of regularity C^{k+1} on D, such that

$$\rho_\mu(h) = df = \overline{\partial} f + \partial f.$$

Since

$$\rho_\mu(h) = h + \mu h,$$

by considering the types, we obtain

$$h = \partial f \text{ and } \mu h = \overline{\partial} f.$$

Therefore

$$\overline{\partial} f = \mu \partial f.$$

□

8. Conclusions and generalizations

Our method have several generalizations and interesting geometric applications. First, the method used in Section 3 can be applied to treat the deformation theory of many other structures, such as pseudogroup structures [**15**], holomorphic vector bundles [**7**], and general differential graded Lie algebra [**8**] etc.

In [**21**], by using the construction briefly reviewed in Section 5, we provide a closed formula for certain canonical sections of Hodge bundles on marked and polarized moduli spaces of projective manifolds. Especially, for the case of Teichmüller space of Riemann surface, this gives a very clean formula which should have applications in geometry of moduli space of Riemann surfaces. Furthermore, although we only consider canonical form i.e. holomorphic $(n,0)$-forms in this paper (cf. Section 5), our method also works for a general (p,q)-form σ_0 with $d\sigma_0 = 0$. In [**32**], we construct the variation formula for d-closed (p,q)-forms on compact complex manifolds with mild condition which simplifies the approach in [**25**].

Finally, an interesting problem is to prove the invariance of plurigenera for compact Kähler manifolds [**26**] by using the method in Section 4.

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