The geometric Airy curve flow on $\mathbb{R}^n$

Chuu-Lian Terng

Dedicated to Professor S. T. Yau on his 70th Birthday

1. Introduction

The following third order curve flow in $\mathbb{R}^n$ was introduced in [**20**],

$$\gamma_t = -\nabla^\perp_{e_1} H(\gamma(\cdot, t)), \tag{1.1}$$

where $e_1 = \gamma_x/||\gamma_x||$, $\nabla^\perp$ is the induced normal connection and $H(\gamma(\cdot,t))$ is the mean curvature vector field of the curve $\gamma(\cdot,t)$. It is easy to see that

$$\gamma_t = -(\frac{1}{2}||H||^2 e_1 + \nabla^\perp_{e_1} H(\gamma(\cdot,t))) \tag{1.2}$$

is a curve flow in $\mathbb{R}^n$ that preserves arc-length parameter. Note that (1.2) and (1.1) only differ by a tangent vector field, so they are essentially the same flows. In fact, the curve at time t of (1.2) is the same curve at time t of (1.1) reparametrized by the arc-length parameter. Since the linear Airy equation is

$$y_t = y_{xxx},$$

we call (1.2) the *geometric Airy curve flow*.

Curve flow (1.2) was considered by Langer and Perline in [**15**]. They proved the following:

(i) If γ is a solution of (1.2) then there exists a parallel orthonormal frame $(e_2, \ldots, e_n)(\cdot, t)$ for the normal bundle of $\gamma(\cdot,t)$ with respect to $\nabla^\perp$ for each $\gamma(\cdot,t)$ such that the corresponding principal curvatures $k_i(\cdot,t)$ satisfy the *vector modified KdV equation* (vmKdV$_n$),

$$k_t = -(k_{xxx} + \frac{3}{2}||k||^2 k_x), \tag{1.3}$$

where $k = (k_1, \ldots, k_{n-1})^t$.

(ii) There exist order $(2j-1)$ curve flow in $\mathbb{R}^n$ whose principal curvatures satisfy the $(2j-1)$-th vmKdV$_n$ flow.

Some main results of these paper are:

(1) We solve the Cauchy problem for the curve flow (1.2) with initial data γ_0 whose principal curvatures are rapidly decaying functions on $\mathbb{R}$. We also solve the periodic Cauchy problem for initial data with trivial normal holonomy.
(2) There is a Poisson structure on the space

$$\mathcal{M}_n = \{\gamma : S^1 \to \mathbb{R}^n \mid ||\gamma_x|| \equiv 1\} \tag{1.4}$$

so that the geometric Airy curve flow (1.2) is Hamiltonian.
(3) Equation (1.2) admits a sequence of commuting Hamiltonians, hence it has a sequence of constants of the motion.
(4) Bäcklund transformations (BT) are constructed, which generate a family of new solutions from a given solution of (1.2) by solving a linear Lax system.
(5) An algorithm for constructing explicit soliton solutions of (1.2) is given.

We mention some other integrable curve flows in flat spaces: (i) Hasimoto showed that if γ is a solution of the *Vortex Filament equation* (VFE), $\gamma_t = \gamma_x \times \gamma_{xx}$, for curves in $\mathbb{R}^3$, then there exists a parallel frame $g(\cdot, t)$ along $\gamma(\cdot, t)$ so that $k_1 + ik_2$ satisfies the NLS ([**7**]). So we can apply soliton theory of the NLS to study geometric properties of the VFE (cf. [**14**], [**17**], [**20**]). (ii) Pinkall considered in [**13**] the *central affine curve flow,* $\gamma_t = \gamma_{xxx} - 3q\gamma_x$, on $\mathbb{R}^2\backslash 0$, where x is the *central affine arc-length parameter* (i.e., $\det(\gamma, \gamma_x) = 1$), and q is the central affine curvature defined by $\gamma_{xx} = q\gamma$. He proved that if γ is a solution of the central affine curve flow then its central affine curvature is a solution of the KdV, $q_t = q_{xxx} - 6qq_x$. Its Hamiltonian aspect was given in [**3**], and Bäcklund transformations were constructed in [**24**]. The relation between central affine curve flows in $\mathbb{R}^n\backslash 0$ and the Gelfand-Dickey flows on the space of n-th order linear differential operators on the line was given in [**4**] for $n = 3$ and in [**26**] for general n, and Bäcklund transformations were constructed in [**25**]. (iii) It was proved in [**27**] that the *isotropic curve flow,* $\gamma_t = \gamma_{xxx} - q\gamma_x$, in the light cone $\Sigma_{2,1} = \{x \in \mathbb{R}^{2,1} \mid (x, x) = 0\}$ preserves arc-length and its curvature q is a solution of the KdV. For general n, let $\mathcal{M}_{n+1,n}$ denote the space of curves $\gamma : \mathbb{R} \to \mathbb{R}^{n+1,n}$ such that the span of $\gamma, \ldots, \gamma_x^{(n-1)}$ is a Lagrangian subspace for all $x \in \mathbb{R}$ and $(\gamma_x^{(n)}, \gamma_x^{(n)}) \equiv 1$. Isotropic curve flows were constructed in $\mathcal{M}_{n+1,n}$ so that their isotropic curvatures were solutions of Drinfeld-Sokolov's KdV type flows associated to $B_n = o(n+1, n)$ and their Hamiltonian theory were given in [**27**]. Bäcklund transformations for these isotropic curve flows were constructed in [**28**].

It is interesting to note that the geometric Airy curve flow in $\mathbb{R}^n$, the isotropic curve flow in the null cone $\Sigma_{2,1}$ and the central affine curve flow in $\mathbb{R}^2\backslash 0$ are given by the re-parametrization of the third order flow $\gamma_t = \gamma_{xxx}$, where x is the arc-length parameter for curves in $\mathbb{R}^n, \Sigma_{2,1}$ and is the central affine arc-length for curves in $\mathbb{R}^2\backslash 0$.

This paper is organized as follows: We explain how to construct a parallel frame for a solution of (1.2) so that (i) hold and prove result (1) in section 2. We review the soliton theory of (1.3) in section 3. We review the Hamiltonian theory and construct Bäcklund transformations for the vmKdV$_n$. We prove (2) and (3) in section 4. Results (4) to (5) are given in the last section.

2. Geometric Airy flow (1.2) on $\mathbb{R}^n$ and vmKdV$_n$

In this section, we

(i) review the relation between the geometric Airy flow (1.2) in $\mathbb{R}^n$ and the vmKdV$_n$ (1.3),

(ii) solve the Cauchy problem on the line for (1.2) with initial data that has rapidly decaying principal curvatures,

(iii) solve the periodic Cauchy problem for (1.2).

Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve parametrized by its arc-length. We call a smooth map $g = (e_1, e_2, \dots, e_n) : \mathbb{R} \to SO(n)$ a *parallel frame* along γ if $e_1 = \gamma_x$ and e_i's are parallel with respect to the induced normal connection for $2 \le i \le n$. In other words, we have

$$\begin{cases} (e_1)_x = \sum_{i=1}^{n-1} k_i e_{i+1}, \\ (e_{i+1})_x = -k_i e_1, \quad 1 \le i \le n-1. \end{cases} \tag{2.1}$$

Equation (2.1) written in terms of g is

$$g^{-1} g_x = \begin{pmatrix} 0 & -k^t \\ k & 0 \end{pmatrix}, \quad k = (k_1, \dots, k_{n-1})^t. \tag{2.2}$$

The function k_i is the *principal curvature* of γ with respect to the normal field e_{i+1}.

When $n = 2$, a parallel frame for a plane curve is the Frenet frame and k_1 is the curvature of the plane curve. When $n > 2$, principal curvatures of a curve depend on the choice of parallel frames. In fact, if $g = (e_1, \dots, e_n)$ is a parallel frames along γ and $f = \mathrm{diag}(1, h)$ with a constant $h \in SO(n-1)$ then gf is also a parallel frame along γ and the principal curvatures with respect to the new parallel frame gf is $\tilde{k} = h^{-1} k$, where $k = (k_1, \dots, k_{n-1})^t$ and $\tilde{k} = (\tilde{k}_1, \dots, \tilde{k}_{n-1})^t$.

Lemma 2.1. *A curve flow in $\mathbb{R}^n$ of the form $\gamma_t = \sum_{i=1}^n A_i e_i$ preserves arc-length parameter if and only if*

$$e_1(A_1) = \sum_{i=2}^n A_i k_{i-1}, \tag{2.3}$$

where $(e_1(\cdot,t), \dots, e_n(\cdot,t))$ is a parallel frame along $\gamma(\cdot,t)$ and $k_i(\cdot,t)$'s are the corresponding principal curvatures.

Proof. Let $s(x,t)$ denote the arc-length parameter for $\gamma(\cdot,t)$, i.e., $s_x = ||\gamma_x||$. We use (2.1) to compute

$$\frac{1}{2}\langle\gamma_x,\gamma_x\rangle_t = \langle(\gamma_t)_x,\gamma_x\rangle = ||\gamma_x||^2\langle(\gamma_t)_s,e_1\rangle = ||\gamma_x||^2((A_1)_s - \sum_{i=2}^n A_ik_{i-1}).$$

This proves the Lemma. □

The curve flow (1.1) written in terms of parallel frame $(e_1,\ldots,e_n)(\cdot,t)$ is

$$\gamma_t = -\sum_{i=1}^{n-1}(k_i)_s e_{i+1}, \tag{2.4}$$

where $k_i(\cdot,t)$ is the principal curvature of $\gamma(\cdot,t)$ with respect to the parallel normal field $e_{i+1}(\cdot,t)$ for $1\le i\le n-1$. It follows from Lemma 2.1 that

$$\gamma_t = -\frac{1}{2}\left(\sum_{i=1}^{n-1}k_i^2\right)e_1 - \sum_{i=1}^{n-1}(k_i)_x e_{i+1}, \tag{2.5}$$

preserves the arc-length parameter. Note that (2.5) is (1.2). So we have the following:

Lemma 2.2. *The geometric Airy curve flow* (1.2) *in* $\mathbb{R}^n$ *preserves arc-length parameter and is* (2.5) *written in terms of parallel frame.*

Recall an elementary lemma:

Lemma 2.3. *Given smooth maps* $A,B:\mathbb{R}^2\to so(n)$*, then the following linear system*

$$\begin{cases} g_x = gA,\\ g_t = gB,\end{cases} \tag{2.6}$$

is solvable for smooth $g:\mathbb{R}^2\to SO(n)$ *if and only if*

$$A_t = B_x + [A,B]. \tag{2.7}$$

Next we give a proof of the following result in [**15**] and explain how to choose the suitable parallel frame (unique up to a constant in $SO(n-1)$) so that the principal curvature evolves according to vmKdV$_n$.

Theorem 2.4 ([**15**]). *Let* γ *be a solution for* (1.2)*. Then there exists a smooth map* $g:\mathbb{R}^2\to SO(n)$ *such that* $g(\cdot,t)$ *is a parallel frame along* $\gamma(\cdot,t)$ *for each* t *and* g *satisfies*

$$\begin{cases} g^{-1}g_x = \begin{pmatrix} 0 & -k^t\\ k & 0\end{pmatrix},\\[2ex] g^{-1}g_t = \begin{pmatrix} 0 & -z^t\\ z & \xi\end{pmatrix},\end{cases} \tag{2.8}$$

where $k = (k_1, \ldots, k_{n-1})^t$, *and*

$$z = -(k_{xx} + \frac{1}{2}||k||^2 k),$$
$$\xi = (\xi_{ij}), \quad \xi_{ij} = (k_i)_x k_j - k_i(k_j)_x, \quad 1 \le i, j \le n-1.$$

Moreover,

(1) k *satisfies the* $vmKdV_n$ (1.3).
(2) *If* $\tilde{g} : \mathbb{R}^2 \to SO(n)$ *satisfies* (2.8) *with* $\tilde{k}$, *then there exists a constant* $c \in SO(n-1)$ *such that* $\tilde{g} = g\,\mathrm{diag}(1, c)$ *and* $\tilde{k} = c^{-1}k$.

Proof. Choose a smooth $h = (e_1, v_2, \ldots, v_n) : \mathbb{R}^2 \to SO(n)$ such that $h(\cdot, t)$ is a parallel frame along $\gamma(\cdot, t)$ for each t. Let $\mu_1, \ldots, \mu_{n-1}$ be the principal curvatures given by h. Then

$$h^{-1}h_x = \begin{pmatrix} 0 & -\mu^t \\ \mu & 0 \end{pmatrix}, \quad \mu = (\mu_1, \ldots, \mu_{n-1})^t.$$

Write $h^{-1}h_t = \begin{pmatrix} 0 & -z^t \\ z & C \end{pmatrix}$ with $z = (z_1, \ldots, z_{n-1})^t$, and $C = (c_{ij}) \in so(n-1)$. It follows from (2.5) and (2.2) that we have

$$\langle (e_1)_t, v_{i+1} \rangle = \langle (\gamma_t)_x, v_{i+1} \rangle = -((\mu_i)_{xx} + \frac{1}{2}||\mu||^2 \mu_i).$$

Hence

$$z_i = -((\mu_i)_{xx} + \frac{1}{2}||\mu||^2 \mu_i). \tag{2.9}$$

Let $A := h^{-1}h_x$ and $B = h^{-1}h_t$. By Lemma 2.3, we have

$$A_t = B_x + [A, B].$$

Equate the (22) block of the above equation to get

$$(C_{ij})_x = -\mu_i(\mu_j)_{xx} + (\mu_i)_{xx}\mu_j = (-\mu_i(\mu_j)_x + (\mu_i)_x\mu_j)_x.$$

Hence $C_{ij}(x, t) = (-\mu_i(\mu_j)_x + (\mu_i)_x\mu_j)(x, t) + \eta(t)$ for some $\eta(t) \in o(n-1)$.

Let $f : \mathbb{R} \to SO(n-1)$ be a solution of $f_t = -\eta(t)f$, and

$$g = (e_1, \ldots, e_n) := h\,\mathrm{diag}(1, f).$$

Then $g(\cdot, t)$ again a parallel frame along $\gamma(\cdot, t)$. The principal curvatures of $\gamma(\cdot, t)$ with respect to the new parallel frame g is $k(x, t) = f(t)^{-1}\mu(x, t)$. A direct computation implies that g satisfies (2.8).

Let P and Q be the right hand sides of (2.8). By Lemma 2.3, we have

$$P_t = Q_x + [P, Q]. \tag{2.10}$$

Equate the (21) block of (2.10) to see that k satisfies (1.3).

If $\tilde{g}(\cdot, t)$ is another parallel frame along $\gamma(\cdot, t)$, then there exists $h : \mathbb{R} \to SO(n-1)$ such that $\tilde{g}(x, t) = g(x, t)\,\mathrm{diag}(1, h(t))$ and the principal curvatures with respect to $\tilde{g}$ is $\tilde{k}(x,t) = h(t)^{-1}k(x,t)$. Note that

$$\begin{aligned}\tilde{g}^{-1}\tilde{g}_t &= \operatorname{diag}(1,h^{-1})g^{-1}g_t \operatorname{diag}(1,h) + \begin{pmatrix} 0 & 0 \\ 0 & h^{-1}h_t \end{pmatrix} \\ &= \operatorname{diag}(1,h^{-1}) \begin{pmatrix} 0 & -z(k)^t \\ z(k) & \xi(k) \end{pmatrix} \operatorname{diag}(1,h) + \begin{pmatrix} 0 & 0 \\ 0 & h^{-1}h_t \end{pmatrix}.\end{aligned}$$

But $\tilde{k}(x,t) = h^{-1}(t)k(x,t)$ implies that

$$z(\tilde{k}) = h^{-1}z(k), \quad \xi(\tilde{k}) = h^{-1}\xi(k)h.$$

Hence we have

$$\tilde{g}^{-1}\tilde{g}_t = \begin{pmatrix} 0 & -z(\tilde{k})^t \\ z(\tilde{k}) & \xi(\tilde{k}) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & h^{-1}h_t \end{pmatrix}.$$

By assumption $\tilde{g}^{-1}\tilde{g}_t = \begin{pmatrix} 0 & -z(\tilde{k})^t \\ z(\tilde{k}) & \xi(\tilde{k}) \end{pmatrix}$. So we have $h^{-1}h_t = 0$, which implies that h is a constant. This proves (2). □

The proof of Theorem 2.4 also gives the following:

COROLLARY 2.5. *Given smooth $k : \mathbb{R}^2 \to \mathbb{R}^{n-1}$, then the linear system (2.8) is solvable for $g : \mathbb{R}^2 \to SO(n)$ if and only if k satisfies (1.3).*

Conversely, given a solution of vmKdV$_n$ we can construct a solution of the geometric Airy flow (1.1):

PROPOSITION 2.6. *Let k be a solution of vmKdV$_n$ (1.3). Then*

(1) *there exists smooth $g = (e_1, \dots, e_n) : \mathbb{R}^2 \to SO(n)$ satisfying (2.8),*
(2) *let*

$$Z = -(\frac{||k||^2}{2}e_1 + \sum_{i=1}^{n-1}(k_i)_x e_{i+1}),$$

$$c(t) = \int_0^t Z(0,t)\mathrm{d}t.$$

Then

$$\gamma(x,t) = c(t) + \int_0^x e_1(s,t)\mathrm{d}s, \tag{2.11}$$

is a solution of (1.2).

PROOF. (1) follows from Corollary 2.5.
The second equation of (2.8) implies that

$$(e_1)_t = -\sum_{i=1}^{n-1}((k_i)_{xx} + \frac{1}{2}||k||^2 k_i)e_{i+1}.$$

The first equation of (1.3) is (2.1). Use (2.1) and a direct computation to see that

$$-\sum_{i=1}^{n-1}((k_i)_{xx} + \frac{1}{2}||k||^2 k_i)e_{i+1} = Z_x.$$

Since $c'(t) = Z(0,t)$, we have

$$\gamma_t = c'(t) + \int_0^x (e_1)_t \mathrm{d}s = c'(t) + \int_0^x Z_s \mathrm{d}s = Z(x,t).$$

So γ satisfies (2.5), which is (1.2). □

Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve parametrized by arc-length. If the principal curvatures with respect to a parallel frame g are rapidly decaying then the principal curvatures with respect to other parallel frame are also rapidly decaying.

The Cauchy problem for the vmKdV$_n$ with rapidly decaying initial data $k^0 : \mathbb{R} \to \mathbb{R}^n$ can be solved by the inverse scattering method (cf. [**1**]). The periodic Cauchy problem for (1.3) is solved in [**2**]. Next we use these solutions to solve the Cauchy problems for the geometric Airy flow (1.2).

Theorem 2.7 (Cauchy Problem for (1.2) on the line). *Let $\gamma_0 : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve parametrized by its arc-length, $g_0(x) = (e_1^0(x), \ldots, e_n^0(x))$ a parallel frame along γ_0, and $k_1^0, \ldots, k_{n-1}^0$ the corresponding principal curvatures. Let $k : \mathbb{R}^2 \to \mathbb{R}^{n-1}$ be the solution of the Cauchy problem*

$$\begin{cases} k_t = -(k_{xxx} + \frac{3}{2}||k||^2 k_x), \\ k(x,0) = k^0(x), \end{cases} \tag{2.12}$$

where $k^0 = (k_1^0, \ldots, k_{n-1}^0)$. Then there exists a unique $g = (e_1, \ldots, e_n) : \mathbb{R}^2 \to SO(n)$ satisfying (2.8) with $g(0,0) = g^0(0)$. Moreover, let $c(t)$ be as in Proposition 2.6. Then

$$\gamma(x,t) = \gamma_0(0) + c(t) + \int_0^x e_1(s,t)\mathrm{d}s \tag{2.13}$$

is the solution of (1.2) with $\gamma(x,0) = \gamma_0(x)$.

Theorem 2.8 (Periodic Cauchy Problem for (1.2)). *Let $\gamma_0 : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve parametrized by its arc-length. Assume that γ_0 is periodic with period 2π and the holonomy of the induced normal connection along the closed curve γ_0 is trivial (so parallel frame g^0 and its principal curvature k^0 are periodic). If $k(x,t)$ is the solution of (2.12) and is periodic in x, then we have the following:*

(1) *Let $g = (e_1, \ldots, e_n) : \mathbb{R}^2 \to SO(n+1)$ be the solution of (2.8) with $g(0,0) = g^0(0)$. Then $g(x,t)$ is periodic in x and $g(x,0) = g^0(x)$.*
(2) *The solution γ of (1.2) defined by (2.13) is periodic in x and $\gamma(x,0) = \gamma_0(x)$.*

PROOF. Corollary 2.5 gives the existence of g. Note that $g(x,0)$ and $g^0(x)$ satisfy the same linear ODE and have the same initial data $y(0) = \gamma_0(0)$. Uniqueness of solutions of ODE implies that $g(x,0) = g^0(x)$.

Claim (A): $g(x,t)$ is periodic in x.

To see this, we consider

$$y(t) = g(2\pi, t) - g(0,t).$$

Recall that

$$g_t = gB, \quad \text{where } B = \begin{pmatrix} 0 & -z^t \\ z & \xi \end{pmatrix}, \tag{2.14}$$

and $z = z(k)$ and $\xi = \xi(k)$ are given as in Theorem 2.4. By assumption $k(x,t)$ is periodic in x. So we have

$$B(x+2\pi, t) = B(x,t) \tag{2.15}$$

for all $x, t \in \mathbb{R}$. Use (2.14) and (2.15) to see that

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= g(2\pi,t)B(2\pi,t) - g(0,t)B(0,t) = g(2\pi,t)B(0,t) - g(0,t)B(0,t) \\ &= y(t)B(0,t). \end{aligned}$$

Since k is periodic in x, $B(x,t)$ is periodic in x. Note that $g(x,0) = g^0(x)$ is periodic. So we have $y(0) = 0$. Then the uniqueness of ODE implies that $y(t) \equiv 0$, i.e., y is periodic. This proves Claim (A).

Claim (B): $\gamma(x,0) = \gamma_0(x)$.

We have proved $g(x,0) = g^0(x)$ is a parallel frame along γ_0. So $e_1(x,0) = (\gamma_0)_x$. By (2.13), we have $\gamma_x = e_1$ and $\gamma(0,0) = \gamma_0(0)$. This shows that $\gamma(x,0) = \gamma_0(x)$.

Claim (C): $\gamma(x,t)$ defined by (2.13) is periodic in x.

It follows from (2.13) that

$$\eta(t) = \gamma(2\pi,t) - \gamma(0,t) = \oint e_1(s,t)\mathrm{d}s.$$

Compute directly to see that

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \oint (e_1)_t \mathrm{d}s = \oint (\gamma_s)_t \mathrm{d}s = \oint (\gamma_t)_s \mathrm{d}s \\ &= -\oint \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{||k||^2}{2} e_1 + \sum_{i=1}^{n-1} (k_i)_s e_{i+1}\right) \mathrm{d}s. \end{aligned}$$

Since $e_1(s,t), \ldots, e_n(s,t)$ and $k_i(s,t)$'s are periodic in s, we have $\frac{\mathrm{d}\eta}{\mathrm{d}t} = 0$. So $\eta(t)$ is a constant. Note that $c(0) = 0$, $\eta(0) = \gamma(2\pi,0) - \gamma(0,0) = \gamma_0(2\pi) - \gamma_0(0) = 0$. Hence $\eta(t) = 0$ for all t and Claim (C) is proved.

Proposition 2.6 implies that γ defined by (2.13) is the solution of (1.2) with $\gamma(x,0) = \gamma_0(x)$. $\square$

3. The vmKdV hierarchy

In this section, we review the construction of the vmKdV$_n$ hierarchy, explain its Hamiltonian theory, and construct Bäcklund transformations.

Let I_{n+1} denote the identity in $GL(n+1)$, and

$$\mathrm{I}_{1,n} = \mathrm{diag}(1, -\mathrm{I}_n).$$

Let τ and σ be involutions of $so(n+1, \mathbb{C})$ defined by

$$\tau(\xi) = \bar{\xi}, \quad \sigma(\xi) = \mathrm{I}_{1,n}\xi\mathrm{I}_{1,n}^{-1}.$$

Note that the fixed point set of τ is $so(n+1)$, $\tau\sigma = \sigma\tau$, and the $1, -1$-eigenspaces of σ on $so(n+1)$ are

$$\mathcal{K} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \,\middle|\, y \in so(n) \right\},$$
$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & -z^t \\ z & 0 \end{pmatrix} \,\middle|\, z \in \mathbb{R}^{n\times 1} \right\}.$$

Then we have $so(n+1) = \mathcal{K} \oplus \mathcal{P}$, and

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] \subset \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}] \subset \mathcal{K}. \tag{3.1}$$

The loop algebras we need to construct the vmKdV$_n$ hierarchy are:

$$\mathcal{L} = \{\xi(\lambda) = \sum_{i \le n_0} \xi_i \lambda^i \,|\, n_0 \text{ some integer}, \ \xi_{2i} \in \mathcal{K}, \xi_{2i-1} \in \mathcal{P}\},$$
$$\mathcal{L}_+ = \{\xi(\lambda) = \sum_{i \ge 0} \xi_i \lambda^i \in \mathcal{L}\},$$
$$\mathcal{L}_- = \{\xi(\lambda) = \sum_{i < 0} \xi_i \lambda^i \in \mathcal{L}\}.$$

Note that

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$$

as linear subspaces. We call $(\mathcal{L}_+, \mathcal{L}_-)$ a *splitting* of $\mathcal{L}$.

Definition 3.1. We say a map ξ from $\mathbb{C}$ to $so(n+1, \mathbb{C})$ or to $SO(n+1, \mathbb{C})$ satisfies the (τ, σ)-*reality condition* if

$$\overline{\xi(\bar{\lambda})} = \xi(\lambda), \quad \mathrm{I}_{1,n}\xi(-\lambda)\mathrm{I}_{1,n}^{-1} = \xi(\lambda). \tag{3.2}$$

Note that $\xi(\lambda) = \sum_i \xi_i \lambda^i$ with $\xi_i \in so(n+1, \mathbb{C})$ is in $\mathcal{L}$ if and only ξ satisfies the (τ, σ)-reality condition (3.2).

A *vacuum sequence* is a linearly independent commuting sequence in $\mathcal{L}_+$. Let

$$a = e_{21} - e_{12}, \tag{3.3}$$

$$J_{2j-1}(\lambda) = a\lambda^{2j-1}, \quad j \ge 1. \tag{3.4}$$

Since $a \in \mathcal{P}$, $\{J_{2j-1} \,|\, j \ge 1\}$ is a vacuum sequence in $\mathcal{L}_+$.

Next we use the standard method given in [6], [21] to construct a soliton hierarchy from the splitting of $(\mathcal{L}_+, \mathcal{L}_-)$ of $\mathcal{L}$ and the vacuum sequence $\{J_{2j-1} \mid j \geq 1\}$.

Given $z \in \mathbb{R}^{n-1}$, henceforth we will use the following notation:

$$\Psi(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -z^t \\ 0 & z & 0 \end{pmatrix}. \tag{3.5}$$

Then

$$Y = [a\lambda, \mathcal{L}_-]_+ = [a, \mathcal{P}] = \{\Psi(z) \mid z \in \mathbb{R}^{(n-1)\times 1}\Big\}. \tag{3.6}$$

The following Theorem is known (cf. [6], [23]). We include a proof here for completeness.

THEOREM 3.2. *Let a be defined by (3.3), and Y by (3.6). Given $u \in C^\infty(\mathbb{R}, Y)$, then there exists a unique $Q(u, \lambda) = a\lambda + \sum_{i\geq 0} Q_i(u)\lambda^{-i}$ in $\mathcal{L}$ satisfying*

$$\begin{cases} [\partial_x + a\lambda + u, Q(u, \lambda)] = 0, \\ Q(u, \lambda) \text{ is conjugate to } a\lambda. \end{cases} \tag{3.7}$$

Moreover, the $Q_i(u)$'s are differential polynomials of u in x.

PROOF. Since $Q_{2i} \in \mathcal{K}$, $Q_{2i-1} \in \mathcal{P}$, we can write

$$Q_{2j-1} = \begin{pmatrix} 0 & -y_{2j-1} & -\eta_{2j-1}^t \\ y_{2j-1} & 0 & 0 \\ \eta_{2j-1} & 0 & 0 \end{pmatrix}, \tag{3.8}$$

$$Q_{2j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -z_{2j}^t \\ 0 & z_{2j} & \xi_{2j} \end{pmatrix}, \tag{3.9}$$

where $y_{2j-1} : \mathbb{R} \to \mathbb{R}$, $\eta_{2j-1}, z_{2j} : \mathbb{R} \to \mathbb{R}^{n-1}$ and $\xi : \mathbb{R} \to so(n-1)$.

Compare coefficients of λ^{-i} of (3.7) to get the following recursive formula:

$$(Q_i)_x + [u, Q_i] + [a, Q_{i+1}] = 0. \tag{3.10}$$

Assume that $u = \Psi(k)$ for some $k \in C^\infty(\mathbb{R}, \mathbb{R}^{n-1})$. Then $y_{2j-1}, \eta_{2j-1}, z_{2j}$ and ξ_{2j} are differential polynomials of k. We use (3.10) to see that

$$(\eta_{2j-1})_x + y_{2j-1}k - z_{2j} = 0, \tag{3.11}$$

$$(y_{2j-1})_x - k^t\eta_{2j-1} = 0, \tag{3.12}$$

$$(z_{2j})_x - \xi_{2j}k + \eta_{2j+1} = 0, \tag{3.13}$$

$$(\xi_{2j})_x - kz_{2j}^t + z_{2j}k^t = 0. \tag{3.14}$$

We use the above equations, induction on j, and direct computation to solve $Q_j(u)$ and see that entries of $Q_j(u)$ are differential polynomials of u.

Since $u = \Psi(k)$, $y_{2j-1}, \eta_{2j-1}, z_{2j}, \xi_{2j}$ are differential polynomials in k. In particular, we obtain

$$z_0 = k, \quad \xi_0 = 0, \tag{3.15}$$

$$y_1 = -\frac{||k||^2}{2}, \quad \eta_1 = -k_x, \tag{3.16}$$

$$z_2 = -(k_{xx} + \frac{1}{2}||k||^2 k), \quad \xi_2 = -kk_x^t + k_x k^t, \tag{3.17}$$

$$y_3 = k^t k_{xx} - \frac{||k_x||^2}{2} + \frac{3}{8}||k||^4, \quad \eta_3 = k_{xxx} + \frac{3}{2}||k||^2 k_x, \tag{3.18}$$

where $k = (k_1, \ldots, k_{n-1})^t$. □

It follows from (3.10) that we have

$$(Q_{2j-2}(u))_x + [u, Q_{2j-2}(u)] = [Q_{2j-1}(u), a] \in C^\infty(S^1, Y).$$

So the following is an evolution equation on $C^\infty(\mathbb{R}, Y)$,

$$u_t = [\partial_x + u, Q_{2j-2}(u)] = [Q_{2j-1}(u), a]. \tag{3.19}$$

Equation (3.19) for u written in terms of k is

$$k_t = (z_{2j-2}(k))_x - \xi_{2j-2}(k)k, \tag{3.20}$$

where $u = \Psi(k)$, and $z_{2j-2}(k)$ and $\xi_{2j-2}(k)$ are the (32) and (33) blocks of $Q_{2j-2}(u)$ respectively. Note that for $j = 2$, (3.20) is $k_t = (z_2)_x - \xi_2 k$, where z_2, ξ_2 are defined by (3.17). So it is the vmKdV$_n$ (1.3).

Definition 3.3. We call (3.19) (or (3.20)) the $(2j-1)$-*th vmKdV$_n$ flow* and this sequence of flows the *vmKdV$_n$ hierarchy*.

It follows from (3.10) that the coefficients of λ^i with $i > 0$ of

$$[\partial_x + a\lambda + u, (Q(u,\lambda)\lambda^{2j-2})_+]$$

are zero. So we have the following well-known existence of Lax pair.

Proposition 3.4. *The following statements for smooth $u : \mathbb{R}^2 \to Y$ are equivalent:*

(1) *u is a solution of the $(2j-1)$-th vmKdV$_n$ flow* (3.19).
(2) *The following linear system is solvable for $g : \mathbb{R}^2 \to SO(n)$*

$$\begin{cases} g_x = gu, \\ g_t = gQ_{2j-2}(u). \end{cases} \tag{3.21}$$

(3) *u satisfies*

$$u_t = [\partial_x + a\lambda + u, (Q(u,\lambda)\lambda^{2j-2})_+], \tag{3.22}$$

(4) *The following linear system is solvable for $E(x,t,\lambda) \in O(n+1,\mathbb{C})$*

$$\begin{cases} E^{-1}E_x = a\lambda + u, \\ E^{-1}E_t = (Q(u,\lambda)\lambda^{2j-2})_+, \\ \overline{E(x,t,\bar\lambda)} = E(x,t,\lambda), \quad \mathrm{I}_{1,n}E(x,t,-\lambda)\mathrm{I}_{1,n}^{-1} = E(x,t,\lambda). \end{cases} \tag{3.23}$$

DEFINITION 3.5. We call (3.23) a *Lax system* for (3.19), and a solution $E(x,t,\lambda)$ of (3.23) a *frame* of a solution u of (3.19) provided $E(x,t,\lambda)$ is holomorphic for all $\lambda \in \mathbb{C}$.

3.6. **Bäcklund transformations for vmKdV$_n$**

We use the loop group factorization method given in [**22**] to construct Bäcklund transformations for the vmKdV$_n$ hierarchy. Let L_+ denote the group of holomorphic maps $f : \mathbb{C} \to SO(n+1,\mathbb{C})$ that satisfies the (τ,σ)-reality condition (3.2), and $\mathfrak{R}_-$ the group of rational maps $f : \mathbb{C} \cup \{\infty\} \to SO(n+1,\mathbb{C})$ that satisfies the (τ,σ)-reality condition (3.2) and $f(\infty) = \mathrm{I}_{n+1}$.

First we recall the following results in [**22**].

THEOREM 3.7 ([**22**]).

(1) *Given $g \in \mathfrak{R}_-$ and $f \in L_+$, then there exists unique $\tilde{g} \in \mathfrak{R}_-$ and $\tilde{f} \in L_+$ such that $gf = \tilde{f}\tilde{g}$.*
(2) *Let $E(x,t,\lambda)$ be a frame of a solution u of the $(2j-1)$-th vmKdV$_n$ flow (3.19), $g \in \mathfrak{R}_-$, $\tilde{E}(x,t,\cdot) \in L_+$, and $\tilde{g}(x,t,\cdot) \in \mathfrak{R}_+$ satisfying*

$$g(\lambda)E(x,t,\lambda) = \tilde{E}(x,t,\lambda)\tilde{g}(x,t,\lambda).$$

Expand

$$\tilde{g}(x,t,\lambda) = \mathrm{I} + \tilde{g}_{-1}(x,t)\lambda^{-1} + \dots .$$

Then

$$\tilde{u} = u + [a, \tilde{g}_{-1}] \tag{3.24}$$

is again a solution of (3.19) and $\tilde{E}(x,t,\lambda)$ is a frame of $\tilde{u}$.

We call an element f in $\mathfrak{R}_-$ *a simple element* if f cannot be written as a product of two elements in $f\mathbb{R}_-$. Given a simple element $g \in \mathfrak{R}_-$ and an $f \in L_+$, if we can write down an explicit formula for the factorization of $gf = \tilde{f}\tilde{g}$ with $\tilde{f} \in \mathfrak{R}_-$ and $\tilde{g} \in L_+$, then $u \mapsto \tilde{u}$ of Theorem 3.7 (2) gives a Bäcklund transformation for (3.19).

Let $s \in \mathbb{R}\backslash 0$, π a Hermitian projection of $\mathbb{C}^{n+1}$ satisfying

$$\bar{\pi} = \mathrm{I}_{1,n}\pi\mathrm{I}_{1,n}^{-1}, \quad \pi\bar{\pi} = \bar{\pi}\pi = 0, \tag{3.25}$$

or equivalently,

$$\bar{V} = JV, \quad \langle V, JV \rangle = 0. \tag{3.26}$$

REMARK 3.8. $v = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in \mathbb{C}^{n+1}$ with $y_0 \in \mathbb{C}$ and $y_1 \in \mathbb{C}^n$ satisfying

$$\bar{v} = \mathrm{I}_{1,n}v, \quad v^t v = 0 \tag{3.27}$$

if and only if $y_0 \in \mathbb{R}$, $y_1 = i\mathbf{c}$ for some $\mathbf{c} \in \mathbb{R}^n$ and $||\mathbf{c}|| = |y_0|$. In particular, if $v \neq 0$ then $y_0 \neq 0$. Note that v satisfies (3.27) if and only if the Hermitian projection π onto $\mathbb{C}v$ satisfies (3.25).

LEMMA 3.9. *Given $f \in L_+$ and a non-zero $v \in \mathbb{C}^{n+1}$ satisfying (3.27), let $\tilde{v} := f(-is)^{-1}v$. Then $\tilde{v}$ satisfies (3.27), $\tilde{v} = \begin{pmatrix} \tilde{y}_0 \\ i\tilde{\mathbf{c}} \end{pmatrix}$ for some $y_0 \in \mathbb{R}\backslash 0$, $\mathbf{c} \in \mathbb{R}^n$, and $||\mathbf{c}|| = |\tilde{y}_0|$.*

PROOF. Note that if $\in L_+$ then $f : \mathbb{C} \to SO(n+1, \mathbb{C})$. So we have

$$f(\bar{\lambda})^* f(\lambda) = \mathrm{I}_{n+1}. \tag{3.28}$$

Use the reality condition (3.2), (3.28) and (3.27) to compute

$$\mathrm{I}_{1,n}\bar{\tilde{v}} = \mathrm{I}_{1,n}\overline{f(-is)^{-1}}\bar{v} = \mathrm{I}_{1,n} f(is)^t \bar{v} = f(-is)^t \mathrm{I}_{1,n}\bar{v} = f(-is)^{-1}v = \tilde{v}.$$

Similarly,

$$\begin{aligned} \langle \bar{\tilde{v}}, \tilde{v}\rangle &= \langle \mathrm{I}_{1,n}\tilde{v}, \tilde{v}\rangle = \langle \mathrm{I}_{1,n} f(-is)^{-1}v, \tilde{v}\rangle = \langle \mathrm{I}_{1,n} f(is)^* v, \tilde{v}\rangle \\ &= \langle v, f(is)\mathrm{I}_{1,n}\tilde{v}\rangle = \langle v, \mathrm{I}_{1,n} f(-is)\tilde{v}\rangle = \langle v, \mathrm{I}_{1,n} v\rangle = \langle v, \bar{v}\rangle = 0. \end{aligned}$$

The rest of the Lemma follows from Remark 3.8. □

The following Proposition gives simple elements in $\mathfrak{R}_-$ and a permutability formula (i.e., a relation among simple elements) for $\mathfrak{R}_-$. This type result was first proved for the group of rational maps $f : \mathbb{C} \to GL(n, \mathbb{C})$ that satisfies (3.28) in [**22**].

PROPOSITION 3.10 ([**5**]). *Let $s \in \mathbb{R}\backslash 0$ and π a Hermitian projection of $\mathbb{C}^{n+1}$ satisfying (3.25). Then*

$$\phi_{is,\pi} = (\mathrm{I} + \frac{2is}{\lambda - is}\bar{\pi}^\perp)(\mathrm{I} - \frac{2is}{\lambda + is}\pi^\perp) = \mathrm{I} + \frac{2is}{\lambda - is}\pi - \frac{2is}{\lambda + is}\bar{\pi}. \tag{3.29}$$

is in $\mathfrak{R}_-$. Moreover, we have the following:

(1) $\phi_{is,\pi}^{-1} = \phi_{-is,\pi}$.

(2) *Given $f \in L_+$, let $\tilde{V} = f(-is)^{-1}(\mathrm{Im}\,\pi)$, and $\tilde{\pi}$ the Hermitian projection onto $\tilde{V}$. Then $\tilde{\pi}$ satisfies (3.25) and*

$$\tilde{f} := \phi_{is,\pi} f \phi_{is,\tilde{\pi}}^{-1} \tag{3.30}$$

is in L_+. In other words, we have factored $\phi_{is,\pi} f = \tilde{f}\phi_{is,\tilde{\pi}}$ with $\tilde{f} \in L_+$ and $\phi_{is,\tilde{\pi}} \in \mathfrak{R}_-$.

(3) *Let $s_1^2 \neq s_2^2 \in \mathbb{R}\backslash 0$, π_1, π_2 Hermitian projections of $\mathbb{C}^{n+1}$ satisfying (3.25), and τ_1, τ_2 Hermitian projections onto $\mathbb{C}^{n+1}$. Then*

$$\phi_{is_2,\tau_2}\phi_{is_1,\pi_1} = \phi_{is_1,\tau_1}\phi_{is_2,\pi_2} \tag{3.31}$$

if and only if

$$\begin{cases} \mathrm{Im}\,\tau_1 = \phi_{-is_2,\pi_2}(-is_1)(\mathrm{Im}\,\pi_1), \\ \mathrm{Im}\,\tau_2 = \phi_{-is_1,\pi_1}(-is_2)(\mathrm{Im}\,\pi_2). \end{cases} \tag{3.32}$$

As a consequence of Theorem 3.7 and Proposition 3.10(2), we obtain BTs for the $(2j-1)$-th flow (3.19).

THEOREM 3.11 (Bäcklund Transformation for (3.19)). *Let $s \in \mathbb{R}\backslash 0$, π a Hermitian projection of $\mathbb{C}^{n+1}$ satisfying (3.25), and $\phi_{is,\pi}$ defined by (3.29). Let u be a solution of the $(2j-1)$-th vmKdV$_n$ flow (3.19), and $E(x,t,\lambda)$ a frame of u. Let $\tilde{\pi}(x,t)$ be the Hermitian projection of $\mathbb{C}^{n+1}$ onto*

$$\tilde{V}(x,t) = E(x,t,-is)^{-1}(\operatorname{Im}\pi).$$

Then

$$\tilde{u} = u + 2is[a, \tilde{\pi} - \overline{\tilde{\pi}}] \tag{3.33}$$

is a new solution of (3.19) and

$$\tilde{E}(x,t,\lambda) = \phi_{is,\pi}(\lambda)E(x,t,\lambda)\phi^{-1}_{is,\tilde{\pi}(x,t)} \tag{3.34}$$

is a frame of $\tilde{u}$.

3.12. **Permutability for BTs of (3.19)**
First we recall the following result.

THEOREM 3.13 ([**22**]). *Let $E(x,t,\lambda)$ be the frame of a solution u of the $(2j-1)$-th vmKdV$_n$ flow (3.19) such that $E(0,0,\mathrm{I}) = \mathrm{I}_{n+1}$. Let $f \in \mathfrak{R}_-$, and $\tilde{u}$ the solution constructed in Theorem 3.11 from E and f. Then $f \bullet u = \tilde{u}$ defines an action of $\mathfrak{R}_-$ on the space of solutions of (3.19).*

THEOREM 3.14. *Let s_i, π_i, τ_i with $i = 1, 2$ be as in Proposition 3.10(3), and E the frame of a solution u of (3.19) with $E(0,0,\lambda) = \mathrm{I}_{n+1}$. Let $\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\tau}_1, \tilde{\tau}_2$ be the Hermitian projection of $\mathbb{C}^{n+1}$ such that*

$$\begin{aligned}
\operatorname{Im}(\tilde{\pi}_i(x,t)) &= E(x,t,-is_i)^{-1}(\operatorname{Im}\pi_i), \quad i = 1, 2,\\
\operatorname{Im}(\tilde{\tau}_1(x,t)) &= \phi_{-is_2,\tilde{\pi}_2(x,t)}(\operatorname{Im}\tilde{\pi}_1(x,t)),\\
\operatorname{Im}(\tilde{\tau}_2(x,t)) &= \phi_{-is_1,\tilde{\pi}_1(x,t)}(\operatorname{Im}\tilde{\pi}_2(x,t)).
\end{aligned}$$

Then

$$u_{12} = u_1 + 2is_2[a, \tilde{\tau}_2 - \overline{\tilde{\tau}_2}], \quad u_{21} = u_2 + 2is_1[a, \tilde{\tau}_1 - \overline{\tilde{\tau}_1}]$$

are solutions of (3.19) and $u_{12} = u_{21}$, where $u_i = u + 2is_i[a, \tilde{\pi}_i - \overline{\tilde{\pi}_i}]$ for $i = 1, 2$. Moreover,

$$E_{12} = \phi_{is_2,\tau_2}\phi_{is_1,\pi_1}E\phi^{-1}_{is_1,\tilde{\pi}_1}\phi_{is_2,\tilde{\tau}_2}$$

is the frame of u_{12} with $E_{12}(0,0,\lambda) = \mathrm{I}_{n+1}$.

PROOF. Let $u_j = \phi_{is_j,\pi_j} \bullet u$. Since $\bullet$ is an action, we have

$$\begin{aligned}
u_{12} &== \phi_{is_2,\tau_2} \bullet u_1 = \phi_{is_2,\tau_2} \bullet (\phi_{is_1,\pi_1} \bullet u) = (\phi_{is_2,\tau_2}\phi_{is_1,\pi_1}) \bullet u,\\
u_{21} &= \phi_{is_1,\tau_1} \bullet (\phi_{is_1,\pi_2} \bullet u) = \phi_{is_1,\tau_1} \bullet u_2 = (\phi_{is_1,\tau_1}\phi_{is_2,\pi_2}) \bullet u.
\end{aligned}$$

So we have $u_{12} = u_{21}$. It follows from Theorem 3.11 that $E_j = \phi_{is_j,\pi_j}E\phi^{-1}_{is_j,\tilde{\pi}_j}$ is the frame of u_j with $E_j(0,0,\lambda) = \mathrm{I}$. Let $\theta_1(x,t)$ and $\theta_2(x,t)$ be Hermitian projections onto $E_2(x,t,-is_1)(\mathrm{Im}\,\tau_1)$ and $E_1(x,t,-is_2)(\mathrm{Im}\,\tau_2)$ respectively. It follows from Theorem 3.11 that we have

$$E_{12} = \phi_{is_2,\tau_2} E_1 \phi^{-1}_{is_2,\theta_2} = \phi_{is_2,\tau_2}\phi_{is_1,\pi_1} E \phi^{-1}_{is_1,\tilde{\pi}_1}\phi^{-1}_{is_2,\theta_2},$$
$$E_{21} = \phi_{is_1,\tau_1} E_2 \phi^{-1}_{is_1,\theta_1} = \phi_{is_1,\tau_1}\phi_{is_2,\pi_2} E \phi^{-1}_{is_2,\tilde{\pi}_2}\phi^{-1}_{is_1,\theta_1}$$

are the frame for u_{12} and u_{21} with $E_{12}(0,0,\lambda) = E_{21}(0,0,\lambda) = \mathrm{I}$. Since $u_{12} = u_{21}$, we have $E_{12} = E_{21}$. This implies that

$$\phi_{is_2,\theta_2}\phi_{is_1,\tilde{\pi}_1} = \phi_{is_1,\theta_1}\phi_{is_2,\tilde{\pi}_2}.$$

Proposition 3.10(3) implies that $\theta_i = \tilde{\tau}_i$ for $i = 1, 2$. □

Example 3.15 (Soliton solutions of (3.19)). Note that $u = 0$ is a solution of the $(2j-1)$-th vmKdV$_n$ flow (3.19) and $E(x,t,\lambda) = \exp(a\lambda x + a\lambda^{2j-1}t)$. So

$$E(x,t,\lambda) = e^{a\lambda x + a\lambda^3 t} = \begin{pmatrix} \cos(\lambda x + \lambda^3 t) & -\sin(\lambda x + \lambda^3 t) & 0 \\ \sin(\lambda x + \lambda^3 t) & \cos(\lambda x + \lambda^3 t) & 0 \\ 0 & 0 & \mathrm{I}_{n-1} \end{pmatrix}$$

is the frame of $u = 0$ with $E(0,0,\lambda) = \mathrm{I}_{n+1}$. Let $s_1, \ldots, s_k \in \mathbb{R}\backslash 0$ satisfying $s_i^2 \neq s_j^2$ for all $1 \leq i \neq j \leq k$, and $\pi_1, \ldots, \pi_k$ Hermitian projections satisfying (3.25). We apply Bäcklund Transform Theorem 3.11 to $u = 0$ with E and ϕ_{is_j,π_j} to obtain explicit 1-soliton solution $u_j = \phi_{is_j,\pi_j} \bullet 0$ of (3.19) and its frame

$$E_j(x,t,\lambda) = \phi_{is_j,\pi_j} E(x,t,\lambda)\phi^{-1}_{is_j,\tilde{\pi}_j(x,t)}(\lambda).$$

We apply the Permutability formula (Theorem 3.14) to u_i, u_j to obtain explicit 2-soliton solutions u_{ij} algebraically from u_i and u_j, etc.

3.16. A Poisson structure for the vmKdV$_n$ hierarchy

We use a known general method (cf. [6], [19]) to construct a Poisson structure for the vmKdV$_n$ flow and its commuting Hamiltonians.

Identify $\mathbb{R}^{n-1}$ as Y via (3.5). Then

$$\langle \Psi(z_1), \Psi(z_2)\rangle := -\frac{1}{2}\oint \mathrm{tr}(\Psi(z_1)\Psi(z_2))\mathrm{d}x = \oint z_1^t z_2 \mathrm{d}x = \langle z_1, z_2\rangle.$$

So $\langle\,,\rangle$ is the standard L^2 inner product on $C^\infty(S^1, \mathbb{R}^{n-1})$.

Given $F : C^\infty(S^1, \mathbb{R}^n) \to \mathbb{R}$ and $k \in C^\infty(S^1, \mathbb{R}^{n-1})$, $\nabla F(k)$ is defined by

$$\mathrm{d}F_k(z) = \langle \nabla F(k), z\rangle$$

for all $z \in C^\infty(S^1, \mathbb{R}^{n-1})$.

Given $k, z \in C^\infty(S^1, \mathbb{R}^{n-1})$, let

$$\Xi_k(z) = z_x - \xi k, \quad \text{where } \xi_x = kz^t - zk^t. \tag{3.35}$$

Then

$$\{H_1, H_2\}(k) = \langle \Xi_k(\nabla H_1(k)), \nabla H_2(k)\rangle \tag{3.36}$$

is a Poisson structure on $C^\infty(S^1, \mathbb{R}^{n-1})$ and the Hamiltonian flow for a functional H is

$$k_t = \Xi_k(\nabla H(k))]. \tag{3.37}$$

THEOREM 3.17 ([19]). *Let a, $u = \Psi(k)$, and $Q_i(u)$ be as given in Theorem 3.2, and $y_{2j-1}(k)$ the (21) block of $Q_{2j-1}(u)$ as defined by (3.8). Let*

$$F_{2j-1}(k) = -\frac{1}{2j-1} \oint y_{2j-1}(k) \mathrm{d}x. \tag{3.38}$$

Then we have

(1) $\nabla F_{2j-1}(k) = z_{2j-2}(k)$,
(2) *the Hamiltonian flow for F_{2j-1} is the $(2j-1)$-th vmKdV$_n$ flow* (3.20),
(3) $\{F_{2j-1}, F_{2\ell-1}\} = 0$,

where $z_{2j-2}(k)$ is the (32)-block of $Q_{2j-2}(u)$ as defined by (3.9).

For example, when $j = 2$, we have

$$F_3(k) = -\frac{1}{3} \oint y_3(k) \mathrm{d}x = -\frac{1}{3} \oint k^t k_{xx} - \frac{||k_x||^2}{2} + \frac{3}{8} ||k||^4 \mathrm{d}x.$$

$\nabla F_3(u) = z_2(k) = -(k_{xx} + \frac{1}{2}||k||^2 k)$. By (3.14), we have $(\xi_2)_x = k z_2^t - z_2 k^t$. This implies that $\Xi_k(\nabla F_3(k)) = (z_2(k))_x - \xi_2 k$, which is the vmKdV$_n$, the third flow.

It follows from $\{F_{2j-1}, F_{2\ell-1}\} = 0$ that we have

PROPOSITION 3.18.

(i) *F_{2j-1} is a constant of the motion for the $(2\ell-1)$-th vmKdV$_n$ flow,*
(ii) *the Hamiltonian flows for F_{2j-1} and $F_{2\ell-1}$ commute, i.e., the $(2j-1)$-th and $(2\ell-1)$-th vmKdV$_n$ flows commute.*

4. The $(2j-1)$-th Airy curve flow

In this section, we

(1) construct a parallel frame for a solution of the order $(2j-1)$ curve flow in $\mathbb{R}^n$ for all $j \geq 1$ so that the principal curvatures satisfy the $(2j-1)$-th vmKdV$_n$-flow (3.19),
(2) give a Poisson structure for the space $\mathcal{M}_n$ so that the $(2j-1)$-th Airy curve flow is Hamiltonian,
(3) these curve Airy curve flows commute, and admit a sequence of commuting Hamiltonians.

PROPOSITION 4.1. *Let $g : \mathbb{R} \to \mathbb{R}^n$ be parameterized by arc-length, and $g = (e_1, \ldots, e_n)$ a parallel frame along γ, k_i the principal curvature of γ with respect to e_{i+1} for $1 \leq i \leq n-1$, and $k = (k_1, \ldots, k_{n-1})^t$. Let $u = \Psi(k)$ as* *defined by* (3.5)*, and* $y_{2j-3}(k), \eta_{2j-3}(k)$ *the* (21) *and* (31) *block of* $Q_{2j-3}(u)$ *defined by* (3.8)*. Then*

$$\gamma_t = y_{2j-3}e_1 + (e_2, \ldots, e_n)\eta_{2j-3} \tag{4.1}$$

is a well-defined arc-length preserving curve flow on $\mathbb{R}^n$*.*

PROOF. First we claim that the right hand side of (4.1) does not depend on the choice of parallel frames. If g_1 is another parallel frame along γ, then there exists a constant $C \in SO(n-1)$ such that $g_1 = g\,\mathrm{diag}(1, C)$ and the corresponding curvature

$$\tilde{k} = C^{-1}k. \tag{4.2}$$

So $\tilde{u} = \Psi(\tilde{k}) = h^{-1}uh$, where $h = \mathrm{diag}(1, 1, C)$. Write $\tilde{g} = (\tilde{e}_1, \ldots, \tilde{e}_n)$. Then we have

$$\tilde{e}_1 = e_1, \quad (\tilde{e}_2, \ldots, \tilde{e}_n) = (e_2, \ldots, e_n)C \tag{4.3}$$

Since h is a constant, it follows from the uniqueness of Theorem 3.2 that we have

$$Q(\tilde{u}, \lambda) = Q(h^{-1}uh) = hQ(u, \lambda)h^{-1}.$$

This implies that $Q_{2j-3}(\tilde{u}) = h^{-1}Q_{2j-2}(u)h$. Recall that $h = \mathrm{diag}(1, 1, C)$. So we have

$$y_{2j-3}(\tilde{u}) = y_{2j-3}(u), \tag{4.4}$$

$$\eta_{2j-3}(\tilde{k}) = C^{-1}\eta_{2j-3}(k). \tag{4.5}$$

It follows from (4.2), (4.3), and (4.4) that $\tilde{y}_{2j-3}(\tilde{u})\tilde{e}_1 + (\tilde{e}_2, \ldots, \tilde{e}_n)\eta_{2j-3}(\tilde{u})$ is equal to $y_{2j-3}(u)e_1 + (e_2, \ldots, e_n)\eta_{2j-3}(u)$. This proves the claim. Hence (4.1) defines a curve flow on $\mathbb{R}^n$.

By (3.12), we have $(y_{2j-3})_x = k^t\eta_{2j-3}$. It now follows from Lemma 2.1 that (4.1) preserves the arc-length parameter. □

DEFINITION 4.2. We call (4.1) the $(2j-1)$*-th Airy curve flow on* $\mathbb{R}^n$*.*

EXAMPLE 4.3. Since

$$(\nabla^{\perp}_{e_1})^{\ell}H = \sum_{i=1}^{n-1}(k_i)^{(\ell)}_x e_{i+1},$$

we can write (4.1) in terms of $(\nabla^{\perp}_{e_1})^i H$ with $i \geq 0$. For example, when $j = 1$, we have $Q_1(u) = a$. So (4.1) is the translation flow,

$$\gamma_t = e_1 = \gamma_x.$$

For $j = 2$, we have

$$Q_3(u) = \begin{pmatrix} 0 & \frac{||k||^2}{2} & k^t_x \\ -\frac{||k||^2}{2} & 0 & 0 \\ -k_x & 0 & 0 \end{pmatrix},$$

and the third Airy curve flow (4.1) is geometric Airy flow (1.2).

The fifth Airy curve flow is $\gamma_t = y_5(k)e_1 + (e_2, \ldots, e_n)\eta_5(k)$. Use (3.18) to rewrite the fifth in terms of $H, \nabla^\perp_{e_1} H, \cdots, (\nabla^\perp_{e_1})^{(3)} H$ as follows:

$$\gamma_t = (\langle H, (\nabla^\perp_{e_1})^2 H\rangle - \frac{1}{2}||\nabla^\perp_{e_1}||^2 + \frac{3}{2}||H||^4)e_1 \\ + ((\nabla^\perp_{e_1})^3 H + \frac{3}{8}||H||^2 \nabla^\perp_{e_1} H)$$

THEOREM 4.4. *If $\gamma : \mathbb{R}^2 \to \mathbb{R}^n$ is a solution of (4.1), then there exists $g = (e_1, \ldots, e_n) : \mathbb{R}^2 \to SO(n)$ such that $g(\cdot, t)$ is a parallel frame along $\gamma(\cdot, t)$ for each t, and g satisfies*

$$\begin{cases} g^{-1}g_x = \begin{pmatrix} 0 & -k^t \\ k & 0 \end{pmatrix}, \\ g^{-1}g_t = \begin{pmatrix} 0 & -z^t_{2j-2}(k) \\ z_{2j-2}(k) & \xi_{2j-2}(k) \end{pmatrix}, \end{cases} \tag{4.6}$$

where z_{2j-2} and ξ_{2j-2} are the (32) *and* (33) *blocks of $Q_{2j-1}(u)$ defined by* (3.8)*. Then*

(1) *$u = \Psi(k)$ (defined by* (3.5)*) is a solution of* (3.19),
(2) *if g_1 is another map satisfies* (4.6) *with $\tilde{k}$ then there exists $c \in SO(n-1)$ such that $g_1 = g\,\mathrm{diag}(1, c)$ and $\tilde{k} = c^{-1}k$.*

PROOF. The proof of this Theorem is similar to that of Theorem 2.4. We will use same notations. (2) can be proved exactly the same way. For (1), let $h = (e_1, v_2, \ldots, v_n) : \mathbb{R}^2 \to SO(n)$ such that $h(\cdot, t)$ is a parallel frame along $\gamma(\cdot, t)$, and $\mu_1, \ldots, \mu_{n-1}$ the corresponding principal curvatures. Then

$$A := h^{-1}h_x = \begin{pmatrix} 0 & -\mu^t \\ \mu & 0 \end{pmatrix}.$$

Let

$$\hat{u} = \Psi(\mu),$$

where Ψ is defined by (3.5). Let $B = (b_{ij}) := h^{-1}h_t$. Since γ satisfies (4.1), we compute directly to see that

$$b_{i+1,1} = \langle (e_1)_t, v_{i+1}\rangle = \langle (\gamma_t)_x, v_{i+1}\rangle = y_{2j-3}(\mu)\mu_i + ((\eta_{2j-3}(\mu))_i)_x.$$

By (3.11), $b_{i+1,1} = (z_{2j}(\mu))_i$, where $z_{2j}(\mu)$ is the (32) block of $Q_{2j}(\hat{u})$ as in (3.9). So we can write $B = \begin{pmatrix} 0 & -z^t_{2j} \\ z_{2j} & S \end{pmatrix}$, for some $S = S(k) \in so(n-1)$. Compare the (22) block of $A_t = B + [A, B]$ to see that $S_x = \mu z^t_{2j}(\mu) - z_{2j}(\mu)\mu^t$. It follows from (3.14), $S_x = (\xi_{2j}(\mu))_x$. So there exists $\eta(t) \in so(n-1)$ such that $S(x,t) = \xi_{2j}(\mu(x,t)) + \eta(t)$. Let $f(t) \in SO(n-1)$ satisfying $f_t = -\eta(t)f$. Then $g = h\,\mathrm{diag}(1, f)$ is a parallel frame, $k = f^{-1}\mu$ is its principal curvature, and g satisfies (4.6).

It follows from Lemma 2.3 that k is a solution of (3.20). □

Conversely, given a solution of the $(2j-1)$-th vmKdV$_n$ flow (3.20) we can construct a solution of the $(2j-1)$-th Airy curve flow (4.1).

PROPOSITION 4.5. *Let k be a solution of $(2j-1)$-th vmKdV$_n$ flow* (3.20). *Then:*

(1) *There is $g = (e_1, \ldots, e_n) : \mathbb{R}^2 \to SO(n)$ satisfying* (4.6).
(2)
$$\gamma(x,t) = \int_0^t Z_{2j-3}(0,t_1)\mathrm{d}t_1 + \int_0^x e_1(s,t)\mathrm{d}s$$
is a solution of (4.1), *where*
$$Z_{2j-3} = y_{2j-3}e_1 + (e_2, \ldots, e_n)\eta_{2j-3},$$
$y_{2j-3}(k)$, $\eta_{2j-3}(k)$ are the (21) *and* (31) *blocks of $Q_{2j-3}(u)$ as in* (3.8), *and $u = \Psi(k)$.*

PROOF. (1) follows from Proposition 3.4.

(2) The proof is similar to that of Proposition 2.6. Note that $\gamma_x = e_1$. The second equation of (4.6) gives
$$(e_1)_t = (e_2, \ldots, e_n)z_{2j-2}.$$
Use the first equation of (4.6) (which is (2.1)) and (3.12) to compute directly to see that
$$(e_2, \ldots, e_n)z_{2j-2} = (Z_{2j-3})_x.$$
Hence
$$\begin{aligned}\gamma_t &= Z_{2j-3}(0,t) + \int_0^x (e_2, \ldots, e_n)z_{2j-2}\mathrm{d}s \\ &= Z_{2j-3}(0,t) + \int_0^x (Z_{2j-3})_s\mathrm{d}s = Z_{2j-3},\end{aligned}$$
which is the $(2j-1)$-th Airy curve flow (4.1). □

Cauchy problem on the line for the $(2j-1)$-th Airy curve flow (4.1) can be proved in the same way as for the geometric Airy curve flow, and similarly for the periodic case.

4.6. **Poisson structure on $\mathcal{M}_n$ and commuting Airy curve flows**

Let $\mathcal{M}_n$ be as in (1.4). We have seen that if both $g, \tilde{g}$ are parallel frames along $\gamma \in \mathcal{M}_n$ and $k, \tilde{k}$ are the corresponding principal curvatures, then there exists a constant $c \in SO(n-1)$ such that $\tilde{g} = gc$ and $\tilde{k} = c^{-1}k$. Note that the group $SO(n-1)$ acts on $C^\infty(S^1, \mathbb{R}^{n-1})$ by $c\dot{k} = c^{-1}k$. So
$$\Gamma : \mathcal{M}_n \to C^\infty(S^1, \mathbb{R}^{n-1})/SO(n-1)$$
defined by $\Gamma(\gamma) =$ the $SO(n-1)$-orbit of k is well-defined, where k is a the curvature along some parallel frame g along γ.

LEMMA 4.7. *Let $k, z \in C^\infty(S^1, \mathbb{R}^{n-1})$, $c \in SO(n-1)$, and Ξ the operator defined by* (3.37). *Then*

(4.7) $$\Xi_{c^{-1}k}(c^{-1}z) = c^{-1}\xi_k(z).$$

Proof. By (3.37), we have $\Xi_k(z) = z_x - \xi k$, where $\xi_x = kz^t - zk^t$. Let $\tilde{k} = c^{-1}k$, $\tilde{z} = c^{-1}z$, and $\tilde{\xi} = c^{-1}\xi c$. A direct computation implies that $\Xi_{\tilde{k}} = \tilde{k}_x - \tilde{\xi}\tilde{k}$ and $\tilde{\xi}_x = \tilde{k}\tilde{z}^t - \tilde{z}\tilde{k}^t$. This proves (4.7). □

Corollary 4.8. *If H_1, H_2 are functionals on $C^\infty(S^1, \mathbb{R}^{n-1})$ invariant under the action of $SO(n-1)$, then $\{H_1, H_2\}$ is also invariant under the action of $SO(n-1)$.*

Formula (4.4) in the proof of Proposition 4.1 implies the following:

Proposition 4.9. *If $H : C^\infty(S^1, \mathbb{R}^{k-1}) \to \mathbb{R}$ is a functional, then $\hat{H} : \mathcal{M}_n \to \mathbb{R}$ defined by $\hat{H} = H(\Gamma(\gamma))$ is a well-defined functional.*

Hence $\{\,,\}$ defined by (3.36) can be viewed as a Poisson structure on the orbit space $C^\infty(S^1, \mathbb{R}^{n-1})/SO(n-1)$. Let $\{\,,\}^\wedge$ denote the pull back of $\{\,,\}$ on $\mathcal{M}_n$ by the map Γ, i.e.,

$$\{\hat{H}_1, \hat{H}_2\}^\wedge(\gamma) = \{H_1, H_2\}(k),$$

where $\hat{H}_i = H_i(\Gamma(\gamma))$.

Proposition 4.10. *Let H be a functional on $C^\infty(S^1, \mathbb{R}^{n-1})$, and $\nabla H(k) = z = (z_1, \ldots, z_{n-1})^t$. Then the Hamiltonian vector field for $\hat{H}$ with respect to $\{\,,\}^\wedge$ is*

$$\delta\gamma = A_0 e_1 + \sum_{i=1}^{n-1} A_i e_{i+1}$$

where $A_0, A_1, \ldots, A_{n-1}$ satisfy

$$\begin{cases} (A_0)_x = \sum_{i=1}^{n-1} k_i A_i, \\ (A_i)_x + A_0 k_i = z_i, \end{cases} \tag{4.8}$$

$g = (e_1, \ldots, e_n)$ is a parallel frame along γ and k is the principal curvature with respect to g.

Proof. Let $\delta k = \Xi_k(\nabla H(k))$ denote the Hamiltonian vector field of H with respect to $\{\,,\}$. Since the Poisson structure $\{\,,\}^\wedge$ is the pull back of $\{\,,\}$ by Γ, we have $d\Gamma_\gamma(\delta\gamma) = \delta k$.

Use $g^{-1}g_x = \begin{pmatrix} 0 & -k^t \\ k & 0 \end{pmatrix}$ and a simple computation to see that

$$[\partial_x + \begin{pmatrix} 0 & -k^t \\ k & 0 \end{pmatrix}, g^{-1}\delta g] = \begin{pmatrix} 0 & -(\delta k)^t \\ \delta k & 0 \end{pmatrix}, \tag{4.9}$$

where δg the variation of parallel frames when we vary γ. Since $g \in SO(n)$, $g^{-1}\delta g$ is $so(n)$-valued. Write

$$g^{-1}\delta g = \begin{pmatrix} 0 & -\eta^t \\ \eta & \zeta \end{pmatrix}.$$

Then (4.9) implies that $(\zeta)_x = kz^t - z^t k$ and $\Xi_k(\eta) = \delta k = \Xi_k(\nabla H(k))$. Hence $\eta = z$. So we have

$$g^{-1}\delta g = \begin{pmatrix} 0 & -z^t \\ z & \zeta \end{pmatrix}, \tag{4.10}$$

where $z = \nabla H(u)$. The first column of (4.10) implies that

$$\delta e_1 = g \begin{pmatrix} 0 \\ z \end{pmatrix}.$$

Compute directly to see that

$$\begin{aligned} \delta e_1 = (\delta\gamma)_x &= (A_0 e_1 + \sum_{i=1}^{n-1} A_i e_{i+1})_x \\ &= ((A_0)_x - \sum_{i=1}^{n-1} A_i k_i) e_1 + \sum_{i=1}^{n-1} A_0 k_i + (A_i)_x) e_{i+1} = \sum_{i=1}^{n-1} z_i e_{i+1}. \end{aligned}$$

This proves (4.8). □

Proposition 4.11. *Let F_{2j-1} be as defined by (3.38). Then the Hamiltonian flow for $\hat{F}_{2j-1}$ on $\mathcal{M}_n$ is the $(2j-1)$-th Airy curve flow (4.1).*

Proof. Let y_{2j-3}, η_{2j-3} be as in (3.8), and z_{2j-2}, ξ_{2j-2} as in (3.9). By Theorem 3.17, we have $\nabla F_{2j-1}(k) = z_{2j-2}$. It follows from (3.11) and (3.12) that $A_0 = y_{2j-3}$ and $A_i = \eta_{2j-3}$ satisfy (4.8). This proves the Proposition. □

Since $\{\hat{F}_{2j-1}, \hat{F}_{2\ell-1}\}^\wedge = \{F_{2j-1}, F_{2\ell-1}\} = 0$, we have the following:

Corollary 4.12.

(1) *The $(2j-1)$-th and the $(2\ell-1)$-th Airy curve flows commute.*
(2) *If $\gamma(x,t)$ is a solution of the $(2j-1)$-th Airy flow, then $\hat{F}_{2\ell-1}(\gamma(\cdot,t))$ is constant in t.*

5. BTs for the geometric Airy flow on $\mathbb{R}^n$

In this section, we

(1) construct solutions of the $(2j-1)$-th Airy curve flow (4.1) from frames of $(2j-1)$-th vmKdV$_n$ (3.19),
(2) give BT for the $(2j-1)$-th Airy curve flow (4.1),
(3) write down explicit 1-soliton solutions for the geometric Airy flow on $\mathbb{R}^2$.

The following Theorem shows that the construction given by Pohlmeyer ([**16**]) and Sym ([**18**]) in soliton theory gives solutions of the $(2j-1)$-th Airy curve flow (4.1) from frames of solutions of vmKdV$_n$ flow (3.19).

THEOREM 5.1. *Let $u=\Psi(k)$ be a solution of the $(2j-1)$-th vmKdV$_n$ flow* (3.19)*, $E(x,t,\lambda)$ the frame of u with $E(0,0,\lambda)=\mathrm{I}_{n+1}$. Let*

$$\zeta(x,t):=E_\lambda E^{-1}(x,t,0).$$

Then

(1) $\zeta=\begin{pmatrix}0 & -\gamma^t\\ \gamma & 0\end{pmatrix}$ *for some $\gamma:\mathbb{R}^2\to\mathbb{R}^n$,*
(2) $E(x,t,0)=\mathrm{diag}(1,g(x,t))$ *for some $g:\mathbb{R}^2\to SO(n)$, $g(\cdot,t)$ is a parallel frame of $\gamma(\cdot,t)$ and $k(x,t)$ is the principal curvature of $\gamma(\cdot,t)$ with respect to $g(\cdot,t)$.*
(3) *γ is a solution of the $(2j-1)$-th Airy curve flow* (4.1).

PROOF. (1) Note that the fixed point set of τ is $SO(n+1)$ and the connected component of the fixed point set of τ on $SO(n+1)$ is $K=\{\mathrm{diag}(1,f)\,|\,f\in SO(n)\}$. The (τ,σ)-reality condition and $E(0,0,0)=\mathrm{I}_{n+1}$ implies that $E(x,t,0)\in K$. So

$$f(x,t):=E(x,t,0)=\mathrm{diag}(1,g(x,t))$$

for some $g(x,t)\in SO(n)$. The (τ,σ)-reality condition also implies that

$$\zeta:=E_\lambda E^{-1}\,|_{\,\lambda=0}$$

lies in $\mathcal{P}$. So there exist $\gamma:\mathbb{R}^2\to\mathbb{R}^n$ such that

$$f(x,t):=E(x,t,0)=\mathrm{diag}(1,g(z,t)),$$
$$\zeta=\begin{pmatrix}0 & -\gamma^t\\ \gamma & 0\end{pmatrix}.$$

(2) Since E is a solution of (3.23), we have

$$\begin{cases}E_x=E(a\lambda+u),\\ E_t=E(a\lambda^{2j-1}+u\lambda^{2j-2}+\cdots+Q_{2j-3}(u)\lambda+Q_{2j-2}(u)),\end{cases}\tag{5.1}$$

Use (5.1), $E(x,t,0)=f=\mathrm{diag}(1,g)$, and a direct computation to get

$$\zeta_x=\begin{pmatrix}0 & -\gamma_x^t\\ \gamma_x & 0\end{pmatrix}=faf^{-1}=\mathrm{diag}(1,g)a\,\mathrm{diag}(1,g^{-1})=\begin{pmatrix}0 & -e_1^t\\ e_1 & 0\end{pmatrix},\tag{5.2}$$

$$\zeta_t=\begin{pmatrix}0 & -\gamma^t\\ \gamma_t & 0\end{pmatrix}=(E_t)_\lambda E^{-1}-E_\lambda E^{-1}E_tE\big|_{\lambda=0}=fQ_{2j-3}(u)f^{-1}\tag{5.3}$$

$$=\begin{pmatrix}1 & 0\\ 0 & g\end{pmatrix}\begin{pmatrix}0 & -y_{2j-3} & -\eta_{2j-3}^t\\ y_{2j-3} & 0 & 0\\ \eta_{2j-3} & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & g^{-1}\end{pmatrix}.\tag{5.4}$$

(5.2) implies that $\gamma_x=e_1$ the first column of g. The first equation of (5.1) at $\lambda=0$ gives $g^{-1}g_x=\begin{pmatrix}0 & -k^t\\ k & 0\end{pmatrix}$. So $g(\cdot,t)$ is a parallel frame and $k(\cdot,t)$ the principal curvature of $\gamma(\cdot,t)$.

(3) Write $g=(e_1,\ldots,e_n)$. (3) follows from (5.4). □

COROLLARY 5.2. *Let γ be a solution of (4.1), and g, k as in Theorem 4.4. Let E be a frame of the solution $u = \Psi(k)$ with $E(x,t,0) = g(0,0)$. Then there is a constant $p_0 \in \mathbb{R}^n$ such that* $E_\lambda E^{-1} = \begin{pmatrix} 0 & -\gamma^t - p_0^t \\ \gamma + p_0 & 0 \end{pmatrix}$.

PROOF. Theorem 4.4 implies that $E_\lambda E^{-1} = \begin{pmatrix} 0 & -\hat{\gamma}^t \\ \hat{\gamma} & 0 \end{pmatrix}$, $\hat{\gamma}$ is a solution of (4.1), $g(x,t) = E(x,t,0)$ is a parallel frame for $\hat{\gamma}$. But g is also a parallel frame for γ. So $\gamma_x = \hat{\gamma}_x$. Both $\hat{\gamma}$ and γ are solutions of (4.1) implies that

$$\hat{\gamma}_t = \gamma_t = y_{2j-3}(k)e_1 + (e_2, \ldots, e_n)\eta_{2j-3}(k).$$

Hence $\hat{\gamma} - \gamma$ is a constant p_0. □

THEOREM 5.3 (Bäcklund transformation for Airy curve flows). *Let γ be a solution of the $(2j-1)$-th Airy curve flow (4.1), and $g(\cdot,t)$ parallel frame and $k(\cdot,t)$ the corresponding principal curvature of $\gamma(\cdot,t)$ as in Theorem 4.4. Let E the frame of the solution $u = \Psi(k)$ of the $(2j-1)$-th vmKdV$_n$ (3.19) with initial data $E(0,0,\lambda) = \mathrm{diag}(1, g(0,0))$, $s \in \mathbb{R}\backslash 0$, $\mathbf{c} \in \mathbb{R}^n$ a unit vector, and* $v = \begin{pmatrix} 1 \\ i\mathbf{c} \end{pmatrix}$. *Let $\tilde{v}(x,t) := E(x,t,-is)^{-1}v$. Then we have the following:*

(1) *$E(x,t,0) = \mathrm{diag}(1, g(x,t))$.*
(2) *$\tilde{v} = (y_0, iy_1, \ldots, iy_n)^t$ for some $y_0, y_1, \ldots, y_n : \mathbb{R}^2 \to \mathbb{R}$, y_0 never vanishes, and $\sum_{i=1}^n y_i^2 = y_0^2$.*
(3) *Write $g = (e_1, \ldots, e_n)$, and set*

$$\tilde{\mathbf{c}} = \frac{1}{y_0}(y_1, \ldots, y_n)^t, \tag{5.5}$$

$$\tilde{\gamma} = -(\mathrm{I}_n - 2\mathbf{c}\mathbf{c}^t)(\gamma - \frac{2}{sy_0}\sum_{i=1}^n y_i e_i), \tag{5.6}$$

$$\tilde{g} = (\mathrm{I}_n - 2\mathbf{c}\mathbf{c}^t)g(\mathrm{I}_n - 2\tilde{\mathbf{c}}\tilde{\mathbf{c}}^t), \tag{5.7}$$

$$\tilde{k} = k - \frac{2s}{y_0}(y_2, \ldots, y_n)^t. \tag{5.8}$$

Then $\tilde{\gamma}$ is a solution of (4.1), $\tilde{g}$ is a parallel frame of $\tilde{\gamma}$ with corresponding principal curvatures $\tilde{k}$

PROOF. (1) Since $E(x,t,\lambda) \in SO(n+1)$ and satisfies the reality condition (3.2), $E(x,t,0) = \mathrm{diag}(1, h(x,t))$ for some $h : \mathbb{R}^2 \to SO(n)$. It follows from the fact that E is a frame of $u = \Psi(k)$ that h satisfies (4.6). So both g and h satisfies the same linear system (4.6) and $g(0,0) = h(0,0)$. This proves that $g = h$.

(2) follows from Remark 3.8. Note that $||\tilde{\mathbf{c}}(x,t)||^2 = 1$ for all $(x,t) \in \mathbb{R}^2$.

(3) Let π, $\tilde{\pi}(x,t)$ be the Hermitian projection onto $\mathbb{C}v$ and $\mathbb{C}\tilde{v}(x,t)$ respectively. So we have

$$\pi = \frac{1}{2}\begin{pmatrix} 1 & -i\mathbf{c}^t \\ i\mathbf{c} & \mathbf{c}\mathbf{c}^t \end{pmatrix}, \quad \tilde{\pi} = \frac{1}{2}\begin{pmatrix} 1 & -i\tilde{\mathbf{c}}^t \\ i\tilde{\mathbf{c}} & \mathbf{c}\mathbf{c}^t \end{pmatrix},$$

where $\tilde{\mathbf{c}}$ is defined by (5.5). Let $\phi = \phi_{is,\pi}$. Then

$$\phi(0) = \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{I}_n - 2\mathbf{c}\mathbf{c}^t \end{pmatrix}, \quad \phi_\lambda \phi^{-1}(0) = \begin{pmatrix} 0 & -\frac{2}{s}\mathbf{c}^t \\ \frac{2}{s}\mathbf{c} & 0 \end{pmatrix}.$$

Apply Theorem 3.11 to u with frame E and $\phi_{is,\pi}$ to see that

$$\tilde{u} = u + 2is[a, \tilde{\pi} - \overline{\tilde{\pi}}] \tag{5.9}$$

is a new solution of (3.19). Write $\tilde{u} = \Psi(\tilde{k})$, and $u = \Psi(k)$. Then (5.9) gives (5.8).

By Theorem 3.11, $\tilde{E} = \phi_{is,\pi} E \phi_{is,\tilde{\pi}}^{-1}$ is a frame of $\tilde{u}$. Apply Theorem 5.1 to $\tilde{E}$ to see that

$$\tilde{E}_\lambda \tilde{E}^{-1}|_{\lambda=0} = \begin{pmatrix} 0 & -\hat{\gamma}^t \\ \hat{\gamma} & 0 \end{pmatrix}$$

and $\hat{\gamma}$ is again a solution of (4.1). By Corollary 5.2, there is a constant $p_0 \in \mathbb{R}^n$ such that

$$E_\lambda E^{-1} = \begin{pmatrix} 0 & -(\gamma + p_0)^t \\ \gamma + p_0 & 0 \end{pmatrix}.$$

Use $\tilde{E} = \phi E \tilde{\phi}^{-1}$ to compute $\tilde{E}_\lambda \tilde{E}^{-1}$ directly to obtain

$$\hat{\gamma} = \frac{2\mathbf{c}}{s} - A(\gamma + p_0) + \frac{2}{s} A g \tilde{\mathbf{c}},$$

where $A = \mathrm{I}_n - 2\mathbf{c}\mathbf{c}^t$. Note that $\frac{2\mathbf{c}}{s} - Ap_0$ is a constant and the $(2j-1)$-th Airy flow (4.1) is invariant under the translation. Hence $\tilde{\gamma}$ defined by (5.6) is a solution of (4.1).

Theorem 5.1 implies that $\tilde{E}(x,t,0) = \mathrm{diag}(1, \tilde{g}(x,t))$ and $\tilde{g}(x,t)$ is a parallel frame for $\hat{\gamma}$. So it is a parallel frame for $\tilde{\gamma}$. □

Example 5.4 (Explicit soliton solutions of (1.2) in $\mathbb{R}^2$). Note that $k = 0$ is a solution of the mKdV, and

$$E(x,t,\lambda) = \exp(a(\lambda x + \lambda^3 t))$$

is a frame of the solution $u = \Psi(0) = 0$ of the third vmKdV$_2$ flow. Since $E_\lambda E^{-1}|_{\lambda=0} = ax$, it follows from Theorem 5.1 that

$$\gamma(x,t) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

is a stationary solution of the geometric Airy curve flow on $\mathbb{R}^2$.

Let π be the Hermitian projection onto $\mathbb{C}v$, where

$$v = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i\mathbf{c} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that

$$E(x,t,-is)^{-1} = \begin{pmatrix} \cosh D & -i \sinh D & 0 \\ i \sinh D & \cosh D & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$D = sx - s^3 t.$$

We apply Theorem 5.3 to γ with $k = 0$ and E and $\phi_{is,\pi}$ and use the same notation as in Theorem 5.3. A direct computation implies that

$$\tilde{c}(x,t) = (\tanh(sx - s^3 t), \operatorname{sech}(sx - s^3 t))^t, \quad A = \operatorname{diag}(-1, 1),$$

and

$$\tilde{\gamma}(x,t) = -\begin{pmatrix} 0 \\ \frac{2}{s} \end{pmatrix} + \begin{pmatrix} x - \frac{2}{s}\tanh(sx - s^3 t) \\ \frac{2}{s}\operatorname{sech}(sx - s^3 t) \end{pmatrix} \tag{5.10}$$

is a solution of the geometric Airy flow on $\mathbb{R}^2$ and

$$\tilde{g} = \begin{pmatrix} 1 - 2\operatorname{sech}^2 D & 2\operatorname{sech} D \tanh(sx - s^3 t) \\ -2\operatorname{sech} D \tanh D & 1 - 2\operatorname{sech}^2 D \end{pmatrix}$$

is a parallel frame with principal curvature

$$\tilde{k}(x,t) = -2s\operatorname{sech}(sx - s^3 t),$$

where $D = sx - s^3 t$. Note that $\tilde{k}$ is a 1-soliton solution of the mKdV.

If $\tilde{\gamma}$ is a solution of the geometric Airy flow on $\mathbb{R}^2$, then so is $\tilde{\gamma} + p_0$ for some constant $p_0 \in \mathbb{R}^2$. Hence

$$\gamma_1(x,t) = \begin{pmatrix} x - \frac{2}{s}\tanh(sx - s^3 t) \\ \frac{2}{s}\operatorname{sech}(sx - s^3 t) \end{pmatrix}$$

is also a solution and $\tilde{k}$ is its curvature. Note that

$$\gamma_1(x, t_0) = \gamma(x - s^2 t_0, 0) + (s^2 t_0, 0).$$

So γ_1 is a self-similar solution of the geometric curve flow on $\mathbb{R}^2$. Note that the profile of 1-soliton solution γ_1 when $s = 1$ of the geometric Airy flow is the plane curve $\gamma_0(x) = (x - 2\tanh x, 2\operatorname{sech} x)^t$ (see graph of the curve in Figure 1), and γ_1 moves to the right but keeps its shape.

Example 5.5. Since $k = 0$ is a trivial solution of the mKdV$_n$, $E(x,t,\lambda) = e^{ax + a^{2j-1}t}$ is the frame of the solution $u = \Psi(0) = 0$ of the third flow (3.19). Then $E_\lambda E^{-1}|_{\lambda=0} = ax$. By Theorem 5.1, $\gamma(x,t) = (x, 0, \ldots, 0)^t$ is a solution of (1.2) $\mathbb{R}^n$ with $g(x,t) = \mathrm{I}_n$ as parallel frame and $k = 0$ as the corresponding principal curvature. Note that

$$E^{-1}(x,t,-is) = \begin{pmatrix} \cosh A & -\sinh A & 0 \\ \sinh A & \cosh A & 0 \\ 0 & 0 & \mathrm{I}_{n-2} \end{pmatrix}, \quad A = sx - s^3 t.$$

So we can use Theorem 5.3 to write down explicit new solutions $\tilde{\gamma}$ with parallel frame $\tilde{g}$ and principal curvature $\tilde{k}$ as in Example 5.4.

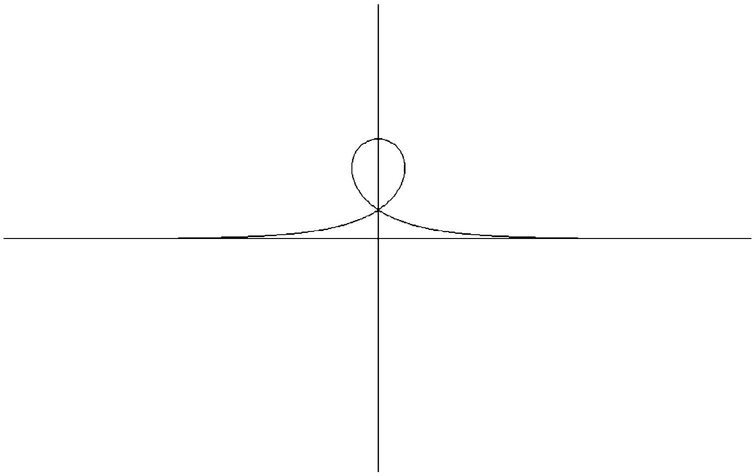

FIGURE 1. Profile of a 1-soliton solution of the geometric Airy flow in the plane.

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Department of Mathematics, University of California at Irvine, Irvine, CA 92697-3875

E-mail address: cterng@math.uci.edu