3d spectral networks and classical Chern-Simons theory

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To Chern, who taught us all

Abstract. We define the notion of spectral network on manifolds of dimension ≤ 3. For a manifold X equipped with a spectral network, we construct equivalences between Chern-Simons invariants of flat $SL_2\mathbb{C}$-bundles over X and Chern-Simons invariants of flat $\mathbb{C}^\times$-bundles over ramified double covers $\widetilde{X}$. Applications include a new viewpoint on dilogarithmic formulas for Chern-Simons invariants of flat $SL_2\mathbb{C}$-bundles over triangulated 3-manifolds, and an explicit description of Chern-Simons lines of flat $SL_2\mathbb{C}$-bundles over triangulated surfaces. Our constructions heavily exploit the locality of Chern-Simons invariants, expressed in the language of extended (invertible) topological field theory.

Contents

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1. Introduction

A classical formula of Lobachevsky-Milnor-Thurston [**T2**, Chapter 7] expresses the volume of a tetrahedron, i.e., 3-simplex, in hyperbolic space in terms of a dilogarithm function. It follows that the volume of a triangulated hyperbolic 3-manifold is a sum of real parts of dilogarithms. Thurston observed that the Chern-Simons invariant of the associated flat $PSL_2\mathbb{C}$-connection has real part equal to the volume, and Meyerhoff [**Me**] extended this to hyperbolic 3-manifolds with cusps. These ideas have been refined and extended since their introduction in the late 1970's and early 1980's, as we briefly review in Section 2. After much work, in particular by Neumann [**Neu**], by the early 2000's the Chern-Simons invariant of a flat $\mathrm{PSL}_2\mathbb{C}$-connection on a closed oriented 3-manifold was expressed as a sum of *complex* dilogarithms. In a closely related development over the past 20 years, Fock and Goncharov [**FG1**] studied special *cluster* coordinate systems on the moduli space of flat bundles on a compact oriented 2-manifold with punctures. The moduli space is symplectic, and the overlap functions—cluster transformations—between different coordinate systems are generated by essentially the same complex dilogarithms. These dilogarithms also serve as transition functions defining a canonical prequantum line bundle over the moduli space [**FG2**]. In this paper we introduce new perspectives and techniques into this circle of ideas. Our work is inspired by two distinct sources: *spectral networks* and *invertible field theories*. Both originated in physics and both have well-developed mathematical underpinnings.

Spectral networks on 2-manifolds were introduced by Gaiotto-Moore-Neitzke [**GMN1**] as part of their study of four-dimensional supersymmetric gauge theories. For our purposes the key point is that, given a spectral network on a surface Y, one can define the notion of stratified abelianization [**GMN1, HN**]: this is a linkage between a flat $\mathrm{GL}_N\mathbb{C}$-connection on Y and a $\mathbb{C}^\times$-connection on a ramified covering $\widetilde{Y} \to Y$. This notion has been useful in various contexts, e.g. in exact WKB analysis and hyperkähler geometry of moduli of Higgs bundles; it also gives a reinterpretation of the cluster coordinates of Fock-Goncharov. In Section 4 we extend the notion of a spectral network and stratified abelianization from 2 dimensions to all dimensions ≤ 3. In particular, in §4.2 we express the data of a spectral network on a smooth manifold as a certain type of stratification together with a double cover over a dense subset and a section of the double cover over a certain codimension one subset. We use it to set up stratified abelianization for the rank one complex Lie groups $\mathrm{GL}_2\mathbb{C}$, $\mathrm{SL}_2\mathbb{C}$, and $\mathrm{PSL}_2\mathbb{C}$. In particular, we construct canonical spectral networks associated to triangulations and ideal triangulations of 2- and 3-manifolds.[1]

The Chern-Simons invariant was introduced in 1971 [**CS1, CS2**], and it was fairly quickly expressed by Cheeger-Simons [**ChS**] in terms of their novel *differential characters*, an amalgam of integral cohomology and differential forms. For flat connections, which are our main focus here, the differential characters are induced from a cohomology class on a classifying space [**Ch, D, DS**]. With the advent of *quantum* Chern-Simons invariants [**W**], it became clear that the *classical* Chern-Simons invariants share the locality properties of the quantum invariants [**F1, F6, F2, RSW**]. Furthermore, this locality of the classical invariants is similar to the locality of the integral of a differential form on a smooth manifold M: if M is expressed

[1]We allow the intermediate case of *semi-ideal triangulations* in which both ideal and interior vertices are allowed; see Definitions 4.26 and 4.36.

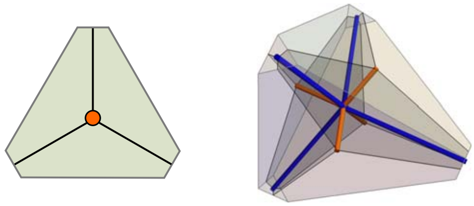


FIGURE 1. Left: the canonical 2d spectral network in a triangle. Right: the canonical 3d spectral network in a tetrahedron. Its restriction to each face is the canonical 2d network in a triangle.

as a union $M = \bigcup_i M_i$ of manifolds with corners glued along positive codimension submanifolds with corners, then the integral over M is the sum of the integrals over the M_i. The fullest expression of this locality is in terms of *invertible field theories*. They are constructed using the theory of *generalized differential cocycles* [**HS, BNV, ADH**], and that theory in turn is a fully local version of the Cheeger-Simons differential characters. We give brief introductions to these ideas in Appendices A and B.

These two lines of development lead to the motivating idea behind our theorems: a stratified abelianization of classical $SL_2\mathbb{C}$ Chern-Simons theory for flat connections. For a manifold X equipped with stratified abelianization data (defined in §4), this amounts to an equivalence between the Chern-Simons invariant of an $SL_2\mathbb{C}$-bundle over X and that of a $\mathbb{C}^\times$-bundle over a ramified double cover $\widetilde{X}$. We develop two main applications: (1) a formula for the Chern-Simons line of a flat $SL_2\mathbb{C}$-connection on a closed[2] oriented 2-manifold Y, derived from the simpler and more explicit $\mathbb{C}^\times$ Chern-Simons theory applied to a branched double covering manifold $\widetilde{Y}$ (Theorem 7.81); and (2) a derivation of the formula for the Chern-Simons invariant of a flat $SL_2\mathbb{C}$-connection on a closed[3] 3-manifold M as a sum of complex dilogarithms (Theorem 8.8).

Here is the rough strategy for (1), which we develop in Section 7. Let Y be a closed 2-manifold equipped with a flat principal $SL_2\mathbb{C}$-bundle $P \to Y$. First, choose a triangulation and, over each vertex, a line in the fiber of the complex 2-plane bundle associated to $P \to Y$; require that this data satisfy a genericity condition (Assumption 4.32). The stratified abelianization derived from the spectral network associated to the triangulation yields an isomorphism of the Chern-Simons line $\mathscr{F}_{SL_2\mathbb{C}}(Y;P)$ with the Chern-Simons line $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y};A)$, where $A \to \widetilde{Y}$ is a flat $\mathbb{C}^\times$-bundle over a branched double cover $\widetilde{Y}$ of Y. We give a concrete description of $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y};A)$ in terms of various auxiliary data: orientations of the edges of the triangulation, a

[2]More generally, we treat flat $SL_2\mathbb{C}$-connections on a compact oriented 2-manifold with boundary whose holonomies around boundary components are unipotent.

[3]with extensions as in footnote [2]

nonzero vector in the line at each vertex, etc. Each set of choices trivializes $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y};A)$, and we deploy $\mathbb{C}^\times$ Chern-Simons theory in its local form to compute explicit formulæ for the ratio of trivializations under changes of auxiliary data. Out of this we construct a groupoid whose points are sets of auxiliary data and whose morphisms are changes of the data. In this form our description of the Chern-Simons lines makes contact with dilogarithmic constructions of line bundles over cluster varieties in the literature; see the discussion in §7.5.5.

For our result (2), which is the subject of Section 8, we proceed as in (1) to choose a triangulation and lines over the vertices.[4] We excise an open ball from the center of each tetrahedron in the triangulation of the 3-manifold M. The boundary 2-sphere of each ball is ramified double covered by a 2-torus in the standard way: there are 4 branch points. The $\mathbb{C}^\times$ Chern-Simons invariant on the branched double cover localizes with a contribution from each tetrahedron that we identify as a complex dilogarithm. This led us to a new construction of the dilogarithm function in terms of (classical) $\mathbb{C}^\times$ Chern-Simons theory, which we worked out in [**FN**] and which we apply here.

Spin structures are used in our stratified abelianization for a simple reason. The generating level of $\mathrm{SL}_2\mathbb{C}$ Chern-Simons theory, when restricted to the maximal torus $\mathbb{C}^\times \subset \mathrm{SL}_2\mathbb{C}$, is *half*[5] the usual generating level of $\mathbb{C}^\times$ Chern-Simons theory; see equation (5.13). The division by 2 is effected by passing to spin manifolds. Just as on oriented manifolds Chern-Simons is a secondary invariant of characteristic classes in integer cohomology, on spin manifolds there are secondary invariants of characteristic classes in KO-theory. Here we use a simple 2-stage Postnikov truncation of KO-theory that we describe in §5.2. The $\mathrm{SL}_2\mathbb{C}$ Chern-Simons theory does not require a spin structure, so necessarily the results of our computations are independent of the choice of spin structure on the base, but the intermediate formulæ on the ramified double cover require us to keep careful track of spin structures there.

The stratified abelianization—the production of a flat $\mathbb{C}^\times$-connection from a flat $\mathrm{SL}_2\mathbb{C}$-connection—gives new geometric meaning to some aspects of standard constructions. For example, the *shape parameters* in Thurston's theory [**T2**, §4.1] are now holonomies of the flat $\mathbb{C}^\times$-connection around certain loops in the total space of the branched double cover. Furthermore, Thurston's *gluing equations* [**T2**, §4.2] are a simple relation in the first homology group of that manifold; see Remark 4.56. Neumann's "combinatorial flattenings" [**Neu**, §3] correspond to global sections of the principal $\mathbb{C}^\times$-bundle over the branched double cover $\widetilde{M}$ (with balls excised).

[4]For ideal vertices we choose a flat section of the associated $\mathbb{CP}^1$-bundle over the corresponding boundary component of M.

[5]There is a minus sign at stake here: see Convention 5.1 and Convention 5.28 for our choices.

In the course of our work we produced computer programs to implement our formulas for the Chern-Simons invariants of 3-manifolds. We have made those programs and computations available as ancillary files in the arXiv version of this paper.

We conclude this introduction with a brief roadmap to the parts of the paper not yet discussed. Section 3 is a brief recollection of the Chern-Simons invariant in 3 dimensions, including its status as the partition function of an invertible field theory. As a theory of a single flat connection, this field theory is *topological*; as such it has a formulation in homotopy theory. However, it is not topological as a theory of *families* of flat connections, and for that reason it requires the setting indicated in Appendix B. Section 5 begins with cohomological computations relating levels of Chern-Simons theory for different subgroups of $SL_2\mathbb{C}$, both in the oriented and spin cases. Then we review the role of differential cochains and prove an important result (Theorem 5.61) which essentially says that the Chern-Simons theory is unchanged as connections move in unipotent directions in $SL_2\mathbb{C}$. We also prove some theorems about the spin $\mathbb{C}^\times$ Chern-Simons theory that are important for our computations. Section 5 concludes with a global—as opposed to stratified—abelianization theorem. Section 6 introduces the auxiliary data we impose on a triangulated manifold. Then we prove important technical results which underpin the abelianization of the Chern-Simons line. As stated earlier, our main theorems are in Sections 7 and 8. We conclude in Section 9 with suggestions for ambitious readers who would like to extend our work in new directions. Finally, Appendix C takes up additional $\mathbb{Z}/2\mathbb{Z}$-gradings in spin Chern-Simons theory which we suppress in the main text; there we prove a spin-statistics result which justifies that suppression.

Marché's approach in [**M**] is a close cousin to our derivation of the formula for the Chern-Simons invariant in §8. Our stratified abelianization is a *classical* version of a *quantum* abelianization proposed by Cecotti-Córdova-Vafa [**CCV**, §7].

Over the long period in which this work was carried out we benefited from the comments and insights of many colleagues, including Clay Córdova, Tudor Dimofte, Stavros Garoufalidis, Matthias Goerner, Alexander Goncharov, Pavel Safronov, Joerg Teschner, Christian Zickert. We warmly thank them all, named and unnamed.

2. Hyperbolic volumes and Chern-Simons invariants

As a warmup suppose Y^2 is a complete hyperbolic 2-manifold with finite area and finitely generated fundamental group. Then the Gauss-Bonnet theorem states that $\mathrm{Area}(Y) = -2\pi\,\mathrm{Euler}(Y)$ is a topological invariant [**Ro**]. Furthermore, Y is the interior of a compact surface. The classification of surfaces shows that the possible areas form a discrete subset of $\mathbb{R}$.

Now suppose X^3 is a complete oriented hyperbolic 3-manifold with finite volume and finitely generated fundamental group. Then X is the interior of a compact 3-manifold whose boundary is a union of tori [**T2**, Proposition 5.11.1]. Mostow rigidity [**Mo, Pr**] asserts that $\mathrm{Vol}(X)$ is again a topological invariant. Jorgenssen-Thurston proved basic properties of this invariant [**T1**]. For example, the set of hyperbolic volumes is a well-ordered subset of $\mathbb{R}$, and there is a finite set of hyperbolic 3-manifolds of a given volume. The volume is an important invariant which orders hyperbolic 3-manifolds by complexity. The "simplest" is the Weeks manifold of volume $0.9427\dots$, the minimal volume closed orientable hyperbolic 3-manifold [**GMM**]. Further analytic properties of the set of hyperbolic volumes were explored early on in [**NZ, Y**].

There is a classical formula for the volume of an ideal tetrahedron $\Delta \subset \mathbb{H}^3$ in hyperbolic 3-space; it can be used to compute the volume of an ideally triangulated hyperbolic 3-manifold. Suppose the vertices of Δ are distinct points $Z_0, Z_1, Z_2, Z_3 \in \mathbb{CP}^1 = \partial\mathbb{H}^3$. Introduce the *Bloch-Wigner dilogarithm function* [**Z**, §3]

$$\begin{aligned} D\colon \mathbb{CP}^1 \setminus \{0,1,\infty\} &\longrightarrow \mathbb{R} \\ z &\longmapsto \operatorname{Im} \mathrm{Li}_2(Z) + \log|z| \arg(1-z), \end{aligned} \tag{2.1}$$

where Li_2 is the classical dilogarithm, defined for $|z| < 1$ by the power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \tag{2.2}$$

and analytically continued to $\mathbb{C} \setminus [1,\infty)$. Let

$$z = \frac{(Z_0 - Z_2)(Z_1 - Z_3)}{(Z_0 - Z_3)(Z_1 - Z_2)} \tag{2.3}$$

be the cross-ratio of the vertices of Δ.

THEOREM 2.4 (Lobachevsky, Milnor-Thurston [**T2**, Chapter 7]). $\mathrm{Vol}(\Delta) = |D(z)|$.

In his PhD thesis Meyerhoff [**Me**] initiated the detailed study of the Chern-Simons invariant $CS(X) \in \mathbb{R}/\mathbb{Z}(1)$ of the Levi-Civita connection Θ_{LC} of a closed oriented hyperbolic 3-manifold. Here and throughout we deploy the notation

$$\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z}, \qquad \mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n} = (2\pi\sqrt{-1})^n\mathbb{Z}, \qquad n \in \mathbb{Z}^{\geq 1}. \tag{2.5}$$

This *real* Chern-Simons invariant is the real part of the *complex* Chern-Simons invariant of the associated flat $\mathrm{PSL}_2\mathbb{C}$-connection Θ. Recall that the SO_3-bundle $\mathcal{B}_{\mathrm{SO}}(X) \to X$ of frames carries not only the Levi-Civita connection Θ_{LC} but also the $\mathbb{R}^3$-valued "soldering form" θ; the complex combination $\Theta = \Theta_{LC} + \sqrt{-1}\,\theta$ is a *flat* connection on the associated principal

$\mathrm{PSL}_2\mathbb{C}$-bundle:

$$\begin{array}{ccc} \mathcal{B}_{SO}(X) & \hookrightarrow & P \\ & {\scriptstyle SO_3}\searrow \quad \swarrow{\scriptstyle \mathrm{PSL}_2\mathbb{C}} & \\ & X & \end{array}$$

The exponentiated *complex* Chern-Simons invariant, which we review in §3, satisfies

$$\mathscr{F}_{\mathrm{PSL}_2\mathbb{C}}(X;\Theta) = \exp\big(\mathrm{Vol}(X) + \sqrt{-1}CS(X)\big) \ \in \mathbb{C}^\times. \tag{2.6}$$

Our focus in this paper is the complex Chern-Simons invariant of arbitrary flat connections, mainly for structure group $\mathrm{SL}_2\mathbb{C}$. (In the intrinsic case of connections on the frame bundle, the passage from $\mathrm{PSL}_2\mathbb{C}$ to $\mathrm{SL}_2\mathbb{C}$ is the introduction of a spin structure.)

Just as the real volume is related to a real dilogarithm (2.1), so too the complex Chern-Simons invariant is related to a complex dilogarithm, the *enhanced Rogers dilogarithm*. Let

$$\mathcal{M} = \{(z_1, z_2) \in \mathbb{C}^\times \times \mathbb{C}^\times : z_1 + z_2 = 1\} \tag{2.7}$$

and

$$\widehat{\mathcal{M}} = \{(u_1, u_2) \in \mathbb{C} \times \mathbb{C} : e^{u_1} + e^{u_2} = 1\}. \tag{2.8}$$

Then $\mathcal{M} \approx \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ and $\widehat{\mathcal{M}} \to \mathcal{M}$ is a universal abelian covering map with Galois group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. The dilogarithm in question [**ZG**, §4], [**Z**, §II.1.B], is the unique function

$$L\colon \widehat{\mathcal{M}} \longrightarrow \mathbb{C}/\mathbb{Z}(2), \tag{2.9}$$

which satisfies the differential equation

$$dL = (u_1\, du_2 - u_2\, du_1)/2 \tag{2.10}$$

and $\lim L(u_1, u_2) = 0$ as $u_1 \to \infty$ and $u_2 \to 0$. (We encounter variants in §§6–8.) The imaginary part of L is the Bloch-Wigner function (2.1) plus $\mathrm{Im}(\overline{u_1}\, u_2)/2$. See [**FN**] for a construction of the enhanced Rogers dilogarithm using Chern-Simons invariants for $\mathbb{C}^\times$-connections.

Let $B\mathrm{SL}_2\mathbb{C}^\delta$ denote the classifying space of flat $\mathrm{SL}_2\mathbb{C}$-bundles.[6] The universal Chern-Simons class for flat bundles

$$\hat{c}_2 \in H^3(B\mathrm{SL}_2\mathbb{C}^\delta; \mathbb{C}/\mathbb{Z}(1)) \tag{2.11}$$

was constructed by Cheeger-Simons [**ChS**] and is known as the Cheeger-Chern-Simons class. It has an expression in terms of the dilogarithm (2.9), going back to work of Dupont and collaborators in the 1980's; see [**D, DS**]. The most precise relationship can be found in [**DZ**, §4], which is based on [**Neu**]; see both papers for exact statements, history, and extensive references.

[6] We use standard terminology: 'flat' is a *structure*—a flat connection—on a principal bundle.

In the early 2000's, the formula for the Chern-Simons invariant of a flat connection on a 3-manifold as a sum of dilogarithms was taken up again in such works as [**Neu, DZ, Zi, GTZ, DGG**]. The formula for flat $PSL_2\mathbb{C}$-connections on closed 3-manifolds is in [**Neu**]; the formula for flat $SL_2\mathbb{C}$-connections on closed 3-manifolds is in [**DZ**]. The formula for boundary-unipotent flat $SL_N\mathbb{C}$-connections appears in [**GTZ**]; for boundary-unipotent flat $PSL_2\mathbb{C}$-connections it is in the earlier paper [**Zi**].

REMARK 2.12. These previous works rely on global ordering data/conditions on the vertices of a triangulation or ideal triangulation of the 3-manifold. By contrast, in our work we only use edge orientations with no constraints. As a consequence, our formula in Theorem 8.8 is a bit more complicated: it involves four variants of the dilogarithm, and also some cube roots of unity enter from a μ_3-symmetry not present in earlier approaches.

3. Chern-Simons as a topological field theory

The integral of a differential form over a smooth manifold M is *local*: if $M = \bigcup M_i$ is a finite union of submanifolds, possibly with boundaries and corners, and if $M_i \cap M_j$ has measure zero for all $i \neq j$, then the integral over M is the sum of the integrals over M_i. The exponentiated Chern-Simons invariant of a connection on a principal bundle $P \to M$ is not the integral of a differential form on M, yet it still satisfies strong locality properties: it is the *partition function* of an *invertible field theory*. We review this aspect of Chern-Simons invariants. See [**FN**, §2] for an exposition of the theory with gauge group $\mathbb{C}^\times$.

Let G be a Lie group with finitely many components, called the *gauge group*, and let $\mathfrak{g}$ be its Lie algebra. In this section we make the simplifying assumption that G is simply connected. Let $\pi\colon P \to M$ be a principal G-bundle with connection[7] $\Theta \in \Omega^1_P(\mathfrak{g})$. Suppose

$$\langle -, - \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \tag{3.1}$$

is a G-invariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$. Chern-Simons [**CS2**] define a scalar 3-form on the total space P,

$$\eta(\Theta) = \langle \Theta \wedge \Omega \rangle - \frac{1}{6}\langle \Theta \wedge [\Theta \wedge \Theta] \rangle \ \in \Omega^3_P, \tag{3.2}$$

where $\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta] \in \Omega^2_P(\mathfrak{g})$ is the curvature of Θ. If $\dim M \leq 3$, then $\eta(\Theta)$ is closed. Also, in that case the simple connectivity of G ensures the existence of sections $s\colon M \to P$ of $\pi\colon P \to M$. If $M = X$ is a closed oriented 3-manifold, then

$$\int_X s^*\eta(\Theta) \ \in \mathbb{C} \tag{3.3}$$

is unchanged under a homotopy of s, since $\eta(\Theta)$ is closed. The space of sections is generally not connected, so to ensure that (3.3) is independent of s we

[7] We use 'G-connection' as a shorthand for 'principal G-bundle with connection'.

make two modifications: (i) we impose an integrality hypothesis on $\langle -,-\rangle$, and (ii) we reduce the integral to $\mathbb{C}/\mathbb{Z}(1)$. The integrality condition lies in topology if G is compact or G is complex, which we now assume. Namely, the vector space of forms $\langle -,-\rangle$ is canonically isomorphic[8] to $H^4(BG;\mathbb{C})$, where BG is the classifying space of G. The image of $H^4(BG;\mathbb{Z})\to H^4(BG;\mathbb{C})$ is a lattice of integral forms.[9] Then if $\langle -,-\rangle$ is an integral form,

$$2\pi\sqrt{-1}\int_X s^*\eta(\Theta) \mod \mathbb{Z}(1) \tag{3.4}$$

is independent of s. Define the *exponentiated Chern-Simons invariant*[10]

$$\mathscr{F}_G(X;\Theta)=\exp\left(2\pi\sqrt{-1}\int_X s^*\eta(\Theta)\right)\in\mathbb{C}^\times. \tag{3.5}$$

REMARK 3.6. The exponentiated Chern-Simons invariant is defined without the simple connectivity assumption on G. In that case the form $\langle -,-\rangle$ is replaced by a class in $H^4(BG;\mathbb{Z})$, called the *level*. See [**F4**, Appendix] for the general construction.

EXAMPLE 3.7 ($G=\mathrm{SL}_2\mathbb{C}$). The special linear group $G=\mathrm{SL}_2\mathbb{C}$ is a matrix group with Lie algebra $\mathfrak{g}=\mathfrak{sl}_2\mathbb{C}$ the space of 2×2 traceless complex matrices. There is an isomorphism $H^4(B\mathrm{SL}_2\mathbb{C};\mathbb{Z})\cong\mathbb{Z}$ with generator

$$\langle A,B\rangle=-\frac{1}{8\pi^2}\,\mathrm{trace}(AB),\qquad A,B\in\mathfrak{sl}_2\mathbb{C}, \tag{3.8}$$

the complex image of $-c_2\in H^4(B\mathrm{SL}_2\mathbb{C};\mathbb{Z})$. On the trivial bundle over X, an $\mathrm{SL}_2\mathbb{C}$-connection is a traceless matrix of 1-forms $A\in\Omega^1_X(\mathfrak{sl}_2\mathbb{C})$ and the Chern-Simons invariant (3.4) is

$$\frac{1}{4\pi\sqrt{-1}}\int_X \mathrm{trace}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\mod \mathbb{Z}(1). \tag{3.9}$$

We remark that our choice of $-c_2$ is motivated by (2.6); if we chose c_2 instead, we would have an extra minus sign in that equation.

Now suppose X' is a compact 3-manifold with boundary, and let Θ' be a G-connection on X' for G a simply connected Lie group. We define $\mathscr{F}_G(X';\Theta')$ so that if $X=X_1\cup_N X_2$ is a decomposition of a closed oriented 3-manifold X along an embedded closed codimension one oriented submanifold N, then

$$\mathscr{F}_G(X;\Theta)=\mathscr{F}_G(X_1;\Theta_1)\cdot\mathscr{F}_G(X_2;\Theta_2), \tag{3.10}$$

[8]The isomorphism maps a form $\langle -,-\rangle$ to the de Rham cohomology class of $\langle\Omega,\Omega\rangle$, where Ω is the curvature of a universal principal G-connection over BG.

[9]There is also a distinguished cone of forms whose restriction to a maximal compact Lie subalgebra is positive definite. For G connected, the map $H^4(BG;\mathbb{Z})\to H^4(BG;\mathbb{C})$ is injective: $H^4(BG;\mathbb{Z})$ is torsionfree.

[10]The notation is deficient, as it does not include the form $\langle -,-\rangle$, but the choice should be clear from the context.

where $\Theta_i = \Theta\big|_{X_i}$. If $\partial X' \neq \emptyset$ then (3.4) is not independent of s; it depends on $s\big|_{\partial X'}$. That dependence satisfies a cocycle relation which leads to the construction of a complex line $\mathscr{F}_G(\partial X'; \partial\Theta')$ which only depends on $\partial\Theta' = \Theta'\big|_{\partial X'}$. The exponentiated Chern-Simons invariant $\mathscr{F}_G(X';\Theta')$ is an element of $\mathscr{F}_G(\partial X';\partial\Theta')$, and (3.10) is satisfied if the dot on the right hand side is interpreted as the pairing of this line with its dual; see [**F1**, §2].

We summarize the situation in the language of field theory. Let $\mathrm{Bord}_{\langle 2,3\rangle}(\mathrm{GL}_3^+\mathbb{R} \times G^\nabla)$ be the bordism category where objects are closed oriented 2-manifolds Y equipped with a G-connection Θ_Y. (The notation indicates the structure group of the manifold, and the superscript ∇ evokes the connection on the principal G-bundle.) A morphism $(Y_0,\Theta_0) \to (Y_1,\Theta_1)$ in $\mathrm{Bord}_{\langle 2,3\rangle}(\mathrm{GL}_3^+\mathbb{R}\times G^\nabla)$ is then a compact oriented 3-manifold X equipped with a G-connection Θ_X, together with a diffeomorphism $-Y_0 \amalg Y_1 \xrightarrow{\cong} \partial X$, and an isomorphism $\Theta_0 \amalg \Theta_1 \xrightarrow{\cong} \partial\Theta_X$. (These diffeomorphisms need to be on collar neighborhoods—or germs of collar neighborhoods—of the boundary.) As usual in bordism categories, composition of morphisms is defined by gluing bordisms, and there is a symmetric monoidal structure given by disjoint union. Let $\mathrm{Line}_{\mathbb{C}}$ denote the groupoid whose objects are 1-dimensional complex vector spaces, and whose morphisms are invertible linear maps. It is a Picard groupoid under tensor product of lines.

THEOREM 3.11. *The exponentiated Chern-Simons invariant is a symmetric monoidal functor*

$$\mathscr{F}_G\colon\ \mathrm{Bord}_{\langle 2,3\rangle}(\mathrm{GL}_3^+\mathbb{R} \times G^\nabla) \longrightarrow \mathrm{Line}_{\mathbb{C}}\,. \tag{3.12}$$ □

So $\mathscr{F}_G$ is an *invertible field theory*, called *classical Chern-Simons theory*; see [**HS, F2**].

Our interest in this paper is the restriction to *flat* G-connections

$$\mathscr{F}_G\colon\ \mathrm{Bord}_{\langle 2,3\rangle}(\mathrm{GL}_3^+\mathbb{R} \times G^\delta) \longrightarrow \mathrm{Line}_{\mathbb{C}}\,. \tag{3.13}$$

This restricted theory is *topological* in a restricted sense—at least on single manifolds (see Remark 3.15 below). Namely, the domain bordism category has no continuously varying parameters.[11] There is a well-developed mathematical theory of topological field theories. In this topological case it is technically easier to implement strong locality in the form of an *extended* field theory. For *invertible* topological theories, homotopy-theoretic methods can be brought to bear [**FHT, FH1**]: an invertible topological field theory can be realized as a map of spectra. The domain is a bordism spectrum and the codomain a spectrum of "higher lines". In that context, for $G = \mathrm{SL}_2\mathbb{C}$ the extended version of (3.13) is realized as the composition

$$MSO \wedge BG^\delta_+ \xrightarrow{\mathrm{id}\wedge(-\hat{c}_2)} MSO \wedge (H\mathbb{C}/\mathbb{Z}(1)_3)_+ \xrightarrow{\int} \Sigma^3 H\mathbb{C}/\mathbb{Z}(1). \tag{3.14}$$

[11]More precisely, for any $t \in \mathbb{A}^1$ the restriction map $\mathrm{Bun}_{G^\delta}(\mathbb{A}^1 \times M) \to \mathrm{Bun}_{G^\delta}(\{t\} \times M)$ on *flat* connections is an equivalence of stacks.

Here MSO is the Thom spectrum of oriented manifolds, $H\mathbb{C}/\mathbb{Z}(1)$ is the Eilenberg-MacLane spectrum associated to the abelian group $\mathbb{C}/\mathbb{Z}(1)$, the cohomology class $\hat{c}_2$ is introduced in (2.11), and $(H\mathbb{C}/\mathbb{Z}(1)_3)_+$ denotes the 3-space of the spectrum $H\mathbb{C}/\mathbb{Z}(1)_3$. The first map is the Cheeger-Simons class (2.11), and the second is integration; see [**HS**, §4.10]. The induced map on π_3 is a bordism invariant of closed oriented 3-manifolds equipped with a *flat* connection. The map (3.14) extends this bordism invariant to an invertible topological field theory, thereby exhibiting its full locality.

Remark 3.15. Our analysis in this paper involves parametrized families of flat connections, that is, connections on the total space of a fiber bundle $\pi\colon M \to S$ that are flat along the fibers of π. Such connections need not be flat on M, and for that reason we need more than the homotopy theoretic map (3.14), since the latter only incorporates families of flat connection in which the connection is flat on the total space M. It is in this broader sense that the invertible field theory (3.13) is not topological. We explain this further in Appendix B.

Remark 3.16. In codimension 1—on closed surfaces—we wrote in (3.13) that the theory $\mathscr{F}_G$ has values complex lines. Similarly, in codimension 2—on closed 1-manifolds—we take the values of the theory to be $\mathbb{V}$-*lines*, i.e., invertible modules over the tensor category $\mathbb{V}$ of complex vector spaces.

There is a spin variant of Chern-Simons theory, which we discuss in §5.2 in a special case.

4. Stratified abelianization and spectral networks

We begin in §4.1 with an elementary concrete example of stratified abelianization which motivates all that follows. Here one explicitly sees the monodromy around branch points (Lemma 4.12) and the unipotent automorphism when crossing a wall (Equation (4.11)). We abstract this into a general definition in §4.2. The data of a spectral network is the specification of a particular type of stratified manifold. This is all for rank one Lie groups. In §4.3 we construct a spectral network and stratified abelianization for a triangulated surface, and in §4.4 we do the same for a triangulated 3-manifold. An important example is a 2-sphere triangulated by the boundary of a tetrahedron—this is the boundary of a 3-simplex, which we encounter at the center of each 3-dimensional tetrahedron in the triangulation of a 3-manifold—and we prove an important relation in the stratified abelianization in Proposition 4.43. Our setup here makes contact with cross ratios (Remark 4.49) and the Thurston gluing equations (Remark 4.56).

4.1. 2-dimensional spectral networks: motivation. To motivate stratified abelianization, begin with an invertible 2×2 complex matrix $A \in \mathrm{GL}_2\mathbb{C}$. For a geometric take, let $E \to S^1$ be a rank 2 flat complex vector bundle with holonomy A. Then A is diagonalizable if and only if there exist

$$L \longrightarrow S^1 \amalg S^1 \tag{4.1}$$

$$\pi_* L \xrightarrow{\cong} E, \tag{4.2}$$

where $\pi\colon S^1 \amalg S^1 \to S^1$ is the product[12] double cover, (4.1) is a (flat) line bundle, and (4.2) is an isomorphism. If so, and if A is not a scalar matrix, then the projectivization $\mathbb{P}E \to S^1$ has two distinguished horizontal sections; the line bundle $L \to S^1 \amalg S^1$ is isomorphic to the restriction of the tautological line bundle $\mathcal{L} \to \mathbb{P}E$ to the union of the images of those sections. If A is not diagonalizable, then the existence of an eigenvector of A implies that $\mathbb{P}E \to S^1$ has a unique flat section.

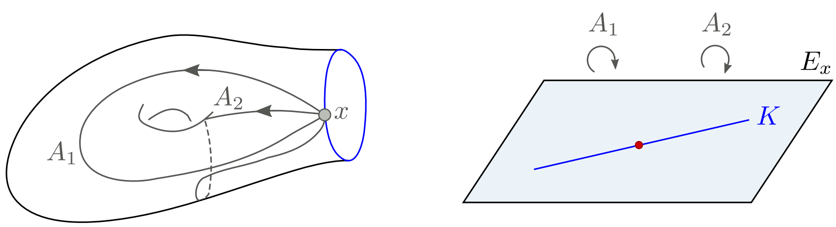


FIGURE 2. A flat bundle over the one-holed torus.

Now consider two invertible matrices $A_1, A_2 \in \mathrm{GL}_2\mathbb{C}$. If $A_1A_2 = A_2A_1$, then there is a flat rank 2 complex vector bundle $E \to S^1 \times S^1$ with holonomies A_1, A_2 about chosen based loops generating $\pi_1(S^1 \times S^1)$. Then—assuming each of A_1, A_2 is diagonalizable—there is a global abelianization based on the product double cover of $S^1 \times S^1$. Our story begins when $A_1A_2 \neq A_2A_1$. In this situation the matrices A_1, A_2 determine a flat rank 2 complex vector bundle $E \to Y$ over the compact surface $Y = S^1 \times S^1 \setminus D^2$, as depicted in Figure 2. Let $x \in \partial Y$ be a basepoint. There is no hope of a global abelianization. Instead, consider the ideal triangulation of Y depicted in Figure 3. If we collapse the boundary ∂Y, there are 2 triangles, glued along 3 edges; each vertex is the point at "infinity" in $Y/\partial Y$. We interpret Figure 3 as a stratification

$$Y = Y_0 \amalg Y_{-1} \amalg Y_{-2}. \tag{4.3}$$

The codimension 2 stratum Y_{-2} consists of two points, one interior to each face. The codimension 1 stratum Y_{-1} is the union of six line segments, joining the codimension 2 stratum to the vertices. The generic stratum Y_0 is the complement of the lower dimensional strata.

The first step in stratified abelianization is the choice of a parallel section of the associated projective bundle over ∂Y, equivalently an eigenline $K \subset E_x$ of the commutator $A_1A_2A_1^{-1}A_2^{-1}$. The generic stratum $Y_0 = R^{(1)} \amalg R^{(2)} \amalg R^{(3)}$ has three contractible components, and for each $i \in \{1, 2, 3\}$

[12] Note the sheets are not ordered.

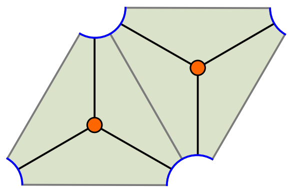


FIGURE 3. The stratification $Y = Y_0 \amalg Y_{-1} \amalg Y_{-2}$. The orange points make up Y_{-2}; the black segments (walls) make up Y_{-1}; the rest (including the gray edges of the triangles and the blue boundary arcs) is Y_0.

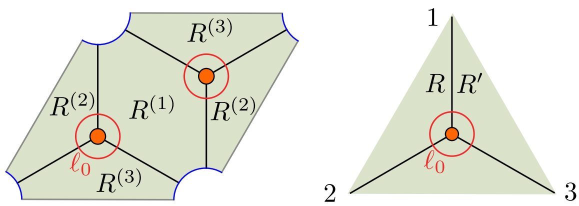


FIGURE 4. Computing the unipotent gluing and holonomy.

the intersection $R^{(i)} \cap \partial Y$ has two contractible components; see Figure 4. By parallel transport from ∂C we obtain for each $i \in \{1, 2, 3\}$ two parallel sections of $\mathbb{P}E\,\big|_{R^{(i)}} \to R^{(i)}$. In the second drawing of Figure 4 the two sections in each component R of Y_0 are labeled by the two vertices in the closure of R.

ASSUMPTION 4.4 (genericity). *For each $i \in \{1, 2, 3\}$, these sections are distinct.*

Then, as in the 1-dimensional case, construct a global abelianization over the generic stratum:

$$\pi\colon \widetilde{Y}_0 \longrightarrow Y_0 \qquad \text{double cover} \tag{4.5}$$

$$L \longrightarrow \widetilde{Y}_0 \qquad \text{flat line bundle} \tag{4.6}$$

$$\pi_* L \xrightarrow{\;\cong\;} E\big|_{Y_0} \qquad \text{isomorphism of flat bundles} \tag{4.7}$$

The map π is the restriction of $\mathbb{P}E \to Y$ over the image of the two sections, and the line bundle (4.6) is the restriction of the tautological line bundle $\mathcal{L} \to \mathbb{P}E$ to $\widetilde{Y}_0 \subset \mathbb{P}E$. The genericity assumption allows us to construct (4.7) from the embedding $\mathcal{L} \to \mathbb{P}E \times E$.

As a preliminary, suppose ℓ_1, ℓ_2, ℓ_3 are three distinct lines in a 2-dimensional vector space F. Define

$$\mathrm{proj}_{\ell_1}\colon \ell_2 \longrightarrow \ell_3 \tag{4.8}$$

as the composition $\ell_2 \hookrightarrow F \xrightarrow{\ell_1} \ell_3$, where the second map is projection with kernel ℓ_1; the composition is an isomorphism.

Our task is to extend the abelianization to a structure over the lower strata. Fix a component I of Y_{-1} and let R, R' be the components of Y_0 on either side of I. The intersection point $I \cap \partial Y$ picks out contiguous components of $R \cap \partial Y$ and $R' \cap \partial Y$. Glue the corresponding sheets of the double cover (4.5) along I; there is a distinguished sheet along I from the contiguous components. In this manner extend (4.5) to a double cover

$$\pi\colon \widetilde{Y}_{\ge -1} \longrightarrow Y_{\ge -1} \tag{4.9}$$

together with a section s of π over Y_{-1}. Next, extend (4.6) to a flat line bundle

$$L \longrightarrow \widetilde{Y}_{\ge -1} \tag{4.10}$$

as follows. (We refer to Figure 4.) In passing from R to R', on the sheet obtained by parallel transport from vertex 1 glue $L \to \widetilde{Y}_0$ via the identity. Cover the identification of the sheet 2 in R and the sheet 3 in R' with the isomorphism (4.8) of the line bundle $L \to \widetilde{Y}_0$ across the segment in $\widetilde{Y}_{-1}$.

We compare the isomorphisms (4.7) on each side of Y_{-1}. By construction, the unipotent automorphism passing from region R to region R' is

$$\begin{aligned} \ell_1 \oplus \ell_2 &\longrightarrow \ell_1 \oplus \ell_3 \\ \xi_1 + \xi_2 &\longmapsto \xi_1 + \mathrm{proj}_{\ell_1}(\xi_2) \end{aligned} \tag{4.11}$$

Lemma 4.12. *Let λ be the link of $Y_{-2} \subset Y$ and $\lambda_0 \subset \lambda$ a component of λ.*

(i) *The restriction of the double cover (4.9) to λ_0 is nontrivial.*
(ii) *The holonomy of (4.10) about $\pi^{-1}(\lambda_0)$ is $-\mathrm{id}$.*

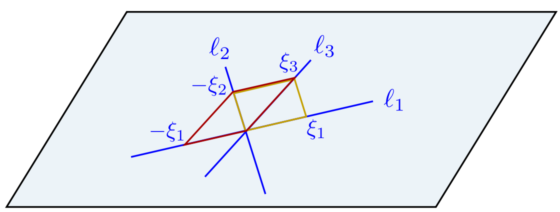


Figure 5. Computing holonomy by composing projections.

Proof. The proof of (i) is straightforward; we leave it to the reader. For (ii), the holonomy about $\pi^{-1}(\lambda_0)$ is the composition

$$\ell_1 \xrightarrow{\mathrm{id}} \ell_1 \xrightarrow{\mathrm{proj}_{\ell_2}} \ell_3 \xrightarrow{\mathrm{id}} \ell_3 \xrightarrow{\mathrm{proj}_{\ell_1}} \ell_2 \xrightarrow{\mathrm{id}} \ell_2 \xrightarrow{\mathrm{proj}_{\ell_3}} \ell_1. \tag{4.13}$$

Fix $\xi_1 \in \ell_1$, and let $\xi_2 \in \ell_2$, $\xi_3 \in \ell_3$ be the unique vectors such that $\xi_1 = \xi_2 + \xi_3$. Then under (4.13)

$$\xi_1 \longmapsto \xi_1 \longmapsto \xi_3 \longmapsto \xi_3 \longmapsto -\xi_2 \longmapsto -\xi_2 \longmapsto -\xi_1. \tag{4.14}$$ ∎

4.2. Stratifications, spectral networks, and abelianization. The double cover (4.9) together with the section over Y_{-1} is called a *spectral network* (subordinate to the stratification (4.3)). Components of Y_{-1} are the *walls* and Y_{-2} is the *branch locus*. Notice that Lemma 4.12(i) implies that $\widetilde{Y}_{\ge -1} \to Y_{\ge -1}$ extends to a *branched* double cover $\widetilde{Y} \to Y$ with branch locus Y_{-2}. The stratified abelianization of $E \to Y$ is the data:

- the flat line bundle $L \to \widetilde{Y}_{\ge -1}$
- the isomorphism (4.7) on Y_0
- the unipotent gluing (4.11) on Y_{-1}

In this subsection we give formal definitions of this structure which apply in some generality.

Two-dimensional spectral networks were introduced by Gaiotto-Moore-Neitzke [**GMN1**] in their study of supersymmetric 4-dimensional gauge theories. They have motivated many mathematical constructions and conjectures since, related to hyperkähler geometry, enumerative invariants, and asymptotic analysis of complex ODE, among others.

4.2.1. *Stratifications.* We use the definition [**L**, 4.3.2]. In that approach a type $\vec{\mathfrak{X}}$ of stratified manifold of dimension n is defined from the *top down*. Namely, begin with a geometric structure[13] for the generic stratum in codimension 0. Then for each $1 \le k \le n$ specify the geometric structure and link of a codimension k stratum; the link is an $\vec{\mathfrak{X}}$-stratified $(k-1)$-dimensional manifold. An $\vec{\mathfrak{X}}$-stratified manifold of dimension $\le n$ is built from the *bottom up*: first the highest codimension strata are specified, then higher strata with the proper link are glued in successively. This heuristic depiction is fleshed out precisely in [**L**, 4.3.2], and the heuristic specifications of the following definition can easily be formulated in that precise framework.

Definition 4.15. An *SN-stratification* on a manifold-with-corners of dimension ≤ 3 has the following specifications.

(i) *codimension 0*: a codimension 0 smooth manifold;
(ii) *codimension 1*: a codimension 1 submanifold—the link is a 0-sphere;
(iii) *codimension 2*: a *Type*[14] *a* codimension 2 stratum has link a circle with an arbitrary codimension 1 submanifold consisting of a finite set of points; a *Type b* codimension 2 stratum has link a circle with a codimension 1 submanifold consisting of 3 points;
(iv) *codimension 3*: a *Type a* codimension 3 stratum has link a 2-sphere with an SN-stratification consisting of a codimension 1 trivalent graph whose vertices are of Type a; a *Type b* codimension 3 stratum has link a 2-sphere with the standard SN-stratification of the boundary of a tetrahedron (Construction 4.37 below).

[13]That is, a topological space $\mathfrak{X}$ equipped with a continuous map $\mathfrak{X} \to BO_n$. An n-manifold M with an $\mathfrak{X}$-structure is equipped with a lift $M \to \mathfrak{X}$ of the classifying map of its tangent bundle.

[14]Mnemonic: Type a is "anodyne", Type b is "branch."

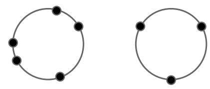


FIGURE 6. Links of codimension 2 strata. For Type a the link can contain an arbitrary number of points of M_{-1}, while for Type b it must contain exactly three.

We use the term *SN-stratified manifold* for a manifold equipped with an SN-stratification.

REMARK 4.16. There is a generalization of Definition 4.15 to manifolds with boundary and corners. The key point is that the SN-strata intersect boundaries and corners transversely.

In §§4.3, 4.4 we define canonical SN-stratifications associated to semi-ideal triangulations of 2- and 3-manifolds.

REMARK 4.17. An SN-stratified manifold M is decomposed as a disjoint union

$$M = M_0 \amalg M_{-1} \amalg M_{-2\mathrm{a}} \amalg M_{-3\mathrm{a}} \amalg M_{-2\mathrm{b}} \amalg M_{-3\mathrm{b}} \tag{4.18}$$

where $M_{-2\mathrm{a}}$ is the union of codimension 2 strata of Type a, $M_{-2\mathrm{b}}$ is the union of codimension 2 strata of Type b, and likewise $M_{-3\mathrm{a}}$, $M_{-3\mathrm{b}}$ are the unions of codimension 3 strata. This unusual notation is convenient for subsequent definitions: the Type a strata of codimension 2,3 behave as singular parts of the codimension 1 strata. Hence define

$$\begin{aligned} M_{\geq -3\mathrm{a}} &= M_0 \amalg M_{-1} \amalg M_{-2\mathrm{a}} \amalg M_{-3\mathrm{a}}, \\ M_{\geq -2\mathrm{b}} &= M_0 \amalg M_{-1} \amalg M_{-2\mathrm{a}} \amalg M_{-3\mathrm{a}} \amalg M_{-2\mathrm{b}}. \end{aligned} \tag{4.19}$$

4.2.2. *Rank one Lie groups.* Spectral networks and abelianization data are conveniently formalized in terms of a triple of complex Lie groups $G \supset H \supset T$ in which T is a (complex) maximal torus of G and H its normalizer. In this paper we restrict to the groups $\mathrm{GL}_2\mathbb{C}$, $\mathrm{SL}_2\mathbb{C}$, and very occasionally $\mathrm{PSL}_2\mathbb{C}$. For $G = \mathrm{GL}_2\mathbb{C}$ we choose $T \cong \mathbb{C}^\times \times \mathbb{C}^\times$ the subgroup of diagonal matrices; then its normalizer is

$$H = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\} \subset \mathrm{GL}_2\mathbb{C}, \tag{4.20}$$

a 2-component Lie group with identity component T. Choose the diagonal matrices to be the maximal torus of $\mathrm{SL}_2\mathbb{C}$ and its image in $\mathrm{PSL}_2\mathbb{C}$ to be the maximal torus in the projective linear group; in each case the normalizer H of T has two components. Let $U \subset \mathrm{GL}_2\mathbb{C}$ be the subgroup

$$U = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\} \tag{4.21}$$

of upper triangular unipotent matrices. Then $U \subset \mathrm{SL}_2\mathbb{C}$ as well, and U projects to a unipotent subgroup of $\mathrm{PSL}_2\mathbb{C}$.

REMARK 4.22. Each of the three groups G acts on the projective line $\mathbb{P}(\mathbb{C}^2) = \mathbb{CP}^1$. In each case H is the stabilizer subgroup of the 2-point subset $A \subset \mathbb{CP}^1$ of the axes in $\mathbb{C}^2$, and T is the subgroup of elements of H that act as the identity on A. The stabilizer of the first axis $\ell \subset \mathbb{C}^2$ is a Borel subgroup $B \subset G$, and there is a diffeomorphism $G/B \approx \mathbb{CP}^1$. Then for $G = \mathrm{GL}_2\mathbb{C}$ or $\mathrm{SL}_2\mathbb{C}$, the Borel B acts linearly on ℓ and $\mathbb{C}^2/\ell$, and $U \subset B$ is the subgroup of elements that act trivially on both ℓ and $\mathbb{C}^2/\ell$.

4.2.3. *Definition of spectral networks and stratified abelianization data.* Assume $G \supset H \supset T$ is one of the three triples defined in §4.2.2. We refer to it as the pair (G, T), since H is determined as the normalizer of $T \subset G$. Some notation: If $Q \to M$ is a principal H-bundle, then we denote by $\iota(Q) = Q \times_H G \to M$ its "inflation" to a principal G-bundle. Also, if $w \subset M_{-1}$ is a wall (a component), and $R \to w$ is a principal T-bundle, then there is an associated fiber bundle of groups

$$U_w = R \times_T U \longrightarrow w, \tag{4.23}$$

where T acts on U by conjugation.

The following definition applies to rank one groups, as does Definition 4.15; there are stratifications, spectral networks, and stratified abelianization data in higher rank as well [**GMN1, GMN2, LP, IM**].

DEFINITION 4.24. Let M be a compact manifold of dimension ≤ 3 with boundary. Suppose M is equipped with an SN-stratification $M \setminus \partial M = M_0 \amalg M_{-1} \amalg M_{-2\mathrm{a}} \amalg M_{-3\mathrm{a}} \amalg M_{-2\mathrm{b}} \amalg M_{-3\mathrm{b}}$.

(i) A *spectral network* $\mathcal{N} = (\pi, s)$ *subordinate to the stratification of* M is:
- a double cover $\pi\colon \widetilde{M}_{\geq -3\mathrm{a}} \to M_{\geq -3\mathrm{a}}$ which restricts nontrivially to the link of each point in $M_{-2\mathrm{b}}$
- a section s of π over $M_{-1} \amalg M_{-2\mathrm{a}} \amalg M_{-3\mathrm{a}}$

(ii) *Stratified abelianization data* $\mathcal{A} = (P, Q, \mu, \theta)$ *of type* (G, T) *over* $(M, \mathcal{N})$ is the data:
- a principal G-bundle $P \to M$ with flat connection
- a principal H-bundle $Q \to M_{\geq -3\mathrm{a}}$ with flat connection
- an isomorphism of double covers $\mu\colon \widetilde{M}_{\geq -3\mathrm{a}} \to Q/T$ over $M_{\geq -3\mathrm{a}}$
- a flat isomorphism $\theta\colon \iota(Q) \to P$ over M_0

We require that the discontinuity of θ lie in $U_w \to w$ as we cross a point of the wall $w \subset M_{-1}$.

Observe that the section s reduces the restriction of $Q \to M_{\geq -3\mathrm{a}}$ over $M_{-1} \amalg M_{-2\mathrm{a}} \amalg M_{-3\mathrm{a}}$ to a principal T-bundle; on a wall $w \subset M_{-1}$ the fiber bundle of groups $U_w \to w$ is defined in (4.23). Stratified abelianization data over a given $(M, \mathcal{N})$ form a category; we leave to the reader the definition of the morphisms. Our usage of the term 'spectral network' often includes the underlying SN-stratification.

Remark 4.25. Definition 4.24 is adequate for our purposes but does not capture the most general rank one spectral networks which can occur in nature, e.g. from trajectory structures of meromorphic or holomorphic quadratic differentials on Riemann surfaces; see [**GMN2, HN, Fe**], for example.

4.3. 2-dimensional spectral networks from triangulations. Let $\mathbb{A}^2$ denote the standard affine plane. Denote the convex hull of a subset $T \subset \mathbb{A}^2$ as $\mathrm{Conv}(T)$. An affine triangle $\hat{\Delta}$ is the convex hull $\mathrm{Conv}(p_0, p_1, p_2)$ of three non-collinear points p_0, p_1, p_2. Fix some $\varepsilon \in (0, \frac{1}{2})$, say $\varepsilon = \frac{1}{10}$. The truncated triangle $\Delta \subset \hat{\Delta}$ is the convex hull of the six points $(1-\varepsilon)p_i + \varepsilon p_j$ for $i \neq j$, as shown in Figure 7. We will sometimes refer to "edges" or "vertices" of Δ, meaning the corresponding edges or vertices of $\hat{\Delta}$.

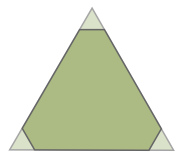


Figure 7. The truncated affine triangle Δ contained in the affine triangle $\hat{\Delta}$.

Let S° be the quotient of a finite union of disjoint truncated affine triangles $\{\Delta_i\}_{i \in I}$ whose edges are identified in pairs via affine isomorphisms. Then S° can be given the structure of a smooth compact 2-manifold with boundary.[15] The gluing of edges induces an equivalence relation on the $3N$ vertices of the N triangles Δ_i; each equivalence class of vertices corresponds to a boundary component of S°, with the topology of a circle. We sometimes call such an equivalence class a "glued vertex" or simply a "vertex". Partition the glued vertices into two subsets of *interior* and *ideal* vertices. Let S be a space obtained by gluing a copy of the standard disc to each boundary component of S° corresponding to an interior vertex. S is a smooth compact 2-manifold with boundary; $\pi_0(\partial S)$ is canonically identified with the set of ideal vertices.

Definition 4.26. Let Y be a compact 2-manifold with boundary. A *semi-ideal triangulation* of Y is a diffeomorphism $S \to Y$, where S is a space of the sort just described. The semi-ideal triangulation is called *ideal* if all vertices are ideal, and just a *triangulation* if all vertices are interior.

Construction 4.27 (SN-stratification of a truncated triangle). A truncated affine triangle Δ carries a canonical SN-stratification $\Delta = \Delta_0 \amalg \Delta_{-1} \amalg \Delta_{-2\mathrm{b}}$ as follows. Let $c = (p_0 + p_1 + p_2)/3$ be the barycenter of Δ. Set $\Delta_{-2\mathrm{b}} = \{c\}$; the stratum Δ_{-1} is the union of the three line segments

[15] Indeed, since edges are identified in pairs, a neighborhood of any point on a glued edge is a disc; moreover the link of a vertex is easily seen to be a circle.

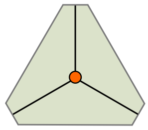


FIGURE 8. The SN-stratification of a truncated affine triangle.

$(\mathrm{Conv}(p_i, c) \cap \Delta) \setminus \Delta_{-2\mathrm{b}}$, $i = 0, 1, 2$; and Δ_0 is the complement of $\Delta_{\le -1}$. This SN-stratification is depicted in Figure 8.

CONSTRUCTION 4.28 (SN-stratification of the standard disc). Let D be the standard closed disc. For any finite subset $W \subset \partial D$ we obtain an SN-stratification $D = D_0 \amalg D_{-1} \amalg D_{-2\mathrm{a}}$ as follows. Let c be the center of D. Then $D_{-2\mathrm{a}} = \{c\}$; D_{-1} is the union of line segments connecting c to each point of W; and D_0 is the complement of $D_{\le -1}$. See Figure 9.

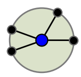


FIGURE 9. An SN-stratification of the disc, determined by a finite subset of the boundary circle.

CONSTRUCTION 4.29 (SN-stratification of a triangulated surface). Let Y be a compact 2-manifold with boundary equipped with a semi-ideal triangulation $\mathcal{T}$. The transport of the SN-stratifications on the truncated triangles and the discs around interior vertices defines an SN-stratification of Y. Figure 3 is an example where there are no interior vertices. See Figure 10 for an example with an interior vertex.

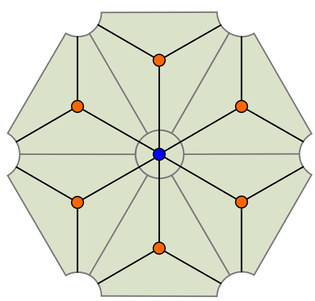


FIGURE 10. A portion of the SN-stratification of a semi-ideally triangulated closed 2-manifold with an interior vertex (center).

CONSTRUCTION 4.30 (spectral network on a triangulated surface). Let $\Delta = \Delta_0 \amalg \Delta_{-1} \amalg \Delta_{-2\mathrm{b}}$ be the canonical SN-stratification of a truncated affine triangle (Figure 8). The boundary of each component of Δ_0 contains precisely one edge of Δ; its boundary consists of two distinguished vertices. Let $\pi\colon \widetilde{\Delta}_0 \to \Delta_0$ be the (trivializable) double cover whose fiber consists of those two distinguished vertices. Each component of Δ_{-1} is in the closure of two components of Δ_0, with one vertex in common. Glue the corresponding sheets of the double cover to define $\pi\colon \widetilde{\Delta}_{\geq -1} \to \Delta_{\geq -1}$ together with a section s over Δ_{-1}, i.e., a spectral network on Δ. This construction glues across edges and extends to discs around interior vertices, and thus transports to give a spectral network $\mathcal{N}^{\mathfrak{T}}$ over a semi-ideally triangulated surface $(Y, \mathfrak{T})$.

Construction 4.31 (stratified abelianization data on a semi-ideally triangulated surface). Consider first $G = \mathrm{GL}_2\mathbb{C}$ or $G = \mathrm{SL}_2\mathbb{C}$. Assume that the compact 2-manifold Y has no closed components and is equipped with a semi-ideal triangulation $\mathfrak{T}$. Let $P \to Y$ be a flat principal G-bundle. On each component of ∂Y, choose a flat section of the associated $\mathbb{CP}^1$-bundle $P/B\big|_{\partial Y} \to \partial Y$, as in §4.1; see Remark 4.22 for the definition of the Borel subgroup $B \subset G$. Also choose an element of the fiber of P/B over each interior vertex. Use parallel transport—as in §4.1—to obtain two flat sections s, s' of $P/B\big|_{Y_0} \to Y_0$. The following is a generalization of Genericity Assumption 4.4:

Assumption 4.32 (genericity). *The sections s, s' are nowhere equal.*

Identify $P \to Y$ as a bundle of bases of a rank 2 complex vector bundle $E \to Y$. The submanifold of bases contained in the lines defined by the sections s, s' determines a reduction of the principal G-bundle $P \to Y_0$ to a principal H-bundle $Q \to Y_0$. For $c \in Y_{-1}$, the limits of s, s' from the two sides of $Y_{-1} \subset Y_{\geq -1}$ give three points $\ell_1, \ell_2', \ell_2'' \in (P/B)_c$ in the projective line $\mathbb{P}E_c$ over c. One of the sections has the same limit ℓ_1 on both sides; the other has two possibly distinct limits. Let $B_{\ell_1} \subset \operatorname{Aut} P_c$ be the subgroup of elements which fix ℓ_1. Then $(P/B)_c$ is the projectivization $\mathbb{P}E_c$ of the 2-dimensional vector space E_c, the group B_{ℓ_1} acts linearly on E_c, and we define φ_c to be the unipotent element (4.11). Glue using φ_c at each $c \in Y_{-1}$ to construct a flat principal H-bundle $Q \to Y_{\geq -1}$. This gives most of the stratified abelianization data Definition 4.24(ii). We leave the rest to the reader, as do we the slight modifications for $G = \mathrm{PSL}_2\mathbb{C}$.

Remark 4.33. The projection $Q \to \widetilde{Y}_{\geq -2\mathrm{a}}$, defined via the isomorphism μ of double covers, is a principal T-bundle. If $G = \mathrm{SL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{C}$, then $T \cong \mathbb{C}^\times$. If $G = \mathrm{GL}_2\mathbb{C}$, there is an associated principal $\mathbb{C}^\times$-bundle from the character $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \mapsto \lambda_1$ of T. Let $L \to \widetilde{Y}_{\geq -2\mathrm{a}}$ be the associated flat line bundle. Lemma 4.12 holds in this more general situation.

We conclude with a theorem about stratified abelianizations over a single triangle Δ equipped with the standard spectral network $\mathcal{N}$ depicted in Figure 8. Specialize to $G = \mathrm{SL}_2\mathbb{C}$ and the corresponding subgroups T, H, B, U. In this case there is a unique stratified abelianization, whose automorphism group is μ_2, in the following sense.

PROPOSITION 4.34. *Let $\mathcal{A} = (P, Q, \mu, \theta)$ and $\mathcal{A}' = (P', Q', \mu', \theta')$ be stratified abelianization data over $(\Delta, \mathcal{N})$. Then there is an isomorphism $\mathcal{A} \to \mathcal{A}'$, unique up to composition with the simultaneous action of -1 on P and Q.*

PROOF. First we construct a map of flat bundles $\varphi_Q : Q \to Q'$. The monodromy of Q around $\partial\Delta$ lies in $H \setminus T$, since $Q/T \simeq \widetilde{\Delta}$ is the nontrivial double cover, and likewise for Q'. But now recall that all elements of $H \setminus T$ are conjugate in H. It follows that there exists an isomorphism $\varphi_Q : Q \to Q'$ of flat H-bundles, unique up to composition with an automorphism of $Q \to \Delta$.

The automorphism group of $Q \to \Delta$ is the commutant of the monodromy, which is a cyclic group of order 4; either generator acts nontrivially on Q/T, and the order 2 element acts by $-1 \in H$. Thus, by composing with an automorphism of $Q \to \Delta$ if necessary, we may arrange that $\mu \circ \varphi_Q = \mu'$, and the remaining freedom in φ_Q is composition with the action of $-1 \in H$.

Next we construct a map of flat bundles $\varphi_P : P \to P'$. Along each wall w we have a section s_w of Q/T. On either side of the wall, $\theta(s_w)$ then gives a section of $P/T \to w$; the condition on the discontinuity of θ ensures that their projections to P/B agree, thus giving a section o_w of $P/B \to w$. Because $G/\{\pm 1\} = \mathrm{PSL}_2\mathbb{C}$ acts simply transitively on triples of distinct points of $G/B \simeq \mathbb{CP}^1$, there exists $\varphi_P : P \to P'$ which maps o_w to o'_w for all three walls w, and such a φ_P is unique up to a sign.

Finally we need to check that on Δ_0 we have (possibly after composing φ_P with the action of $-1 \in G$)

$$\varphi_P = \theta' \circ \varphi_Q \circ \theta^{-1}. \tag{4.35}$$

For this we consider the difference $\xi = \varphi_P^{-1} \circ \theta' \circ \varphi_Q \circ \theta^{-1}$ which is a covariantly constant section of $\mathrm{Aut}(P)|_{\Delta_0}$, with two properties:

- In a component of Δ_0 bounded by two walls w, w', the difference ξ belongs to the subgroup $T_{ww'} \simeq T$ preserving o_w and $o_{w'}$. Thus ξ acts by a constant scalar λ_w on o_w, with $\lambda_{w'} = \lambda_w^{-1}$.
- The discontinuity of ξ across w belongs to the subgroup $U_w \simeq U$. It follows that λ_w is the same on both sides of w.

Labeling the three walls as w_i (with i mod 3), the above properties say $\lambda_{w_{i+1}} = \lambda_{w_i}^{-1}$, which gives $\lambda_{w_{i+3}} = \lambda_{w_i}^{-1}$, so $\lambda_w = \lambda_w^{-1} = \lambda_{w_{i+1}}$ and thus $\xi = \pm 1$. This completes the proof. □

4.4. 3-dimensional spectral networks from triangulations. We begin with a 3-dimensional analog of Definition 4.26. A *tetrahedron* $\hat{\mathbb{A}}$ in $\mathbb{A}^3$ is the convex hull $\mathrm{Conv}(p_0, p_1, p_2, p_3)$ of four points in general position. The truncated tetrahedron $\mathbb{A} \subset \hat{\mathbb{A}}$ is the convex hull of the 12 points $(1-\varepsilon)p_i + \varepsilon p_j$ with $j \neq i$. (Recall $\varepsilon \in (0, 1)$, say $\varepsilon = \frac{1}{10}$.) See Figure 11. We will sometimes refer to "faces", "edges" or "vertices" of $\mathbb{A}$, meaning those of $\hat{\mathbb{A}}$.

Let S° be the quotient of a finite union of disjoint affine truncated tetrahedra $\{\mathbb{A}_i\}_{i \in I}$ whose faces are identified in pairs via affine isomorphisms. Then S° can be given the structure of a smooth compact 3-manifold with

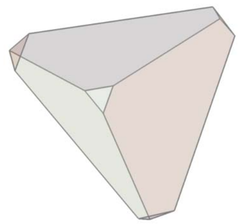


FIGURE 11. A truncated tetrahedron.

boundary. The gluing of faces induces an equivalence relation on the $4N$ vertices of the N tetrahedra Δ_i; each equivalence class of vertices corresponds to a boundary component of S°, which is a compact connected surface. We sometimes call such an equivalence class a "glued vertex" or simply a "vertex". Partition the glued vertices into two subsets of *interior* and *ideal* vertices, subject to the condition that the boundary component corresponding to an interior vertex must be diffeomorphic to S^2. Let S be a space obtained by gluing a copy of the standard 3-disc to each boundary component of S° corresponding to an interior vertex. S is a smooth compact 3-manifold with boundary; $\pi_0(\partial S)$ is canonically identified with the set of ideal vertices.

DEFINITION 4.36. Let Y be a compact 3-manifold with boundary. A *semi-ideal triangulation* of Y is a diffeomorphism $S \to Y$, where S is a space of the sort just described. The semi-ideal triangulation is called *ideal* if all vertices are ideal, and just a *triangulation* if all vertices are interior.

CONSTRUCTION 4.37 (Spectral network on a tetrahedron). Let $\hat{\Delta} = \mathrm{Conv}(p_0, p_1, p_2, p_3)$ be a tetrahedron in $\mathbb{A}^3$. Let $q_i = (p_{i+1}+p_{i+2}+p_{i+3})/3$ be the barycenter of the face opposite p_i, $i = 0, 1, 2, 3$; set $c = (p_0+p_1+p_2+p_3)/4$ the barycenter of Δ. (We use $p_{i+4} = p_i$, $i = 0, 1, 2, 3$.) Figure 12 depicts a

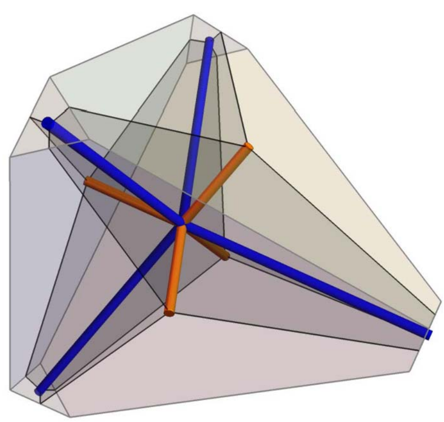


FIGURE 12. The canonical SN-stratification on a truncated affine tetrahedron.

canonical SN-stratification of Δ,

$$
\begin{aligned}
\Delta_{-3\mathrm{b}} &= \{c\} \\
\Delta_{-2\mathrm{b}} &= \bigcup_{i=0}^{4} \mathrm{Conv}(q_i, c) \setminus \Delta_{-3\mathrm{b}} \\
\Delta_{-3\mathrm{a}} &= \emptyset \\
\Delta_{-2\mathrm{a}} &= \bigcup_{i=0}^{4} \mathrm{Conv}(p_i, c) \cap \Delta \setminus \Delta_{-3\mathrm{b}} \\
\Delta_{-1} &= \bigcup_{i=0}^{4} \bigcup_{j=1}^{3} \mathrm{Conv}(p_{i+j}, q_i, c) \cap \Delta \setminus \Delta_{\ge -3\mathrm{a}} \\
\Delta_{0} &= \Delta \setminus \Delta_{\ge -1}
\end{aligned}
\tag{4.38}
$$

The link Y_c of $\Delta_{-3\mathrm{b}}$ is a 2-sphere triangulated as the boundary of a tetrahedron. By Construction 4.30 it has a canonical SN-stratification—the restriction of (4.38) to Y_c—and subordinate spectral network; see Figure 13. The same construction extends the SN-stratification to a 3-dimensional spectral network subordinate to (4.38). Namely, each component U of Δ_0 contains one edge with two vertices, and each component of Δ_{-1} in $\overline{U}$ corresponds to one of those vertices. Let $\pi\colon \widetilde{\Delta}_0 \to \Delta_0$ be the (trivializable) double cover whose fiber over U is the aforementioned set of two vertices, and glue along $\Delta_{-2\mathrm{a}} \amalg \Delta_{-1}$ by identifying the common vertex on each wall. This produces a double cover $\pi\colon \widetilde{\Delta}_{\ge -3\mathrm{a}} \to \Delta_{\ge -3\mathrm{a}}$ with a section s over $\Delta_{-2\mathrm{a}} \amalg \Delta_{-1}$, i.e., a spectral network. There is an extension to a branched double cover $\pi\colon \widetilde{\Delta}_{\ge -2\mathrm{b}} \to \Delta_{\ge -2\mathrm{b}}$ with branch locus $\Delta_{-2\mathrm{b}}$.

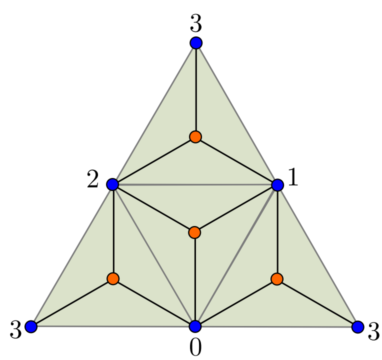


FIGURE 13. The spectral network on $Y = Y_c$.

REMARK 4.39. It will be convenient to excise an open ball about c as well as its inverse image on the branched double cover.

We investigate stratified abelianization on the link $Y = Y_c$ of the barycenter of Δ. The stratum $Y_{-2\mathrm{b}}$ of Y consists of 4 points, and the double cover

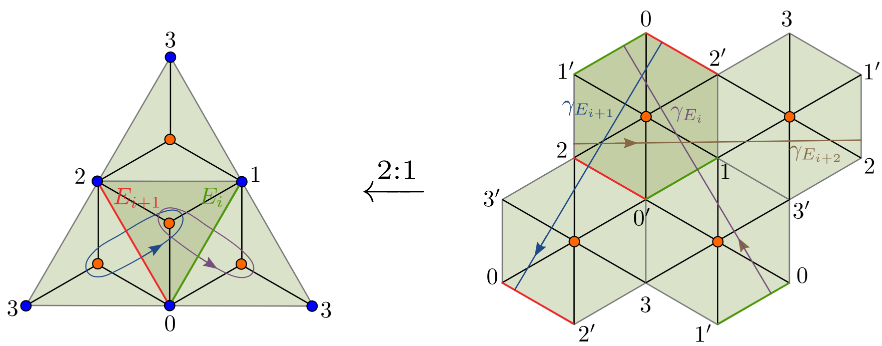


FIGURE 14. The branched double cover of the tetrahedrally triangulated 2-sphere Y_c by a 2-torus $\widetilde{Y}_c$, and a distinguished triple of cycles on $\widetilde{Y}_c$.

$\pi\colon \widetilde{Y}_{\geq -2\mathrm{a}} \to Y_{\geq -2\mathrm{a}}$ extends to a branched double cover $\pi\colon \widetilde{Y} \to Y$ in which $\widetilde{Y}$ is diffeomorphic to a 2-torus. The double cover π is depicted in Figure 14. The boundary of the tetrahedron has been unfolded, as in Figure 13, as has been the covering 2-torus. Assume Y is oriented, and use π to induce an orientation on $\widetilde{Y}$. To each edge E in $\triangle$ associate an element $\Gamma_E \in H_1(\widetilde{Y}_{\geq -2\mathrm{a}})$ as follows. Let $\bar{\lambda}_E \subset Y_{\geq -1}$ be a loop which crosses E twice transversely and encircles the branch points in the faces of $\triangle$ which abut E. Orient $\bar{\lambda}_E$ as the boundary of the region which contains these two branch points. The desired lift λ_E is distinguished from the other lift of $\bar{\lambda}_E$ as follows: the lifts to λ_E of the two intersection points $\bar{\lambda}_E \cap E$ lie on the sheet of the double cover labeled by the closest endpoint of E. Then Γ_E is the homology class of λ_E in $H_1(\widetilde{Y}_{\geq -2\mathrm{a}})$, the homology of the torus with the 4 branch points excised. Let $\gamma_E \in H_1(\widetilde{Y})$ be its image in the homology of the torus. We invite the reader to deduce the following, using Figure 14.

PROPOSITION 4.40.

(1) *The image of* $\Gamma_E \in H_1(\widetilde{Y}_{\geq -2\mathrm{a}})$ *under the deck transformation is* $-\Gamma_E$.
(2) *Opposite edges of* $\triangle$, *such as 13 and 02 in the figure, induce the same homology class in* $H_1(\widetilde{Y}_{\geq -2\mathrm{a}})$.
(3) *The three pairs of opposite edges lead to three homology classes* $\gamma_0, \gamma_1, \gamma_2 \in H_1(\widetilde{Y})$ *which sum to zero.*

Cyclically order the three pairs of opposite edges so that the intersection product $\langle \gamma_i, \gamma_{i+1} \rangle = +1$ for $i \in \mathbb{Z}/3\mathbb{Z}$. Denote the corresponding loops in $\widetilde{Y}_{\geq -2\mathrm{a}}$ as $\lambda_{E_0}, \lambda_{E_1}, \lambda_{E_2}$.

Let $\widetilde{Y}_{-2\mathrm{b}} \subset \widetilde{Y}$ be the set of 4 branch points $\pi^{-1}(Y_{-2\mathrm{b}})$. Since Y is simply connected, any flat G-bundle $P \to Y$ is trivializable. Fix ℓ_i in the fiber of the associated $\mathbb{CP}^1$-bundle $P/B \to Y$ at each vertex $p_i \in \triangle$. Let $\mathcal{E} \approx \mathbb{CP}^1$ be the space of horizontal sections of $P/B \to Y$, and let $\ell_i \in \mathcal{E}$ be the extension of the previous ℓ_i to a horizontal section. Assume that $\ell_0, \dots, \ell_3 \in \mathcal{E}$ are distinct; this implies the Genericity Assumption 4.32. Use Construction 4.31 to produce a stratified abelianization. By Remark 4.33 there is a flat line bundle

$$L \longrightarrow \widetilde{Y}_{\geq -2\mathrm{a}} \tag{4.41}$$

with holonomy -1 around each of the 4 (branch) points in $\widetilde{Y}_{-2\mathrm{b}}$. The isomorphism class of the flat bundle (4.41) is determined by its holonomy, a homomorphism

$$\mathrm{hol}_L \colon H_1(\widetilde{Y}_{\geq -2\mathrm{a}}) \longrightarrow \mathbb{C}^\times. \tag{4.42}$$

Set $z_i = \mathrm{hol}_L(\lambda_{E_i})$ for $i \in \mathbb{Z}/3\mathbb{Z}$.

Proposition 4.43. *The holonomies of L satisfy*

$$z_{i+1} = 1 - \frac{1}{z_i}. \tag{4.44}$$

Proof. We compute as in the proof of Lemma 4.12 using Figure 14 as a guide. The holonomy of $L \to \widetilde{Y}$ around λ_{E_i} is the composition

$$\ell_0 \xrightarrow{\mathrm{proj}_{\ell_3}} \ell_1 \xrightarrow{\mathrm{proj}_{\ell_2}} \ell_0 \tag{4.45}$$

and the holonomy of $L \to \widetilde{Y}$ around $\lambda_{E_{i+1}}$ is the composition

$$\ell_2 \xrightarrow{\mathrm{proj}_{\ell_3}} \ell_0 \xrightarrow{\mathrm{proj}_{\ell_1}} \ell_2 \tag{4.46}$$

(Recall the projections in (4.8).) Choose $\xi_j \in \ell_j^{\neq 0}$, $j = 1$ and then $j = 3, 0, 2$, such that

$$\begin{aligned} \xi_1 &= \xi_3 + \xi_0 \\ &= \xi_2 + z\xi_0 \end{aligned} \tag{4.47}$$

for some $z \in \mathbb{C} \setminus \{0,1\}$. Then $\mathrm{proj}_{\ell_3}(\xi_0) = \xi_1$ and $\mathrm{proj}_{\ell_3}(\xi_2) = (1-z)\xi_0$, etc. Hence the image of ξ_0 under (4.45) is $z\xi_0$, and the image of ξ_2 under (4.46) is $(1 - \frac{1}{z})\xi_2$. Therefore,

$$z_{i+1} = \mathrm{hol}_L(\lambda_{E_{i+1}}) = 1 - \frac{1}{z} = 1 - \frac{1}{\mathrm{hol}_L(\lambda_{E_i})} = 1 - \frac{1}{z_i}. \qquad \square \tag{4.48}$$

Remark 4.49.

(1) One interpretation of $z = \mathrm{hol}_L(\lambda_{E_i})$ is as follows. Recall that 4 distinct points in a projective line $\mathbb{P}F$ are characterized up to isomorphism by their cross-ratio. If ℓ_0, ℓ_1, ℓ_2, ℓ_3 are the corresponding lines in the 2-dimensional vector space F, then the cross-ratio is

$$\frac{(\xi_0 \wedge \xi_3)(\xi_1 \wedge \xi_2)}{(\xi_0 \wedge \xi_2)(\xi_1 \wedge \xi_3)} \in \mathbb{C} \setminus \{0,1\}, \qquad \xi_i \in \ell_i \text{ nonzero}, \tag{4.50}$$

where the numerator and denominator are nonzero elements in $(\mathrm{Det}\, F)^{\otimes 2}$; the ratio is independent of the choice of $\xi_i \in \ell_i^{\neq 0}$. Permuting the lines we obtain numbers z, $1/z$, $1-z$, $1/(1-z)$, $z/(z-1)$,

$(z-1)/z$ for some $z \in \mathbb{C}^\times \setminus \{1\}$. In the case at hand, with the chosen vectors ξ_1, ξ_2, ξ_3, ξ_4 in (4.47), we compute

$$\frac{(\xi_0 \wedge \xi_3)(\xi_1 \wedge \xi_2)}{(\xi_0 \wedge \xi_2)(\xi_1 \wedge \xi_3)} = z = \mathrm{hol}_L(\lambda_{E_i}). \tag{4.51}$$

(2) As a corollary of Proposition 4.43 the product of the holonomies around the loops λ_{E_0}, λ_{E_1}, λ_{E_2} defined after Proposition 4.40 is

$$z\left(1-\frac{1}{z}\right)\left(1-\frac{1}{1-\frac{1}{z}}\right) = -1. \tag{4.52}$$

This leads to a sharpening of Proposition 4.40(3). Let $S \subset \widetilde{Y}_{\ge -2\mathrm{a}}$ be a link of the 4 points $\widetilde{Y}_{-2\mathrm{b}} \subset \widetilde{Y}_{\ge -2\mathrm{a}}$, so $S = \bigsqcup_{k=1}^4 S_k$ is a union of 4 disjoint circles S_k, one surrounding each branch point. Form the commutative diagram

$$\begin{array}{ccccccccc} & & H_1(S) & \longrightarrow & H_1(\widetilde{Y}_{\ge -2\mathrm{a}}) & \longrightarrow & H_1(\widetilde{Y}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mu_2 & \longrightarrow & \widehat{H_1(\widetilde{Y})} & \longrightarrow & H_1(\widetilde{Y}) & \longrightarrow & 0 \end{array} \tag{4.53}$$

in which the homomorphism χ maps a generator of $H_1(S_k)$ to $-1 \in \mu_2$. The bottom row of (4.53) is a central group extension. The refinement of Proposition 4.40(3) is that the product of the images of $[\lambda_{E_i}]$ in $\widehat{H_1(\widetilde{Y})}$ is $-1 \in \mu_2$.

(3) The space $\mathcal{M} = \mathbb{C}\setminus\{0,1\} \approx \mathbb{CP}^1\setminus\{0,1,\infty\}$ is the domain of the real dilogarithm function (2.1), and the total space of an abelian cover $\widehat{\mathcal{M}} \to \mathcal{M}$ is the domain of the enhanced Rogers dilogarithm (2.9). In our current setup $\mathcal{M}$ is a space of flat $\mathbb{C}^\times$-connections on a punctured torus. In §7.2 we introduce an extra twist to get rid of the punctures, and so identify $\mathcal{M}$ as a space of flat $\mathbb{C}^\times$-connections on a torus. See [**FN**] for a development of the dilogarithm function with this starting point.

Construction 4.54 (SN-stratification on a 3-disc). Let D be the standard closed 3-disc. Given an SN-stratification of the boundary $\partial D = S^2$, of the form $\partial D = (\partial D)_0 \amalg (\partial D)_{-1} \amalg (\partial D)_{-2\mathrm{a}}$, we obtain an SN-stratification of D as follows. Let c be the center of D. Then $D_{-3\mathrm{b}} = \emptyset$, $D_{-3\mathrm{a}} = \{c\}$, and each other stratum D_α is the cone over $(\partial D)_\alpha$ with c removed.

Construction 4.55 (Stratified abelianization data on a 3-manifold). Let X be a compact 3-manifold with boundary, and suppose $\mathcal{T}$ is a semi-ideal triangulation. The SN-stratification (4.38) and subordinate spectral network on each truncated tetrahedron transport to X, and extend over the 3-discs around interior vertices. In particular, there is a branched double cover $\pi\colon \widetilde{X} \to X_{\ge -2\mathrm{b}}$ with branch locus $X_{-2\mathrm{b}}$.

Suppose $P \to X$ is a flat principal G-bundle. Assume there exists[16] a flat section of the restriction of the associated $\mathbb{CP}^1$-bundle $P/B \to X$ to ∂X, and furthermore that we can and do choose a section such that Genericity Assumption 4.32 hold. Excise from X open balls about the barycenters of the tetrahedra. Let $X \subset \widetilde{X}$ be the total space of the double cover π with the inverse images of the balls excised. Then $\widetilde{X}$ is a compact manifold with boundary $\partial X \amalg \partial X \amalg \widetilde{S}_1 \amalg \cdots \amalg \widetilde{S}_N$, where each $\widetilde{S}_i$ is a 2-torus. The preceding gives an SN-stratification of X with strata of codimension 0, 1, and 2, and a flat line bundle $L \to X_{\geq -3\mathrm{a}}$. The holonomy around a circle linking $X_{-2\mathrm{b}}$ is -1.

REMARK 4.56 (Thurston gluing equations). Each tetrahedron $\mathbb{A}^{(j)}$ in Construction 4.55 has a shape parameter $z^{(j)} \in \mathbb{C} \setminus \{0,1\}$ which is one of the holonomies defined before Proposition 4.43. (There are three possibilities labeled by the three pairs of opposite edges of $\mathbb{A}^{(j)}$.) Let E be an edge in the triangulation $\mathcal{T}$, and let $S_E \subset \{1,\dots,N\}$ be the set of j such that E is an edge of $\mathbb{A}^{(j)}$. For $j \in S_E$, let γ_j be the loop in the torus $\widetilde{S}_j$ which is called 'γ_E' in the text following Remark 4.39. Then

$$\sum_{j \in S_E} [\gamma_j] = 0 \quad \text{in } H_1(X). \tag{4.57}$$

To prove this relation consider Figure 15. Depicted are the two faces of $\mathbb{A}^{(j)}$, $j \in S_E$, which abut E and the image $\bar{\gamma}_j$ of the corresponding loop γ_j. Now each of the triangular faces occurs in exactly one additional tetrahedron $\mathbb{A}^{(j')}$, $j' \in S_E \setminus \{j\}$, and it does so with the opposite orientation. Hence the halves of $\bar{\gamma}_j$ and $\bar{\gamma}_{j'}$ contained in that face cancel, as do the halves of their lifts γ_j and $\gamma_{j'}$. This leads to (4.57). (The cancellation is in homology; the actual half curves are not strictly opposite.) The relation (4.57) in homology immediately implies the Thurston gluing equation [**T2**, §4.2]

$$\prod_{j \in S_E} z^{(j)} = 1, \tag{4.58}$$

where we choose the edge E to define the shape parameter in each $\mathbb{A}^{(j)}$, $j \in S_E$.

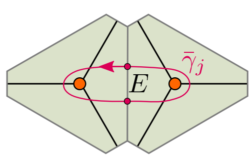


FIGURE 15. The two faces of the j^{th} tetrahedron which abut the edge E.

[16] Existence condition: on each component of ∂X the holonomies around loops at a basepoint have a common eigenline.

5. Levels and Chern-Simons invariants

We begin in §5.1 by proving relations among the Chern-Simons levels of $GL_2\mathbb{C}$, $SL_2\mathbb{C}$, and their various subgroups. This is the topological basis for abelianization. These topological computations imply relations among secondary differential geometric invariants via differential cohomology. We provide a brief introduction to differential cohomology in Appendix A. In §5.2 we introduce the spin refinement of Chern-Simons theory and prove appropriate relations among the "spin levels". We fully embrace differential cohomology in §5.3, where we prove a key result: Theorem 5.61. It states, heuristically, that moving in the unipotent direction does not change Chern-Simons invariants. We also prove results about $\mathbb{C}^\times$ Chern-Simons theory (Theorem 5.83, Corollary 5.89, Corollary 5.103) that are important in our later work. We conclude with a *global* statement, Theorem 5.106, of abelianization. Our main focus, *stratified* abelianization, is the subject of the subsequent §§6–8.

In this section we change notation slightly. Set $\widehat{G} = GL_2\mathbb{C}$ and let $\widehat{H}, \widehat{T}$ be the subgroups defined in §4.2.2. Also, set $G = SL_2\mathbb{C}$ and let H, T be the associated subgroups; the unipotent subgroup U is a subgroup of G, hence too of $\widehat{G}$.

We remind of a choice made in Example 3.7.

Convention 5.1. 3-dimensional Chern-Simons theory $\mathscr{F}_{SL_2\mathbb{C}}$ is based on the level $-c_2 \in H^4(BSL_2\mathbb{C};\mathbb{Z})$.

In that section the level is encoded in a symmetric bilinear form (3.1) on the Lie algebra, and (3.8) is the form that corresponds to $-c_2$. In the next section we compute the restriction of $-c_2$ to the subgroup $H \subset G$, and then we will define Chern-Simons theory on H—or, rather, a spin refinement—in terms of that restricted level.

5.1. Levels and abelianization.

5.1.1. *Levels in* $GL_2\mathbb{C}$. Our goal is to relate Chern-Simons invariants of principal $\widehat{G}$-bundles to Chern-Simons invariants of $\widehat{H}$-, $\widehat{T}$-, and U-bundles, and to do the same for G-bundles. These derive from relationships among appropriate degree four integral cohomology classes on the classifying spaces, which we prove in this section. The inclusions $\widehat{T} \subset \widehat{H} \subset \widehat{G}$ and surjective homomorphism $\widehat{H} \to \mu_2$ lead to a diagram

$$\begin{array}{ccc} B\widehat{T} & & \\ \downarrow{\scriptstyle p} & & \\ B\widehat{H} & \xrightarrow{\;r\;} & B\widehat{G} \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle q} & & \\ B\mu_2 & & \end{array} \tag{5.2}$$

in which p is a double cover and the vertical maps p, q form a fibration sequence. The section s of q is the classifying map of the inclusion $\mu_2 \cong \{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\} \hookrightarrow \widehat{H}$. Let $c_i \in H^{2i}(B\widehat{G};\mathbb{Z})$, $i=1,2$, be the universal Chern classes, and $p_1 = c_1^2 - 2c_2$ the universal first Pontrjagin class. Let $c', c'' \in H^2(B\widehat{T};\mathbb{Z})$ be the first Chern class of the homomorphisms $\widehat{T} \to \mathbb{C}^\times$ indicated by the matrix $\left(\begin{smallmatrix}z'&0\\0&z''\end{smallmatrix}\right) \in \widehat{T}$. Let $a \in H^2(B\mu_2;\mathbb{Z})$ be the generator; note $2a = 0$.

Proposition 5.3. *In diagram* (5.2) *we have the following equality in* $H^4(B\widehat{H};\mathbb{Z})$:

$$p_*(c')^2 = r^*p_1 + q^*a^2. \tag{5.4}$$

Proof of Proposition 5.3. From the Leray-Serre spectral sequence of the vertical fibration in (5.2), we deduce the split short exact sequence[17]

$$0 \longrightarrow H^4(B\mu_2;\mathbb{Z}) \underset{s^*}{\overset{q^*}{\rightleftarrows}} H^4(B\widehat{H};\mathbb{Z}) \xrightarrow{p^*} H^4(B\widehat{T};\mathbb{Z}) \longrightarrow 0 \tag{5.5}$$

Hence a class in $H^4(B\widehat{H};\mathbb{Z})$ is determined by its pullbacks under p^* and s^*.

For $\sigma\colon B\widehat{T} \to B\widehat{T}$ the deck transformation, we have $p^*p_* = 1 + \sigma^*$. Hence

$$p^*p_*(c')^2 = (c')^2 + (c'')^2 = p^*r^*(c_1^2 - 2c_2) = p^*r^*(p_1), \tag{5.6}$$

because c', c'' are the Chern roots of the universal $\widehat{G}$-bundle. Since s induces an isomorphism on π_1, the fiber product of s and p is contractible, from which $s^*p_* = 0$. Also, $s^*r^*(c_1^2 - 2c_2) = (s^*r^*c_1)^2$, since $H^4(B\mu_2;\mathbb{Z})$ is torsion of order two. The composition $r \circ s$ classifies the sum of the complex sign and trivial representations of $\mathbb{Z}/2\mathbb{Z}$, so its first Chern class is the generator $a \in H^2(B\mu_2;\mathbb{Z})$. Combining the preceding with $s^*q^* = \mathrm{id}$ we deduce (5.4). □

Remark 5.7. For $\widehat{G} = \mathrm{GL}_2\mathbb{C}$ a level $mc_1^2 + nc_2$ is parametrized by integers $m, n \in \mathbb{Z}$. By a similar argument to the preceding proof, $p_*c' = r^*c_1 + q^*a$, from which

$$(p_*c')^2 = r^*c_1^2 + q^*a^2. \tag{5.8}$$

Thus we can realize any level with n even by a linear combination of $p_*(c')^2$ and $(p_*c')^2$, up to q^*a^2.

[17]It helps to observe that the action of μ_2 on $H^2(B\widehat{T};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ exchanges the two summands, so the resulting local system on $B\mu_2$ is the pushforward of the trivial local system on its contractible double cover. Hence the cohomology vanishes in positive degrees.

5.1.2. *Levels in* $\mathrm{SL}_2\mathbb{C}$. By restriction we deduce a formula for the "special" subgroups which appear in the diagram

$$\begin{array}{ccc} BT & & \\ \big\downarrow{\scriptstyle p} & & \\ BH & \xrightarrow{\ r\ } & BG \\ \big\downarrow{\scriptstyle q} & & \\ B\mu_2 & & \end{array} \tag{5.9}$$

LEMMA 5.10. *In the diagram* (5.9) *we have* $q^*a^2 = 0$.

PROOF. Let $x \in H^1(B\mu_2; \mathbb{Z}/2\mathbb{Z})$ be the generator. Then $a = \beta(x)$, where $\beta\colon H^q(-;\mathbb{Z}/2\mathbb{Z}) \to H^{q+1}(-;\mathbb{Z})$ is the integral Bockstein, and also $a^2 = \beta(x^3)$. It suffices to prove $q^*x^3 = 0$. Passing to maximal compact subgroups we replace $T \to H \to \mu_2$ by $\mathrm{Spin}_2 \to \mathrm{Pin}_2^- \to \mu_2$. In the Leray-Serre spectral sequence for the fibration sequence $B\mu_2 \to B\mathrm{Pin}_2^- \to BO_2$, the differential $d_2\colon E_2^{0,1} \to E_2^{2,0}$ sends the generator $y \in H^1(B\mu_2;\mathbb{Z}/2\mathbb{Z})$ to $w_1^2 + w_2 \in H^2(BO_2;\mathbb{Z}/2\mathbb{Z})$. (See [**KT**] for a review of pin groups.) Then $d_2(w_1y) = w_1^3 + w_1w_2$. Also, since y transgresses so too do its Steenrod squares, and in particular $d_3(y^2) = d_3(Sq^1y) = Sq^1(d_2y) = Sq^1(w_1^2 + w_2) = w_1w_2$. Hence w_1^3 is killed when pulled back to $B\mathrm{Pin}_2^-$. Conclude by observing that

$$\begin{array}{ccc} \mathrm{Pin}_2^- & \longrightarrow & \mu_2 \\ & \searrow \quad \nearrow{\scriptstyle \det} & \\ & O_2 & \end{array} \tag{5.11}$$

commutes. So the pullback of x equals the pullback of w_1. ☐

The classifying map of the inclusion $i\colon T \hookrightarrow \widehat{T}$ satisfies $(Bi)^*c' = -(Bi)^*c'' = c$ for $c \in H^2\big(BT;\mathbb{Z}\big)$ a generator. Also, $i^*r^*c_1 = 0$. The following is a corollary of Proposition 5.3 and Lemma 5.10.

COROLLARY 5.12. *In diagram* (5.9) *we have the following equality in* $H^4\big(BH;\mathbb{Z}\big)$*:*

$$p_*c^2 = -2r^*c_2. \tag{5.13}$$ ☐

REMARK 5.14. Note the minus sign in (5.13)! We must be mindful of it when we define a $\mathbb{C}^\times$ Chern-Simons theory which is compatible with our Convention 5.1 for $\mathrm{SL}_2\mathbb{C}$ Chern-Simons theory.

5.1.3. *Abelianization of connections.* Let us now focus on $G = \mathrm{SL}_2\mathbb{C}$. If X is a 3-manifold with a flat H-connection, then *global* abelianization of the associated flat $\mathrm{SL}_2\mathbb{C}$-connection is encoded in the commutative diagram

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & B(\mathbb{C}^\times)^\delta & & \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle p} & & \\ X & \longrightarrow & BH^\delta & \xrightarrow{\;r\;} & B(\mathrm{SL}_2\mathbb{C})^\delta \end{array} \tag{5.15}$$

where we write $\mathbb{C}^\times$ for the group of diagonal matrices $T \subset \mathrm{SL}_2\mathbb{C}$. The pullback square defines the (unramified) double cover π. *Stratified* abelianization is encoded in the diagram

$$\begin{array}{ccccc} \widetilde{X}_{\ge -3\mathrm{a}} & \longrightarrow & B(\mathbb{C}^\times)^\delta & & \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle p} & & \\ X_{\ge -3\mathrm{a}} & \longrightarrow & BH^\delta & \xrightarrow{\;r\;} & B(\mathrm{SL}_2\mathbb{C})^\delta \\ \big\downarrow & & & \nearrow & \\ X & & & & \end{array} \tag{5.16}$$

in which the bottom triangle commutes on X_0. In both the global and stratified cases our goal is to compute the Chern-Simons invariant of the flat $\mathrm{SL}_2\mathbb{C}$-connection on X in terms of a Chern-Simons invariant of the flat $\mathbb{C}^\times$-connection on $\widetilde{X}$. The $\mathrm{SL}_2\mathbb{C}$ Chern-Simons invariant is the secondary invariant of $-c_2 \in H^4(B\mathrm{SL}_2\mathbb{C};\mathbb{Z})$; the $\mathbb{C}^\times$ Chern-Simons invariant is the secondary invariant of $c^2 \in H^4(B\mathbb{C}^\times;\mathbb{Z})$. There is a mismatch for abelianization: the factor of -2 in (5.13). To rectify we must divide the $\mathbb{C}^\times$-level by 2 (and include the minus sign). This can be done—a secondary invariant for "$c^2/2$" exists—but at the cost of introducing a new cohomology theory and a spin structure on $\widetilde{X}$, as we explain in §5.2.

Remark 5.17. Levels have a refinement in *differential cohomology*, and the Chern-Simons invariants are nicely located in the differential theory; see [**ChS, HS, F2, FH2**]. We give a précis of differential cochains in Appendix A and use this point of view on Chern-Simons invariants in §5.3; see also [**FN**, Appendix A]. This framework makes clear that cohomology identities immediately imply corresponding relations among secondary invariants.

We conclude our discussion of levels in ordinary cohomology by examining the restriction to the unipotent subgroup U in (4.21). Recall (Definition 4.24(ii)) that the failure of the bottom triangle in (5.16) to commute on all of $X_{\ge -3\mathrm{a}}$ is due to the unipotent gluing along the walls (components of X_{-1}) of the spectral network. Since $U \cong \mathbb{C}$ is contractible, so is BU, and the following is immediate.

Proposition 5.18. *The restriction of any level of* $\mathrm{GL}_2\mathbb{C}$ *or* $\mathrm{SL}_2\mathbb{C}$ *to* U *vanishes.* □

In principle, then, the unipotent gluing does not change the Chern-Simons invariant and essentially allows us to proceed as if the bottom triangle in (5.16) commutes on $X_{\geq -3\mathrm{a}}$, though this heuristic requires a bit of work to make precise; see Theorem 5.61.

5.2. Levels for spin Chern-Simons theory.

5.2.1. *E-cohomology and spin $\mathbb{C}^\times$ Chern-Simons theory.* To divide $c^2 \in H^4(B\mathbb{C}^\times;\mathbb{Z})$ by 2, we pass to a cohomology theory simply denoted E, the nontrivial extension

$$H\mathbb{Z} \xrightarrow{\ i\ } E \xrightarrow{\ j\ } \Sigma^{-2}H\mathbb{Z}/2\mathbb{Z} \tag{5.19}$$

of Eilenberg-MacLane spectra; the k-invariant $\Sigma^{-2}H\mathbb{Z}/2\mathbb{Z} \to \Sigma H\mathbb{Z}$ is $\beta \circ Sq^2$, the composition of the integral Bockstein and the Steenrod square. For any topological space S, the extension (5.19) leads to a long exact sequence of cohomology groups

(5.20)

$$\cdots \longrightarrow H^q(S;\mathbb{Z}) \xrightarrow{\ i\ } E^q(S) \xrightarrow{\ j\ } H^{q-2}(S;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\ \beta\circ Sq^2\ } H^{q+1}(S;\mathbb{Z}) \longrightarrow \cdots$$

Multiplication by 2 on $E^q(S)$ factors through i:

$$\begin{array}{ccc} H^q(S;\mathbb{Z}) & \xrightarrow{\ i\ } & E^q(S) \\ {\scriptstyle 2}\downarrow & \overset{k}{\swarrow} & \downarrow{\scriptstyle 2} \\ H^q(S;\mathbb{Z}) & \xrightarrow{\ i\ } & E^q(S) \end{array} \tag{5.21}$$

For $S = B\mathbb{C}^\times$, a slice of the long exact sequence (5.20) is the *nontrivial* abelian group extension

$$0 \longrightarrow H^4(B\mathbb{C}^\times;\mathbb{Z}) \xrightarrow{\ i\ } E^4(B\mathbb{C}^\times) \longrightarrow H^2(B\mathbb{C}^\times;\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0, \tag{5.22}$$

i.e., $E^4(B\mathbb{C}^\times)$ is infinite cyclic and $i(c^2)$ is twice a generator $\lambda \in E^4(B\mathbb{C}^\times)$. The class λ plays the role of "$c^2/2$". Passing to maximal compact subgroups there is a generalization from $\mathbb{T} \cong SO_2$ to SO_N for any $N \geq 2$. Namely, there is a characteristic class $\lambda \in E^4(BSO_N)$ whose image under $k \oplus j$ is $(p_1, w_2) \in H^4(BSO_N;\mathbb{Z}) \oplus H^2(BSO_N;\mathbb{Z}/2\mathbb{Z})$. Furthermore, λ is additive: for real vector bundles $V', V'' \to X$ over a space X we have

$$\lambda(V' \oplus V'') = \lambda(V') \oplus \lambda(V''). \tag{5.23}$$

The pullback of λ to $E^4(B\mathrm{Spin}_N)$ is the image under i of a class $\tilde{\lambda} \in H^4(B\mathrm{Spin}_N;\mathbb{Z})$ whose double is p_1. Also, $\tilde{\lambda} \equiv w_4 \pmod 2$ if $N \geq 4$. We refer to [**F3**, §1] for background about this cohomology theory E and proofs[18] of these assertions.

[18] Even if the precise statement does not appear in [**F3**], the same techniques apply. The standard fact that $\tilde{\lambda} \neq 0 \pmod 2$ follows since $H^4(B\mathrm{Spin}_N;\mathbb{Z}) \cong \mathbb{Z}$ and $\tilde{\lambda}$ is a generator, if $N \geq 4$.

The characteristic class $\lambda \in E^4(B\mathbb{C}^\times)$ has a lift $\check{\lambda} \in \check{E}^4_{\mathbb{C}}(B_\nabla\mathbb{C}^\times)$ to the *differential* E-cohomology of the classifying object for principal $\mathbb{C}^\times$-connections. (See [**FN**, Appendix A].) Here $B_\nabla\mathbb{C}^\times$ is a simplicial sheaf on smooth manifolds, in the sense of [**FH2**], for example. There is also a simplicial sheaf $B_\delta\mathbb{C}^\times$ which classifies flat $\mathbb{C}^\times$-connections, as well as a map $i\colon B_\delta\mathbb{C}^\times \to B_\nabla\mathbb{C}^\times$. The pullback $i^*\check{\lambda}$ is a flat differential class. Define the spectrum $E\mathbb{C}/\mathbb{Z}(1)$ as the cofiber of the composition

$$E \xrightarrow{\;k\;} H\mathbb{Z} \xrightarrow{\;\pi\sqrt{-1}\;} H\mathbb{C}. \tag{5.24}$$

Its nonzero homotopy groups are $\pi_0 \cong \mathbb{C}/\mathbb{Z}(1)$ and $\pi_{-1} \cong \mathbb{Z}/2\mathbb{Z}$. The topological space $B(\mathbb{C}^\times)^\delta$ is a geometric realization of the simplicial sheaf $B_\delta\mathbb{C}^\times$. Then $i^*\check{\lambda}$ determines a characteristic class

$$\widehat{\lambda} \in E^3\bigl(B(\mathbb{C}^\times)^\delta; \mathbb{C}/\mathbb{Z}(1)\bigr) \tag{5.25}$$

in the cohomology theory $E\mathbb{C}/\mathbb{Z}(1)$.

An *oriented* real vector bundle has a Thom class in integer cohomology, but a Thom class in E-cohomology requires a *spin* structure [**F3**, Proposition 4.4]. In particular, E-cohomology classes can be integrated on compact spin manifolds. This leads immediately to a *fully extended* unitary 3-dimensional topological field theory $\mathscr{S}_{\mathbb{C}^\times}$ on spin manifolds equipped with a *flat* $\mathbb{C}^\times$-connection, analogous to the usual Chern-Simons theory (3.13) on oriented manifolds. It has a fully local version defined as a map of spectra analogous to (3.14):

$$M\mathrm{Spin} \wedge B(\mathbb{C}^\times)^\delta_+ \xrightarrow{\;\mathrm{id}\wedge\widehat{\lambda}\;} M\mathrm{Spin} \wedge (E\mathbb{C}/\mathbb{Z}(1)_3)_+ \xrightarrow{\;\int\;} \Sigma^3 E\mathbb{C}/\mathbb{Z}(1). \tag{5.26}$$

The field theory (5.26) assigns a $\mathbb{Z}/2\mathbb{Z}$-*graded* line to a closed spin 2-manifold with flat $\mathbb{C}^\times$-connection. As noted in Remark 3.15 we need the theory for parametrized families of flat connections, so for nonflat connections.

Remark 5.27. In fact, the grading of the spin Chern-Simons line of a $\mathbb{C}^\times$-connection on a surface is determined by the parity of the degree of the underlying principal $\mathbb{C}^\times$-bundle. For a flat connection that degree is zero, hence the line is even. Also, to a $\mathbb{C}^\times$-connection over a spin 1-manifold, the spin Chern-Simons theory assigns an invertible module over *super* vector spaces. See Appendix C for more details as well as a justification for ignoring these $\mathbb{Z}/2\mathbb{Z}$-gradings in the body of this paper.

As a companion to Convention 5.1 we signpost our choice of sign for the level, which is motivated by Corollary 5.40 below.

Convention 5.28. 3-dimensional spin Chern-Simons theory $\mathscr{S}_{\mathbb{C}^\times}$ is based on the level $\lambda \in E^4(B\mathbb{C}^\times)$.

This spin Chern-Simons theory is developed in some detail in [**FN**]. For future use we recall one particular result: [**FN**, Theorem 3.9(vii)]. Let Y be a closed 2-manifold endowed with a spin structure σ, and fix a principal $\mathbb{C}^\times$-bundle $\pi\colon Q \to Y$ with connection $\Theta \in \Omega^1(Q;\mathbb{C})$. A section t of π produces

$$\tau_t \in \mathscr{S}_{\mathbb{C}^\times}(Y;\Theta;\sigma), \tag{5.29}$$

a nonzero element in the spin Chern-Simons line computed from the $\mathbb{C}^\times$-connection Θ and the spin structure σ. Let $h\colon Y \to \mathbb{C}^\times$ be a smooth function. Then $t' = t \cdot h$ is another section of π, and the ratio of nonzero elements in $\mathscr{S}_{\mathbb{C}^\times}(Y;\Theta;\sigma)$ is

$$\frac{\tau_{t'}}{\tau_t} = \epsilon_{t,h} \exp\left(\frac{1}{4\pi\sqrt{-1}} \int_Y t^*\Theta \wedge \frac{dh}{h}\right), \tag{5.30}$$

where

$$\epsilon_{t,h}(s) = (-1)^{q_\sigma([h])}. \tag{5.31}$$

Here $q_\sigma\colon H^1(Y;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ is the quadratic refinement of the intersection pairing given by the spin structure σ, and $[h] \in H^1(Y;\mathbb{Z}/2\mathbb{Z})$ is the reduction modulo two of the homotopy class of h.

5.2.2. *Levels in E-cohomology.* We revisit Proposition 5.3 and Corollary 5.12 in E-cohomology, so effectively divide (5.4) and (5.13) by 2.

LEMMA 5.32.

(1) *The map*

$$H^4(B\mathrm{SL}_2\mathbb{C};\mathbb{Z}) \xrightarrow{\;i\;} E^4(B\mathrm{SL}_2\mathbb{C}) \tag{5.33}$$

is an isomorphism.

(2) *The group extension*

$$0 \longrightarrow H^4(B\mu_2;\mathbb{Z}) \xrightarrow{\;i\;} E^4(B\mu_2) \xrightarrow{\;j\;} H^2(B\mu_2;\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \tag{5.34}$$

is nontrivial: $E^4(B\mu_2)$ is cyclic of order 4.

(3) *The pullback map $E^4(B\mu_2) \to E^4(B\mathbb{Z})$ is zero.*

PROOF. Statement (1) follows from $H^2(B\mathrm{SL}_2\mathbb{C};\mathbb{Z}/2\mathbb{Z}) = 0$. For (2), we claim

$$\alpha := \lambda(L \oplus L) \in E^4(B\mu_2) \tag{5.35}$$

has order 4, where $L \to B\mu_2$ is the real Hopf line bundle. For this observe $L^{\oplus 2} \to B\mu_2$ is orientable, $j\lambda(L^{\oplus 2}) = w_2(L^{\oplus 2}) = x^2$, and $2\alpha = 2\lambda(L^{\oplus 2}) = \lambda(L^{\oplus 4}) \neq 0$ since $w_2(L^{\oplus 4}) = 0$ and $w_4(L^{\oplus 4}) = x^4 \neq 0$, so $\tilde{\lambda}(L^{\oplus 4}) \neq 0$. (We use the Whitney sum formula (5.23).) Finally, (3) follows immediately from $E^4(B\mathbb{Z}) = 0$. ∎

Observe that $2\alpha = i(a^2)$, where $a \in H^2(B\mu_2;\mathbb{Z}/2\mathbb{Z})$ is the generator.

Remark 5.36. Let $p\colon \mathbb{Z} \to \mu_2$ be the homomorphism $n \mapsto (-1)^n$. Observe that the pullback $(Bp)^*\alpha \in E^4(B\mathbb{Z})$ vanishes, since $B\mathbb{Z} \simeq S^1$ and $E^4(S^1) = 0$. The μ_2 Chern-Simons theory based on α defines invariants of compact oriented manifolds equipped with a double cover. A lift of a double cover to a principal $\mathbb{Z}$-bundle trivializes[19] the μ_2 Chern-Simons invariant.

Let $\lambda_{\widehat{G}} \in E^4(B\widehat{G})$ be the pullback of the generator $\lambda \in E^4(B\mathbb{C}^\times)$ under $\det\colon \widehat{G} \to \mathbb{C}^\times$, where $\widehat{G} = \mathrm{GL}_2\mathbb{C}$; then $2\lambda_{\widehat{G}} = i(c_1^2)$. Let $\lambda' \in E^4(B\widehat{T})$ be the unique class such that $k(\lambda') = c'^2$. (See (5.21) for the definition of k.) Identify $c_2 \in H^4(B\mathrm{SL}_2\mathbb{C};\mathbb{Z})$ with its image under i in $E^4(B\mathrm{SL}_2\mathbb{C})$.

Proposition 5.37. *In diagram (5.2) we have the following equality in $E^4(B\widehat{H})$:*

$$p_*\lambda' = r^*(\lambda_{\widehat{G}} - c_2) + q^*\alpha. \tag{5.38}$$

Proof. In the diagram

$$\begin{array}{ccccc} H^4(B\mu_2;\mathbb{Z}) & \xrightarrow{\ i\ } & E^4(B\mu_2) & \xrightarrow{\ j\ } & H^2(B\mu_2;\mathbb{Z}/2\mathbb{Z}) \\ {\scriptstyle s^*}\uparrow\downarrow{\scriptstyle q^*} & & {\scriptstyle s^*}\uparrow\downarrow{\scriptstyle q^*} & & {\scriptstyle s^*}\uparrow\downarrow{\scriptstyle q^*} \\ H^4(B\widehat{H};\mathbb{Z}) & \xrightarrow{\ i\ } & E^4(B\widehat{H}) & \xrightarrow{\ j\ } & H^2(B\widehat{H};\mathbb{Z}/2\mathbb{Z}) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ H^4(B\widehat{T};\mathbb{Z}) & \xrightarrow{\ i\ } & E^4(B\widehat{T}) & \xrightarrow{\ j\ } & H^2(B\widehat{T};\mathbb{Z}/2\mathbb{Z}) \end{array} \tag{5.39}$$

the rows are exact, and the first and third columns are exact; see (5.5). It follows that the second column is also exact. In other words, a class in $E^4(B\widehat{H})$ is determined by its pullbacks under p^* and s^*. Also, observe that twice (5.38) is (5.4), which implies that the two sides of (5.38) differ by an element of order dividing 2. Since $E^4(B\widehat{T})$ is torsionfree, as can be deduced from (5.22), it follows that the pullback under p^* of the two sides of (5.38) agree. For the pullback under s^* we argue as in the proof of Proposition 5.3: the $\lambda_{\widehat{G}}$-class of the *complex* sign representation is α; see (5.35). □

Corollary 5.40. *In diagram (5.9) we have the following equality in $E^4(B(H))$:*

$$p_*\lambda = -r^*c_2 + q^*\alpha. \tag{5.41}$$ □

Note from Lemma 5.10 that $2q^*\alpha = q^*a^2 = 0$. Corollary 5.40 follows immediately from Proposition 5.37. The minus sign in (5.41) is the spin echo of the minus sign in (5.13); see Remark 5.14.

[19]The Chern-Simons theory is defined using a geometric representative of α, say a map $B\mu_2 \to E_4$, where E_4 is the 4-space in the spectrum E, sometimes denoted $\Omega^{\infty+4}E$. The trivialization is based on a choice of null homotopy of the composition $B\mathbb{Z} \to B\mu_2 \to E_4$.

5.3. Chern-Simons theory and differential cochains. Shortly after the introduction of secondary invariants of connections by Chern-Simons [**CS2**], Cheeger-Simons [**ChS**] recast them in terms of new objects in differential geometry: differential characters. Differential cohomology, which we discuss briefly in Appendix A, introduces cochains into the theory of differential characters; it is the natural home in which to express the full locality of Chern-Simons invariants. In [**FN**, Appendix A] we prove some properties of *spin* $\mathbb{C}^\times$ Chern-Simons theory using *generalized* differential cohomology, and in §5.3.5 we take this up to prove a lemma we need later. Otherwise, in this section we restrict to *ordinary* differential cohomology with complex coefficients and Chern-Simons theory for $G = \mathrm{SL}_2\mathbb{C}$. Our main goal is to prove Theorem 5.61 about the behavior of Chern-Simons invariants under unipotent modifications. We begin with some preliminaries in §§5.3.1–5.3.3. A *global* abelianization theorem appears in §5.3.6.

5.3.1. *The universal* $\mathrm{SL}_2\mathbb{C}$*-connection.* Let G be a Lie group with finitely many components. There is a groupoid-valued sheaf $B_\nabla G$ on the category of smooth manifolds whose value on a test manifold M is the groupoid of G-connections; see [**FH2**] for an introduction and details. The sheaf $B_\nabla G$ classifies G-connections: there is a universal principal G-bundle

$$\pi\colon E_\nabla G \longrightarrow B_\nabla G \tag{5.42}$$

with connection Θ^{univ}, and if $P \to M$ is a principal G-bundle with connection Θ over a smooth manifold M, then there is a unique G-equivariant map $\varphi\colon P \to E_\nabla G$ which satisfies $\varphi^*\Theta^{\mathrm{univ}} = \Theta$. Moreover, the universal connection on (5.42) is a weak equivalence

$$\Theta^{\mathrm{univ}}\colon E_\nabla G \longrightarrow \Omega^1 \otimes \mathfrak{g}, \tag{5.43}$$

where $\Omega^1 \otimes \mathfrak{g}$ is the set-valued sheaf which assigns to a test manifold M the set $\Omega^1_M(\mathfrak{g})$ of $\mathfrak{g}$-valued 1-forms on M. The total space $E_\nabla G$ of (5.42) assigns to M the discrete groupoid of principal G-bundles $Q \to M$ with connection $\Theta \in \Omega^1(Q;\mathfrak{g})$ and section $s\colon M \to Q$; the universal connection (5.43) maps the triple (Q, Θ, s) to the $\mathfrak{g}$-valued 1-form $s^*\Theta$.

The universal Chern-Simons-Weil invariant is a differential cohomology class on $B_\nabla G$. The variant $\check{H}^\bullet_{\mathbb{C}}$ of differential cohomology we need uses *complex* differential forms. The construction of $\check{H}^\bullet_{\mathbb{C}}$ as a homotopy fiber product [**HS, BNV, ADH**] leads to the exact sequence

$$0 \longrightarrow \check{H}^4_{\mathbb{C}}(B_\nabla G) \longrightarrow H^4(BG;\mathbb{Z}) \times \Omega^4_{\mathrm{cl}}(B_\nabla G;\mathbb{C}) \xrightarrow{\;-\;} H^4(BG;\mathbb{C}) \tag{5.44}$$

in which $\Omega^4_{\mathrm{cl}}(B_\nabla G;\mathbb{C})$ denotes the vector space of closed complex differential forms. The main theorem of [**FH2**] computes $\Omega^4_{\mathrm{cl}}(B_\nabla G;\mathbb{C})$ as the vector space of *real* linear G-invariant symmetric bilinear forms $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. For $G = \mathrm{SL}_2\mathbb{C}$ we choose $-c_2 \in H^4(B\mathrm{SL}_2\mathbb{C};\mathbb{Z})$ and the bilinear form

$$\langle A, B\rangle = -\frac{1}{8\pi^2}\,\mathrm{trace}(AB), \qquad A, B \in \mathfrak{sl}_2\mathbb{C}, \tag{5.45}$$

as in Example 3.7; see also Convention 5.1. By (5.44) there is a unique lift $-\check{c}_2 \in \check{H}^4_{\mathbb{C}}(B_\nabla \mathrm{SL}_2\mathbb{C})$, the desired universal Chern-Simons-Weil class. This gives, for each principal G bundle with connection Θ over a smooth manifold M, a differential characteristic class $-\check{c}_2(\Theta) \in \check{H}^4_{\mathbb{C}}(M)$. We need a refinement to a differential cocycle representative of this class in $\check{Z}^4_{\mathbb{C}}(M)$; see §A.1. This depends on a contractible choice; see [**F2**, §3.1] or [**HS**, §3.3] for detailed constructions. We use the same symbol $\check{c}_2$ for the differential cocycle representative, and we make clear whether it denotes the cocycle or cohomology class.

Suppose $P \xrightarrow{\pi} M \xrightarrow{p} S$ is an iterated fiber bundle in which π is a principal $\mathrm{SL}_2\mathbb{C}$-bundle and the fibers of p are manifolds with boundary of dimension $n \le 3$. Assume given an orientation on p, i.e., on the relative tangent bundle $T(M/S) \to M$. Let $\Theta \in \Omega^1(P; \mathfrak{sl}_2\mathbb{C})$ be a connection. We obtain a differential cocycle $-\check{c}_2(\Theta) \in \check{Z}^4_{\mathbb{C}}(M)$. The Chern-Simons invariant of this family of $\mathrm{SL}_2\mathbb{C}$-connections is

$$\mathscr{F}_{\mathrm{SL}_2\mathbb{C}}(M \to S; \Theta) = 2\pi\sqrt{-1} \int_{M/S} (-\check{c}_2(\Theta)), \tag{5.46}$$

a differential cochain in $\check{C}^{4-n}_{\mathbb{C}}(S)$; see §A.4 for the integral. For $n = 3$ and assuming the fibers of $M \xrightarrow{p} S$ are closed, (5.46) is a function $S \to \mathbb{C}/\mathbb{Z}(1)$, as in (3.4). For $n = 2$ and closed fibers (5.46) is a complex line bundle with covariant derivative over M, the Chern-Simons line bundle. For $n = 3$ and a fiber bundle of manifold with boundary, (5.46) is a section of the Chern-Simons line bundle computed from the boundaries; see Theorem A.24.

Consider the pullback $-\pi^*\check{c}_2 \in \check{H}^4_{\mathbb{C}}(E_\nabla \mathrm{SL}_2\mathbb{C})$ to the total space of the universal bundle (5.42). Let $\Omega^\bullet_{\mathbb{Z}} \subset \Omega_{\mathrm{cl}}$ denote the presheaf of closed differential forms with integral periods. The exact sequence

$$0 \longrightarrow \frac{\Omega^3(E_\nabla \mathrm{SL}_2\mathbb{C}; \mathbb{C})}{\Omega^3_{\mathbb{Z}}(E_\nabla \mathrm{SL}_2\mathbb{C}; \mathbb{C})} \longrightarrow \check{H}^4_{\mathbb{C}}(E_\nabla \mathrm{SL}_2\mathbb{C}) \longrightarrow H^4(E_\nabla \mathrm{SL}_2\mathbb{C}; \mathbb{C}) \tag{5.47}$$

from [**HS**, (3.3)] reduces to an isomorphism (the middle map), since $H^4(E_\nabla \mathrm{SL}_2\mathbb{C}; \mathbb{C}) = 0$. Hence $-\pi^*\check{c}_2$ reduces to a 3-form modulo closed 3-forms with integral periods. There is a canonical choice of 3-form,[20] the Chern-Simons form $\eta \in \Omega^3(E_\nabla \mathrm{SL}_2\mathbb{C}; \mathbb{C})$; see (3.2). To a triple $(Q \to M, \Theta, s)$ which represents a map $M \to E_\nabla \mathrm{SL}_2\mathbb{C}$, the pullback of η to M is

$$-\frac{1}{8\pi^2} \operatorname{trace}\bigl(\alpha \wedge d\alpha + \frac{2}{3} \alpha \wedge \alpha \wedge \alpha\bigr) \in \Omega^3(M; \mathbb{C}), \tag{5.48}$$

where $\alpha = s^*\Theta \in \Omega^1(M; \mathfrak{sl}_2\mathbb{C})$. The Chern-Simons invariant of an oriented family $P \xrightarrow{\pi} M \xrightarrow{p} S$ with connection and trivialization $s\colon M \to P$ can be computed by integrating the 3-form (5.48) over the fibers of p. For example, if the fibers of p are closed of dimension 2, then the resulting 1-form on S is the connection form of a trivialized complex line bundle over S.

[20]Use (A.10) to deduce the existence of this 3-form.

5.3.2. *Restriction to the unipotent subgroup.* Recall the unipotent subgroup $U \subset \mathrm{SL}_2\mathbb{C}$ defined in (4.21).

DEFINITION 5.49. Let M be a smooth manifold with boundary.

(1) A flat principal $\mathrm{SL}_2\mathbb{C}$-bundle $P \to M$ is *boundary-unipotent* if its restriction $\partial P \to \partial M$ to the boundary admits a reduction to a flat principal U-bundle.
(2) Such a $P \to M$ is *boundary-reduced* if a reduction is chosen.
(3) Stratified abelianization data (P, Q, μ, θ) over $(M, \mathcal{N})$ is *boundary-reduced* if P is boundary-reduced and $\theta(Q)|_{\partial M_0}$ lies in the U-bundle given by the reduction.

Note that a flat $\mathrm{SL}_2\mathbb{C}$-bundle is boundary-unipotent iff on each boundary component the holonomies around loops at a basepoint have a common eigenline. Moreover, if (P, Q, μ, θ) is boundary-reduced, then $Q|_{\partial M}$ is a trivializable flat bundle, since $H \cap U = \{1\}$.

Let $B_\nabla U$ be the groupoid-valued sheaf of U-connections. Then there is a map $B_\nabla U \to B_\nabla \mathrm{SL}_2\mathbb{C}$.

LEMMA 5.50. *The restriction of the universal second differential Chern class* $\check{c}_2 \in \check{H}^4_{\mathbb{C}}(B_\nabla \mathrm{SL}_2\mathbb{C})$ *to* $B_\nabla U$ *vanishes.*

PROOF. Since U is contractible, the restriction of $c_2 \in H^4(B\,\mathrm{SL}_2\mathbb{C}; \mathbb{Z})$ to $H^4(BU; \mathbb{Z})$ vanishes; also, the restriction of the bilinear form (5.45) to the Lie algebra of U vanishes. □

Recall that we choose a differential cocycle representative of $\check{c}_2$; see the text following (5.45). Now choose a trivialization of its restriction to $B_\nabla U$.

REMARK 5.51. With these choices, the Chern-Simons invariant of a boundary-reduced flat $\mathrm{SL}_2\mathbb{C}$-bundle is trivialized on the boundary. For example, on a compact 2-manifold with boundary, the invariant is a complex line.

5.3.3. *A lemma in differential cohomology.* Let M be an oriented n-manifold with corners, equipped with the extra structure of a bordism outlined in §A.4; let S be a smooth manifold, which plays the role of parameter space; and suppose[21] $\check{\omega} \in \check{Z}^q_{\mathbb{C}}\big(S \times [0,1] \times M\big)$ is a differential cocycle of some degree q. Let $\omega \in \Omega^q_{\mathbb{Z}}\big(S \times [0,1] \times M; \mathbb{C}\big)$ be the "curvature" of $\check{\omega}$, i.e., the differential form underlying the differential cocycle $\check{\omega}$. Let $\partial/\partial t$ denote the standard vector field on $[0,1]$, lifted to $S \times [0,1] \times M$, and let $\iota_{\partial/\partial t}$ denote

[21] $N = [0,1] \times M$ has the structure of a bordism [**FT1**, §A.2]: set

$$\begin{aligned} N_0 &= (0,1) \times M_0 \\ N^0_{-1} &= \{0\} \times M_0 \\ N^1_{-1} &= \{1\} \times M_0 \\ N^\delta_{-j+1} &= M^\delta_{-j}, \qquad j \geq 1, \quad \delta \in \{0,1\}. \end{aligned} \tag{5.52}$$

its action via contraction on differential forms. Theorem A.24 implies the following for M closed.

LEMMA 5.53. *If M is closed, then the integral*

$$\int_{[0,1]\times M} \breve{\omega} \colon \int_{\{0\}\times M} \breve{\omega} \longrightarrow \int_{\{1\}\times M} \breve{\omega} \tag{5.54}$$

is a nonflat isomorphism of the differential cocycles on S computed in the domain and codomain. Its covariant derivative is

$$\int_{[0,1]\times M} \omega \in \Omega^{q-n-1}(S;\mathbb{C}). \tag{5.55}$$

In particular, if $\iota_{\partial/\partial t}\omega = 0$, then (5.54) is a flat isomorphism. □

If M is a manifold with corners—a bordism of positive depth—then the integrals of $\breve{\omega}$ over $\{0\}\times M$ and $\{1\}\times M$ are higher morphisms in a groupoid of differential cochains on S: see §A.3. For example, if M has depth ≤ 2, then the integrals are 2-morphisms in $\breve{\mathcal{G}}_{(q-n+2)}(S)$, as depicted in (A.27).

LEMMA 5.56.

(1) *If M has corners of depth ≤ 2 and $\iota_{\partial/\partial t}\omega = 0$, then $\int_{[0,1]\times M}\breve{\omega}$ is an isomorphism of 2-morphisms in $\breve{\mathcal{G}}_{(q-n+2)}(S)$:*

$$\int_{\{0\}\times M^0_{-2}} \underset{\int_{\{0\}\times M^0_{-1}}}{\overset{\int_{\{0\}\times M^1_{-1}}}{\rightrightarrows}} \int_{\{0\}\times M^1_{-2}} \quad \Uparrow \int_{\{0\}\times M_0}$$

$$\xrightarrow{\ \cong\ } \quad \int_{\{1\}\times M^0_{-2}} \underset{\int_{\{1\}\times M^0_{-1}}}{\overset{\int_{\{1\}\times M^1_{-1}}}{\rightrightarrows}} \int_{\{1\}\times M^1_{-2}} \quad \Uparrow \int_{\{0\}\times M_0} \tag{5.57}$$

(2) *Suppose $\breve{\tau} \in \check{C}^{q-1}(S\times[0,1]\times M)$ is a nonflat trivialization of $\breve{\omega}$ with covariant derivative $\tau \in \Omega^{q-1}(S\times[0,1]\times M)$, and assume $\iota_{\partial/\partial t}\tau = 0$. Then the isomorphism (5.57) preserves the nonflat trivializations and nonflat isomorphisms in Theorem A.29.* □

We omit the integrand '$\breve{\omega}$' in (5.57) for readability.

EXAMPLE 5.58. If $n = q-2$, then $\int_{\{i\}\times M}\breve{\omega}$, $i = 0,1$, is a complex line bundle $L_i \to S$ with connection, and (5.54) is an isomorphism $L_0 \to L_1$ of the underlying line bundles; its usual covariant derivative is the 1-form (5.55).

EXAMPLE 5.59. If $q = 2$ and $M = S^1$, then (5.55) reduces to a well-known formula for the ratio of holonomies of a line bundle with connection around the ends of a cylinder.

REMARK 5.60. If $\breve{\omega}$ is equipped with a nonflat trivialization, then so too are its integrals over M, and then (5.54) becomes an equation in differential forms which follows from the usual Stokes' theorem. The assertion in Lemma 5.56(2) is a variation for manifolds with corners.

5.3.4. *Moving along unipotents.* Theorem 5.61 below is based on the fact that the bilinear form (5.45) vanishes if A is diagonal and B is upper triangular.

Let M be an oriented manifold with corners of depth ≤ 2 and dimension ≤ 3. Suppose $M = M_0 \amalg M_{-1} \amalg M_{-2\text{a}} \amalg M_{-3\text{a}}$ is equipped with an SN-stratification which satisfies $M_{-2\text{b}} = M_{-3\text{b}} = \emptyset$. In other words, $M = M_{\ge -3\text{a}}$. Suppose (π, s) is a subordinate spectral network. Thus $\pi\colon \widetilde{M} \to M$ is a double cover and $s\colon M_{-1} \amalg M_{-2\text{a}} \amalg M_{-3\text{a}} \to \widetilde{M}_{-1} \amalg \widetilde{M}_{-2\text{a}} \amalg \widetilde{M}_{-3\text{a}}$ is a section of π over $M_{-1} \amalg M_{-2\text{a}} \amalg M_{-3\text{a}}$. Let $T \subset \mathrm{SL}_2\mathbb{C}$ denote the diagonal subgroup, and $\iota\colon H \hookrightarrow \mathrm{SL}_2\mathbb{C}$ its normalizer. Suppose $\mathcal{A} = (P, Q, \mu, \theta)$ is stratified abelianization data of type $(\mathrm{SL}_2\mathbb{C}, T)$. Thus

(1) $P \to M$ is a principal $\mathrm{SL}_2\mathbb{C}$-bundle with flat connection Θ_P,
(2) $Q \to M$ is a principal H-bundle with flat connection Θ_Q,
(3) $\mu\colon \widetilde{M} \to Q/T$ is an isomorphism of double covers, and
(4) $\theta\colon \iota(Q) \to P$ is an isomorphism of flat principal $\mathrm{SL}_2\mathbb{C}$-bundles over M_0.

Furthermore, let $U \subset \mathrm{SL}_2\mathbb{C}$ be the subgroup (4.21) of unipotent matrices. Then we require that the discontinuity of θ along M_{-1} lie in U, relative to the reduction of $Q \to M_{-1}$ to a principal T-bundle given by the section s.

Our task is to compute the Chern-Simons invariants (5.46) of the flat $\mathrm{SL}_2\mathbb{C}$-bundles $\iota(Q) \to M$ and $P \to M$. To state the theorem we posit a family of this data over a smooth manifold S. Thus we work over $S \times M$; the connections Θ_P, Θ_Q over $S \times M$ are only assumed flat along M.

THEOREM 5.61. *There is a natural flat isomorphism*

$$\mathscr{F}_{\mathrm{SL}_2\mathbb{C}}\big(S \times M \to S, \iota(\Theta_Q)\big) \xrightarrow{\ \cong\ } \mathscr{F}_{\mathrm{SL}_2\mathbb{C}}\big(S \times M \to S, \Theta_P\big). \tag{5.62}$$

Intuitively, moving a connection in unipotent directions does not affect the $\mathrm{SL}_2\mathbb{C}$ Chern-Simons invariant. In the proof we construct a flat isomorphism which depends on a set of choices, and then we check that the isomorphism is independent of the choices.

PROOF. Fix a smooth function $\phi\colon \mathbb{R}^{\neq 0} \to \mathbb{R}^{\neq 0}$ which is odd and satisfies

$$\phi(x) = \begin{cases} 0, & x \le -1; \\ -\frac{1}{2}, & -\frac{1}{2} \le x < 0 \\ \frac{1}{2}, & 0 < x \le \frac{1}{2} \\ 0, & 1 \le x. \end{cases} \tag{5.63}$$

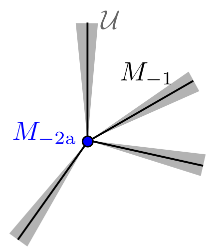


FIGURE 16. The tubular neighborhood $\mathcal{U}$.

Furthermore, require that ϕ be monotonic nonincreasing on $\mathbb{R}^{<0}$ and $\mathbb{R}^{>0}$. Choose a tubular neighborhood of $M_{-1} \subset M$: an open subset $\mathcal{U} \subset M_{\ge -1}$ which contains M_{-1}, a surjective submersion $\rho\colon \mathcal{U} \to M_{-1}$, and an isomorphism of ρ with the normal bundle $\nu \to M_{-1}$ to $M_{-1} \subset M_{\ge -1}$. (See Figure 16.) Fix an inner product on $\nu \to M_{-1}$. Let $R \to M_{-1}$ be the reduction of $Q \to M_{-1}$ to a principal T-bundle; it is defined via the section s and isomorphism μ. Locally, for each orientation of $\nu \to M_{-1}$ the discontinuity in θ along M_{-1} is a section u of the bundle $S \times R \times_T U \to S \times M_{-1}$ of unipotent groups. Under reversal of orientation, u maps to u^{-1}. Globally, write $u = e^X$ for X a section of the bundle of Lie algebras

$$S \times R \times_T \mathfrak{u} \to S \times M_{-1} \tag{5.64}$$

twisted by the orientation bundle of $\nu \to M_{-1}$. Extend X to $\mathcal{U}$ using parallel transport along the fibers of $\rho\colon \mathcal{U} \to M_{-1}$. The inner product on the normal bundle identifies each fiber of ρ with $\mathbb{R}$ after choosing an orientation of the normal bundle. Hence the product ϕX is a well-defined section of the pullback of (5.64) over $S \times (\mathcal{U} \cap M_0)$. It extends[22] by zero to $S \times M_0$.

We now construct a connection Ξ on the principal $\mathrm{SL}_2\mathbb{C}$-bundle

$$\mathcal{Q} = [0,1] \times S \times \iota(Q) \to [0,1] \times S \times M \tag{5.65}$$

whose restriction to $\{0\} \times S \times M$ is isomorphic to $\iota(\Theta_Q)$ and whose restriction to $\{1\} \times S \times M$ is isomorphic to Θ_P. First, set $\Xi\,\big|_{\{0\}\times S\times M} = \iota(\Theta_Q)$. Then over $\{1\} \times S \times M_0$ let φ be the gauge transformation of the restriction of (5.65) which equals $e^{-\phi X}$ on $\{1\} \times S \times (\mathcal{U} \cap M_0) \subset [0,1] \times S \times (\mathcal{U} \cap M_0)$ and is the identity map on $\{1\} \times S \times (M_0 \setminus \mathcal{U})$. Construct an isomorphism

$$\psi\colon \mathcal{Q}\Big|_{S\times\{1\}\times M} \xrightarrow{\;\cong\;} P \tag{5.66}$$

which equals $\theta \circ \varphi$ on $\{1\} \times S \times M_0$; it extends over $\{1\} \times S \times (M_{-1} \amalg M_{-2a} \amalg M_{-3a})$ using the fact that θ jumps by u on $\{1\} \times S \times M_{-1}$. Set $\Xi\,\big|_{\{1\}\times S\times M} =$

[22]Since the codomain of ϕX does not so extend, this is not strictly correct. What we mean simply is that in formulas below replace ϕX by '0' on $S \times (M_0 \setminus \mathcal{U})$.

$\psi^*(\Theta_P)$. Finally, define Ξ on $[0,1]_t \times S \times M$ by affine interpolation between the specified connections Ξ_0, Ξ_1:

$$\Xi_t = \iota(\Theta_Q) + t\alpha, \qquad \alpha = \Xi_1 - \Xi_0. \tag{5.67}$$

Then α is a 1-form on $S \times M$ with values in the adjoint bundle of Lie algebras isomorphic to $\mathfrak{sl}_2\mathbb{C}$; it has support on $S \times \mathcal{U}$. We claim that α takes values in the subbundle (5.64) of nilpotent subalgebras, extended over $S \times \mathcal{U}$ by parallel transport along the fibers of ρ. Namely, on $S \times (\mathcal{U} \cap M_0)$ we have

$$\alpha = \left[\mathrm{Ad}_{e^{-\phi X}}\big(\iota(\Theta_Q)\big) - \iota(\Theta_Q)\right] + d_{\iota(\Theta_Q)}(\phi X). \tag{5.68}$$

The second term clearly lies in the nilpotent subalgebra. For the first, observe

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} = \begin{pmatrix} 0 & -2xy \\ 0 & 0 \end{pmatrix} \tag{5.69}$$

is nilpotent.

Let $\breve{\omega} = \breve{c}_2(\Xi)$ be the Chern-Simons-Weil differential cocycle. As in (5.57),

$$2\pi\sqrt{-1}\int_{[0,1]\times M} \breve{\omega} \colon \mathscr{F}_{\mathrm{SL}_2\mathbb{C}}\big(S \times M \to S, \iota(\Theta_Q)\big) \longrightarrow \mathscr{F}_{\mathrm{SL}_2\mathbb{C}}\big(S \times M \to S, \Theta_P\big) \tag{5.70}$$

is an isomorphism. We claim that it is a *flat* isomorphism. By Lemma 5.56 it suffices to show that $\iota_{\partial/\partial t}\omega = 0$, where

$$\omega = \langle \Omega(\Xi), \Omega(\Xi) \rangle \tag{5.71}$$

is the Chern-Weil 4-form of Ξ. The only nonzero contribution to $\iota_{\partial/\partial t}\omega$ is potentially on $[0,1] \times \operatorname{supp}\alpha \subset [0,1] \times S \times \mathcal{U}$. From (5.67) we compute the curvature

$$\Omega(\Xi) = \iota\big(\Omega(\Theta_Q)\big) + dt \wedge \alpha + t d_{\iota(\Theta_Q)}\alpha + \frac{t^2}{2}[\alpha \wedge \alpha]. \tag{5.72}$$

The last term vanishes since the Lie algebra $\mathfrak{u}$ of the unipotent group is abelian. The first term in (5.72) takes values in the diagonal subalgebra $\mathfrak{t} \subset \mathfrak{sl}_2\mathbb{C}$ and the other terms take values in the nilpotent subalgebra $\mathfrak{u} \subset \mathfrak{sl}_2\mathbb{C}$. It follows that

$$\iota_{\partial/\partial t}\omega = 2\langle \alpha \wedge \iota\big(\Omega(\Theta_Q)\big)\rangle + 2t\langle \alpha \wedge d_{\iota(\Theta_Q)}\alpha \rangle = 0. \tag{5.73}$$

It remains to prove that (5.70) is independent of the choices of $\phi, \mathcal{U}, \rho$ and the isomorphism of ρ with the normal bundle. Any two sets of choices can be joined by a path, so we extend the previous setup by taking the Cartesian product with $[0,1]_r$, where r is the parameter along the path. If $\tilde{\omega}$ is the resulting Chern-Weil 4-form, then $\iota_{\partial/\partial r}\tilde{\omega} = 0$ by a similar argument. ∎

5.3.5. *A theorem in $T = \mathbb{C}^\times$ Chern-Simons theory.* Recall from §5.2.1 and [**FN**, Appendix A] the characteristic class $\lambda \in E^4(B\mathbb{C}^\times)$ and its differential refinement $\check{\lambda} \in \check{E}^4_{\mathbb{C}}(B_\nabla\mathbb{C}^\times)$, the universal differential class. The class λ is the image of the generator under a quadratic function

$$q\colon H^2(B\mathbb{C}^\times;\mathbb{Z}) \longrightarrow E^4(B\mathbb{C}^\times) \tag{5.74}$$

which for $c, c' \in H^2(B\mathbb{C}^\times;\mathbb{Z})$ satisfy

$$2q(c) = c \smile c = c^2 \tag{5.75}$$

$$jq(c) = \bar{c} \tag{5.76}$$

$$q(c + c') = q(c) + q(c') + i(c \smile c'), \tag{5.77}$$

where $j\colon E^4(B\mathbb{C}^\times) \to H^2(BC;\mathbb{Z}/2\mathbb{Z})$ and $i\colon H^4(B\mathbb{C}^\times;\mathbb{Z}) \to E^4(B\mathbb{C}^\times)$ are the maps in (5.19), and we denote $\bar{c} = c \pmod 2$. The differential refinement

$$\check{q}\colon \check{H}^2_{\mathbb{C}}(B_\nabla\mathbb{C}^\times) \longrightarrow \check{E}^4_{\mathbb{C}}(B_\nabla\mathbb{C}^\times) \tag{5.78}$$

satisfies analogous properties. We implicitly use refinements of $q, \check{q}$ to cochains. Recall that integration[23] of E-cocycles over a manifold M requires a spin structure σ on M. Furthermore, if $\delta \in H^1(M;\mathbb{Z}/2\mathbb{Z})$ is the class of a double cover over M, and we write the shifted spin structure as[24] $\sigma \to \sigma + \delta$, then

$$\int_{M,\sigma+\delta} q(c) = \int_{M,\delta} q(c) + \frac{1}{2}\int_M \delta \smile \bar{c}, \tag{5.79}$$

where $\frac{1}{2}\colon \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{C}/\mathbb{Z}$; see [**F3**, Proposition 4.4] and [**FN**, Theorem 3.9 (ii)]. For $\dim M = 3$ and M closed, (5.79) is an equation in $\mathbb{C}/\mathbb{Z}$; for manifolds with boundary and manifolds of lower dimension it is a canonical isomorphism of cochains in E-cohomology theory. We use the differential refinement of (5.79).

Not only do double covers shift spin structures, but they also shift $\mathbb{C}^\times$-bundles via the homomorphism $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{C}^\times$. The corresponding shift of Chern classes is via the integer Bockstein

$$\beta\colon H^1(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}). \tag{5.80}$$

The differential refinement shifts $\mathbb{C}^\times$-connections by a flat $\mathbb{C}^\times$-connection of order two.

The following is a restatement of part of Lemma 5.32.

LEMMA 5.81. *$\check{E}^4_{\mathbb{C}}(B\mu_2) \cong \mathbb{Z}/4\mathbb{Z}$ with generator $\check{q}\big(\check{\beta}(\delta)\big)$ for $\delta \in H^1(B\mu_2;\mathbb{Z}/2\mathbb{Z})$ the nonzero class. Also, $2\check{q}\big(\check{\beta}(\delta)\big)$ is the image of δ^3 under the map $\phi\colon H^3(B\mu_2;\mathbb{Z}/2\mathbb{Z}) \to \check{E}^4_{\mathbb{C}}(B\mu_2)$.*

[23] In (5.79) we also use the integral symbol for the pairing of a mod 2 cohomology class with the fundamental class.

[24] Our notation conflates a double cover and its equivalence class, an overload we also deploy in this section for spin structures and $\mathbb{C}^\times$-connections.

PROOF. The main theorem in [**FH2**] implies $\check{E}^4_{\mathbb{C}}(B\mu_2) \to E^4(B\mu_2)$ is an isomorphism. Now apply Lemma 5.32(2). For the last assertion apply (5.75). □

Our main result in this section expresses the change of spin $\mathbb{C}^\times$ Chern-Simons invariants under the simultaneous shift of spin structure and $\mathbb{C}^\times$-connection by a double cover. We express our result as a relation among 3-dimensional invertible field theories whose background fields are independent choices of: a spin structure σ, a double cover δ, and a $\mathbb{C}^\times$-connection $\check{c}$. The partition functions which define the theories are:

(5.82)
$$\begin{aligned} \alpha_1(\sigma,\delta,\check{c}) &= \text{spin } \mathbb{C}^\times \text{ Chern-Simons invariant of } \check{c} \text{ in spin structure } \sigma \\ \alpha_2(\sigma,\delta,\check{c}) &= \text{spin } \mathbb{C}^\times \text{ Chern-Simons invariant of } \check{c}+\check{\beta}(\delta) \\ &\qquad \text{in spin structure } \sigma+\delta \\ \alpha_3(\sigma,\delta,\check{c}) &= \text{integral of } 3\check{q}\bigl(\check{\beta}(\delta)\bigr) \text{ in spin structure } \sigma \end{aligned}$$

The theory α_3 is topological (of order 4). The Chern-Simons invariants in α_1, α_2 are based on λ, and so are computed by integrating q, the quadratic function (5.74).

The proofs in the rest of this section draw on the material in Appendix B.

THEOREM 5.83. *There is an isomorphism $\alpha_1 \otimes \alpha_3 \cong \alpha_2$ of invertible field theories.*

PROOF. By (B.12) the curvatures of α_1 and α_2 are equal. Therefore $\alpha_1^{-1} \otimes \alpha_2 \otimes \alpha_3^{-1}$ is a flat invertible field theory, so it is topological in the strong sense. To verify that it is trivializable, it suffices to check the partition function on a closed 3-manifold X. The quadratic property (5.77) implies

$$\check{q}\bigl(\check{c}+\check{\beta}(\delta)\bigr) = \check{q}(\check{c}) + \check{q}\bigl(\check{\beta}(\delta)\bigr) + i\bigl(\check{c}\smile\check{\beta}(\delta)\bigr), \tag{5.84}$$

and (5.79) implies that its integral in spin structure $\sigma+\delta$ is the integral of

$$\check{q}(\check{c}) + \check{q}\bigl(\check{\beta}(\delta)\bigr) + i\bigl(\check{c}\smile\check{\beta}(\delta)\bigr) + \phi\left(\frac{1}{2}(\delta\smile\bar{c} \;+\; \delta\smile\delta^2)\right) \tag{5.85}$$

in spin structure σ. Here $\phi\colon H^3(X;\mathbb{Z}/2\mathbb{Z}) \to \check{E}^4_{\mathbb{C}}(X)$ is the inclusion of flat elements of order two. (The first two terms of (5.85) lie in $\check{E}^4_{\mathbb{C}}(X)$ and their integral uses the spin structure; the last two terms lie in $H^3(X;\mathbb{C}/\mathbb{Z})$ and no spin structure is used to integrate.) The difference of $\check{q}(\check{c})$ and (5.85) computes the partition function in the theory $\alpha_1^{-1}\otimes\alpha_2$:

(5.86)
$$\begin{aligned} \check{q}\bigl(\check{\beta}(\delta)\bigr) + \phi\left(\frac{1}{2}(\bar{c}\smile\delta \;+\; \delta\smile\bar{c} \;+\; \delta\smile\delta^2)\right) &= \check{q}\bigl(\check{\beta}(\delta)\bigr) + \phi\left(\frac{1}{2}(\delta^3)\right) \\ &= 3\check{q}\bigl(\check{\beta}(\delta)\bigr) \end{aligned}$$

where we use Lemma 5.81. Since the integral of this last quantity is by definition the partition function in the theory α_3, we see that the partition function in the theory $\alpha_1^{-1} \otimes \alpha_2 \otimes \alpha_3^{-1}$ is trivial. $\square$

EXAMPLE 5.87. Let $X = \mathbb{RP}^3$ equipped with either spin structure σ and the nontrivial double cover δ. The spin Chern-Simons partition function of the product $\mathbb{C}^\times$-connection $\check{c}$ is 1; the partition function of the $\mathbb{C}^\times$-connection $\check{c} + \delta$ in either spin structure is a primitive 4^{th} root of unity.

A principal $\mathbb{Z}$-bundle δ^∞ has a mod 2 reduction δ which is a double cover. The invertible field theories $\alpha_1, \alpha_2, \alpha_3$ in (5.82) lift to invertible field theories α_1^∞, α_2^∞, α_3^∞ with background fields $(\sigma, \delta^\infty, \check{c})$.

THEOREM 5.88. *α_3^∞ is isomorphic to the trivial theory.*

PROOF. The integer Bockstein of the mod 2 reduction is trivial. $\square$

COROLLARY 5.89. *There exist isomorphisms $\zeta\colon \alpha_1^\infty \xrightarrow{\cong} \alpha_2^\infty$.*

For our work in §6 we need an isomorphism which satisfies a particular property that we specify below in (5.102). We proceed to construct it. To begin, fix a *flat*[25] isomorphism

$$\zeta\colon \alpha_1^\infty \xrightarrow{\cong} \alpha_2^\infty. \tag{5.90}$$

Pull back the theories α_1^∞, α_2^∞ to invertible theories ${}^{\text{tt}}\alpha_1^\infty$, ${}^{\text{tt}}\alpha_2^\infty$ with background fields $(\sigma, \delta^\infty, \check{c}, t)$ in which t is a nonflat trivialization of the $\mathbb{C}^\times$-connection $\check{c}$, i.e., t is a section of the underlying principal $\mathbb{C}^\times$-bundle. Then t induces a *nonflat* trivialization of the theories ${}^{\text{tt}}\alpha_1^\infty$, ${}^{\text{tt}}\alpha_2^\infty$: they are *topologically trivial*. In the formalism of Appendix B we omit the spin structure, and the remaining fields are sections of the sheaf[26] $B_\nabla\mathbb{Z} \times E_\nabla\mathbb{C}^\times$ on Man. It is convenient to replace $B_\nabla\mathbb{Z}$ with the representable sheaf S^1. This amounts to specifying a classifying map for each principal $\mathbb{Z}$-bundle. Then the topologically trivialized theories ${}^{\text{tt}}\alpha_i^\infty$, $i = 1, 2$, give rise to differential forms (see (B.9))

$$\eta_i\colon S^1 \times E_\nabla\mathbb{C}^\times \longrightarrow \Omega^3_{\mathbb{C}}. \tag{5.91}$$

Let $\omega \in \Omega^1_{S^1}$ be the rotation-invariant closed 1-form which integrates to 1, and let $a \in \Omega^1_{\mathbb{C}}(E_\nabla\mathbb{C}^\times)$ be $\frac{\sqrt{-1}}{2\pi}$ times the universal connection 1-form; the latter gives the equivalence $E_\nabla\mathbb{C}^\times \to \Omega^1_{\mathbb{C}}$. Then from (5.75) we deduce

$$\begin{aligned} \eta_1 &= \frac{1}{2} a \wedge da \\ \eta_2 &= \frac{1}{2}(a + \frac{1}{2}\omega) \wedge d(a + \frac{1}{2}\omega) = \eta_1 + \frac{1}{4} d(a \wedge \omega). \end{aligned} \tag{5.92}$$

[25] Isomorphisms of invertible field theories may be flat or nonflat; compare §A.2.

[26] A principal $\mathbb{Z}$-bundle has a unique connection, so the '∇' in '$B_\nabla\mathbb{Z}$' is redundant; the latter is better denoted '$B_\bullet\mathbb{Z}$'

The topologically trivialized theories defined by η_1, η_2 are isomorphic as invertible theories (forgetting the topological trivialization)—as they must be by Corollary 5.89—since the 3-forms differ by an exact 3-form.

Our constructions yield two isomorphisms ${}^{\mathrm{tt}}\alpha_1^\infty \to {}^{\mathrm{tt}}\alpha_2^\infty$. First, the flat isomorphism $\zeta\colon \alpha_1^\infty \to \alpha_2^\infty$ lifts to a *flat* isomorphism

$$ {}^{\mathrm{tt}}\zeta\colon {}^{\mathrm{tt}}\alpha_1^\infty \xrightarrow{\;\cong\;} {}^{\mathrm{tt}}\alpha_2^\infty. \tag{5.93}$$

Second, the topological trivializations induce a *nonflat* isomorphism

$$\lambda\colon {}^{\mathrm{tt}}\alpha_1^\infty \xrightarrow{\;\cong\;} {}^{\mathrm{tt}}\alpha_2^\infty. \tag{5.94}$$

From (5.92) we compute that the curvature of λ is $\frac{1}{4}d(a\wedge\omega)$. The ratio ${}^{\mathrm{tt}}\zeta/\lambda$ is a 2-dimensional invertible theory on spin manifolds with a background field in $S^1 \times E_\nabla\mathbb{C}^\times$, and its curvature is the 3-form $-\frac{1}{4}d(a\wedge\omega)$. Let β be the 2-dimensional invertible field theory defined by the 2-form

$$-\frac{1}{4}\,a\wedge\omega, \tag{5.95}$$

and define the *flat* 2-dimensional theory γ by

$${}^{\mathrm{tt}}\zeta = \beta\gamma\lambda. \tag{5.96}$$

LEMMA 5.97. *The abelian group of topological invertible 2-dimensional theories with background fields $(\sigma,\delta^\infty,\check{c},t)$ is isomorphic to the Klein group $\mu_2\times\mu_2$. Furthermore, each theory depends only on the spin structure σ and the double cover δ induced by the principal $\mathbb{Z}$-bundle δ^∞.*

The partition functions of these four theories on a closed 2-manifold Σ are

$$1,\quad (-1)^{\mathrm{Arf}(\sigma)},\quad (-1)^{\mathrm{Arf}(\sigma+\delta)},\quad (-1)^{\mathrm{Arf}(\sigma+\delta)-\mathrm{Arf}(\sigma)}, \tag{5.98}$$

where Arf is the Arf invariant of the spin structure.

PROOF. Since $E_\nabla\mathbb{C}^\times \cong \Omega^1_{\mathbb{C}}$ is contractible, the group of invertible theories is isomorphic[27] to the group of characters of

$$\pi_2(M\mathrm{Spin}\wedge S^1_+) \cong \pi_2(M\mathrm{Spin}) \oplus \pi_1(M\mathrm{Spin}) \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}. \tag{5.99}$$

One can see that the theories listed in (5.98) exhaust the possibilities, or can check that

$$M\mathrm{Spin}\wedge S^1_+ \longrightarrow M\mathrm{Spin}\wedge\mathbb{RP}^\infty_+ \tag{5.100}$$

induces an isomorphism on π_2. □

[27] Both unitary and nonunitary theories are discussed in [**FH1**]. Here we do not assume unitarity, but the background fields are for 3-manifolds, even for the 2-dimensional theory which is the ratio of isomorphisms of 3-dimensional theories, hence the domain should at first glance have $\Sigma^3 MT\mathrm{Spin}_3$ in place of $M\mathrm{Spin}$. However, the obstruction theory argument in the proof of [**FH1**, Theorem 7.22] allow us to replace $\Sigma^3 MT\mathrm{Spin}_3$ with $M\mathrm{Spin}$ in (5.100).

It follows that the theory γ in (5.96) depends only on (σ,δ). Therefore, replace (5.90) by the isomorphism

$$\widetilde{\zeta}=\gamma^{-1}\zeta\colon \alpha_1^\infty \xrightarrow{\;\cong\;} \alpha_2^\infty. \tag{5.101}$$

For this choice of isomorphism we have

$$^{\mathrm{tt}}\widetilde{\zeta}=\beta\lambda, \tag{5.102}$$

where recall that β is defined by the 2-form (5.95).

We summarize with this refinement of Corollary 5.89.

Corollary 5.103. *There exists an isomorphism*

$$\zeta\colon \alpha_1^\infty \xrightarrow{\;\cong\;} \alpha_2^\infty \tag{5.104}$$

such that the induced isomorphism $^{\mathrm{tt}}\zeta\colon {}^{\mathrm{tt}}\alpha_1^\infty \to {}^{\mathrm{tt}}\alpha_2^\infty$ *of theories which include a nonflat trivialization of the* $\mathbb{C}^\times$*-connection satisfies*

$$\frac{^{\mathrm{tt}}\zeta}{\lambda}=\beta, \tag{5.105}$$

where λ *is the isomorphism* (5.94) *and* β *is defined by the differential form* (5.95)*.*

This is the isomorphism we use in §6.

5.3.6. *Global abelianization.* Diagram (5.15) illustrates global abelianization of an $\mathrm{SL}_2\mathbb{C}$-connection. We apply Corollary 5.40 to deduce an isomorphism of Chern-Simons invariants, expressed as an isomorphism among three invertible 3-dimensional field theories[28] $\epsilon_{\mathrm{SL}_2\mathbb{C}}$, $\epsilon_{\mathbb{C}^\times}$, and ϵ_{μ_2}. Each is defined on the bordism multicategory of dimension ≤ 3 manifolds with corners equipped with a spin structure σ and a flat H-connection Θ. The first theory $\epsilon_{\mathrm{SL}_2\mathbb{C}}$ uses only the underlying orientation of σ, and it evaluates the Chern-Simons theory $\mathscr{F}_{\mathrm{SL}_2\mathbb{C}}$ at level $-i(c_2)$ on the flat $\mathrm{SL}_2\mathbb{C}$-connection $r(\Theta)$. The second theory $\epsilon_{\mathbb{C}^\times}$ maps a spin manifold with flat H-connection to the total space of the associated μ_2-bundle with its induced spin structure and flat $\mathbb{C}^\times$-connection, and then evaluates this data using spin Chern-Simons theory $\mathscr{S}_{\mathbb{C}^\times}$ at level λ. The third theory ϵ_{μ_2} evaluates the spin Chern-Simons theory $\mathscr{S}_{\mu_2}$ at level α on the associated μ_2-connection[29] $q(\Theta)$.

Theorem 5.106. *There is an isomorphism* $\epsilon_{\mathrm{SL}_2\mathbb{C}}\cong\epsilon_{\mathbb{C}^\times}\otimes\epsilon_{\mu_2}$*.*

Proof. As in the proof of Theorem 5.83, it suffices to check equality of partition functions on a closed oriented 3-manifold X equipped with a flat

[28] As in Theorem 5.83 we restrict to flat connections, so to topological invertible field theories.

[29] Of course, this is simply a double cover, but we have endeavored to use consistent and transparent notation.

H-connection. For this, apply the secondary invariant version of (5.41) to the following slight enlargement of the diagram (5.15):

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & B(\mathbb{C}^\times)^\delta & & \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} & & \\ X & \longrightarrow & B(H)^\delta & \xrightarrow{r} & B(SL_2\mathbb{C})^\delta \\ & & \downarrow{\scriptstyle q} & & \\ & & B\not\mu_2 & & \end{array} \tag{5.107}$$

□

Let $\epsilon^\infty_{\mathrm{SL}_2\mathbb{C}}, \epsilon^\infty_{\mathbb{C}^\times}, \epsilon^\infty_{\not\mu_2}$ denote the pullbacks of $\epsilon_{\mathrm{SL}_2\mathbb{C}}, \epsilon_{\mathbb{C}^\times}, \epsilon_{\not\mu_2}$ to the bordism multicategory of dimension ≤ 3 manifolds with corners equipped with a spin structure, a flat H-connection, and a lift of the associated $\not\mu_2$-bundle to a principal $\mathbb{Z}$-bundle. Then Lemma 5.32(3) immediately implies

Theorem 5.108. *$\epsilon^\infty_{\not\mu_2}$ is isomorphic to the trivial theory.*

Corollary 5.109. *A trivialization of $\epsilon^\infty_{\not\mu_2}$ determines an isomorphism $\nu : \epsilon^\infty_{\mathrm{SL}_2\mathbb{C}} \xrightarrow{\cong} \epsilon^\infty_{\mathbb{C}^\times}$.*

In Appendix C we constrain the trivialization, based on considerations in §6.5.

6. Abelianization of Chern-Simons lines

Throughout this section we take $G = \mathrm{SL}_2\mathbb{C}$.

So far we have discussed generalities about the Chern-Simons theories $\mathscr{F}_G$, $\mathscr{S}_{\mathbb{C}^\times}$, and their relation to one another via stratified abelianization. Now we begin discussing applications.

Suppose Y is a compact 2-manifold, equipped with a boundary-reduced flat G-bundle $P \to Y$. In this section and the next we give a new description of the line $\mathscr{F}_G(Y;P)$. The idea is to identify $\mathscr{F}_G(Y;P)$ with $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y};Q^\epsilon_{\mathrm{tw}};\sigma^\epsilon_{\mathrm{tw}}) \otimes \mathcal{L}(Y,\epsilon)$, where Q^ϵ_{tw} is a flat $\mathbb{C}^\times$-bundle over a branched double cover $\widetilde{Y} \to Y$, and $\mathcal{L}(Y,\epsilon)$ is a universal line which does not depend on P. In the rest of this section we give a sketch of the construction.

The bundle Q^ϵ_{tw} will be constructed as follows. First, we fix a semi-ideal triangulation $\mathcal{T}$ of Y, and let $\widetilde{Y} \to Y$ be the associated branched double cover, and $\mathcal{N}^{\mathcal{T}}$ the associated spectral network (Construction 4.30). Next, we fix a section of P/B over each vertex of Y (as usual, for ideal vertices this means a flat section over the corresponding boundary component), obeying the genericity Assumption 4.32. From these data, by Construction 4.31 we obtain stratified abelianization data (P,Q,μ,θ).

Now, suppose we ignore the branch locus $Y_{-2\mathrm{b}}$ for a moment, i.e. we work just over $Y_{\geq -2\mathrm{a}}$. Then, according to Theorem 5.61, $\mathscr{F}_G(Y_{\geq -2\mathrm{a}};P)$ is naturally isomorphic to $\mathscr{F}_H(Y_{\geq -2\mathrm{a}};Q)$; and by Corollary 5.109, if we choose a spin structure σ on Y, and a lift of $\widetilde{Y}_{\geq -2a} \to Y_{\geq -2a}$ to a $\mathbb{Z}$-bundle $Y^{\epsilon,\infty} \to Y_{\geq -2a}$, $\mathscr{F}_H(Y_{\geq -2a}; Q)$ is naturally isomorphic to $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}_{\geq -2a}; Q; \pi^*\sigma)$. Composing these two we would get an isomorphism

$$\mathscr{F}_G(Y_{\geq -2a}; P) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}_{\geq -2a}; Q; \pi^*\sigma) \ . \tag{6.1}$$

This is the kind of statement we are after, but to extend the right side over the full Y it needs to be modified. The complication is that $Q \to \widetilde{Y}_{\geq -2a}$ has holonomy -1 around these points, as noted in Lemma 4.12, and $\pi^*\sigma$ does not extend over them either. One could think of this holonomy as a kind of singularity, and try to define a modified version of the theory $\mathscr{S}_{\mathbb{C}^\times}$ which works directly with these singular objects. Here we take an alternative path: we twist both $Q \to \widetilde{Y}_{\geq -2a}$ and $\pi^*\sigma$ by a μ_2-bundle $\widetilde{Y}^{\epsilon,4} \to \widetilde{Y}_{\geq -2a}$, which cancels the unwanted holonomy. Fortunately, Corollary 5.103 ensures that this twisting does not change the Chern-Simons theory away from the branch locus, i.e., we get an isomorphism

$$\mathscr{F}_G(Y_{\geq -2a}; P) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}_{\geq -2a}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}) \ . \tag{6.2}$$

Now both sides extend over the branch locus. The isomorphism however does not: there is a mismatch between the two Chern-Simons theories over the branch locus. We measure this mismatch by a P-independent line we call $\mathcal{L}(Y,\epsilon)$. Thus ultimately what we get is an isomorphism

$$\mathscr{F}_G(Y; P) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}) \otimes \mathcal{L}(Y,\epsilon) \ . \tag{6.3}$$

In the above we needed to make various choices: a semi-ideal triangulation $\mathcal{T}$ of Y, a $\mathbb{Z}$-bundle $\widetilde{Y}^\infty \to Y$, and a μ_2-bundle $\widetilde{Y}^{\epsilon,4} \to \widetilde{Y}_{\geq -2a}$. It turns out that both $\widetilde{Y}^\infty \to Y$ and $\widetilde{Y}^{\epsilon,4} \to \widetilde{Y}_{\geq -2a}$ can be conveniently built from the data of edge-orientations on $\mathcal{T}$; this is the data we call ϵ.

In §6.1-§6.5 we discuss the necessary twisting and the properties of the difference line $\mathcal{L}(Y,\epsilon)$; in particular, we compute the action of rotations of a triangle on this line. Note also Appendix C in which we fix a choice in the construction so that the super line $\mathcal{L}(Y,\epsilon)$ is *even*. (Recall the discussion in Remark 5.27.) In §6.6 we develop the stratified abelianization map (6.3), as Construction 6.51. In the remaining sections §6.7-§6.8 we discuss some aspects of the dependence of stratified abelianization on the edge-orientations ϵ, which will be used in the explicit calculations to follow.

6.1. Edge-orientations on a triangle. Let $(\Delta, \mathcal{T})$ be a triangle. Choose an orientation ϵ_E for each $E \in \mathbf{edges}(\mathcal{T})$, and let $\epsilon = (\epsilon_E)_{E \in \mathbf{edges}(\mathcal{T})}$. Let

$$\epsilon_E(v,v') = \begin{cases} +1 & \text{if the edge } E \text{ with vertices } v,\ v' \text{ is oriented from } v \text{ to } v', \\ -1 & \text{otherwise.} \end{cases} \tag{6.4}$$

CONSTRUCTION 6.5. ϵ determines a lift of the μ_2-bundle $\widetilde{\Delta}_{\geq -2a} \to \Delta_{\geq -2a}$ to a $\mathbb{Z}$-bundle $\widetilde{\Delta}^{\epsilon,\infty} \to \Delta_{\geq -2a}$.

PROOF. Δ_0 has three connected components Δ_0^E, each containing one edge E. Fix E with vertices v, v'. Over Δ_0^E we define the fiber of $\widetilde{\Delta}^{\epsilon,\infty}$ as the $\mathbb{Z}$-torsor

$$(\{v, v'\} \times \mathbb{Z})/\sim, \qquad (v, n) \sim (v', n + \epsilon_E(v, v')). \tag{6.6}$$

Each wall in Δ_{-1} lies in the boundary of two domains Δ_0^E, $\Delta_0^{E'}$, where the edges E, E' have one vertex w in common. We glue $\widetilde{\Delta}^{\epsilon,\infty}$ across such a wall by identifying $[(w, n)]$ on one side with $[(w, n)]$ on the other side.

The μ_2-bundle $\widetilde{\Delta}^{\epsilon,\infty}/2\mathbb{Z}$ is isomorphic to $\widetilde{\Delta}_{\geq -2a}$, via the map which takes $[(v, n)] \mapsto v$ when $n \in 2\mathbb{Z}$. Thus $\widetilde{\Delta}^{\epsilon,\infty}$ is indeed a lift of $\widetilde{\Delta}_{\geq -2a}$ to a $\mathbb{Z}$-bundle as claimed. $\square$

Note that the clockwise monodromy of $\widetilde{\Delta}^{\epsilon,\infty}$ around $\partial\Delta$ is $n_+ - n_- \in \{3, 1, -1, -3\}$ where n_+ (n_-) is the number of edges oriented clockwise (counterclockwise). Reducing mod 2, we recover the fact that the monodromy of the double cover $\widetilde{\Delta}_{\geq -2a} \to \Delta_{\geq -2a}$ around $\partial\Delta$ is the nontrivial element of μ_2.

6.2. Edge-orientations on a triangulated surface. In the last subsection we considered a single triangle. More generally, suppose we have a semi-ideally triangulated surface $(Y, \mathcal{T})$ and edge-orientations $\epsilon = (\epsilon_E)_{E \in \mathbf{edges}(\mathcal{T})}$. All of our constructions glue canonically across edges, and thus we obtain a $\mathbb{Z}$-bundle

$$\widetilde{Y}^{\epsilon,\infty} \to Y_{\geq -2a}. \tag{6.7}$$

The action of $2\mathbb{Z}$ on $\widetilde{Y}^{\epsilon,\infty}$ commutes with the projection $\widetilde{Y}^{\epsilon,\infty} \to \widetilde{Y}_{\geq -2a}$, so $\widetilde{Y}^{\epsilon,\infty}$ is also a $2\mathbb{Z}$-bundle over $\widetilde{Y}_{\geq -2a}$. Let

$$\widetilde{Y}^{\epsilon,4} = \widetilde{Y}^{\epsilon,\infty}/4\mathbb{Z}\,. \tag{6.8}$$

This is a μ_2-bundle over $\widetilde{Y}_{\geq -2a}$, since $2\mathbb{Z}/4\mathbb{Z} = \mu_2$. We can describe its holonomies around cycles explicitly, as follows.

DEFINITION 6.9. For any $E \in \mathbf{edges}(\mathcal{T})$, let $\mathcal{Q}_E$ be the quadrilateral formed by the two triangles containing E. The ϵ-sign of E is $(-1)^n$, where n is the number of edges of $\mathcal{Q}_E$ which are oriented clockwise by ϵ. (Replacing "clockwise" by "counterclockwise" here would give the same definition.)

We recall the class $\gamma_E \in H_1(\widetilde{Y})$ defined before Proposition 4.40. This class depends on an orientation of Y, and reversing the orientation sends $\gamma_E \mapsto -\gamma_E$; the assertions in the rest of this section hold independent of the choice of orientation.

PROPOSITION 6.10. *The holonomy* $\mathrm{hol}_{\widetilde{Y}^{\epsilon,4}}(\gamma_E)$ *is the ϵ-sign of E as defined in Definition 6.9.*

PROOF. Over each domain Y_0^E, the fiber over the sheet labeled by vertex w of E is $\{[(w,n)] : n \in 2\mathbb{Z}\}$. Thus $\widetilde{Y}^{\epsilon,\infty}$ is the trivial $2\mathbb{Z}$-bundle on each connected component of $\widetilde{Y}_0$. The gluing across preimages of walls is as follows. Each wall runs into a vertex v. On the sheet labeled by the vertex v, $[(v,n)]$ is glued to $[(v,n)]$. The other sheet is labeled by a vertex v' on one side and v'' on the other. There the gluing takes $[(v',n)]$ to $[(v'',n+k)]$, where

$$k = \epsilon(v',v) + \epsilon(v,v'') \in \{-2,0,2\}\,. \tag{6.11}$$

In traversing γ_E, referring to Figure 17 we see that we cross two walls where the gluing is nontrivial (the horizontal walls in the figure). Summing their contributions, $\mathrm{hol}_{\widetilde{Y}^{\epsilon,\infty}}(\gamma_E)$ is a shift by $\epsilon(0,1)+\epsilon(1,2)+\epsilon(2,3)+\epsilon(3,0)$, which agrees mod 4 with $2n$, where n was defined in Definition 6.9. Reducing mod 4 gives the desired statement. □

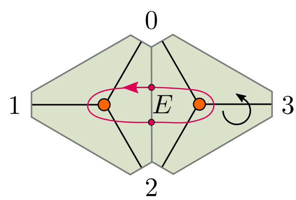


FIGURE 17. The quadrilateral $\mathcal{Q}_E$, with vertices labeled and the class γ_E shown.

CONSTRUCTION 6.12. Consider two edge-orientations ϵ, ϵ' which differ by reversing the orientation on a single edge E. The difference μ_2-bundle

$$\widetilde{Y}^{\epsilon,4} \otimes_{\mu_2} \widetilde{Y}^{\epsilon',4} \to \widetilde{Y}_{\geq -2\mathrm{a}} \tag{6.13}$$

admits a lift to a $\mathbb{Z}$-bundle (canonical up to isomorphism)

$$\varpi^E \to \widetilde{Y}_{\geq -2\mathrm{a}} \tag{6.14}$$

which extends over $\widetilde{Y}$.

For any class $\mu \in H_1(\widetilde{Y})$

$$\mathrm{hol}_{\varpi^E}(\mu) = \langle \gamma_E, \mu \rangle\,. \tag{6.15}$$

PROOF. We first describe the difference bundle (6.13), as we did in the proof of Proposition 6.10. It is the trivial μ_2-bundle on each connected component of $\widetilde{Y}_0$. The gluing across preimages of walls is as follows. Consider a wall ending on a vertex v. On the sheet labeled by v, the gluing is given by the identity element in μ_2. On the other sheet, the gluing is given by the nontrivial element in μ_2 if v is a vertex of E, and otherwise by the identity.

Now we can define the lift ϖ^E: it is the trivial $\mathbb{Z}$-bundle on each connected component of $\widetilde{Y}_0$, with gluing across preimages of walls as follows. Consider a wall ending on a vertex v. On the sheet labeled by v, the gluing is given by $0 \in \mathbb{Z}$. On the other sheet, if v is a vertex of E, then the gluing in the direction away from E is given by $+1 \in \mathbb{Z}$; if v is not a vertex of E, then the gluing is given by $0 \in \mathbb{Z}$.

The holonomy of ϖ^E around a loop in $\widetilde{Y}_{\geq -2\mathrm{a}}$ encircling a branch point comes to $-1+1=0$, so ϖ^E extends across the branch points, and thus over $\widetilde{Y}$ as desired. Finally, the formula $\mathrm{hol}_{\varpi^E}\,\mu = \langle \mu, \gamma_E\rangle$ is obtained directly by evaluating both sides on arcs μ crossing the quadrilateral $\mathcal{Q}_E$. □

6.3. Twistings over a triangulated surface. Now let $(Y,\epsilon,\sigma,\mathcal{A})$ be a triangulated surface, with edge-orientations, spin structure, and stratified abelianization data $\mathcal{A} = (P,Q,\mu,\theta)$. Then define the $\mathbb{C}^\times$-bundle

$$Q^\epsilon_{\mathrm{tw}} = Q \otimes_{\mu_2} \widetilde{Y}^{\epsilon,4} \to \widetilde{Y}_{\geq -2\mathrm{a}}. \tag{6.16}$$

Q^ϵ_{tw} has trivial monodromy around each point of $\widetilde{Y}_{-2\mathrm{b}}$, and thus extends to $\widetilde{Y}$, unlike Q. We use the name Q^ϵ_{tw} also for the extension.

The spin structure $\pi^*\sigma$ on $\widetilde{Y}_{\geq -2\mathrm{a}}$ is non-bounding on a circle around a branch point and thus does not extend from $\widetilde{Y}_{\geq -2\mathrm{a}}$ to $\widetilde{Y}$, but its twist

$$\sigma^\epsilon_{\mathrm{tw}} = \pi^*\sigma \otimes_{\mu_2} \widetilde{Y}^{\epsilon,4} \tag{6.17}$$

does extend to a spin structure over $\widetilde{Y}$.

It will be useful below to have some concrete information about this spin structure. We recall a convenient bit of notation first. Given a spin structure σ on a surface, and a simple closed curve λ, we define

$$\sigma(\lambda) = \begin{cases} 1 & \text{if } \sigma|_\lambda \text{ extends to a spin structure on the disc,} \\ -1 & \text{otherwise.} \end{cases} \tag{6.18}$$

We also recall the class $\gamma_E \in H_1(\widetilde{Y})$ defined before Proposition 4.40.

PROPOSITION 6.19. *$\sigma^\epsilon_{\mathrm{tw}}(\gamma_E)$ is the ϵ-sign of E as defined in Definition 6.9.*

PROOF. Since $\pi_*\gamma_E$ can be represented by the boundary of a disc in Y, $\pi^*\sigma(\gamma_E) = +1$. Thus $\sigma^\epsilon_{\mathrm{tw}}(\gamma_E)$ is the monodromy of the μ_2-bundle $\widetilde{M}^{\epsilon,4} \to \widetilde{Y}$ around γ_E, which we computed in Proposition 6.10. □

6.4. Stratified abelianization. Fix a manifold X of dimension ≤ 3 with corners, with $X_{-2\mathrm{b}} = \emptyset$. Suppose X is equipped with a spin structure σ, a spectral network $\mathcal{N}$, and stratified abelianization data $\mathcal{A} = (P,Q,\mu,\theta)$ over $(X,\mathcal{N})$. Also suppose given a $\mathbb{Z}$-bundle $\widetilde{X}^\infty$ whose mod 2 reduction is the double cover $\widetilde{X}$. Then let $\widetilde{X}^4$ denote the mod 4 reduction of $\widetilde{X}^\infty$. $\widetilde{X}^4 \to \widetilde{X}$ is a μ_2-bundle.

THEOREM 6.20. *There is a canonical isomorphism*

$$\chi(X;\mathcal{A};\sigma;\widetilde{X}^\infty) : \mathscr{F}_G(X;P) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{X}; Q \otimes_{\mu_2} \widetilde{X}^4; \pi^*\sigma \otimes_{\mu_2} \widetilde{X}^4)\,. \tag{6.21}$$

We remark that the isomorphism (6.23) is based on a choice of trivialization in Corollary 5.109. We constrain that choice in Appendix C.

Proof. We combine ingredients as follows. First, Theorem 5.61 (triviality of Chern-Simons in unipotent directions) gives

$$\rho(X;\mathcal{A}) : \mathscr{F}_G(X;P) \to \mathscr{F}_H(X;Q)\,. \tag{6.22}$$

Second, Corollary 5.109 (identity between Chern-Simons for H-bundles over X and $\mathbb{C}^\times$-bundles over $\widetilde{X}$) gives

$$\nu(X;Q;\widetilde{X}^\infty) : \mathscr{F}_H(X;Q) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{X};Q;\pi^*\sigma)\,. \tag{6.23}$$

Finally, Corollary 5.103 (invariance of spin $\mathbb{C}^\times$ Chern-Simons under μ_2-twists) gives

(6.24)

$$\zeta(\widetilde{X};Q;\pi^*\sigma;\widetilde{X}^\infty) : \mathscr{S}_{\mathbb{C}^\times}(\widetilde{X};Q;\pi^*\sigma) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{X};Q\otimes_{\mu_2}\widetilde{X}^4;\pi^*\sigma\otimes_{\mu_2}\widetilde{X}^4)\,.$$

The composition of these three is the desired isomorphism. □

The concrete nature of the isomorphism $\chi(X)$ in Theorem 6.20 depends on the nature of X. In general, $\chi(X)$ is an isomorphism between objects in appropriate diagram categories. For instance, if X is a closed 2-manifold, $\chi(X)$ is an isomorphism of lines; if X is a 3-manifold with boundary, $\chi(X)$ is an isomorphism of lines together with an isomorphism of objects in those lines; if X is a closed 3-manifold, $\chi(X)$ is just an equation.

In what follows we will need to know that χ has good gluing properties. These properties are most succinctly summarized as follows: they are just as if χ came from an isomorphism of 3-dimensional topological field theories, defined on a bordism category of oriented manifolds X equipped with a spectral network, stratified abelianization data, and a lift of the double cover $\widetilde{X}$ to a $\mathbb{Z}$-bundle. We will apply this below to various individual manifolds X carrying this data. Of the three ingredients above, two of them were formulated as isomorphisms of topological field theories; the third, Theorem 5.61, was not formulated in this language, but it was constructed in a fully local and canonical way. This is sufficient to imply the desired gluing properties.

6.5. The difference line for a triangle. Let $(\Delta,\epsilon,\sigma,\mathcal{A})$ be an oriented triangle with edge-orientations, spin structure, and stratified abelianization data $\mathcal{A} = (P,Q,\mu,\theta)$

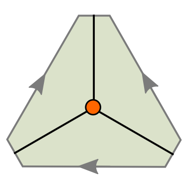


Figure 18. A triangle Δ with edge-orientations ϵ and the canonical SN-stratification.

Using G Chern-Simons theory on Δ gives an object in the $\mathbb{V}$-line associated to the boundary $\partial\Delta$,

$$\mathscr{F}_G(\Delta; P) \in \mathscr{F}_G(\partial\Delta; \partial P)\,. \tag{6.25}$$

Using $\mathbb{C}^\times$ Chern-Simons theory on $\widetilde{\Delta}$ likewise gives an object in the $\mathbb{V}$-line associated to $\partial\widetilde{\Delta}$,

$$\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}) \in \mathscr{S}_{\mathbb{C}^\times}(\partial\widetilde{\Delta}; \partial Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}})\,. \tag{6.26}$$

Because $\Delta_{-2\mathrm{b}} \neq \emptyset$, we cannot use Theorem 6.20 to identify these two objects. However, $\Delta_{-2\mathrm{b}}$ does not intersect $\partial\Delta$, so Theorem 6.20 gives an equivalence of $\mathbb{V}$-lines,

$$\chi(\partial\Delta; \partial\mathcal{A}; \sigma; \widetilde{\Delta}^{\epsilon,\infty}) : \mathscr{F}_G(\partial\Delta; \partial P) \to \mathscr{S}_{\mathbb{C}^\times}(\partial\widetilde{\Delta}; \partial Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}})\,. \tag{6.27}$$

Now we can compare the two objects: we define a line

$$\mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A}) = \chi(\partial\Delta; \partial\mathcal{A}; \sigma; \widetilde{\Delta}^{\epsilon,\infty})(\mathscr{F}_G(\Delta; P)) \otimes (\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}))^*. \tag{6.28}$$

We explain in Appendix C how to make a choice of isomorphism χ in (6.21) so that the super line $\mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A})$ is *even*; without further argument it could be odd.

The line $\mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A})$ depends only on (Δ, ϵ), in the following sense.

Proposition 6.29. *Suppose* $(\Delta, \epsilon, \sigma, \mathcal{A})$ *and* $(\Delta', \epsilon', \sigma', \mathcal{A}')$ *are triangles with edge-orientations, spin structure, and stratified abelianization data. An orientation-preserving affine-linear isomorphism* $f : \Delta \to \Delta'$ *which carries* ϵ *to* ϵ' *induces a canonical map*

$$f_* : \mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A}) \to \mathcal{L}(\Delta', \epsilon', \sigma', \mathcal{A}')\,. \tag{6.30}$$

An orientation-reversing affine-linear isomorphism $f : \Delta \to \Delta'$ *which carries* ϵ *to* ϵ' *induces a canonical map*

$$f_* : \mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A}) \to \mathcal{L}(\Delta', \epsilon', \sigma', \mathcal{A}')^*\,. \tag{6.31}$$

Proof. By uniqueness of spin structures on Δ, we can lift f to an isomorphism $\sigma \to f^*\sigma'$. By Proposition 4.34 we can lift f to an isomorphism $\mathcal{A} \to f^*\mathcal{A}'$. Finally, since $f^*\epsilon' = \epsilon$ we can lift f to an isomorphism $\widetilde{\Delta}^{\epsilon,\infty} \to \widetilde{\Delta}^{f^*\epsilon',\infty}$. All of our constructions are canonical and depend only on these data, so we obtain a map f_* as desired. It only remains to check that this map is independent of the choices we made in lifting f. To see this, we need to show that the nontrivial automorphism of the spin structure—the *spin flip*—and the automorphism of $\mathcal{A}$ induced by the action of $-1 \in G$ and $-1 \in \mathbb{C}^\times$ both act trivially on $\mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A})$.

The argument that the spin flip acts trivially is the subject of Appendix C.

To show that the automorphism -1 of $\mathcal{A}$ acts trivially we argue as follows. Our definition of $\mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A})$ can equally well be made using $G = \mathrm{GL}_2\,\mathbb{C}$ rather than $\mathrm{SL}_2\,\mathbb{C}$. In this case the whole center $Z(\mathrm{GL}_2\,\mathbb{C}) \simeq \mathbb{C}^\times$ acts on $\mathcal{A}$ and thus on $\mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A})$. Now, to compute the action of the element $\lambda \in \mathbb{C}^\times$ on $\mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A})$ we consider the mapping torus $\Delta \times S^1$, with abelianization data $\mathcal{A}_\lambda = (P_\lambda, Q_\lambda, \mu_\lambda, \theta_\lambda)$ obtained by gluing $\mathcal{A}$ to itself with the action of λ. The action of $\lambda \in \mathbb{C}^\times$ is given by the ratio

$$\frac{\chi(\partial\Delta \times S^1; \partial\hat{\mathcal{A}}; \sigma \times S^1; \widetilde{\Delta}^{\epsilon,\infty} \times S^1)(\mathscr{F}_G(\Delta \times S^1; P_\lambda))}{\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta} \times S^1; Q^\epsilon_{\lambda,\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}} \times S^1)} . \tag{6.32}$$

Now we consider the dependence on λ. Since P_λ is a flat G-connection for each λ, and the curvature of $\mathscr{F}_G$ vanishes when evaluated on a 1-parameter family of flat connections, it follows that $\mathscr{F}_G(\Delta \times S^1; P_\lambda)$ gives a covariantly constant section of the bundle over $\mathbb{C}^\times$ with fiber $\mathscr{F}_G(\partial\Delta \times S^1; \partial P_\lambda)$. Likewise $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta} \times S^1; Q^\epsilon_{\lambda,\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}} \times S^1)$ is covariantly constant. Finally, the isomorphism $\chi(\partial\Delta \times S^1; \partial\hat{\mathcal{A}}; \sigma \times S^1; \widetilde{\Delta}^{\epsilon,\infty} \times S^1)$ is flat, i.e. it is an isomorphism of line bundles with connection. Thus the ratio (6.32) is locally constant as a function of λ. But at $\lambda = 1$ it gives the action of 1, which is trivial; thus it must be 1 for all λ, as desired. $\square$

COROLLARY 6.33. *The line $\mathcal{L}(\Delta, \epsilon, \sigma, \mathcal{A})$ depends only on (Δ, ϵ) up to canonical isomorphism.*

PROOF. Given (Δ, ϵ) and two different data $(\sigma, \mathcal{A})$ and $(\sigma', \mathcal{A}')$ we apply Proposition 6.29 taking $f : \Delta \to \Delta$ to be the identity map. This gives the desired isomorphism $f_* : \mathcal{L}(\Delta, \sigma, \mathcal{A}, \epsilon) \to \mathcal{L}(\Delta, \sigma, \mathcal{A}', \epsilon')$. $\square$

With this corollary in mind we just write the line as $\mathcal{L}(\Delta, \epsilon)$. The most important feature of this line for concrete computations is that it transforms nontrivially under the μ_3 rotational symmetry of Δ, as measured by the following proposition.

PROPOSITION 6.34. *Suppose ϵ induces a consistent orientation of $\partial\Delta$, and f is a positively oriented rotation by $\frac{2\pi}{3}$ with respect to the orientation of Δ. Then f_* acts on $\mathcal{L}(\Delta, \epsilon)$ as multiplication by* $\exp(2\pi\sqrt{-1}/3)$.

PROOF. Fix stratified abelianization data $\mathcal{A} = (P, Q, \mu, \theta)$ and a spin structure σ over Δ. Lift the action of f to $\mathcal{A}$ and σ, in such a way that $f^3 = 1$. (This is possible, since each of $\mathcal{A}$ and σ is unique up to isomorphism and has only a single nontrivial automorphism ρ; an arbitrary lift of f will have either $f^3 = 1$ or $f^3 = \rho$, and in the latter case we replace the lift by $f \circ \rho$.) Also lift f to the μ_2-bundle $\widetilde{\Delta}^{\epsilon,4} \to \widetilde{\Delta}$, again in such a way that $f^3 = 1$. Combining this lift with the actions of f on Q and σ gives actions of f on Q^ϵ_{tw} and $\sigma^\epsilon_{\mathrm{tw}}$.

We consider the $\mathbb{V}$-lines associated to the boundary, $\mathscr{F}_G(\partial\Delta; \partial P)$ and $\mathscr{S}_{\mathbb{C}^\times}(\partial\widetilde{\Delta}; \partial Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}})$. f gives actions of μ_3 on both $\mathbb{V}$-lines, the equivalence $\chi(\partial\Delta; \partial\mathcal{A}; \sigma; \widetilde{\Delta}^{\epsilon,\infty})$ is μ_3-equivariant, and $\mathscr{F}_G(\Delta; P)$ and $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}})$ are μ_3-invariant objects. The line

$$\mathcal{L}(\Delta, \epsilon) = \mathrm{Hom}(\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}), \chi(\partial\Delta; \partial\mathcal{A}; \sigma; \widetilde{\Delta}^{\epsilon,\infty})(\mathscr{F}_G(\Delta; P))) \tag{6.35}$$

is thus acted on by μ_3, and we want to compute this action.

We will use an explicit picture of the μ_3-equivariant equivalence $\chi(\partial\Delta; \partial\mathcal{A}; \sigma; \widetilde{\Delta}^{\epsilon,\infty})$, obtained by chopping $\partial\Delta$ into three segments, ending at the midpoints of the three edges. First, we fix an f-invariant trivialization of the restriction of $\mathcal{A}$ to the midpoints. (There is a T-torsor worth of freedom in this choice, which we will fix below.) Given this trivialization, we may factorize each of our $\mathbb{V}$-lines as a tensor product of three $\mathbb{V}$-lines associated to the three segments, and factorize the equivalence $\chi(\partial\Delta; \partial\mathcal{A}; \sigma; \widetilde{\Delta}^{\epsilon,\infty})$ into a tensor product of three equivalences: for each segment S, we have

$$\chi(S; \partial\mathcal{A}|_S; \sigma; \widetilde{\Delta}^{\epsilon,\infty}|_S) : \mathscr{F}_G(S; P|_S) \to \mathscr{S}_{\mathbb{C}^\times}(\pi^{-1}(S); Q^{\epsilon}_{\text{tw}}|_{\pi^{-1}(S)}; \sigma^{\epsilon}_{\text{tw}})\,. \tag{6.36}$$

Now comes the crucial technical step: for the purposes of our computation we may replace the equivalences $\chi(S; \partial\mathcal{A}|_S; \sigma; \widetilde{\Delta}^{\epsilon,\infty}|_S)$ by any other equivalences ξ_S between the same $\mathbb{V}$-lines, compatible with the f-action. Indeed, any two equivalences differ by tensorization with a line L_S, and the f-equivariance identifies the three lines L_S with a single line L, so the effect of changing from $\chi(S; \partial\mathcal{A}|_S; \sigma; \widetilde{\Delta}^{\epsilon,\infty}|_S)$ to ξ_S would be to replace $\mathcal{L}(\Delta, \epsilon)$ by $\mathcal{L}(\Delta, \epsilon) \otimes L^3$; the μ_3-action on L^3 induced by cyclic permutation of the factors is trivial, so the μ_3-action we want to compute is insensitive to this replacement.

We construct a convenient ξ_S as follows. We extend the trivializations of P and Q^{ϵ}_{tw} from the midpoints to sections s_P and s_Q of $P|_{\partial\Delta}$ and $Q^{\epsilon}_{\text{tw}}|_{\partial\widetilde{\Delta}}$ respectively, in an f-invariant way. On each segment S this gives trivializations of our two $\mathbb{V}$-lines, and we choose ξ_S to intertwine these trivializations. Tensoring the ξ_S we get

$$\xi : \mathscr{F}_G(\partial\Delta; \partial P) \to \mathscr{S}_{\mathbb{C}^\times}(\partial\widetilde{\Delta}; \partial Q^{\epsilon}_{\text{tw}}; \sigma^{\epsilon}_{\text{tw}})\,. \tag{6.37}$$

We may choose the trivialization of $\mathcal{A}$ at the midpoints in such a way that the parallel transport of Q^{ϵ}_{tw} along any of the 6 preimages of segments is given by $1 \in \mathbb{C}^\times$. (Indeed, for an arbitrary f-invariant trivialization, the parallel transport of Q along each segment is given by some fixed element $h \in H \setminus T$; changing the trivialization by $t \in T$ at each vertex conjugates this transport by t, and by so doing we can set the off-diagonal entries to ± 1 as needed.) From now on we fix such a choice. Having done so, we can choose s_Q to be covariantly constant.

However, we cannot choose s_P to be covariantly constant. Indeed, the parallel transport of P along an edge is given, relative to the trivializations at the midpoints, by an element $g = hb \in G$, where $h \in H \setminus T$ and $b \in B$; in particular, $g \neq 1$. The f-invariance implies that g is independent of the choice of edge. Moreover, $g^3 = 1$, since the holonomy of P around $\partial\Delta$ is trivial. We choose s_P as follows. Let t be a covariantly constant section of $\partial P \to \partial\Delta$ (necessarily not f-invariant). Then choose $s_P = \phi t$, where

$\phi : \partial\Delta \to G$ obeys

$$\phi(f(y)) = g\,\phi(y)\,. \tag{6.38}$$

Now, to compute the action of f on $\mathcal{L}(\Delta, \epsilon)$ we consider the mapping torus of f,

$$\Delta_f = (\Delta \times \mathbb{R}) \,/\, [(y,x) \sim (f(y), x+1)]\,, \tag{6.39}$$

and the mapping torus $\widetilde{\Delta}_f$ of the lift of f to $\widetilde{\Delta}$. The f-equivariant flat bundles $P \to \Delta$ and $Q^{\epsilon}_{\mathrm{tw}} \to \widetilde{\Delta}$ induce flat bundles $P_f \to \Delta_f$ and $Q^{\epsilon}_{\mathrm{tw},f} \to \widetilde{\Delta}_f$ respectively. Moreover, the f-equivariant equivalence ξ of $\mathbb{V}$-lines induces an isomorphism of lines, $\xi_f : \mathscr{F}_G(\partial\Delta_f; \partial P_f) \to \mathscr{S}_{\mathbb{C}^\times}(\partial\widetilde{\Delta}_f; \partial Q^{\epsilon}_{\mathrm{tw},f}; \sigma^{\epsilon}_{\mathrm{tw}})$. The μ_3-action on $\mathcal{L}(\Delta, \epsilon)$ is multiplication by

$$\frac{\xi_f(\mathscr{F}_G(\Delta_f; P_f))}{\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta}_f; Q^{\epsilon}_{\mathrm{tw},f}; \sigma^{\epsilon}_{\mathrm{tw}})}\,. \tag{6.40}$$

To compute this, first note that, being f-invariant, s_P and s_Q induce sections s_{P_f} and s_{Q_f} of $\partial P_f \to \partial\Delta_f$ and $\partial Q^{\epsilon}_{\mathrm{tw},f} \to \partial\widetilde{\Delta}_f$ respectively. The resulting trivializations of the boundary lines have, essentially by definition of ξ,

$$\xi_f(\tau_{s_{P_f}}) = \tau_{s_{Q_f}}\,. \tag{6.41}$$

Our task now is to compute the numerator and denominator relative to these trivializations.

To compute $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta}_f; Q^{\epsilon}_{\mathrm{tw},f}; \sigma^{\epsilon}_{\mathrm{tw}})$ we note that s_{Q_f} is covariantly constant on $\partial\widetilde{\Delta}_f$, and it can be extended to a covariantly constant section over the full $\widetilde{\Delta}_f$; thus the $\mathbb{C}^\times$ Chern-Simons form vanishes, and $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{\Delta}_f; Q^{\epsilon}_{\mathrm{tw},f}; \sigma^{\epsilon}_{\mathrm{tw}}) = \tau_{s_{Q_f}}$.

To compute $\mathscr{F}_G(\Delta_f; P_f)$ is more interesting. We choose an arbitrary extension s of s_{P_f} to the solid torus Δ_f. Let A denote the connection form in P_f relative to the section s; then using (3.9)

$$\mathscr{F}_G(\Delta_f; P_f) = \tau_{s_{P_f}} \exp\left[\frac{1}{4\pi\sqrt{-1}} \int_{\Delta_f} \operatorname{trace}\left(A \wedge dA + \frac{2}{3}\, A \wedge A \wedge A\right)\right]. \tag{6.42}$$

To compute explicitly, we pull back to a triple cover $p : \Delta \times S^1 \to \Delta_f$,

$$\Delta \times S^1 = (\Delta \times \mathbb{R}) \,/\, [(y,x) \sim (y, x+3)]\,, \quad p([(y,x)]) = [(y,x)]\,. \tag{6.43}$$

The covariantly constant section t of $\partial P \to \partial\Delta$ induces a covariantly constant section of $\partial(p^*P) \to \partial\Delta \times S^1$,[30] which we can further extend to a

[30] Since t is not f-invariant, it would not induce a section of $P_f \to \partial\Delta_f$; this is the reason why we had to pull back to the triple cover.

covariantly constant section $\hat{t}$ of $p^*P \to \Delta \times S^1$. Then $p^*s = \phi\hat{t}$ for some $\phi : \Delta \times S^1 \to G$, and the invariance of p^*s under the μ_3 deck group gives

$$\phi(f(y), x+1) = g\,\phi(y, x)\,. \tag{6.44}$$

The connection form in p^*P relative to p^*s is $A = \phi^{-1}\mathrm{d}\phi$, and this form is pulled back from Δ_f; thus the integral over Δ_f in (6.42) can be rewritten as one-third of a more explicit integral over $\Delta \times S^1$,

$$\mathscr{F}_G(\Delta_f; P_f) = \tau_{s_{P_f}} \exp\left[\frac{1}{3}\frac{1}{12\pi\sqrt{-1}}\int_{\Delta\times S^1} \mathrm{trace}(\phi^{-1}\mathrm{d}\phi)^3\right]. \tag{6.45}$$

Moreover, $s|_{\partial\Delta\times S^1}$ is covariantly constant along translation in the x-direction holding $y \in \partial\Delta$ fixed, while t is covariantly constant in every direction. Thus $\phi|_{\partial\Delta\times S^1}$ is constant in the x-direction. Attaching a solid torus $S^1 \times D^2$ to $\Delta \times S^1$ along this direction we obtain a closed 3-manifold $M \simeq S^3$. The map ϕ naturally extends to the added $S^1 \times D^2$, by choosing it to be constant along the D^2 factor. Then $(\phi^{-1}\mathrm{d}\phi)^3 = 0$ there, and thus we can replace the domain of integration in (6.45) by M. This integral gives $\exp(\frac{2\pi\sqrt{-1}}{3}k)$, where k is the degree of the map $\phi : M \to G$ (by which we mean the degree of the retraction of ϕ from G to $\mathrm{SU}(2)$, when we equip $\mathrm{SU}(2)$ with the orientation for which $-\,\mathrm{trace}(h^{-1}\mathrm{d}h)^3$ is a positive 3-form.) To compute this degree, we will use only the fact that ϕ commutes with certain μ_3-actions on M and $\mathrm{SL}(2,\mathbb{C})$, as follows.

First, we can identify M with $\{|\alpha|^2 + |\beta|^2 = 1\} = S^3 \subset \mathbb{C}^2$ as follows. On the torus $\partial\Delta \times S^1$, we fix coordinates $\alpha = \frac{1}{\sqrt{2}}\mathrm{e}^{\sqrt{-1}\theta}$, where θ parameterizes $\partial\Delta$ (positively with respect to the boundary orientation), and $\beta = \frac{1}{\sqrt{2}}\mathrm{e}^{2\pi\sqrt{-1}x/3}$. These coordinates naturally extend to the two solid tori $\Delta \times S^1$ and $S^1 \times D^2$, identifying them respectively as the loci $|\alpha| < \frac{1}{2}$ and $|\beta| < \frac{1}{2}$ in S^3. The orientation of M matches the standard orientation of S^3. The μ_3-action $(y, x) \mapsto (f(y), x+1)$ on $\Delta \times S^1$ becomes in these coordinates $(\alpha, \beta) \mapsto (\mathrm{e}^{2\pi\sqrt{-1}/3}\alpha, \mathrm{e}^{2\pi\sqrt{-1}/3}\beta)$ (and thus extends to a fixed-point-free action on the whole S^3).

Next, parameterize $\mathrm{SU}(2)$ by $h = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & -\bar{\alpha} \end{pmatrix}$. This gives $\mathrm{SU}(2) \simeq S^3 \subset \mathbb{C}^2$. Computing $-\,\mathrm{trace}(h^{-1}\mathrm{d}h)^3$ in this parameterization we see that it is positive for the standard orientation on S^3. By composing ϕ with an inner automorphism of $\mathrm{SL}(2,\mathbb{C})$ we may assume $g = \mathrm{diag}(\mathrm{e}^{2\pi\sqrt{-1}/3}, \mathrm{e}^{-2\pi\sqrt{-1}/3})$. The μ_3-action $h \mapsto gh$ then acts by $(\alpha, \beta) \mapsto (\mathrm{e}^{2\pi\sqrt{-1}/3}\alpha, \mathrm{e}^{2\pi\sqrt{-1}/3}\beta)$.

We have shown how to identify both M and $\mathrm{SU}(2)$ with S^3, in such a way that the μ_3-actions and orientations agree with the standard ones for S^3. Using (6.44), ϕ intertwines the μ_3-action on M with the μ_3-action on $\mathrm{SU}(2)$. Then using Lemma 6.46 below completes the proof. □

LEMMA 6.46. *Any continuous map $\phi : S^3 \to S^3$ commuting with the standard μ_3-action has degree equal to* 1 *mod* 3.

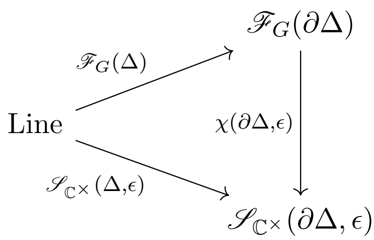
PROOF. Such a ϕ descends to a map $\bar{\phi} : S^3/\mu_3 \to S^3/\mu_3$ which lifts to the μ_3-bundle $S^3 \to S^3/\mu_3$; this bundle has a nonzero characteristic class lying in $H^3(S^3/\mu_3; \mu_3) \simeq \mu_3$ (because the inclusion $S^3/\mu_3 \to B\mu_3 = S^\infty/\mu_3$ induces an isomorphism on $H^3(\cdot; \mu_3)$), and thus $\bar{\phi}^*$ must act trivially on $H^3(S^3/\mu_3; \mu_3)$, i.e. the degree of $\bar{\phi}$ is 1 mod 3. Since the degree of ϕ agrees with that of $\bar{\phi}$, this finishes the proof. □

To finish this section we remark on a diagrammatic perspective on $\mathcal{L}(\Delta, \epsilon)$ which will be useful in some of the arguments to follow. Here we suppress most of the background fields to reduce clutter. We regard Δ as a morphism in the bordism category

$$\emptyset \xrightarrow{\Delta} \partial\Delta\,. \tag{6.47}$$

Applying $\mathscr{F}_G$ and $\mathscr{S}_{\mathbb{C}^\times}$ to this diagram, and including the map $\chi(\partial\Delta, \epsilon)$, we get

(6.48)



Here, and in various diagrammatic arguments to follow, we freely identify morphisms Line $\to \mathcal{C}$ with objects of $\mathcal{C}$. Then the composition

$$\mathcal{L}(\Delta, \epsilon) = \mathscr{S}_{\mathbb{C}^\times}(\Delta, \epsilon)^{-1} \circ \chi(\partial\Delta, \epsilon) \circ \mathscr{F}_G(\Delta)\,. \tag{6.49}$$

6.6. Abelianization of Chern-Simons over triangulated surfaces. Now suppose given an oriented surface Y with a semi-ideal triangulation $\mathfrak{T}$, edge-orientations ϵ, and a spin structure σ. Also fix boundary-reduced stratified abelianization data $\mathcal{A} = (P, Q, \mu, \theta)$ over $(Y, \mathcal{N}^{\mathfrak{T}})$. As we have discussed in §5.3.2, because P is boundary-reduced, the $\mathbb{V}$-line $\mathscr{F}_G(\partial Y; \partial P)$ is canonically trivial, and thus $\mathscr{F}_G(Y; P)$ is a line. The $\mathbb{V}$-line $\mathscr{S}_{\mathbb{C}^\times}(\partial\widetilde{Y}; \partial Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}})$ is also canonically trivial, since Q^ϵ_{tw} has trivial holonomy around the boundary components; thus $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}})$ is also a line.

Define the difference line

$$\mathcal{L}(Y, \epsilon) = \bigotimes_{\Delta \in \mathbf{faces}(\mathfrak{T})} \mathcal{L}(\Delta, \epsilon|_\Delta)\,. \tag{6.50}$$

CONSTRUCTION 6.51. There is a canonical isomorphism of lines

$$\chi^\epsilon_Y : \mathscr{F}_G(Y; P) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}) \otimes \mathcal{L}(Y, \epsilon)\,. \tag{6.52}$$

Proof. First for simplicity suppose that there are no ideal vertices, so Y is a closed triangulated surface.

For each $\Delta \in \mathbf{faces}(\mathcal{T})$ we consider the dilation $\mu : \Delta \to \Delta$ which rescales distance from the barycenter by $\frac{1}{2}$. Then we have a decomposition $Y = Y_{\mathrm{out}} \cup Y_{\mathrm{in}}$, where Y_{in} is the union of the rescaled triangles $\mu(\Delta)$.

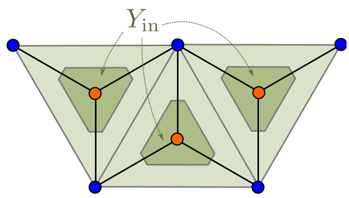


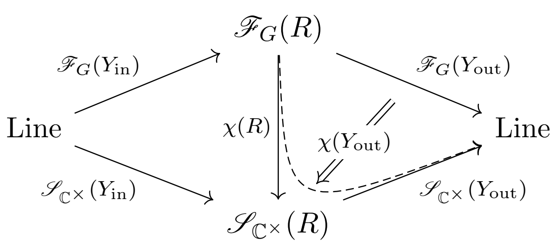
Figure 19. A portion of the triangulated surface Y with its SN-stratification, and the decomposition $Y = Y_{\mathrm{out}} \cup Y_{\mathrm{in}}$.

Let $R = \partial Y_{\mathrm{in}}$. To condense the notation we will just write the manifolds, suppressing all the extra background fields, including the dependence on ϵ (since ϵ is held fixed throughout this proof). Then we have a diagram in the bordism category,

$$\emptyset \xrightarrow{Y_{\mathrm{in}}} R \xrightarrow{Y_{\mathrm{out}}} \emptyset \tag{6.53}$$

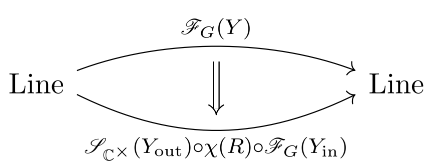
Applying $\mathscr{F}_G$ and $\mathscr{S}_{\mathbb{C}^\times}$ to this diagram, and inserting the maps $\chi(R)$ and $\chi(Y_{\mathrm{out}})$ provided by Theorem 6.20, we get the diagram below:

(6.54)



Here the dashed arrow indicates the composition $\mathscr{S}_{\mathbb{C}^\times}(Y_{\mathrm{out}}) \circ \chi(R)$. Now whiskering $\chi(Y_{\mathrm{out}})$ by $\mathscr{F}_G(Y_{\mathrm{in}})$ we get a 2-morphism

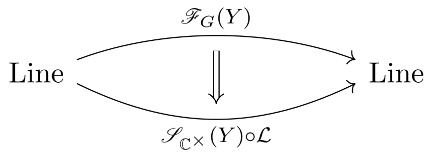
(6.55)



Finally, defining

$$\mathcal{L} = \mathscr{S}_{\mathbb{C}^\times}(Y_{\mathrm{in}})^{-1} \circ \chi(R) \circ \mathscr{F}_G(Y_{\mathrm{in}}), \tag{6.56}$$

we have $\chi(R) \circ \mathscr{F}_G(Y_{\text{in}}) = \mathscr{S}_{\mathbb{C}^\times}(Y_{\text{in}}) \circ \mathcal{L}(Y, \epsilon)$, so we can rewrite the diagram as

(6.57)


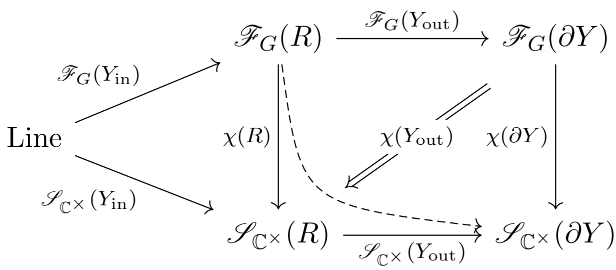
Composition of maps Line $\to$ Line is tensor product, so this is a map of lines

$$\mathscr{F}_G(Y) \to \mathscr{S}_{\mathbb{C}^\times}(Y) \otimes \mathcal{L} \tag{6.58}$$

as desired; what remains is to identify $\mathcal{L}$ with the $\mathcal{L}(Y, \epsilon)$ we defined in (6.50). This follows directly from (6.49) and the decomposition of Y_{in} into the disjoint union of triangles.

So far we discussed only the case where there are no ideal vertices. In the general case one has to draw slightly more complicated diagrams:

$$\emptyset \xrightarrow{Y_{\text{in}}} R \xrightarrow{Y_{\text{out}}} \partial Y \tag{6.59}$$

(6.60)


However, because $\mathcal{A}$ is assumed boundary-reduced, we have trivializations of $\mathscr{F}_G(\partial Y)$ and $\mathscr{S}_{\mathbb{C}^\times}(\partial Y)$, which are intertwined by $\chi(\partial Y)$; using these trivializations the diagram (6.60) reduces to (6.57), and then we can proceed just as above. This completes the proof. $\square$

We may also consider a family of boundary-reduced stratified abelianization data $\mathcal{A} = (P_s, Q_s, \mu_s, \theta_s)$ over a fixed $(Y, \mathcal{N})$, varying with a parameter $s \in S$. All of our constructions can be applied to such a family: then we obtain two invariants $\mathscr{F}_G(Y; P_s)$ and $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^\epsilon_{\text{tw},s}; \sigma^\epsilon_{\text{tw}})$ both of which vary over S, and an isomorphism relating them. In particular, if Y is a semi-ideally triangulated surface as above, then both $\mathscr{F}_G(Y; P_s)$ and $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^\epsilon_{\text{tw},s}; \sigma^\epsilon_{\text{tw}})$ are line bundles over S with connection, and $\chi^\epsilon_Y : \mathscr{F}_G(Y; P_s) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^\epsilon_{\text{tw},s}; \sigma^\epsilon_{\text{tw}}) \otimes \mathcal{L}(Y, \epsilon)$ is an isomorphism of line bundles with connection.

6.7. Reversing an edge orientation on a triangle. Now suppose $(\Delta, \epsilon, \sigma, \mathcal{A})$ is an oriented triangle with edge-orientations, spin structure, and boundary-reduced stratified abelianization data. We first consider the restriction of all the data to a single oriented edge E. Let ϵ be the given orientation of E and ϵ' the opposite orientation. Then in the bordism category we have the diagram

$$\emptyset \xrightarrow{E} \partial E \tag{6.61}$$

Once again we suppress some background fields in the notation: we just keep the manifolds and (where necessary) the edge-orientations. We will also use freely the fact that the $\mathbb{Z}$-bundles $\widetilde{\Delta}^{\epsilon,\infty}$ and $\widetilde{\Delta}^{\infty,\epsilon'}$ are canonically trivial over ∂E.

Then we have a diagram in the 2-$\mathbb{V}$-line $\mathscr{S}_{\mathbb{C}^\times}(\partial E)$:

$$\begin{array}{ccc} & & \mathscr{S}_{\mathbb{C}^\times}(E,\epsilon) \\ & \nearrow^{\chi(E,\epsilon)} & \\ \chi(\partial E)\circ \mathscr{F}_G(E) & & \Big\downarrow {\scriptstyle \zeta(E,\varpi^E,\epsilon)} \\ & \searrow_{\chi(E,\epsilon')} & \\ & & \mathscr{S}_{\mathbb{C}^\times}(E,\epsilon') \end{array}$$

We define

$$\Xi(E) = \chi(E,\epsilon')^{-1} \circ \zeta(E,\varpi^E,\epsilon) \circ \chi(E,\epsilon) \tag{6.62}$$

which is an automorphism of the object $\chi(\partial E)\circ \mathscr{F}_G(E) \in \mathscr{S}_{\mathbb{C}^\times}(\partial E)$, or equivalently a line.

Now suppose $(\Delta,\epsilon,\sigma,\mathcal{A})$ is an oriented triangle with edge-orientations, spin structure, and stratified abelianization data. Also suppose E is an edge such that ϵ_E agrees with the boundary orientation, and ϵ' is obtained from ϵ by reversing the orientation on E. Let $\tilde{E} = -(\overline{\partial\Delta \setminus E})$, so now we have

$$\emptyset \underset{E}{\overset{\tilde{E}}{\rightrightarrows}} \partial E, \qquad \Delta\Downarrow \tag{6.63}$$

Then $\zeta(\cdot,\varpi^E,\epsilon)$ provides an isomorphism of 2-morphisms
(6.64)

$$\left(\mathbb{V}\mathrm{Line} \underset{\mathscr{S}_{\mathbb{C}^\times}(E,\epsilon)}{\overset{\mathscr{S}_{\mathbb{C}^\times}(\tilde{E})}{\rightrightarrows}} \mathscr{S}_{\mathbb{C}^\times}(\partial E),\ \mathscr{S}_{\mathbb{C}^\times}(\Delta,\epsilon)\Downarrow\right) \longrightarrow \left(\mathbb{V}\mathrm{Line} \underset{\mathscr{S}_{\mathbb{C}^\times}(E,\epsilon')}{\overset{\mathscr{S}_{\mathbb{C}^\times}(\tilde{E})}{\rightrightarrows}} \mathscr{S}_{\mathbb{C}^\times}(\partial E),\ \mathscr{S}_{\mathbb{C}^\times}(\Delta,\epsilon')\Downarrow\right)$$

while $\chi(\cdot, \epsilon)$ gives a similar isomorphism of diagrams, but without the inner arrow,

(6.65)

$$\mathbb{V}\mathrm{Line} \underset{\mathscr{F}_G(E)}{\overset{\mathscr{F}_G(\tilde{E})}{\rightrightarrows}} \mathscr{F}_G(\partial E) \qquad \longrightarrow \qquad \mathbb{V}\mathrm{Line} \underset{\mathscr{S}_{\mathbb{C}^\times}(E,\epsilon)}{\overset{\mathscr{S}_{\mathbb{C}^\times}(\tilde{E})}{\rightrightarrows}} \mathscr{S}_{\mathbb{C}^\times}(\partial E)$$

and $\chi(\cdot, \epsilon')$ gives the same isomorphism of diagrams with ϵ replaced by ϵ'. Combining these, we can whisker the isomorphism (6.64) into an isomorphism between the lines we previously defined in (6.49), (6.62),

$$\rho^{\epsilon',\epsilon}(\Delta) : \mathcal{L}(\Delta, \epsilon') \to \mathcal{L}(\Delta, \epsilon) \otimes \Xi(E)^{-1} \,. \tag{6.66}$$

This map describes the effect of reversing the orientation on one edge of Δ.

Now we want to consider reversing orientation on two edges of Δ. We introduce a bit of notation that will be convenient below:

Definition 6.67. If E_1 and E_2 are edges of an oriented triangle Δ,

(6.68)

$$\langle E_1, E_2 \rangle = \begin{cases} +1 & \text{if } E_1 \text{ is ahead of } E_2 \text{ in the boundary orientation,} \\ -1 & \text{if } E_1 \text{ is behind } E_2 \text{ in the boundary orientation,} \\ 0 & \text{if } E_1 = E_2. \end{cases}$$

Suppose ϵ_{E_1}, ϵ_{E_2} both agree with the boundary orientation, let ϵ'_i be obtained from ϵ by reversing ϵ_{E_i}, and let ϵ''_{12} be obtained from ϵ by reversing both ϵ_{E_1} and ϵ_{E_2}.

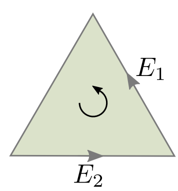


Figure 20. An oriented triangle with two marked edges, with $\langle E_1, E_2 \rangle = 1$, and edge orientations agreeing with the boundary orientation. The third edge orientation is arbitrary.

We want to compare the two 2-morphisms appearing in the diagram:

$$\begin{array}{ccc} \mathscr{S}_{\mathbb{C}^\times}(\partial\Delta, \epsilon) & \xrightarrow{\zeta(E_1,\varpi_{E_1})\circ_h \zeta(E_2,\varpi_{E_2})\circ_h \mathrm{id}} & \mathscr{S}_{\mathbb{C}^\times}(\partial\Delta, \epsilon''_{12}) \\ & \overset{\zeta(\Delta,\varpi_{E_1})\circ\zeta(\Delta,\varpi_{E_2})}{\underset{\zeta(\Delta,\varpi_{E_2})\circ\zeta(\Delta,\varpi_{E_1})}{\Rrightarrow}} & \\ {\scriptstyle \mathscr{S}_{\mathbb{C}^\times}(\Delta,\epsilon)} \nwarrow & \mathrm{Line} & \nearrow {\scriptstyle \mathscr{S}_{\mathbb{C}^\times}(\Delta,\epsilon''_{12})} \end{array}$$

LEMMA 6.69. *In this situation,*

$$\frac{\zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon_1'}; \sigma_{\mathrm{tw}}^{\epsilon_1'}; \varpi_{E_2}) \circ \zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon}; \sigma_{\mathrm{tw}}^{\epsilon}; \varpi_{E_1})}{\zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon_2'}; \sigma_{\mathrm{tw}}^{\epsilon_2'}; \varpi_{E_1}) \circ \zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon}; \sigma_{\mathrm{tw}}^{\epsilon}; \varpi_{E_2})} = \exp\left(-\frac{2\pi\sqrt{-1}}{4}\langle E_1, E_2\rangle\right). \tag{6.70}$$

PROOF. We use (5.105). For this purpose we choose a section s of $Q_{\mathrm{tw}}^{\epsilon}$ and maps $p_1, p_2 : \widetilde{\Delta} \to S^1$ representing ϖ_{E_1}, ϖ_{E_2}, with p_i constant on $\partial\widetilde{\Delta}$ except for $\widetilde{E}_i$. Then we have another section $s_{12}'' = e^{\pi\sqrt{-1}(p_1+p_2)} s$ of $Q_{\mathrm{tw}}^{\epsilon_{12}''}$, and in terms of these sections

$$\begin{aligned}&\zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon_i'}; \sigma_{\mathrm{tw}}^{\epsilon_i'}; \varpi_{E_j}) \circ \zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon}; \sigma_{\mathrm{tw}}^{\epsilon}; \varpi_{E_i})\tau_s \\ &\quad = \exp\left(-\frac{1}{4}\int_{\widetilde{\Delta}} \left(s^*\alpha + \pi\sqrt{-1}\, dp_i\right) \wedge dp_j + s^*\alpha \wedge dp_i\right) \tau_{s_{12}''}\end{aligned} \tag{6.71}$$

so that the desired ratio is

$$\exp\left(-\frac{2\pi\sqrt{-1}}{4}\int_{\widetilde{\Delta}} dp_1 \wedge dp_2\right) = \exp\left(-\frac{2\pi\sqrt{-1}}{4}\langle E_1, E_2\rangle\right). \tag{6.72}$$

□

PROPOSITION 6.73. *In this situation,*

$$\rho^{\epsilon_2', \epsilon_{12}''}(\Delta) \circ \rho^{\epsilon, \epsilon_2'}(\Delta) = \exp\left(-\frac{2\pi\sqrt{-1}}{4}\langle E_1, E_2\rangle\right) \rho^{\epsilon_1', \epsilon_{12}''}(\Delta) \circ \rho^{\epsilon, \epsilon_1'}(\Delta). \tag{6.74}$$

PROOF. Since ρ is constructed by whiskering ζ we get

$$\frac{\rho^{\epsilon_2', \epsilon_{12}''}(\Delta) \circ \rho^{\epsilon, \epsilon_2'}(\Delta)}{\rho^{\epsilon_1', \epsilon_{12}''}(\Delta) \circ \rho^{\epsilon, \epsilon_1'}(\Delta)} = \frac{\zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon_2'}; \sigma_{\mathrm{tw}}^{\epsilon_2'}; \varpi_{E_1}) \circ \zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon}; \sigma_{\mathrm{tw}}^{\epsilon}; \varpi_{E_2})}{\zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon_1'}; \sigma_{\mathrm{tw}}^{\epsilon_1'}; \varpi_{E_2}) \circ \zeta(\widetilde{\Delta}; Q_{\mathrm{tw}}^{\epsilon}; \sigma_{\mathrm{tw}}^{\epsilon}; \varpi_{E_1})}. \tag{6.75}$$

Then use Lemma 6.69. □

Similarly we can consider reversing the orientation on an edge E where the initial orientation ϵ_E is opposite to the boundary orientation. Then the same constructions as above give an isomorphism

$$\rho^{\epsilon, \epsilon'}(\Delta) : \mathcal{L}(\Delta, \epsilon') \to \mathcal{L}(\Delta, \epsilon) \otimes \Xi(E). \tag{6.76}$$

6.8. Reversing an edge orientation on a triangulated surface. Now suppose $(Y, \mathcal{T})$ is a semi-ideally triangulated surface, with boundary-reduced stratified abelianization data (P, Q, μ, θ). Suppose ϵ, ϵ' are edge-orientations on $\mathcal{T}$, differing by reversing the orientation on one edge E. We would like to compare the maps χ_Y^{ϵ} and $\chi_Y^{\epsilon'}$ from Construction 6.51. Their ratio is an isomorphism

$$\chi_Y^{\epsilon} \circ (\chi_Y^{\epsilon'})^{-1} : \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q_{\mathrm{tw}}^{\epsilon'}; \sigma_{\mathrm{tw}}^{\epsilon'}) \otimes \mathcal{L}(Y, \epsilon') \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q_{\mathrm{tw}}^{\epsilon}; \sigma_{\mathrm{tw}}^{\epsilon}) \otimes \mathcal{L}(Y, \epsilon). \tag{6.77}$$

We already have an isomorphism

$$\zeta(\widetilde{Y}; Q_{\mathrm{tw}}^{\epsilon'}; \sigma_{\mathrm{tw}}^{\epsilon'}; \varpi^E) : \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q_{\mathrm{tw}}^{\epsilon'}; \sigma_{\mathrm{tw}}^{\epsilon'}) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q_{\mathrm{tw}}^{\epsilon}; \sigma_{\mathrm{tw}}^{\epsilon}) \tag{6.78}$$

given by Corollary 5.103. Thus we can write

$$\chi_Y^{\epsilon} \circ (\chi_Y^{\epsilon'})^{-1} = \zeta(Y; Q_{\mathrm{tw}}^{\epsilon'}; \varpi^E) \otimes (\lambda^{\epsilon, \epsilon'}(Y))^{-1} \tag{6.79}$$

for some

$$\lambda^{\epsilon,\epsilon'}(Y) : \mathcal{L}(Y,\epsilon) \to \mathcal{L}(Y,\epsilon'). \tag{6.80}$$

This map is determined in terms of the edge-reversal maps for the two triangles Δ_1, Δ_2 abutting E, as follows.

PROPOSITION 6.81. $\lambda^{\epsilon,\epsilon'}(Y) = \rho^{\epsilon,\epsilon'}(\Delta_1) \otimes \rho^{\epsilon,\epsilon'}(\Delta_2)$.

PROOF. We decompose Y as indicated in the figure:

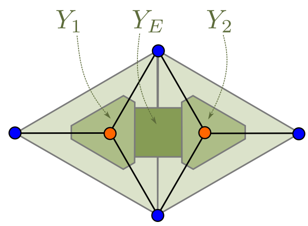


FIGURE 21. The decomposition of a triangulated surface Y associated to an edge E.

$$Y = Y_{\text{out}} \cup Y_1 \cup Y_2 \cup Y_E \tag{6.82}$$

Now we want to describe $\zeta(Y; Q^{\epsilon'}_{\text{tw}}; \varpi^E)^{-1} \circ \chi^{\epsilon}_Y \circ (\chi^{\epsilon'}_Y)^{-1}$ relative to this decomposition. We will content ourselves with a heuristic description, leaving the full diagrammatics to the reader. On Y_{out} all the background fields associated to ϵ and ϵ' are canonically isomorphic, and the background field ϖ_E is canonically trivial, so likewise $\zeta(Y; Q^{\epsilon'}_{\text{tw}}; \varpi^E)^{-1} \circ \chi^{\epsilon} \circ (\chi^{\epsilon'})^{-1}$ is trivial. On Y_i, $\zeta(Y; Q^{\epsilon'}_{\text{tw}}; \varpi^E)^{-1} \circ \chi^{\epsilon} \circ (\chi^{\epsilon'})^{-1}$ is described by the map $(\rho^{\epsilon,\epsilon'}(\Delta_i))^{-1} : \mathcal{L}(\Delta_i, \epsilon') \otimes \Xi(E_i)^{\pm 1} \to \mathcal{L}(\Delta_i, \epsilon)$ where E_i is the edge of Δ_i corresponding to E. Finally, on Y_E, $\zeta(Y; Q^{\epsilon'}_{\text{tw}}; \varpi^E)^{-1} \circ \chi^{\epsilon} \circ (\chi^{\epsilon'})^{-1}$ gives the isomorphism $\Xi(E_1) \to \Xi(E_2)$ induced by identifying these two edges. Combining these gives the desired result. □

PROPOSITION 6.83. *The composition*

$$\lambda^{\epsilon',\epsilon}(Y) \circ \lambda^{\epsilon,\epsilon'}(Y) : \mathcal{L}(Y,\epsilon) \to \mathcal{L}(Y,\epsilon) \tag{6.84}$$

acts by multiplication by the ϵ-sign of the edge E, as defined in Definition 6.9.

PROOF. First note that from (6.79) it follows directly that $\lambda^{\epsilon',\epsilon}(Y) \circ \lambda^{\epsilon,\epsilon'}(Y)$ acts by $\zeta(\widetilde{Y}; Q^{\epsilon'}_{\text{tw}}; \sigma^{\epsilon'}_{\text{tw}}; \varpi_E) \circ \zeta(\widetilde{Y}; Q^{\epsilon}_{\text{tw}}; \sigma^{\epsilon}_{\text{tw}}; \varpi_E)$. Using (5.105) and (5.30) we see that this is multiplication by $\sigma^{\epsilon}_{\text{tw}}([\varpi_E]) = \sigma^{\epsilon}_{\text{tw}}(\gamma_E)$; by Proposition 6.19 this is the ϵ-sign of E. □

7. Gluing description of the Chern-Simons line

We continue with the setup of the last section. There we used abelianization to produce an isomorphism

$$\psi_D : \mathscr{F}_G(Y; P) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^{\epsilon}_{\text{tw}}; \sigma^{\epsilon}_{\text{tw}}) \otimes \mathcal{L}(Y,\epsilon) \tag{7.1}$$

depending on various choices encapsulated in D. In this section we process this further to turn it into an explicit description of $\mathscr{F}_G(Y;P)$. The final result is Theorem 7.81, which asserts the existence of nonzero elements $\tau_{\hat{D}} \in \mathscr{F}_G(Y;P)^*$ depending on choices $\hat{D}$ (including an ideal triangulation, edge-orientations, and certain choices of logarithms), and gives the cocycle expressing how $\tau_{\hat{D}}$ changes when the data $\hat{D}$ are changed.

7.1. Abelianization of Chern-Simons over interpolating 3-manifolds. Suppose M is a spin 3-manifold with boundary, equipped with a spectral network $\mathcal{N}$ such that $M_{-3} = \emptyset$, the restriction of $\mathcal{N}$ to $Y = \partial M$ is the spectral network $\mathcal{N}^{\mathfrak{T}}$ for a triangulation $\mathfrak{T}$ of Y, and $M_{-2\mathrm{b}}$ has no closed components.

Each component B of $M_{-2\mathrm{b}}$ is a closed interval, whose two ends lie on $Y_{-2\mathrm{b}}$, in two triangles $\Delta_0, \Delta_1 \in \mathbf{faces}(\mathfrak{T})$, with edge-orientations ϵ_{Δ_0}, ϵ_{Δ_1}. Moreover, a tubular neighborhood of B can be identified with $\Delta \times [0,1]$, for a triangle Δ with its standard spectral network. This in particular gives an orientation-reversing identification $f : \Delta_0 \to \Delta_1$.

Now suppose given edge-orientations ϵ over Y, and a $\mathbb{Z}$-bundle $\widetilde{M}^\infty \to \widetilde{M}_{\geq -3\mathrm{a}}$, which restricts to $\widetilde{Y}^{\epsilon,\infty} \to \widetilde{Y}_{\geq -3\mathrm{a}}$ over the boundary, and also restricts to $\widetilde{\Delta}^{\epsilon_B,\infty} \times [0,1]$ over the tubular neighborhood of each component $B \subset M_{-2\mathrm{b}}$, for some edge-orientations ϵ_B on Δ. In particular ϵ_B restricts to match ϵ at the two ends, and so $f_*(\epsilon|_{\Delta_0}) = \epsilon|_{\Delta_1}$. Thus using Proposition 6.29 we get an isomorphism $\mathcal{L}(\Delta, \epsilon_\Delta) \to \mathcal{L}(\Delta', \epsilon_{\Delta'})^*$, i.e. an element $\beta \in \mathcal{L}(\Delta, \epsilon_\Delta) \otimes \mathcal{L}(\Delta', \epsilon_{\Delta'})$. Tensoring over all components $B \subset M_{-2\mathrm{b}}$ gives a canonical element $\beta_M \in \mathcal{L}(Y, \epsilon)$. This element should be thought of as just implementing the matching-up of triangles provided by $M_{-2\mathrm{b}}$.

Finally, suppose we have stratified abelianization data $\mathcal{A} = (P, Q, \mu, \theta)$ over $(M, \mathcal{N})$.

Proposition 7.2. *In this situation,*

(7.3)
$$\chi_Y^\epsilon(\mathscr{F}_G(M;P)) = \mathscr{S}_{\mathbb{C}^\times}(\widetilde{M}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}) \otimes \beta_M \in \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; \partial Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}) \otimes \mathcal{L}(Y, \epsilon).$$

Proof. Decompose $M = M_{\mathrm{out}} \cup M_{\mathrm{in}}$, where M_{in} is a small tubular neighborhood of $M_{-2\mathrm{b}}$. M_{in} is a 3-manifold with 1-dimensional corners R; R is a union of circles, which divides ∂M_{in} into a union of cylinders B and a union of discs $Y_{\mathrm{in}} \subset Y$. Likewise $\partial M_{\mathrm{out}}$ is divided by R into $-B$ and $Y_{\mathrm{out}} \subset Y$.

Now we need to apply $\mathscr{F}_G$ and $\mathscr{S}_{\mathbb{C}^\times}$ to the various parts of this decomposition. To lighten the notation a bit we write the $\mathbb{V}$-lines (dimension 1)

$$\mathcal{C}_G = \mathscr{F}_G(R; P|_R), \quad \mathcal{C}_{\mathbb{C}^\times} = \mathscr{S}_{\mathbb{C}^\times}(\widetilde{R}; Q^\epsilon_{\mathrm{tw}}|_{\widetilde{R}}; \sigma^\epsilon_{\mathrm{tw}}), \tag{7.4}$$

with objects (dimension 2)

$$\mathcal{O}_{G,\mathrm{in}} = \mathscr{F}_G(Y_{\mathrm{in}}; P|_{Y_{\mathrm{in}}}), \quad \mathcal{O}_{\mathbb{C}^\times,\mathrm{in}} = \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}_{\mathrm{in}}; Q^\epsilon_{\mathrm{tw}}|_{\widetilde{Y}_{\mathrm{in}}}; \sigma^\epsilon_{\mathrm{tw}}). \tag{7.5}$$

Thus $\mathcal{O}_{G,\mathrm{in}} \in \mathcal{C}_G$, $\mathcal{O}_{\mathbb{C}^\times,\mathrm{in}} \in \mathcal{C}_{\mathbb{C}^\times}$, and likewise we have $\mathcal{O}_{G,\mathrm{out}} \in \mathcal{C}_G^*$, $\mathcal{O}_{\mathbb{C}^\times,\mathrm{out}} \in \mathcal{C}_{\mathbb{C}^\times}^*$, and $\mathcal{O}_{G,B} \in \mathcal{C}_G$, $\mathcal{O}_{\mathbb{C}^\times,B} \in \mathcal{C}_{\mathbb{C}^\times}$. Finally, we have the elements (dimension 3)

$$\psi_{G,\mathrm{in}} = \mathscr{F}_G(M_{\mathrm{in}}; P|_{M_{\mathrm{in}}}) \in \mathcal{O}_{G,\mathrm{in}} \otimes \mathcal{O}_{G,B}^* , \tag{7.6}$$

$$\psi_{G,\mathrm{out}} = \mathscr{F}_G(M_{\mathrm{out}}; P|_{M_{\mathrm{out}}}) \in \mathcal{O}_{G,\mathrm{out}} \otimes \mathcal{O}_{G,B} , \tag{7.7}$$

$$\psi_{\mathbb{C}^\times,\mathrm{in}} = \mathscr{S}_{\mathbb{C}^\times}(M_{\mathrm{in}}; Q^\epsilon_{\mathrm{tw}}|_{M_{\mathrm{in}}}; \sigma^\epsilon_{\mathrm{tw}}) \in \mathcal{O}_{\mathbb{C}^\times,\mathrm{in}} \otimes \mathcal{O}^*_{\mathbb{C}^\times,B} , \tag{7.8}$$

$$\psi_{\mathbb{C}^\times,\mathrm{out}} = \mathscr{S}_{\mathbb{C}^\times}(M_{\mathrm{out}}; Q^\epsilon_{\mathrm{tw}}|_{M_{\mathrm{out}}}; \sigma^\epsilon_{\mathrm{tw}}) \in \mathcal{O}_{\mathbb{C}^\times,\mathrm{out}} \otimes \mathcal{O}_{\mathbb{C}^\times,B} . \tag{7.9}$$

We can use Theorem 6.20 to construct various canonical abelianization maps: an equivalence of $\mathbb{V}$-lines (associated to dimension 1),

$$\chi^\epsilon_{\partial B} : \mathcal{C}_G \to \mathcal{C}_{\mathbb{C}^\times} , \tag{7.10}$$

homs (associated to dimension 2),

$$\chi^\epsilon_{Y_{\mathrm{out}}} : \chi^{\epsilon *}_{\partial B}(\mathcal{O}_{G,\mathrm{out}}) \to \mathcal{O}_{\mathbb{C}^\times,\mathrm{out}} , \tag{7.11}$$

$$\chi^\epsilon_{Y_{\mathrm{in}}} : \chi^\epsilon_{\partial B}(\mathcal{O}_{G,\mathrm{in}}) \to \mathcal{O}_{\mathbb{C}^\times,\mathrm{in}} \otimes \mathcal{L}(Y,\epsilon) , \tag{7.12}$$

$$\chi^\epsilon_B : \chi^\epsilon_{\partial B}(\mathcal{O}_{G,B}) \to \mathcal{O}_{\mathbb{C}^\times,B} \tag{7.13}$$

and an equation (associated to dimension 3),

$$((\chi^\epsilon_{Y_{\mathrm{out}}} \otimes \chi^{\epsilon *}_B) \circ \Omega_{Y_{\mathrm{out}},\partial B,B})(\psi_{G,\mathrm{out}}) = \psi_{\mathbb{C}^\times,\mathrm{out}} \tag{7.14}$$

where

$$\Omega_{Y_{\mathrm{out}},\partial B,B} : \mathcal{O}_{G,\mathrm{out}} \otimes \mathcal{O}_{G,B} \to \chi^{\epsilon *}_{\partial B}(\mathcal{O}_{G,\mathrm{out}}) \otimes \chi^\epsilon_{\partial B}(\mathcal{O}_{G,B}) \tag{7.15}$$

is the canonical map. Now, we define an element $\delta_M \in \mathcal{L}(Y,\epsilon)$ by

$$((\chi^\epsilon_{Y_{\mathrm{in}}} \otimes \chi^\epsilon_B) \circ \Omega_{Y_{\mathrm{in}},\partial B,B})(\psi_{G,\mathrm{in}}) = \psi_{\mathbb{C}^\times,\mathrm{in}} \otimes \delta_M . \tag{7.16}$$

This is a measurement of the difference between the G and $\mathbb{C}^\times$ theories on the cylinders M_{in}, analogous to our definition of $\mathcal{L}(Y,\epsilon)$, but one dimension up, so it gives an element rather than a line. Tensoring (7.14) and (7.16) gives

$$\chi^\epsilon_Y(\psi_G) = \psi_{\mathbb{C}^\times} \otimes \delta_M \tag{7.17}$$

so what remains to prove (7.3) is to show that $\delta_M = \beta_M$. Each component of M_{in} is a cylinder, with ends on two discs carrying the standard spectral network for a triangle; call these discs Δ, Δ'. By a diffeomorphism we can identify this cylinder with the mapping cylinder I_f of a map $f : \Delta \to \Delta'$. Moreover, f lifts to the spectral networks, spin structures and stratified abelianization data. Thus we can transport the computation of δ_M to the union of mapping cylinders; on each mapping cylinder I_f this gives the action of $f_* : \mathcal{L}(\Delta,\epsilon) \to \mathcal{L}(\Delta',\epsilon')$, and then tensoring over the cylinders gives β_M as desired. □

In Proposition 7.2 we only considered the case of a 3-manifold M with boundary a closed triangulated surface Y. In applications we sometimes want to use Y with boundary and a semi-ideal or ideal triangulation. In this case M will have to have extra boundary components extending ∂Y, and corners around the ideal vertices. As long as we always work with boundary-reduced stratified abelianization data over Y, these extra boundary components and corners do not introduce additional complications in the formal structure: the statement and its proof are unchanged except for a bit more notation, which we omit here.

7.2. The dilogarithm in abelianization on one tetrahedron. Suppose $Y = S^2$, identified as the boundary of a tetrahedron Δ, with the induced triangulation $\mathcal{T}$. Let σ denote a spin structure on Y, and $\mathcal{A} = (P, Q, \mu, \theta)$ stratified abelianization data over $(Y, \mathcal{N}^{\mathcal{T}})$. Then P extends to $\hat{P} \to \Delta$ (uniquely up to isomorphism) and applying $\mathscr{F}_G$ gives an element

$$\mathscr{F}_G(\Delta; \hat{P}) \in \mathscr{F}_G(Y; P)\,. \tag{7.18}$$

Fix edge-orientations ϵ on Δ. The goal of this section is to describe the image $\chi_Y^\epsilon(\mathscr{F}_G(\Delta; \hat{P}))$ in terms of a relative of the dilogarithm function.

Choose a pair of opposite edges in $\mathcal{T}$; as discussed above Proposition 4.40, this determines classes $\gamma_1, \gamma_2, \gamma_3 \in H_1(\widetilde{Y})$. Also choose a section t of the $\mathbb{C}^\times$-bundle Q^ϵ_{tw} and let

$$u_i = \oint_{\gamma_i} t^* \alpha \tag{7.19}$$

where α denotes the connection form in the $\mathbb{C}^\times$-bundle $Q \to \widetilde{Y}$. Thus u_i is a logarithm of $\mathrm{hol}_{Q^\epsilon_{\mathrm{tw}}}(\gamma_i)$. Changing the choice of section t shifts each u_i by an integer multiple of $2\pi\sqrt{-1}$. Also let

$$\eta_i = \sigma^\epsilon_{\mathrm{tw}}(\gamma_i) \in \{\pm 1\}\,. \tag{7.20}$$

Equivalently, η_i is the ϵ-sign of the edge E_i. Thus (η_1, η_2) measure the isomorphism class of the spin structure $\sigma^\epsilon_{\mathrm{tw}}$ on the torus $\widetilde{Y}$, and are determined by (but have less information than) the six edge-orientations ϵ.

Proposition 7.21. *We have the relation*

$$\eta_1 \mathrm{e}^{-u_1} + \eta_2 \mathrm{e}^{u_2} = 1\,. \tag{7.22}$$

Proof. Theorem 4.43 shows that the holonomies $z_i = \mathrm{hol}_Q(\gamma_i)$ obey $z_1^{-1} + z_2 = 1$, and (6.16) gives $\mathrm{e}^{u_i} = \mathrm{hol}_{Q^\epsilon_{\mathrm{tw}}}(\gamma_i) = \eta_i z_i$; combining these gives (7.22). □

Now let $\eta = (\eta_1, \eta_2)$,

$$S^\eta = \{(u_1, u_2) : \eta_1 \mathrm{e}^{-u_1} + \eta_2 \mathrm{e}^{u_2} = 1\} \subset \mathbb{C}^2, \tag{7.23}$$

and let

$$\ell^\eta : S^\eta \to \mathbb{C}/4\pi^2\mathbb{Z} \tag{7.24}$$

be any function obeying

$$\mathrm{d}\ell^\eta = \frac{1}{2}(u_2\,\mathrm{d}u_1 - u_1\,\mathrm{d}u_2). \tag{7.25}$$

Each ℓ^η is a variant of the dilogarithm function; see (7.69)–(7.72) for concrete examples.

Proposition 7.26.

$$\frac{\chi_Y^\epsilon(\mathscr{F}_G(\mathbb{A};\hat{P}))}{\tau_t} = c^\epsilon \exp\left[\frac{1}{2\pi\sqrt{-1}}\ell^\eta(u_1,u_2)\right], \tag{7.27}$$

for some $c^\epsilon \in \mathcal{L}(Y,\epsilon)$ *independent of* u_1, u_2.

Proof. We first note that (P,Q,μ,θ) fits into a family of stratified abelianization data $(P_{u_1,u_2},Q_{u_1,u_2},\mu,\theta)$ over $(Y,\mathcal{N}^{\mathfrak{T}})$, parameterized by $(u_1,u_2)\in S^\eta$, with the property $\mathrm{e}^{u_i}=\eta_i\,\mathrm{hol}_{Q_{u_1,u_2}}(\gamma_i)$. To construct this family we can use Construction 4.31, choosing the trivial flat bundle $P_{u_1,u_2}\to S^2$, and taking the 4 elements $s_i\in\mathbb{CP}^1$ at the 4 vertices to have cross-ratio $\eta_1\mathrm{e}^{u_1}$; then the calculation in Remark 4.49 shows that the holonomies of Q_{u_1,u_2} will be as desired. To show that our given (P,Q,μ,θ) is indeed isomorphic to a member of this family, we use the fact that stratified abelianization data is determined up to isomorphism by the holonomies of Q (for this calculation see e.g. [**HN**]).

Because $P_{u_1,u_2}\to S^2$ is trivial and its extension $\hat{P}_{u_1,u_2}\to\mathbb{A}$ is unique, the elements $\mathscr{F}_G(\mathbb{A};\hat{P}_{u_1,u_2})$ sweep out a covariantly constant section of the line bundle with fibers $\mathscr{F}_G(Y;P_{u_1,u_2})$ over S^η. Using the compatibility of χ_Y^ϵ with the connection in this bundle, it follows that $\chi_Y^\epsilon(\mathscr{F}_G(\mathbb{A};\hat{P}_{u_1,u_2}))$ is a covariantly constant section of the line bundle with fibers $\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y};Q^\epsilon_{\mathrm{tw},u_1,u_2};\sigma^\epsilon_{\mathrm{tw}})\otimes\mathcal{L}(Y,\epsilon)$ over S^η. Moreover the section t of the given Q^ϵ_{tw} deforms to a section t_{u_1,u_2} of $Q^\epsilon_{\mathrm{tw},u_1,u_2}$, unique up to homotopy. The computations in Section 4 of [**FN**] then show that the function

$$f(u_1,u_2)=\frac{\chi_Y^\epsilon(\mathscr{F}_G(\mathbb{A};\hat{P}_{u_1,u_2}))}{\tau_{t_{u_1,u_2}}} \tag{7.28}$$

on S^η obeys[31]

$$\mathrm{d}\log f=\frac{1}{4\pi\sqrt{-1}}(u_2\mathrm{d}u_1-u_1\mathrm{d}u_2)\,. \tag{7.29}$$

This proves (7.27). □

7.3. Flipping an edge. Suppose given a boundary-reduced flat G-bundle $P\to Y$. Suppose $\mathfrak{T}_0$, $\mathfrak{T}_1$ are two semi-ideal triangulations of Y, which differ by flipping an edge $E\in\mathbf{edges}(\mathfrak{T}_0)$ to an edge $E'\in\mathbf{edges}(\mathfrak{T}_1)$. Also suppose given a choice of sections of G/P over the interior vertices of $\mathfrak{T}$, such that the genericity Assumption 4.32 holds for both $\mathfrak{T}_0$ and $\mathfrak{T}_1$. Then by Construction 4.31 we obtain boundary-reduced stratified abelianization data $(P,Q_0,\dots)$ and $(P,Q_1,\dots)$ over $(Y,\mathcal{N}^{\mathfrak{T}_0})$ and $(Y,\mathcal{N}^{\mathfrak{T}_1})$ respectively.

Let ϵ_0, ϵ_1 be systems of edge-orientations on $\mathfrak{T}_0$ and $\mathfrak{T}_1$, which agree on all common edges, i.e. all edges except for E and E'. We would like to

[31] In [**FN**] we discussed only the case $\eta=(+1,+1)$, but this computation is independent of η. Also, beware that u_1 in this paper is $-u_1$ in [**FN**], which leads to a sign flip in comparing.

compare $\chi_Y^{\mathcal{T}_0,\epsilon_0}$ and $\chi_Y^{\mathcal{T}_1,\epsilon_1}$. Their ratio is an isomorphism

$$\begin{aligned}\chi_Y^{\mathcal{T}_1,\epsilon_1} \circ (\chi^{\mathcal{T}_0,\epsilon_0})_Y^{-1} : \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}_0; Q_{0,\mathrm{tw}}^{\epsilon_0}; \sigma_{\mathrm{tw}}^{\epsilon_0}) \otimes \mathcal{L}(Y, \epsilon_0) \\ \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}_1; Q_{1,\mathrm{tw}}^{\epsilon_1}; \sigma_{\mathrm{tw}}^{\epsilon_1}) \otimes \mathcal{L}(Y, \epsilon_1).\end{aligned} \tag{7.30}$$

To describe this map we consider the 3-manifold

$$M = Y \times [0,1]\,. \tag{7.31}$$

Let $p : M \to Y$ denote the projection, and let $\hat{P} = p^* P$. On M we construct a SN-stratification, spectral network and stratified abelianization, as follows.

Let Y_{rest} be the union of all the triangles in Y except for the two abutting E. Then we have a decomposition $Y = Y_{\mathrm{rest}} \cup \mathcal{Q}_E$. The stratified abelianizations $(P, Q_0, \dots)|_{Y_{\mathrm{rest}}}$ and $(P, Q_1, \dots)|_{Y_{\mathrm{rest}}}$ are canonically isomorphic, and thus extend to boundary-reduced stratified abelianization data over $(Y_{\mathrm{rest}} \times [0,1], \mathcal{N}_{\mathrm{rest}} \times [0,1])$. Over $Y_{\mathrm{rest}} \times [0,1]$ we also have a $\mathbb{Z}$-bundle $p^* \widetilde{Y}^{\epsilon,\infty}|_{Y_{\mathrm{rest}}}$.

Next we consider $\mathcal{Q}_E \times [0,1]$. Collapsing the vertical intervals in $\partial\mathcal{Q}_E \times [0,1]$ gives a surjective map $\partial\phi : \partial(\mathcal{Q}_E \times [0,1]) \to \partial\mathbb{A}$, where $\mathbb{A}$ is a tetrahedron, and this map extends to

$$\phi : \mathcal{Q}_E \times [0,1] \to \mathbb{A} \tag{7.32}$$

which is a homeomorphism on the interior. On $\mathbb{A}$ we have a standard SN-stratification and spectral network as described in Construction 4.37; pulling this back by ϕ gives a SN-stratification and spectral network on $\mathcal{Q}_E \times [0,1]$, which glue to the ones we already have on $Y_{\mathrm{rest}} \times [0,1]$. Using Construction 4.55 we obtain stratified abelianization data over $\mathbb{A}_{\geq -2\mathrm{b}}$, which likewise pulls back to $\mathcal{Q}_E \times [0,1]$ and glues to the stratified abelianization data we already have on $Y_{\mathrm{rest}} \times [0,1]$.

Thus altogether we have:

- A SN-stratification and spectral network $\mathcal{N}$ over M, with isomorphisms $\mathcal{N}|_{Y\times\{0\}} \simeq \mathcal{N}^{\mathcal{T}_0}$, $\mathcal{N}|_{Y\times\{1\}} \simeq \mathcal{N}^{\mathcal{T}_1}$,
- Stratified abelianization data $(\hat{P}, \hat{Q}, \hat{\mu}, \hat{\theta})$ over $(M, \mathcal{N})$, which restricts to $(P, Q_0, \mu_0, \theta_0)$, $(P, Q_1, \mu_1, \theta_1)$ over Y_0, Y_1 respectively.
- A $\mathbb{Z}$-bundle $\widetilde{M}^\infty \to M_{\geq -3\mathrm{a}}$, which restricts to $\widetilde{Y}^{\epsilon_0,\infty}$, $\widetilde{Y}^{\epsilon_1,\infty}$ over Y_0, Y_1.

We need to extend our μ_2-twisted objects from Y_0 and Y_1 to M: define $\widetilde{M}^4 = \widetilde{M}^\infty / 4\mathbb{Z}$, and twist $\hat{Q} \to M_{\geq -3\mathrm{a}}$ to $\hat{Q}_{\mathrm{tw}} = \hat{Q} \otimes_{\mu_2} \widetilde{M}^4$ and $\pi^* p^* \sigma$ to $\sigma_{\mathrm{tw}} = \pi^* p^* \sigma \otimes_{\mu_2} \widetilde{M}^4$.

The stratum $M_{-3\mathrm{b}}$ consists of a single point p, the barycenter of the tetrahedron $\mathbb{A}$. Let B_p be a small ball around p. By radial projection centered at p, the sphere $H = \partial B_p$ acquires a triangulation $\mathcal{T}_H$, whose four faces correspond naturally to the two faces of $\mathcal{Q}_E$ and two of $\mathcal{Q}_{E'}$. $\mathcal{T}_H$ comes with edge-orientations ϵ_H, induced by ϵ_0 and ϵ_1 (using the fact that ϵ_0 and ϵ_1

agree on the common edges). Moreover, the restriction $\mathcal{N}|_H$ is the spectral network $\mathcal{N}^{\mathfrak{T}_H}$, and $\widetilde{H}^{\epsilon_H,\infty}$ is the restriction of $\widetilde{M}^\infty$ to H.

Now we can describe the effect of a flip on the abelianization maps $\chi_Y^{\mathfrak{T},\epsilon}$:

Proposition 7.33. *The map*

$$\begin{aligned}\chi_Y^{\mathfrak{T}_1,\epsilon_1}\circ(\chi_Y^{\mathfrak{T}_0,\epsilon_0})^{-1} &: \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y};Q_{0,\mathrm{tw}}^{\epsilon_0};\sigma_{\mathrm{tw}}^{\epsilon_0})\otimes\mathcal{L}(Y,\epsilon_0)\\ &\to\mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y};Q_{1,\mathrm{tw}}^{\epsilon_1};\sigma_{\mathrm{tw}}^{\epsilon_1})\otimes\mathcal{L}(Y,\epsilon_1)\end{aligned}\tag{7.34}$$

is the product

$$\mathscr{S}_{\mathbb{C}^\times}(M\setminus B_p;\hat{Q}_{\mathrm{tw}}|_{M\setminus B_p};\sigma_{\mathrm{tw}})\otimes\chi_H^{\epsilon_H}(\mathscr{F}_G(B_p;\hat{P}|_{B_p}))\otimes\beta_{M\setminus B_p}\,.\tag{7.35}$$

Proof. We apply Proposition 7.2 to the 3-manifold $M\setminus B_p$, noting that what was called Y there is here the disconnected boundary $\partial(M\setminus B_p)=Y_0\cup -Y_1\cup -H$, which carries a semi-ideal triangulation $\mathfrak{T}_0\cup\mathfrak{T}_1\cup\mathfrak{T}_H$ and edge-orientations $\epsilon_0\cup\epsilon_1\cup\epsilon_H$.

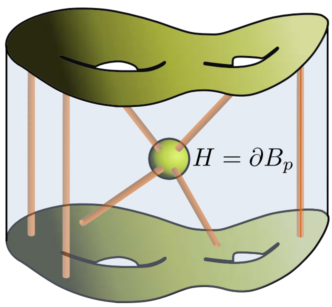


Figure 22. The 3-manifold $M=Y\times[0,1]$ which we use to study a flip of the semi-ideal triangulation on Y. The branch locus $M_{-2\mathrm{b}}$ is shown in orange. In this example each triangulation has five triangles, and thus there are five points of $M_{-2\mathrm{b}}$ on each of Y_0 and Y_1.

This gives the formula

$$\begin{aligned}&\chi_{Y_0\cup -Y_1\cup -H}^{\mathfrak{T}_0\cup\mathfrak{T}_1\cup\mathfrak{T}_H,\epsilon_0\cup\epsilon_1\cup\epsilon_H}(\mathscr{F}_G(M\setminus B_p;\hat{P}|_{M\setminus B_p}))\\ &\qquad=\mathscr{S}_{\mathbb{C}^\times}(M\setminus B_p;\hat{Q}_{\mathrm{tw}}|_{M\setminus B_p};\sigma_{\mathrm{tw}})\otimes\beta_{M\setminus B_p}\,.\end{aligned}\tag{7.36}$$

Tensoring with $\chi_H^{\epsilon_H}(\mathscr{F}_G(B_p;\hat{P}|_{B_p}))$ then gives

$$\begin{aligned}&\chi_{Y_0\cup -Y_1}^{\mathfrak{T}_0\cup\mathfrak{T}_1,\epsilon_0\cup\epsilon_1}(\mathscr{F}_G(M;\hat{P}))\\ &\quad=\mathscr{S}_{\mathbb{C}^\times}(M\setminus B_p;\hat{Q}_{\mathrm{tw}}|_{M\setminus B_p};\sigma_{\mathrm{tw}})\otimes\beta_{M\setminus B_p}\otimes\chi_H^{\epsilon_H}(\mathscr{F}_G(B_p;P|_{B_p}))\,.\end{aligned}\tag{7.37}$$

But since $M=Y\times[0,1]$ and $\hat{P}=p^*P$, $\mathscr{F}_G(M;\hat{P})$ is the identity map on $\mathscr{F}_G(Y;P)$; the desired result follows by rearranging factors. ∎

Using the identification between the 4 triangles in $\mathcal{Q}_E$, $\mathcal{Q}_{E'}$ and the 4 triangles in H, we get from Proposition 6.29 an element $\beta : \mathcal{L}(\mathcal{Q}_E, \epsilon_0) \otimes \mathcal{L}(H, \epsilon_H) \to \mathcal{L}(\mathcal{Q}_{E'}, \epsilon_1)$. Contracting that with the normalization constant $c^{\epsilon_H} \in \mathcal{L}(H, \epsilon_H)$ defined in Proposition 7.26, we obtain an isomorphism

$$\kappa_E^{\epsilon_0,\epsilon_1} = \beta(c^{\epsilon_H}) : \mathcal{L}(\mathcal{Q}_E, \epsilon_0) \to \mathcal{L}(\mathcal{Q}_{E'}, \epsilon_1)\,. \tag{7.38}$$

7.4. Gluing the Chern-Simons line. Let Y be a compact oriented surface with boundary. Suppose $P \to Y$ is a flat boundary-reduced G-bundle. In this section we use abelianization to give a description of the line $\mathscr{F}_G(Y; P)$.

Construction 7.39. We consider tuples $D = (\mathfrak{T}, \epsilon, s, x)$, where:

- $\mathfrak{T}$ is a semi-ideal triangulation of Y,
- $\epsilon = (\epsilon_E)_{E \in \mathbf{edges}(\mathfrak{T})}$ is a system of edge-orientations for $\mathfrak{T}$,
- $s = (s_v)_{v \in \mathbf{vertices}(\mathfrak{T})}$ is a flat section of $P/U = P \times_G (\mathbb{C}^2 \setminus \{0\})$ over each v, such that if v is an ideal vertex then the projection of s_v to P/B agrees with the boundary reduction,
- $x = (x_E)_{E \in \mathbf{edges}(\mathfrak{T})}$ is a collection of complex numbers, where

$$\exp(x_E) = \epsilon_E(v, v') s_{v'} \wedge s_v \tag{7.40}$$

  if E is an edge with vertices v, v'. Here s_v means the continuation of s from v by parallel transport along the edge E, similarly $s_{v'}$, and the wedge product is evaluated at any point along E; here and below, we use the fact that P is an $\mathrm{SL}_2\mathbb{C}$-bundle and thus there is a canonical volume form in P/U.

For each such tuple there is a canonical isomorphism

$$\psi_D : \mathscr{F}_G(Y; P) \to \mathcal{L}(Y, \epsilon). \tag{7.41}$$

Proof. Fix $P \to Y$ and D as above. Construction 4.31 extends P to stratified abelianization data (P, Q, μ, θ) over $(Y, \mathcal{N}^{\mathfrak{T}})$. Then Construction 6.51 gives an isomorphism

$$\chi_Y^{\mathfrak{T},\epsilon} : \mathscr{F}_G(Y; P) \to \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}}) \otimes \mathcal{L}(Y, \epsilon). \tag{7.42}$$

Using the data (s, x) we determine a section $\widetilde{s}_x$ of the line bundle $Q^\epsilon_{\mathrm{tw}} \to \widetilde{Y}$ (up to homotopy), as follows. Recall that each $v \in \mathbf{vertices}(\mathfrak{T})$ has two preimages (v, v) and $(v, \overline{v})$ in $\widetilde{Y}$. Each s_v determines an element $\widetilde{s}(v, v)$ of the fiber of Q^ϵ_{tw}. We also determine an element $\widetilde{s}(v, \overline{v})$ by the condition $\theta(\widetilde{s}(v, v)) \wedge \theta(\widetilde{s}(v, \overline{v})) = 1$. Now suppose $E \in \mathbf{edges}(\mathfrak{T})$ with vertices v, v'. The parallel transport of Q^ϵ_{tw} along preimages of E takes

$$\widetilde{s}(v, v) \mapsto \exp(x_E)\widetilde{s}(v', \overline{v}'), \quad \widetilde{s}(v, \overline{v}) \mapsto \exp(-x_E)\widetilde{s}(v', v')\,. \tag{7.43}$$

The choice of a logarithm x_E thus determines (up to homotopy) an extension $\widetilde{s}_x$ of $\widetilde{s}$ over the preimages of E. Finally, by construction $\widetilde{s}_x$ has zero winding around the preimage of the boundary of each triangle, because the summands x_E and $-x_E$ cancel over each of the three edges; thus $\widetilde{s}_x$ can be extended to a section $\widetilde{s}_x$ of $Q^\epsilon_{\mathrm{tw}} \to \widetilde{Y}$, uniquely up to homotopy. The section $\widetilde{s}_x$ determines a trivialization $\tau_{\widetilde{s}_x} \in \mathscr{S}_{\mathbb{C}^\times}(\widetilde{Y}; Q^\epsilon_{\mathrm{tw}}; \sigma^\epsilon_{\mathrm{tw}})$. Then we define

$$\psi_D = \frac{\chi_Y^{\mathfrak{T},\epsilon}}{\tau_{\widetilde{s}_x}}\,. \tag{7.44}$$ □

We remark that the existence of the data x_E above requires $s_v \wedge s_{v'} \neq 0$, i.e. that the genericity Assumption 4.32 is satisfied.

Proposition 7.45. *We have relations among the maps associated to data $D = (\mathfrak{T}, \epsilon, s, x)$ and $D' = (\mathfrak{T}', \epsilon', s', x')$ as follows:*

(1) *Suppose D and D' differ only by a change involving a single vertex v: $s'_v = s_v \exp(t)$ and $x'_E = x_E + t$ for all edges E incident on v. Then*

$$\psi_{D'} = \psi_D. \tag{7.46}$$

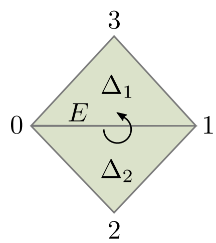


Figure 23. The quadrilateral $\mathcal{Q}_E$.

(2) *Suppose D and D' differ only by a change involving a single edge E: $\epsilon'_E = -\epsilon_E$ and $x'_E = x_E + \pi\sqrt{-1}$. With labeling as in Figure 23 let*

$$u = x_{E_{30}} - x_{E_{02}} + x_{E_{21}} - x_{E_{13}}\,. \tag{7.47}$$

Then

$$\psi_{D'} = \exp\left[\frac{1}{4}u\right] \left(\rho^{\epsilon,\epsilon'}(\Delta_1) \otimes \rho^{\epsilon,\epsilon'}(\Delta_2)\right) \psi_D. \tag{7.48}$$

(3) *Suppose D and D' differ only in that $\mathfrak{T}'$ is obtained from $\mathfrak{T}$ by flipping edge E to obtain a new edge E', with some orientation $\epsilon'_{E'}$ and a choice of logarithm $x_{E'}$. Labeling the vertices as in Figure 24, let*

$$u_1 = x_{E_{30}} - x_{E_{02}} + x_{E_{21}} - x_{E_{13}}\,, \quad u_2 = x_{E_{32}} - x_{E_{21}} + x_{E_{10}} - x_{E_{03}}\,, \tag{7.49}$$

$$\eta_1 = (-1)^{\frac{1}{2}(\epsilon(3,0)+\epsilon(0,2)+\epsilon(2,1)+\epsilon(1,3))}\,, \quad \eta_2 = (-1)^{\frac{1}{2}(\epsilon(3,2)+\epsilon(2,1)+\epsilon(1,0)+\epsilon(0,3))}\,. \tag{7.50}$$

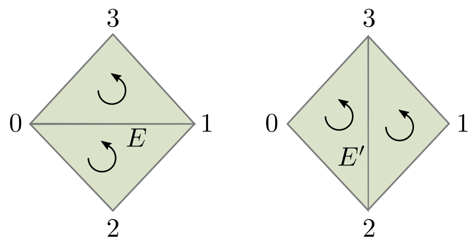


FIGURE 24. The quadrilaterals $\mathcal{Q}_E$ in $\mathcal{T}$ and $\mathcal{Q}_{E'}$ in $\mathcal{T}'$, related by a flip.

Then

$$\psi_{D'} = \exp\left[\frac{1}{2\pi\sqrt{-1}}\ell^{\eta}(u_1,u_2)\right]\kappa_E^{\epsilon,\epsilon'} \circ \psi_D\,. \tag{7.51}$$

PROOF. We treat the three relations in turn:

(1) In this case the sections $\widetilde{s}_x$ and $\widetilde{s}'_{x'}$ are homotopic (the homotopy obtained by continuously varying from 0 to t) and thus they induce the same trivialization of $\mathscr{S}_{\mathbb{C}^\times}(Q^\epsilon_{\mathrm{tw}};\sigma_{\mathrm{tw}})$.

(2) By (7.44) and (6.79) we have $\psi_{D'}\cdot\tau_{\widetilde{s}'_{x'}} = (\lambda^{\epsilon,\epsilon'}(Y)\otimes\zeta(\widetilde{Y};Q^\epsilon_{\mathrm{tw}};\varpi^E))\circ \psi_D\cdot\tau_{\widetilde{s}_x}$. We use Proposition 6.81 to expand the λ factor. For the ζ factor, we use (5.105), which gives

$$\zeta(\widetilde{Y};Q^\epsilon_{\mathrm{tw}};\varpi^E)(\tau_{\widetilde{s}_x}) = \exp\left[-\frac{1}{4}\int_{\widetilde{Y}}\widetilde{s}_x^*\alpha\wedge\varpi^E\right]\tau_{\widetilde{s}'_{x'}} \tag{7.52}$$

where α denotes the connection form on Q^ϵ_{tw}. By (6.15) the multiplicative factor here is

$$\exp\left[-\frac{1}{4}\oint_{\gamma_E}\widetilde{s}_x^*\alpha\right] \tag{7.53}$$

and comparing (7.47) to the definition of γ_E, we have $\oint_{\gamma_E}\widetilde{s}_x^*\alpha = -u$, so the factor becomes

$$\exp\left[\frac{1}{4}u\right] \tag{7.54}$$

as desired.

(3) We use the setup and notation of §7.3, involving the 3-manifold $M = Y\times[0,1]$. Proposition 7.33 gives $\chi_Y^{\mathcal{T}',\epsilon'}\circ(\chi_Y^{\mathcal{T},\epsilon})^{-1}$ as a tensor product of three ingredients:

(a) First we have $\mathscr{S}_{\mathbb{C}^\times}(M\setminus B_p;\hat{Q}_{\mathrm{tw}};\sigma_{\mathrm{tw}})$. The data of D and D' in particular determine sections $\widetilde{s}_x$ and $\widetilde{s}'_{x'}$ of $\hat{Q}^\epsilon_{\mathrm{tw}}$ on the boundaries Y_0 and Y_1, as discussed in Construction 7.39. There is a section of $\hat{Q}^\epsilon_{\mathrm{tw}}\to M\setminus B_p$ which extends $\widetilde{s}_x$ and $\widetilde{s}'_{x'}$; let $\widetilde{s}_H$ be its restriction to H. The $\mathbb{C}^\times$ Chern-Simons form vanishes for a flat bundle; it follows that

$$\mathscr{S}_{\mathbb{C}^\times}(M \setminus B_p; \hat{Q}_{\mathrm{tw}}; \sigma_{\mathrm{tw}}) = \tau^*_{\widetilde{s}_x} \otimes \tau_{\widetilde{s}'_{x'}} \otimes \tau^*_{\widetilde{s}_H}\,. \tag{7.55}$$

(b) Next we have the factor $\chi^{\epsilon_H}(\mathscr{F}_G(B_p; \hat{P}|_{B_p}))$. Note that u_1 and u_2 defined in (7.49) agree with the u_1 and u_2 defined in §7.2, if we take the section $t = \widetilde{s}_H$. Likewise η_1 and η_2 defined in (7.50) agree with the η_1 and η_2 used there. Then by (7.27),

$$\chi^{\epsilon_H}(\mathscr{F}_G(B_p; \hat{P}|_{B_p})) = c^{\epsilon_H} \exp\left[\frac{1}{2\pi\sqrt{-1}}\ell^\eta(u_1, u_2)\right]\tau_{\widetilde{s}_H}\,.$$

(c) Finally we have the isomorphism $\beta_{M\setminus B_p}$.

Tensoring these ingredients together, and using the definition (7.44) of ψ_D and the definition (7.38) of $\kappa_E^{\epsilon,\epsilon'}$, we get the desired (7.51). ☐

7.5. Explicit formulas. In the last section we gave a description of the line $\mathscr{F}_G(Y; P)$ which is canonical but somewhat inexplicit: the transition maps described by Proposition 7.45 involve the maps $\rho^{\epsilon,\epsilon'}(\Delta)$ and $\kappa_E^{\epsilon,\epsilon'}$, for which we have not yet written down concrete formulas. Roughly speaking, we have fully described the P-dependence in $\mathscr{F}_G(Y; P)$, but left some constant phases undetermined. In this section we rectify this omission, getting concrete formulas in terms of actual numbers, at the cost of making some arbitrary choices.

7.5.1. *Trivializing the difference lines.* Let Δ_s denote the standard triangle with its standard orientation, and vertices labeled 012 in cyclic order given by the orientation. We choose a nonzero element

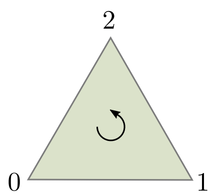


FIGURE 25. The standard triangle.

$$\tau_\epsilon \in \mathcal{L}(\Delta_s, \epsilon) \tag{7.56}$$

for each of the eight possible ϵ. We may choose τ_ϵ so that if $f : \Delta_s \to \Delta_s$ is a rotation which acts on the vertices by $012 \to 120$,[32]

$$f_*(\tau_\epsilon) = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)\tau_{f_*\epsilon}, \tag{7.57}$$

and also if $r : \Delta_s \to \Delta_s$ is the reflection which acts on the vertices by $012 \to 102$,

$$r_*(\tau_\epsilon) = \tau^{-1}_{r_*\epsilon}\,. \tag{7.58}$$

[32]The factor appearing in (7.57) is dictated by Proposition 6.34.

From now on we assume that we make such a choice.

Now suppose $(Y, \mathcal{T}, \epsilon)$ is a general triangulated manifold with edge-orientations. We want to trivialize the line $\mathcal{L}(Y, \epsilon)$; for this we need one more datum:

- A marked edge e_Δ on each $\Delta \in \mathbf{faces}(\mathcal{T})$.

The edge e_Δ determines an orientation-preserving identification $\phi_{e_\Delta} : \Delta_s \to \Delta$, by the condition that ϕ_{e_Δ} takes the edge $(0,1)$ to e_Δ. Then, given the extended data $\hat{D} = (\mathcal{T}, \epsilon, s, x, e)$ we obtain a trivialization

$$\tau_{\hat{D}} = \frac{\psi_D}{\bigotimes_{\Delta \in \mathbf{faces}(\mathcal{T})} (\phi_{e_\Delta})_* \tau_\epsilon} \in \mathscr{F}_G(Y; P)^* . \tag{7.59}$$

What remains is to describe the relations between the elements $\tau_{\hat{D}} \in \mathscr{F}_G(Y; P)^*$ associated to different $\hat{D}$.

7.5.2. *Computing $\rho^{\epsilon,\epsilon'}(\Delta_s)$.* Recall the edge-reversal map ρ from §6.7. Fix arbitrarily a nonzero element ξ in the line $\Xi(E)$ considered there, and then define constants $b(\epsilon, \epsilon') \in \mathbb{C}^\times$ by

$$\rho^{\epsilon,\epsilon'}(\Delta_s)(\tau_\epsilon) = b(\epsilon, \epsilon')\, \tau_{\epsilon'} \otimes \xi^{\pm 1} . \tag{7.60}$$

Our aim now is to compute these constants. We write each ϵ as a 3-tuple $(\epsilon(0,1), \epsilon(1,2), \epsilon(2,0))$.

Proposition 7.61. *We have*

$$b(f_* \epsilon, f_* \epsilon') = b(\epsilon, \epsilon') , \tag{7.62}$$

and for some $p, q \in \mathbb{C}^\times$ we have

(a_1, a_2)	$b((a_1, a_2, +1), (a_1, a_2, -1))$	$b((a_1, a_2, -1), (a_1, a_2, +1))$
$(-1, -1)$	p	$-p^{-1}$
$(-1, +1)$	$\omega^{-1} q$	ωq^{-1}
$(+1, -1)$	ωq	$\omega^{-1} q^{-1}$
$(+1, +1)$	$-p$	p^{-1}

where $\omega = \exp(\frac{2\pi\sqrt{-1}}{8})$.

Proof. Applying f_* to both sides of (7.60) and using (7.57) gives (7.62). Proposition 6.73 gives

$$b((+1,+1,+1),(+1,+1,-1))b((+1,+1,-1),(+1,-1,-1)) = \exp\left(-\frac{2\pi\sqrt{-1}}{4}\right) b((+1,+1,+1),(+1,-1,+1))b((+1,-1,+1),(+1,-1,-1)). \tag{7.63}$$

Applying (7.62) and canceling a common factor $b((+1,+1,+1),(+1,+1,-1))$ gives

$$\begin{aligned} &b((-1,+1,+1),(-1,+1,-1)) \\ &\quad = \exp\left(-\frac{2\pi\sqrt{-1}}{4}\right) b((+1,-1,+1),(+1,-1,-1)) . \end{aligned} \tag{7.64}$$

Next, using Proposition 6.83, we have

$$\begin{aligned} &b((a_1,a_2,+1),(a_1,a_2,-1))\times b((a_3,a_4,-1),(a_3,a_4,+1))\times \\ &\qquad b((a_1,a_2,-1),(a_1,a_2,+1))\times b((a_3,a_4,+1),(a_3,a_4,-1)) \\ &= (-1)^{\frac{1}{2}(a_1+a_2+a_3+a_4)} \end{aligned} \tag{7.65}$$

Finally, using the reflection condition (7.58) gives

$$b(\epsilon,\epsilon')\, b(r_*\epsilon, r_*\epsilon') = 1\,, \tag{7.66}$$

which implies in particular

$$b((+1,-1,+1),(+1,-1,-1))\, b((+1,-1,-1),(+1,-1,+1)) = 1 \tag{7.67}$$

and

$$b((+1,+1,+1),(+1,+1,-1))\, b((-1,-1,-1),(-1,-1,+1)) = 1\,. \tag{7.68}$$

The constraints (7.64), (7.65), (7.67), (7.68) determine the $b(\epsilon,\epsilon')$ up to two undetermined constants as indicated in the table. □

Proposition 7.61 determines all of the $b(\epsilon,\epsilon')$ in terms of the undetermined constants p, q. Fixing p, q is equivalent to fixing the remaining freedom in the trivializations τ_ϵ, up to an overall scale which remains unfixed.

7.5.3. *Fixing the ℓ^η.* In §7.2 we determined the functions L^η up to an overall constant. We now make a definite choice as follows. Let Li_2 denote the principal branch of the dilogarithm function, which has a branch cut along $(1,\infty)$. Then let

$$\ell^{(+1,+1)}(u_1,u_2) = \mathrm{Li}_2(e^{-u_1}) - \frac{1}{2}u_1u_2 - 2\pi\sqrt{-1}\left\lfloor -\frac{\operatorname{Im} u_2}{2\pi} + \frac{1}{2}\right\rfloor u_1, \tag{7.69}$$

$$\begin{aligned} \ell^{(-1,+1)}(u_1,u_2) = \mathrm{Li}_2(-e^{-u_1}) - \frac{1}{2}u_1u_2 \\ - 2\pi\sqrt{-1}\left\lfloor -\frac{\operatorname{Im} u_2}{2\pi} + \frac{1}{2}\right\rfloor (u_1 + \pi\sqrt{-1}), \end{aligned} \tag{7.70}$$

$$\ell^{(+1,-1)}(u_1,u_2) = \mathrm{Li}_2(e^{-u_1}) - \frac{1}{2}u_1u_2 - \pi\sqrt{-1}u_1 - 2\pi\sqrt{-1}\left\lfloor -\frac{\operatorname{Im} u_2}{2\pi}\right\rfloor u_1, \tag{7.71}$$

$$\begin{aligned} \ell^{(-1,-1)}(u_1,u_2) = \mathrm{Li}_2(-e^{-u_1}) - \frac{1}{2}u_1u_2 - \pi\sqrt{-1}u_1 \\ - 2\pi\sqrt{-1}\left\lfloor -\frac{\operatorname{Im} u_2}{2\pi}\right\rfloor (u_1 + \pi\sqrt{-1}). \end{aligned} \tag{7.72}$$

It is routine to check that these formulas indeed define holomorphic functions $\ell^\eta : S^\eta \to \mathbb{C}/4\pi^2\mathbb{Z}$: the discontinuity of Li_2 across its branch cut gets compensated by the discontinuity of the floor function, up to an integer multiple of $4\pi^2$. Using the differential equation obeyed by Li_2, it is also straightforward to check that they obey (7.25) as needed.

7.5.4. *Computing the $\kappa_E^{\epsilon,\epsilon'}$.* Now that we have fixed our choices of ℓ^η and also fixed trivializations of the $\mathcal{L}(\Delta,\epsilon)$ we are in position to express the normalization constants from (7.38),

$$\kappa_E^{\epsilon,\epsilon'} : \mathcal{L}(\mathcal{Q}_E,\epsilon) \to \mathcal{L}(\mathcal{Q}_{E'},\epsilon')\,, \tag{7.73}$$

as concrete numbers: we write

$$\kappa_E^{\epsilon,\epsilon'}((\phi_E)_*\tau_{\epsilon_1} \otimes (\phi_E)_*\tau_{\epsilon_2}) = k_E^{\epsilon,\epsilon'}((\phi_{E'})_*\tau_{\epsilon_1'} \otimes (\phi_{E'})_*\tau_{\epsilon_2'}) \tag{7.74}$$

for some $k_E^{\epsilon,\epsilon'} \in \mathbb{C}^\times$.

To compute the $k_E^{\epsilon,\epsilon'}$, it will be useful to recall their origin in Chern-Simons theory on an oriented sphere H with tetrahedral triangulation $\mathcal{T}$, obtained by gluing $-\mathcal{Q}_E$ to $\mathcal{Q}_{E'}$ along the common boundary. We label the edges of $\mathcal{Q}_E$ and $\mathcal{Q}_{E'}$ as shown in Figure 24, thus identifying H with a standard model. We use ϵ_H to represent the full collection of six edge-orientations on $\mathcal{T}$ induced by (ϵ,ϵ'), and use the condensed notation k^{ϵ_H} for $k_E^{\epsilon,\epsilon'}$.

The next lemma concerns just H, not the original surface Y.

Lemma 7.75. *Suppose ϵ and ϵ' are two systems of edge-orientations on $\mathcal{T}$, which differ by reversing the orientation on some $\tilde{E} \in$ **edges**$(\mathcal{T})$. Fix stratified abelianization data (P,Q,μ,θ) on $(H,\mathcal{N}^{\mathcal{T}})$, and extend $(\mathcal{T},\epsilon)$ to data $D = (\mathcal{T},\epsilon,s,x)$ as in Construction 7.39. Also define $D' = (\mathcal{T},\epsilon',s,x')$ where $x'_{\tilde{E}} = x_{\tilde{E}} + \sqrt{-1}\pi$, and $x'_E = x_E$ for all other edges. Let (u_1,u_2) and (η_1,η_2) be given by (7.49) and (7.50), likewise (u_1',u_2') and (η_1',η_2'). Let Δ_1, Δ_2 denote the two triangles abutting $\tilde{E}$. Then*

$$\begin{aligned}\frac{k^{\epsilon'}}{k^{\epsilon}} = \exp\Bigg[&\frac{1}{2\pi\sqrt{-1}}\left(\ell^\eta(u_1,u_2) - \ell^{\eta'}(u_1',u_2')\right)\\ &+\frac{1}{4\pi\sqrt{-1}}(u_1u_2' - u_2u_1')\Bigg]\, b(\epsilon_{\Delta_1},\epsilon'_{\Delta_1})b(\epsilon_{\Delta_2},\epsilon'_{\Delta_2})\end{aligned} \tag{7.76}$$

with b determined in Proposition 7.61.

Proof. We consider the element $o = \mathscr{F}_G(\mathbb{A};\hat{P})$, which has

$$\psi_D(o) = c^\epsilon \exp\left[\frac{1}{2\pi\sqrt{-1}}\ell^\eta(u_1,u_2)\right] \in \mathcal{L}(H,\epsilon)\,. \tag{7.77}$$

Part (2) of Proposition 7.45 gives the relation between $\psi_D(o)$ and $\psi_{D'}(o)$. The quantity u appearing there is not necessarily the u_1 here, because $\tilde{E}$ is not necessarily E_{01}; one can show however (e.g. by checking the six cases for $\tilde{E}$) that the correct relation is

$$\sqrt{-1}\pi u = u_1u_2' - u_2u_1'\,. \tag{7.78}$$

Thus we get

$$\begin{aligned}\exp\left[\frac{1}{2\pi\sqrt{-1}}\ell^{\eta'}(u_1',u_2')\right]c^{\epsilon'} = \exp\Bigg[&\frac{1}{2\pi\sqrt{-1}}\ell^{\eta}(u_1,u_2) \\ &+\frac{1}{4\pi\sqrt{-1}}(u_1u_2'-u_2u_1')\Bigg](\rho^{\epsilon,\epsilon'}(\Delta_1) \\ &\otimes\rho^{\epsilon,\epsilon'}(\Delta_2))(c^{\epsilon})\,. \end{aligned} \tag{7.79}$$

Now for each $\Delta \in \mathbf{faces}(\mathcal{T})$ let the marked edge e_Δ be either E_{01} or E_{23}, whichever lies on Δ. Dividing both sides by $\bigotimes_\Delta(\phi_{e_\Delta})_*(\tau_{e_\Delta})$ gives the desired result. □

Lemma 7.75 determines the constants k^ϵ up to a single overall multiplicative constant. The remaining constant can be fixed as follows. We consider a triangulated surface $(Y,\mathcal{T})$ with stratified abelianization data (P,Q,μ,θ). Suppose $\mathcal{T}$ contains three triangles which make up a pentagon. Then we consider two possible sequences of flips, involving various triangulations $\mathcal{T}_i$ as indicated in Figure 26, and choose data $\hat{D}_i$ (logarithms, edge-orientations and marked edges) extending the triangulations $\mathcal{T}_i$.

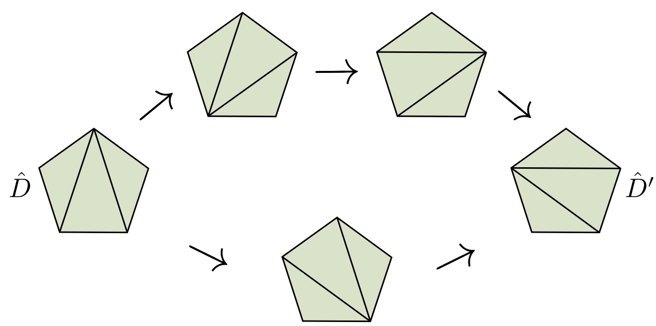


FIGURE 26. Two sequences of flips in a triangulated pentagon.

Since both sequences begin at $\hat{D}$ and end at $\hat{D}'$, we can follow either sequence to compute the ratio $\tau_{\hat{D}'}/\tau_{\hat{D}}$; but one sequence contains 3 flips and the other contains 2, so requiring that they are equal is enough to determine the overall multiplicative constant in k^ϵ. Carrying this computation out we obtain the following (the detailed computation can be found in the file `dilog-compute.nb` included with the arXiv version of this paper.)

LEMMA 7.80. *If we set $p=q=1$, then $k^\epsilon = \exp\left(\frac{2\pi\sqrt{-1}}{24}n(\epsilon)\right)$, with $n(\epsilon)$ given below. We specify ϵ by the tuple $(\epsilon(0,2),\epsilon(2,1),\epsilon(1,3),\epsilon(3,0),\epsilon(1,0),$ $\epsilon(2,3))$.*

$(+1,+1,+1,+1,+1,+1) \to 7$	$(-1,+1,+1,+1,+1,+1) \to 7$
$(+1,+1,+1,+1,+1,-1) \to -5$	$(-1,+1,+1,+1,+1,-1) \to 10$
$(+1,+1,+1,+1,-1,+1) \to -5$	$(-1,+1,+1,+1,-1,+1) \to 10$
$(+1,+1,+1,+1,-1,-1) \to 7$	$(-1,+1,+1,+1,-1,-1) \to 1$
$(+1,+1,+1,-1,+1,+1) \to -11$	$(-1,+1,+1,-1,+1,+1) \to 10$
$(+1,+1,+1,-1,+1,-1) \to -2$	$(-1,+1,+1,-1,+1,-1) \to 10$
$(+1,+1,+1,-1,-1,+1) \to -8$	$(-1,+1,+1,-1,-1,+1) \to -8$
$(+1,+1,+1,-1,-1,-1) \to -11$	$(-1,+1,+1,-1,-1,-1) \to -8$
$(+1,+1,-1,+1,+1,+1) \to 1$	$(-1,+1,-1,+1,+1,+1) \to 1$
$(+1,+1,-1,+1,+1,-1) \to 10$	$(-1,+1,-1,+1,+1,-1) \to 1$
$(+1,+1,-1,+1,-1,+1) \to 10$	$(-1,+1,-1,+1,-1,+1) \to 1$
$(+1,+1,-1,+1,-1,-1) \to 7$	$(-1,+1,-1,+1,-1,-1) \to 1$
$(+1,+1,-1,-1,+1,+1) \to -8$	$(-1,+1,-1,-1,+1,+1) \to -11$
$(+1,+1,-1,-1,+1,-1) \to 10$	$(-1,+1,-1,-1,+1,-1) \to -2$
$(+1,+1,-1,-1,-1,+1) \to -8$	$(-1,+1,-1,-1,-1,+1) \to -8$
$(+1,+1,-1,-1,-1,-1) \to 10$	$(-1,+1,-1,-1,-1,-1) \to -11$
$(+1,-1,+1,+1,+1,+1) \to -11$	$(-1,-1,+1,+1,+1,+1) \to 10$
$(+1,-1,+1,+1,+1,-1) \to -8$	$(-1,-1,+1,+1,+1,-1) \to -8$
$(+1,-1,+1,+1,-1,+1) \to -2$	$(-1,-1,+1,+1,-1,+1) \to 10$
$(+1,-1,+1,+1,-1,-1) \to -11$	$(-1,-1,+1,+1,-1,-1) \to -8$
$(+1,-1,+1,-1,+1,+1) \to 1$	$(-1,-1,+1,-1,+1,+1) \to 7$
$(+1,-1,+1,-1,+1,-1) \to 1$	$(-1,-1,+1,-1,+1,-1) \to 10$
$(+1,-1,+1,-1,-1,+1) \to 1$	$(-1,-1,+1,-1,-1,+1) \to 10$
$(+1,-1,+1,-1,-1,-1) \to 1$	$(-1,-1,+1,-1,-1,-1) \to 1$
$(+1,-1,-1,+1,+1,+1) \to -8$	$(-1,-1,-1,+1,+1,+1) \to -11$
$(+1,-1,-1,+1,+1,-1) \to -8$	$(-1,-1,-1,+1,+1,-1) \to -8$
$(+1,-1,-1,+1,-1,+1) \to 10$	$(-1,-1,-1,+1,-1,+1) \to -2$
$(+1,-1,-1,+1,-1,-1) \to 10$	$(-1,-1,-1,+1,-1,-1) \to -11$
$(+1,-1,-1,-1,+1,+1) \to 1$	$(-1,-1,-1,-1,+1,+1) \to 7$
$(+1,-1,-1,-1,+1,-1) \to 10$	$(-1,-1,-1,-1,+1,-1) \to -5$
$(+1,-1,-1,-1,-1,+1) \to 10$	$(-1,-1,-1,-1,-1,+1) \to -5$
$(+1,-1,-1,-1,-1,-1) \to 7$	$(-1,-1,-1,-1,-1,-1) \to 7$

7.5.5. *The final result.* We summarize our description of the Chern-Simons line, in its most concrete form:

THEOREM 7.81. *Fix a surface Y and a boundary-reduced flat G-bundle $P \to Y$. We consider tuples $\hat{D} = (\mathcal{T}, \epsilon, s, x, e)$, where:*

- *$\mathcal{T}$ is a semi-ideal triangulation of Y,*
- *$\epsilon = (\epsilon_E)_{E \in \mathbf{edges}(\mathcal{T})}$ is a system of edge-orientations for $\mathcal{T}$,*
- *$s = (s_v)_{v \in \mathbf{vertices}(\mathcal{T})}$ is a flat section of $P/U = P \times_G (\mathbb{C}^2 \setminus \{0\})$ over each v, which when projected to P/B agrees with the boundary reduction,*
- *$x = (x_E)_{E \in \mathbf{edges}(\mathcal{T})}$ is a collection of complex numbers, where $\exp(x_E) = \epsilon_E(v, v') s_{v'} \wedge s_v$ if E is an edge with vertices v, v',*

- $e = (e_\Delta)_{\Delta \in \mathbf{faces}(\mathcal{T})}$ *is a system of marked edges, where* e_Δ *is an edge of* Δ.

For each such tuple there is a canonical nonzero element

$$\tau_{\hat{D}} \in \mathscr{F}_G(Y; P)^* . \tag{7.82}$$

They obey relations as follows.

(1) Suppose $\hat{D}$ *and* $\hat{D}'$ *differ only by a change involving a single vertex* v*:* $s'_v = s_v \exp(t)$ *and* $x'_E = x_E + t$ *for all edges* E *incident on* v*. Then*

$$\tau_{\hat{D}'} = \tau_{\hat{D}}. \tag{7.83}$$

(2) Suppose $\hat{D}$ *and* $\hat{D}'$ *differ only by changing the marking* e_Δ *to* e'_Δ *for a single triangle* Δ*. Then*

$$\tau_{\hat{D}'} = \exp\left[\frac{2\pi\sqrt{-1}}{3}\langle e_\Delta, e'_\Delta\rangle\right]\tau_{\hat{D}} . \tag{7.84}$$

(3) Suppose $\hat{D}$ *and* $\hat{D}'$ *differ only by a change involving a single edge* E*:* $\epsilon'_E = -\epsilon_E$ *and* $x'_E = x_E + \pi\sqrt{-1}$*. Labeling the edges of* $\mathcal{Q}_E$ *as in Figure 23, define* u *by (7.47). Then*

$$\tau_{\hat{D}'} = \exp\left[\frac{1}{4}u\right] b(\epsilon_{\Delta_1}, \epsilon'_{\Delta_1}) b(\epsilon_{\Delta_2}, \epsilon'_{\Delta_2})\, \tau_{\hat{D}} , \tag{7.85}$$

where $b(\epsilon, \epsilon')$ *is given by Proposition 7.61 with* $p = q = 1$.

(4) Suppose $\hat{D}$ *and* $\hat{D}'$ *differ only in that* $\mathcal{T}'$ *is obtained from* $\mathcal{T}$ *by flipping edge* E *to obtain a new edge* E'*, with some orientation* $\epsilon'_{E'}$ *and a choice of logarithm* $x_{E'}$*. Suppose that the triangles* Δ *abutting* E *in* $\mathcal{T}$ *both have* $e_\Delta = E$*, and the triangles abutting* E' *in* $\mathcal{T}'$ *both have* $e_{\Delta'} = E'$*. Labeling the edges of* $\mathcal{Q}_E$ *and* $\mathcal{Q}'_E$ *as in Figure 24, define* u_1*,* u_2*,* η_1*,* η_2 *by (7.49), (7.50). Then*

$$\tau_{\hat{D}'} = \exp\left[\frac{1}{2\pi\sqrt{-1}}\ell^\eta(u_1, u_2)\right] k^{\epsilon,\epsilon'}\tau_{\hat{D}} , \tag{7.86}$$

where ℓ^η *is given in (7.69)–(7.72) and* $k^{\epsilon,\epsilon'}$ *is given by Lemma 7.80.*

Proof. The object $\tau_{\hat{D}}$ has been defined in (7.59). The relations it obeys are obtained from Proposition 7.45 by contracting with $\bigotimes_{\Delta\in\mathbf{faces}(\mathcal{T})}(\phi_{e_\Delta})_*\tau_\epsilon$. ∎

Theorem 7.81 closely resembles known patching constructions of a prequantum line bundle L_{cl} over a symplectic leaf of the $\mathrm{SL}_2\mathbb{C}$-character variety of a punctured surface. In particular, in [**FG2**] such a line bundle is constructed in the more general setting of an X-cluster variety, using the dilogarithm as a gluing map. Related constructions appear in the subsequent works [**N, APP, BK, CLT**].[33] In all these cases, the characteristic property

[33] In all of these works the basic dilogarithmic formula for transition functions appears, but the detailed treatment of the constant factors is somewhat different in each case.

of L_{cl} is that it carries a natural connection given concretely in terms of the cluster coordinates on the character variety. From our point of view, this connection is the one provided by the TFT $\mathscr{F}_G$ applied to families of flat G-bundles $P \to Y$, the cluster coordinates are (up to sign) the holonomies of the corresponding flat $\mathbb{C}^\times$-bundles $Q^\epsilon_{\mathrm{tw}} \to \widetilde{Y}$, and the fact that the connection has a simple expression in cluster coordinates is obtained by using χ^ϵ_Y to pass from $\mathscr{F}_G$ to $\mathscr{S}_{\mathbb{C}^\times}$.

8. Computing CS invariants for flat $\mathrm{SL}_2\mathbb{C}$-bundles over 3-manifolds

Let M be an oriented 3-manifold with boundary. Suppose $P \to M$ is a boundary-reduced flat G-bundle. In this section we explain how to use our results to obtain a dilogarithmic formula for the Chern-Simons invariant $\mathscr{F}_G(M; P)$, Theorem 8.8 below, assuming that P satisfies a genericity condition. This formula closely resembles the previously known formulas we recalled in §2, but its precise structure is slightly different, involving several variants of the dilogarithm function and some extra cube roots of unity, and not requiring any orderability constraints on the triangulation of M.

8.1. Abelianization of the CS invariant.

We fix data:

- $\mathfrak{T}$ a semi-ideal triangulation of M, such that the genericity Assumption 4.32 is satisfied.
- $\epsilon = (\epsilon_E)_{E \in \mathbf{edges}(\mathfrak{T})}$ a system of edge-orientations for $\mathfrak{T}$.

According to Construction 4.55, $\mathfrak{T}$ determines a spectral network $\mathcal{N}^{\mathfrak{T}}$ on M, and $P \to M$ extends to boundary-reduced stratified abelianization data $\mathcal{A} = (P, Q, \mu, \theta)$ over $(M, \mathcal{N}^{\mathfrak{T}})$.

The edge-orientations ϵ determine a lift of the double cover $\widetilde{M}_{\geq -3\mathrm{a}} \to M_{\geq -3\mathrm{a}}$ to a $\mathbb{Z}$-bundle over each face of $\mathfrak{T}$ by Construction 6.5. We can then extend by radial projection to get a $\mathbb{Z}$-bundle $\widetilde{M}^{\epsilon,\infty} \to M_{\geq -3\mathrm{a}}$. As before, we use this to twist Q to Q^ϵ_{tw}, and $\pi^*\sigma$ to $\sigma^\epsilon_{\mathrm{tw}}$, both of which extend over $M_{\geq -2\mathrm{b}}$.

For each tetrahedron $\blacktriangle \in \mathbf{tets}(\mathfrak{T})$ we introduce a small ball $B_\blacktriangle$ around the barycenter, and define $H_\blacktriangle = \partial B_\blacktriangle$. As we discussed in §7.3, $H_\blacktriangle$ is naturally triangulated and has edge-orientations induced from those of $\blacktriangle$. Then we have

$$\chi^{\epsilon_\blacktriangle}_{H_\blacktriangle}\left(\mathscr{F}_G(B_\blacktriangle; P|_{B_\blacktriangle})\right) \in \mathscr{S}_{\mathbb{C}^\times}(H_\blacktriangle; Q^{\epsilon_\blacktriangle}_{\mathrm{tw}}|_{H_\blacktriangle}; \sigma^{\epsilon_\blacktriangle}_{\mathrm{tw}}) \otimes \mathcal{L}(H_\blacktriangle, \epsilon_\blacktriangle). \tag{8.1}$$

Moreover, each $\Delta \in \mathbf{faces}(\mathfrak{T})$ is pierced by a component of the branch locus $M_{-2\mathrm{b}}$ which connects a triangle Δ_1 in $H_{\blacktriangle_1}$ to another triangle Δ_2 in $H_{\blacktriangle_2}$, and thus gives an element

$$\beta_\Delta \in \mathcal{L}(\Delta_1, \epsilon) \otimes \mathcal{L}(\Delta_2, \epsilon)\,. \tag{8.2}$$

Tensoring over all faces Δ gives an element

$$\beta : \bigotimes_{\bigtriangleup \in \mathbf{tets}(\mathcal{T})} \mathcal{L}(\bigtriangleup, \epsilon_{\bigtriangleup}) \to \mathbb{C}\,. \tag{8.3}$$

PROPOSITION 8.4. *The Chern-Simons invariant $\mathscr{F}_G(M;P)$ is*

$$\begin{aligned} \mathscr{F}_G(M;P) = \beta \Bigg(& \bigotimes_{\bigtriangleup \in \mathbf{tets}(\mathcal{T})} \chi^{\epsilon_{\bigtriangleup}}_{H_{\bigtriangleup}} \left(\mathscr{F}_G(B_{\bigtriangleup}; P|_{B_{\bigtriangleup}}) \right) \Bigg) \\ & \otimes \mathscr{S}_{\mathbb{C}^\times} \left(M \setminus \cup_{\bigtriangleup} B_{\bigtriangleup}; Q^{\epsilon}_{\text{tw}}|_{M \setminus \cup_{\bigtriangleup} B_{\bigtriangleup}}; \sigma^{\epsilon}_{\text{tw}} \right)\,. \end{aligned} \tag{8.5}$$

PROOF. Apply Proposition 7.2 to the 3-manifold $M \setminus \cup_{\bigtriangleup} B_{\bigtriangleup}$. This gives

$$\chi^{\epsilon}_{\cup_{\bigtriangleup} - H_{\bigtriangleup}}(\mathscr{F}_G(M;P)) = \beta_{M \setminus \cup_{\bigtriangleup} B_{\bigtriangleup}} \otimes \mathscr{S}_{\mathbb{C}^\times} \left(M \setminus \cup_{\bigtriangleup} B_{\bigtriangleup}; Q^{\epsilon}_{\text{tw}}|_{M \setminus \cup_{\bigtriangleup} B_{\bigtriangleup}}; \sigma^{\epsilon}_{\text{tw}} \right). \tag{8.6}$$

Then gluing in a factor $\chi^{\epsilon_{\bigtriangleup}}_{H_{\bigtriangleup}} \left(\mathscr{F}_G(B_{\bigtriangleup}; P|_{B_{\bigtriangleup}}) \right)$ for each $\bigtriangleup$ gives the desired formula. □

8.2. Explicit formulas. To make the formula in Proposition 8.4 more concrete, we fix more data:

- $s = (s_v)_{v \in \mathbf{vertices}(\mathcal{T})}$ a flat section of $P/U = P \times_G (\mathbb{C}^2 \setminus \{0\})$ over each v, lying in the line given by the boundary reduction if v is an ideal vertex,
- $x = (x_E)_{E \in \mathbf{edges}(\mathcal{T})}$ a collection of complex numbers, where $\exp(x_E) = \epsilon_E(v, v')\, s_{v'} \wedge s_v$ if E is an edge with vertices v, v'.
- $e = (e_{\bigtriangleup})_{\bigtriangleup \in \mathbf{tets}(\mathcal{T})}$ a pair of opposite edges on each tetrahedron.

Note that $e_{\bigtriangleup}$ induces a marked edge $e(\Delta, \bigtriangleup)$ for each face Δ of $\bigtriangleup$. Each face Δ has two abutting tetrahedra $\bigtriangleup_1$, $\bigtriangleup_2$, and the marked edges coming from these two tetrahedra need not agree. Suppose we equip Δ with the boundary orientation induced by $\bigtriangleup_1$; then we can consider the pairing $\langle e(\Delta, \bigtriangleup_1), e(\Delta, \bigtriangleup_2) \rangle$ which measures the mismatch. Note that this quantity is independent of which tetrahedron we called $\bigtriangleup_1$, since that choice enters both in the orientation of Δ and in the ordering of the antisymmetric pairing.

Fix a $\bigtriangleup \in \mathbf{tets}(\mathcal{T})$, and let E_1 denote one of the marked edges $e_{\bigtriangleup}$; then let E_2, E_3 be defined by following E_1 around a face, with $\langle E_i, E_{i+1} \rangle = 1$. Then define

$$u_{i\,\bigtriangleup} = \sum_{E' \in \mathbf{edges}(\mathcal{T})} \langle E_i, E' \rangle x_{E'} \tag{8.7}$$

and let $(\eta_{\bigtriangleup})_i$ be the ϵ-sign of the edge E_i. Then we have the following concrete formula for the Chern-Simons invariant:

THEOREM 8.8. *The Chern-Simons invariant* $\mathscr{F}_G(M;P)$ *is*

$$\mathscr{F}_G(M;P) = \left(\prod_{\mathbb{A} \in \mathbf{tets}(\mathcal{T})} k^{\epsilon_{\mathbb{A}}} \exp\left[\frac{1}{2\pi\sqrt{-1}} \ell^{\eta_{\mathbb{A}}}(u_{1\,\mathbb{A}}, u_{2\,\mathbb{A}})\right]\right) \times \left(\prod_{\Delta \in \mathbf{faces}(\mathcal{T})} \exp\left[-\frac{2\pi\sqrt{-1}}{3}\langle e(\Delta, \mathbb{A}_1), e(\Delta, \mathbb{A}_2)\rangle\right]\right),$$

where ℓ^η *is given in* (7.69)–(7.72) *and* k^ϵ *is given by Lemma 7.80.*

PROOF. As in Construction 7.39, the choice of logarithms x_E determines a section of Q^ϵ_{tw} over each face; using radial projection in each tetrahedron, this extends to a section t of $Q^\epsilon_{\mathrm{tw}} \to M_{\geq -2\mathrm{b}}$. Then, using the fact that the $\mathbb{C}^\times$ Chern-Simons form vanishes for a flat bundle, we have

$$\mathscr{S}_{\mathbb{C}^\times}\left(M \setminus \cup_{\mathbb{A}} B_{\mathbb{A}}; Q^\epsilon_{\mathrm{tw}}|_{M\setminus \cup_{\mathbb{A}} B_{\mathbb{A}}}; \sigma^\epsilon_{\mathrm{tw}}\right) = \bigotimes_{\mathbb{A} \in \mathbf{tets}(\mathcal{T})} \tau^*_{t|_{H_{\mathbb{A}}}}. \tag{8.9}$$

We plug this into the formula of Proposition 8.4, which gives

$$\mathscr{F}_G(M;P) = \beta\left(\bigotimes_{\mathbb{A} \in \mathbf{tets}(\mathcal{T})} \frac{\chi^{\epsilon_{\mathbb{A}}}_{H_{\mathbb{A}}}\left(\mathscr{F}_G(B_{\mathbb{A}}; P|_{B_{\mathbb{A}}})\right)}{\tau_{t|_{H_{\mathbb{A}}}}}\right). \tag{8.10}$$

On each $\mathbb{A}$, we have

$$\frac{\chi^{\epsilon_{\mathbb{A}}}_{H_{\mathbb{A}}}\left(\mathscr{F}_G(B_{\mathbb{A}}; P|_{B_{\mathbb{A}}})\right)}{\tau_{t|_{H_{\mathbb{A}}}}} = k^{\epsilon_{\mathbb{A}}} \exp\left[\frac{1}{2\pi\sqrt{-1}} \ell^{\eta_{\mathbb{A}}}(u_{1\,\mathbb{A}}, u_{2\,\mathbb{A}})\right] \bigotimes_{\Delta \in \mathbf{faces}(\mathbb{A})} (\phi_{e(\Delta,\mathbb{A})})_*(\tau_{\epsilon_\Delta}) \tag{8.11}$$

Applying β to this tensor product pairs up the factors corresponding to the same face on different tetrahedra. For a face where the marked edges $e(\Delta, \mathbb{A}_1)$ and $e(\Delta, \mathbb{A}_2)$ agree, this pairing just gives 1, using (7.58). More generally, using (7.57), we see that the pairing is $\exp\left[\frac{2\pi\sqrt{-1}}{3}\langle e(\Delta,\mathbb{A}_1), e(\Delta,\mathbb{A}_2)\rangle\right]$. Plugging this into (8.10) gives the desired result. □

8.3. An example. Let M be the ideally triangulated manifold obtained by gluing two copies $\mathbb{A}_1$, $\mathbb{A}_2$ of the standard oriented tetrahedron across the four triangles $\Delta_1, \ldots, \Delta_4$, with each gluing specified by giving the mapping between three vertices of $\mathbb{A}_1$ and three vertices of $\mathbb{A}_2$:[34]

[34]This manifold is known as the "figure-eight sister," or `m003` in the SnapPea census. To avoid confusion we note that it is not the figure-eight knot complement; the latter is also obtained by gluing two tetrahedra, but it does not admit a boundary-unipotent $\mathrm{SL}(2,\mathbb{C})$-connection — although it does have boundary-unipotent $\mathrm{PSL}(2,\mathbb{C})$-connections, e.g. the one induced by the hyperbolic structure. See [**C**] for a general discussion of the obstruction to lifting a boundary-unipotent $\mathrm{PSL}(2,\mathbb{C})$-connection to a boundary-unipotent $\mathrm{SL}(2,\mathbb{C})$-connection.

$$\begin{aligned}
\Delta_1 &: (\bigtriangleup_1\ 012) \leftrightarrow (\bigtriangleup_2\ 130)\\
\Delta_2 &: (\bigtriangleup_1\ 013) \leftrightarrow (\bigtriangleup_2\ 012)\\
\Delta_3 &: (\bigtriangleup_1\ 023) \leftrightarrow (\bigtriangleup_2\ 231)\\
\Delta_4 &: (\bigtriangleup_1\ 123) \leftrightarrow (\bigtriangleup_2\ 023)
\end{aligned}$$

The induced orientations on each face are such that the vertex orderings shown above are positively oriented on $\bigtriangleup_1$, negatively oriented on $\bigtriangleup_2$. This gluing induces identifications on vertices and edges; after the gluing there is 1 vertex and 2 edges. Choose edge-orientations so that ϵ_{E_1} points from $0 \to 1$ in $\bigtriangleup_1$ and ϵ_{E_2} points from $0 \to 3$ in $\bigtriangleup_1$. See Figure 27.

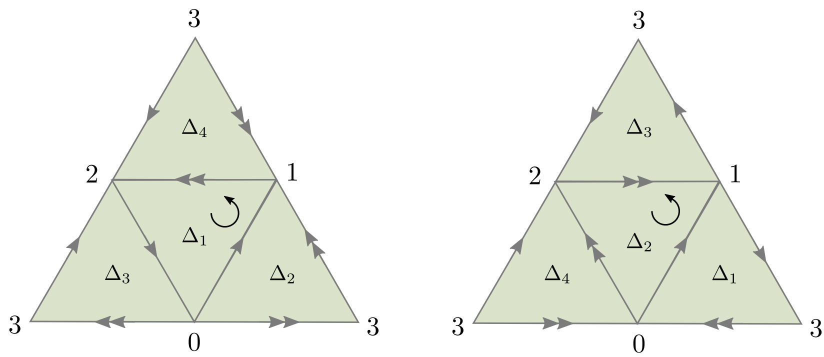


FIGURE 27. The gluing pattern defining the ideally triangulated 3-manifold M. The boundaries of the two tetrahedra are shown, with their boundary orientations. Edges with single arrows form the equivalence class E_1 after gluing; edges with double arrows form the class E_2. The direction of the arrows gives the edge-orientations ϵ_{E_1} and ϵ_{E_2}.

Suppose given a flat G-bundle $P \to M$, with a section s chosen at the vertex. Write $X_E = s_v \wedge s_{v'}$ if E is an edge from v to v' and $\epsilon_E(v, v') = 1$. The X_E corresponding to the edges of a single tetrahedron are constrained by the "Ptolemy relation":[35] if we have four elements $s_0, \dots, s_3$ in a 2-dimensional vector space then

$$(s_0 \wedge s_1)(s_2 \wedge s_3) + (s_0 \wedge s_2)(s_3 \wedge s_1) + (s_0 \wedge s_3)(s_1 \wedge s_2) = 0\,. \tag{8.12}$$

Applying this to $\bigtriangleup_1$ we get the relation

$$-X_1^2 - X_1X_2 + X_2^2 = 0\,. \tag{8.13}$$

[35]These relations are exploited systematically in [**GGZ1**, **GGZ2**] to parameterize flat bundles over triangulated 3-manifolds.

As it happens, in $\triangle_2$ we get the same relation.[36] Assuming that $X_2 \neq 0$, we may choose the scale of s so that $X_2 = 1$, and then using (8.13)

$$X_1 = \frac{1}{2}(-1 \pm \sqrt{5}), \quad X_2 = 1. \tag{8.14}$$

For either choice of the sign in (8.14), we can build a flat boundary-reduced G-bundle $P \to M$, by taking the trivial bundle in each tetrahedron and then gluing across each triangle, requiring that the gluing match up the sections s_v at the three vertices of the triangle. In this way we obtain two inequivalent flat boundary-reduced G-bundles $P \to M$. Moreover, since all $X_E \neq 0$ these G-bundles obey the genericity Assumption 4.32.

In either case, we would like to compute $\mathscr{F}_G(M;P)$ using Theorem 8.8. For this we have to make a choice of a pair of marked edges on each tetrahedron. We choose the edges labeled 01, 23. We also have to choose logarithms x_i of X_i. Then for either $\triangle_1$ or $\triangle_2$ we have

$$u_1 = -x_1 + x_2, \quad u_2 = 2x_1 - 2x_2, \quad \eta = (+1, +1) \tag{8.15}$$

To be completely explicit let us choose $P \to M$ corresponding to the $-$ sign in (8.14) above. Then, using (7.69) and (8.14), it turns out that[37]

$$\begin{aligned} \ell^{(+1,+1)}(u_1, u_2) &= \frac{9\pi^2}{20} \pmod{4\pi^2\mathbb{Z}}, \\ \exp\left(\frac{1}{2\pi\sqrt{-1}}\ell^{(+1,+1)}(u_1,u_2)\right) &= \exp\left(-\frac{9\pi\sqrt{-1}}{20}\right). \end{aligned} \tag{8.16}$$

Next we look up from Lemma 7.80 that for either tetrahedron we have

$$k^{\epsilon_\triangle} = \exp\left(\frac{7\pi\sqrt{-1}}{12}\right). \tag{8.17}$$

Finally, we have to include the cube roots of unity from face gluings. For instance, on face Δ_1 we see from the face gluings above that the edge $e(\Delta_1, \triangle_1)$ (numbered 01 on $\triangle_1$) and the edge $e(\Delta_1, \triangle_2)$ (numbered 01 on $\triangle_2$) have $\langle e(\Delta_1, \triangle_1), e(\Delta_1, \triangle_2)\rangle = 1$. Computing similarly for the other three faces, we get the overall factor

$$\exp\left(\frac{2\pi\sqrt{-1}}{3}(1+0+1+0)\right) = \exp\left(-\frac{2\pi\sqrt{-1}}{3}\right). \tag{8.18}$$

[36]Had we used the figure-eight knot complement instead, we would have gotten a sign flipped here, with the result that the only common solution of the two equations would be $X_1 = X_2 = 0$.

[37]The fact that we get a closed form expression here comes from the fact that, for $z = -\frac{1+\sqrt{5}}{2}$, we have the closed form expression $\mathrm{Li}_2(z) = -\frac{\pi^2}{10} - \log(-z)^2$. Also, in this example $\ell^\eta(u_1, u_2)$ is actually independent $\pmod{4\pi^2\mathbb{Z}}$ of the choice of logarithms x_1, x_2. Both of these are exceptional properties, which do not hold in most examples; more typically, the final result for $\mathscr{F}_G(M;P)$ involves a sum of $\mathrm{Li}_2(z_i)$ for algebraic numbers z_i, and changing the choice of logarithms changes the contributions from individual tetrahedra while leaving the final result unchanged.

Combining the factors (8.16) (for each tetrahedron), (8.17) (for each tetrahedron), and (8.18) we get

$$\mathscr{F}_G(M;P) = \exp\left(\pi\sqrt{-1}\left(-\frac{9}{10}+\frac{7}{6}-\frac{2}{3}\right)\right) = \exp\left(-\frac{2\pi\sqrt{-1}}{5}\right). \tag{8.19}$$

If we take $P \to M$ corresponding to the $+$ sign in (8.14), then (8.16) is replaced by

$$\begin{aligned} &\ell^{(+1,+1)}(u_1,u_2) = \frac{\pi^2}{20} \pmod{4\pi^2\mathbb{Z}}, \\ &\exp\left(\frac{1}{2\pi\sqrt{-1}}\ell^{(+1,+1)}(u_1,u_2)\right) = \exp\left(-\frac{\pi\sqrt{-1}}{20}\right), \end{aligned} \tag{8.20}$$

which when combined with the other factors gives

$$\mathscr{F}_G(M;P) = \exp\left(\pi\sqrt{-1}\left(-\frac{1}{10}+\frac{7}{6}-\frac{2}{3}\right)\right) = \exp\left(\frac{2\pi\sqrt{-1}}{5}\right). \tag{8.21}$$

8.4. Another example. Now let M be the manifold called `m071` in the SnapPea census. M admits an ideal triangulation $\mathfrak{T}$ with 5 tetrahedra, 10 triangles, 5 edges and 1 vertex. Applying Theorem 8.8 in this example we obtain the Chern-Simons invariants $\mathscr{F}_G(M;P)$ for 7 inequivalent boundary-reduced flat G-bundles $P \to M$: numerically they are approximately

$$\begin{array}{lll} 0.697849 + 0.716244\sqrt{-1} & -0.99614 + 0.0877733\sqrt{-1} & -0.26787 + 0.963455\sqrt{-1} \\ -0.948968 - 0.315372\sqrt{-1} & 0.982867 + 0.184319\sqrt{-1} & 1.51835 + 0.0629475\sqrt{-1} \\ 0.65748 + 0.0272577\sqrt{-1} & & \end{array}$$

Details of this computation (along with a few others) are given in the Mathematica notebook `dilog-compute.nb`, available as an ancillary file included with the arXiv preprint version of this paper. In making these computations we made use of the software SnapPy [**CDGW**] and Regina [**BBP**].

9. Future directions

We conclude with a brief description of possible avenues for further exploration.

1. Although some basic setup in §4 and §5 applies to all rank one complex Lie groups, our main results are proved here only for flat $SL_2\mathbb{C}$-connections. One should be able to extend them to the groups $GL_2\mathbb{C}$ and $PSL_2\mathbb{C}$.
2. More ambitious is an extension to higher rank complex Lie groups. Spectral networks in 2 dimensions for higher rank groups are discussed in [**GMN1, GMN2, LP, IM**]. Some clues for spectral networks in 3 dimensions can be found in [**GTZ, DGG**].
3. The flat $SL_2\mathbb{C}$-connections to which we apply stratified abelianization are assumed to be boundary-unipotent, this for both 2- and 3-dimensional

compact manifolds with boundary. The expectation is that the boundary-unipotent assumption could be dropped at the cost of having additional lines or $\mathbb{V}$-lines associated to boundary components.

4. The 3-dimensional spectral networks in this paper are either induced from a triangulation or for the Cartesian product of a triangulated 2-manifold and a closed interval (§7.3). According to [**GMN3**] one obtains a 3-dimensional spectral network from a 1-parameter family of holomorphic quadratic differentials on a Riemann surface. This recovers the spectral network of §7.3, but one can also get another transition of triangulations called a "juggle". It would be interesting to study this kind of 3-dimensional spectral network.
5. One can strive to develop spectral networks (rank one or higher rank) as a topological structure on a smooth manifold, much like an orientation or spin structure. If so, then one could define bordism categories and bordism spectra as domains of field theories on manifolds equipped with a spectral network and stratified abelianization. This should lead to more powerful theorems. Perhaps, then, our main results could be stated and proved as parts of an isomorphism of field theories.
6. One of the most important aspects of 2-dimensional spectral networks is the relationship to the WKB analysis of ordinary differential equations; see e.g. [**KaTa, GMN3**] and references therein. One can inquire whether 3-dimensional spectral networks have a similar relationship to differential equations.
7. We give a construction of the Chern-Simons line of a flat bundle on a 2-manifold via stratified abelianization. In a parametrized family of flat bundles, this gives a construction of the Chern-Simons line bundle with its covariant derivative. This is the same construction that appears in [**FG2, APP, N, CLT**], and therefore the line bundle constructed in those papers must be the Chern-Simons line bundle. One can imagine that this identification of the line bundle will lead to new insights and results.
8. 2d spectral networks on a Riemann surface can be constructed in terms of BPS particles in a corresponding supersymmetric quantum field theory [**GMN1**]. It would be very interesting to know whether one can understand 3d spectral networks in a similar way; candidates for the corresponding quantum field theory are known in the physics literature, beginning with [**DGGu**].
9. 2d spectral networks can be used to construct the quantum trace map (or q-nonabelianization map, or quantum UV-IR map) of [**BW**], as a homomorphism from the $\mathfrak{gl}(2)$ skein algebra of a surface Y to the $\mathfrak{gl}(1)$ skein algebra of a double cover $\widetilde{Y}$ [**G, NY**]. It would be interesting to use the 3d spectral networks defined in this paper to construct a 3-manifold version of the quantum trace, mapping the $\mathfrak{gl}(2)$ skein module of a 3-manifold M to an $\mathfrak{gl}(1)$ skein module of a branched double cover $\widetilde{M}$.

According to the recent proposal of [AGLR], such a 3-manifold quantum trace is one of the necessary ingredients in the formulation of a "length conjecture" for link invariants.

Appendix A. Ordinary differential cochains

Differential cohomology descends on the one hand from the differential characters of Cheeger-Simons [ChS] and on the other from Deligne cohomology [De] in algebraic geometry. The ideas for an explicit model are suggested in [DF, §6.3]. Hopkins and Singer [HS, §3] develop the model that we recall in this appendix. The theory of *generalized differential cohomology* (and differential cocycles) has been further developed in many works, for example [SS, BNV, ADH]. In this brief appendix we also include a few complements needed for Chern-Simons theory.

A.1. Cochain model. Let M be a smooth manifold, A an abelian group, and $C^p(M;A)$ the group of singular p-cochains with values in the abelian group A. For each $q\in\mathbb{Z}^{\ge 0}$ define a cochain complex $\check{C}(q)^\bullet$ by

$$\check{C}(q)^p(M)=\begin{cases}C^p(M;\mathbb{Z})\times C^{p-1}(M;\mathbb{R})\times\Omega^p(M), & p\ge q;\\ C^p(M;\mathbb{Z})\times C^{p-1}(M;\mathbb{R}), & p<q,\end{cases}\tag{A.1}$$

with differential acting on $(c,h,\omega)\in\check{C}(q)^p(M)$, $p\ge q$, or on $(c,h)\in\check{C}(q)^p(M)$, $p<q$, given by

$$\begin{aligned}d(c,h,\omega)&=\quad(\delta c\,,\,\omega-c-\delta h\,,\,d\omega), & p\ge q;\\ d(c,h)&=\begin{cases}(\delta c,-c-\delta h,0), \\ (\delta c,-c-\delta h),\end{cases} & \begin{matrix}p=q-1;\\ p<q-1.\end{matrix}\end{aligned}\tag{A.2}$$

The cohomology of the cochain complex $\check{C}(q)^\bullet$ is

$$\check{H}(q)^p(M)\cong\begin{cases}H^p(M;\mathbb{Z}), & p>q;\\ H^p(M;\mathbb{R}/\mathbb{Z}), & p<q,\end{cases}\tag{A.3}$$

and the diagonal group $\check{H}(q)^q(M)$ is the *differential cohomology.* Since we have great use for the diagonal groups, we introduce the notations

$$\begin{aligned}\check{C}^q(M)&:=\check{C}(q)^q(M)\\ \check{Z}^q(M)&:=\check{Z}(q)^q(M)\\ \check{H}^q(M)&:=\check{H}(q)^q(M),\end{aligned}\tag{A.4}$$

where $\check{Z}(q)^p(M)\subset\check{C}(q)^p(M)$ is the subgroup of cocycles. The group $\check{H}^q(M)$ is isomorphic to the group of Cheeger-Simons differential characters on M. It can be given [BSS], [DGRW, §6] the structure of an infinite dimensional abelian Lie group with Lie algebra and homotopy groups

$$\operatorname{Lie}\check{H}^q(M) \cong \frac{\Omega^{q-1}(M)}{d\Omega^{q-2}(M)},$$

(A.5)
$$\pi_k \check{H}^q(M) \cong \begin{cases} H^q(M;\mathbb{Z}), & k=0; \\ H^{q-1}(M;\mathbb{Z}), & k=1; \\ 0, & k \geq 2 \end{cases}$$

REMARK A.6.

(1) The case $q = 2$ is instructive: $\check{H}^2(M)$ is isomorphic to the group of isomorphism classes of principal $\mathbb{R}/\mathbb{Z}$-bundles with connection over M.
(2) In the main text we use complex differential forms and complex singular cochains in (A.1), for which we use the notation '$\check{C}_{\mathbb{C}}(q)^p(M)$'.
(3) The construction is functorial for smooth maps, so can be phrased in terms of simplicial sheaves on manifolds [**FH2**, **BNV**, **ADH**].
(4) There are alternative models which replace singular cochains with integer coefficients by other models of cochains or by maps to Eilenberg-MacLane spaces. The latter approach is used to define differential versions of generalized cohomology theories, as in [**HS**, §4].

A.2. Curvature, characteristic class, and nonflat trivializations. The projection

$$\check{C}^q(M) \longrightarrow \Omega^q(M) \tag{A.7}$$

is called the *curvature* or *covariant derivative* map, depending on the context. The restriction of (A.7) to differential cocycles $\check{Z}^q(M) \subset \check{C}^q(M)$ factors through $\check{H}^q(M)$ and has image the subgroup of closed differential forms with integral periods. There is also a *characteristic class* homomorphism

$$\check{H}^q(M) \longrightarrow H^q(M;\mathbb{Z}). \tag{A.8}$$

The short exact sequence

$$0 \longrightarrow \check{C}(q)^\bullet(M) \xrightarrow{\ i\ } \check{C}(q-1)^\bullet(M) \longrightarrow \Omega^{q-1}(M)[1-q] \longrightarrow 0 \tag{A.9}$$

induces a long exact sequence in cohomology:

$$\cdots \longrightarrow \check{H}^{q-1}(M) \longrightarrow \Omega^{q-1}(M) \longrightarrow \check{H}^q(M) \xrightarrow{\ i\ } H^q(M;\mathbb{Z}) \longrightarrow 0 \tag{A.10}$$

Let $\breve{\omega} \in \check{Z}^q(M)$ be a differential cocycle.

DEFINITION A.11. A cochain $\breve{\tau} \in \check{C}^{q-1}(M)$ is a *nonflat*[38] *trivialization* of $\breve{\omega}$ if $d\breve{\tau} = i(\breve{\omega})$ holds in $\check{C}(q-1)^q(M)$.

[38]'Nonflat' could be replaced by the more accurate 'not necessarily flat'.

By (A.10), a nonflat trivialization $\breve{\tau}$ produces a differential form $\tau \in \Omega^{q-1}(M)$, the *covariant derivative* of $\breve{\tau}$. (It is the image of $\breve{\tau}$ under (A.7).) For $q = 2$ we can represent $\breve{\omega}$ as a complex line bundle $\pi\colon L \to M$ with connection. Then a nonflat trivialization is a section of π, and its covariant derivative is as usual.

REMARK A.12. See [**F3**, Definition 5.12] for a model of nonflat trivializations which works in generalized differential cohomology theories.

A.3. Higher Picard groupoids of differential cocycles. As a warmup, recall that a cochain complex

$$0 \longrightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \tag{A.13}$$

gives rise to a sequence of higher Picard groupoids. For $q \in \mathbb{Z}^{\ge 0}$, form the truncated complex

$$0 \longrightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \xrightarrow{d} A^q_{\text{closed}} \longrightarrow 0 \tag{A.14}$$

in which $A^q_{\text{closed}} \subset A^q$ is the subgroup of closed elements. Define the Picard q-groupoid $\mathcal{G}_{(q)}$ as follows. The set of objects is A^q_{closed}. The A^{q-1}_{closed}-torsor of 1-morphisms between $a_0, a_1 \in A^q_{\text{closed}}$ is

$$\text{Hom}_1(a_0, a_1) = \bigl\{a' \in A^{q-1} : a_0 + da' = a_1\bigr\}. \tag{A.15}$$

The A^{q-2}_{closed}-torsor of 2-morphisms between $a'_0, a'_1 \in \text{Hom}_1(a_0, a_1)$ is

$$\text{Hom}_2(a'_0, a'_1) = \bigl\{a'' \in A^{q-2} : a'_0 + da'' = a'_1\bigr\}, \tag{A.16}$$

and so on. The homotopy groups of $\mathcal{G}_{(q)}$ are

$$\pi_i\mathcal{G}_{(q)} = \begin{cases} H^{q-i}(A), & i = 0, \dots, q; \\ 0, & i > q. \end{cases} \tag{A.17}$$

A variation of this construction produces higher Picard groupoids out of differential cohomology on a smooth manifold M. Define the Picard q-groupoid $\breve{\mathcal{G}}_{(q)}(M)$ to have as its set of objects $\breve{Z}^q(M) \subset \breve{C}^q(M)$, the set of differential cocycles of degree q. Then for $\breve{\omega}_0, \breve{\omega}_1 \in \breve{Z}^q(M)$, define

$$\text{Hom}_1(\breve{\omega}_0, \breve{\omega}_1) = \bigl\{\breve{\tau} \in \breve{C}^{q-1}(M) : i(\breve{\omega}_0) + d\breve{\tau} = i(\breve{\omega}_1)\bigr\}, \tag{A.18}$$

where as in Definition A.11 the equation $i(\breve{\omega}_0) + d\breve{\tau} = i(\breve{\omega}_1)$ lies in $\breve{C}(q-1)^q(M)$. In other words, $\breve{\tau} \in \text{Hom}_1(\breve{\omega}_0, \breve{\omega}_1)$ is a nonflat trivialization of $\breve{\omega}_1 - \breve{\omega}_0$. Now for $\breve{\tau}_0, \breve{\tau}_1 \in \text{Hom}_1(\breve{\omega}_0, \breve{\omega}_1)$, define

$$\text{Hom}_2(\breve{\tau}_0, \breve{\tau}_1) = \bigl\{\breve{\lambda} \in \breve{C}^{q-2}(M) : i(\breve{\tau}_0) + d\breve{\lambda} = i(\breve{\tau}_1)\bigr\}. \tag{A.19}$$

The homotopy groups of $\breve{\mathcal{G}}_{(q)}(M)$ are the differential cohomology groups:

$$\pi_i\breve{\mathcal{G}}_{(q)}(M) = \begin{cases} \breve{H}^{q-i}(M), & i = 0, \dots, q; \\ 0, & i > q. \end{cases} \tag{A.20}$$

REMARK A.21. In low degrees the $\check{\mathcal{G}}_{(q)}(M)$ have alternative, more geometric, presentations. For example, $\check{\mathcal{G}}_{(2)}(M)$ is equivalent to the following Picard 2-groupoid: an object is a principal $\mathbb{R}/\mathbb{Z}$-bundle $P \to M$ with connection Θ; a morphism $(P_0, \Theta_0) \to (P_1, \Theta_1)$ is a section s of $P_1 \otimes P_0^{-1} \to M$; and a 2-morphism $s_0 \to s_1$ is a function $f\colon M \to \mathbb{R}$ such that $s_0 + \overline{f} = s_1$, where $\overline{f}\colon M \to \mathbb{R}/\mathbb{Z}$ is the mod $\mathbb{Z}$ reduction of f.

A.4. Integration and Stokes' theorem. We repeat and expand upon material from [**HS**, §3.4], slightly specialized. Let $p\colon M \to S$ be a proper fiber bundle whose base S is a smooth manifold and whose total space M is a smooth manifold with boundary; the fibers of p are smooth manifolds with boundary. Suppose an $\check{H}$-orientation is given; see [**HS**, §2.4]. Then if the fibers have dimension k, integration is a homomorphism

$$\int_{M/S}\colon \check{C}(q)^p(M) \longrightarrow \check{C}(q-k)^{p-k}(S). \tag{A.22}$$

We also have the usual integration of differential forms, and the diagram

$$\begin{array}{ccc} \check{C}(q)^p(M) & \longrightarrow & \Omega^p(M) \\ {\scriptstyle \int_{M/S}}\downarrow & & \downarrow{\scriptstyle \int_{M/S}} \\ \check{C}(q-k)^{p-k}(S) & \longrightarrow & \Omega^{p-k}(S) \end{array} \tag{A.23}$$

commutes. Also the integration map (A.22) commutes with base change.

We state a version of Stokes' theorem in this context. It concerns the case $p = q$ in (A.22), restricted to differential cocycles.[39]

THEOREM A.24. *Let $\check{\omega} \in \check{Z}^q(M)$ be a differential cocycle with curvature $\omega \in \Omega^q(M)$. Then the integral $\int_{M/Z} \check{\omega} \in \check{C}^{q-k}(S)$ is a nonflat trivialization of $\int_{\partial M/S} \check{\omega} \in \check{Z}^{q-k+1}(S)$ with covariant derivative $\int_{M/S} \omega \in \Omega^{q-k}(S)$.*

See [**HS**, §3.4] for a more general theorem and proof.

We also need a generalization of Theorem A.24 to manifolds with corners. Here we state the next simplest version—corners of codimension at most 2—and leave to the reader the more general version. For convenience we use manifolds with corners equipped with the extra structure needed for objects and morphisms in bordism multicategories, as in [**FT1**, §A.2]. The data of such a manifold M of dimension k and depth $\le d$ includes manifolds M^δ_{-j} with corners of depth $\le d-j$, $\delta \in \{0,1\}$, $j \in \{1,\dots,d\}$, and embeddings $[0,1]^{j-1} \times M^\delta_{-j} \to \partial M$. The data is designed to fit the formalism of multicategories. For example, if $d = 2$ then a 2-categorical interpretation has objects M^0_{-2}, M^1_{-2}; 1-morphisms $M^0_{-1}, M^1_{-1}\colon M^0_{-2} \to M^1_{-2}$; and M itself is a 2-morphism $M^0_{-1} \to M^1_{-1}$.

[39]We do not vouch for the signs.

Let $p\colon M \to S$ be a proper fiber bundle whose base S is a smooth manifold and whose total space M is a manifold with corners of depth ≤ 2. Assume that M carries the extra structure of [**FT1**, §A.2], fibered over S. In particular, there are fiber bundles $p^{\delta}_{-j}\colon M^{\delta}_{-j} \to S$, $\delta \in \{0,1\}$, $j \in \{1,2\}$. Let $\breve{\omega} \in \check{Z}^q(M)$ be a differential cocycle with curvature $\omega \in \Omega^q(M)$. Define

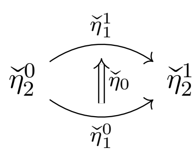
(A.25)
$$\begin{aligned} \breve{\eta}^{\delta}_j &= \int_{M^{\delta}_{-j}/S} \breve{\omega} \qquad \in \check{C}^{q-k+j}(S), \\ \eta^{\delta}_j &= \int_{M^{\delta}_{-j}/S} \omega \qquad \in \Omega^{q-k+j}(S). \end{aligned}$$

These formulas pertain for $j \in \{1,2\}$, $\delta \in \{0,1\}$; for $j = 0$ omit δ.

THEOREM A.26.

(1) *$\breve{\eta}^{\delta}_1$ is a nonflat trivialization of $\breve{\eta}^{1}_2 - \breve{\eta}^{0}_2$ with covariant derivative η^{δ}_1.*
(2) *$\breve{\eta}_0$ is a nonflat trivialization of $\breve{\eta}^{1}_1 - \breve{\eta}^{0}_1$ with covariant derivative η_0.*

The diagram

(A.27)


captures some of Theorem A.26. In the terms of §A.3, the integral of $\breve{\omega}$ over M_0/S is a 2-morphism in $\check{\mathcal{G}}_{(q-k+2)}(S)$. We leave generalizations to greater depths to the reader.

We also use a generalization of Theorem A.26 for the integral of a 1-morphism in $\check{\mathcal{G}}_{(q)}(M)$. For simplicity, consider a 1-morphism of the form $\breve{0} \xrightarrow{\breve{\tau}} \breve{\omega}$. In other words, $\breve{\tau} \in \check{C}^{q-1}(M)$ is a nonflat trivialization of $\breve{\omega} \in \check{Z}^q(M)$. Let $\tau \in \Omega^{q-1}(M)$ be the covariant derivative of $\breve{\tau}$. As in (A.25), set

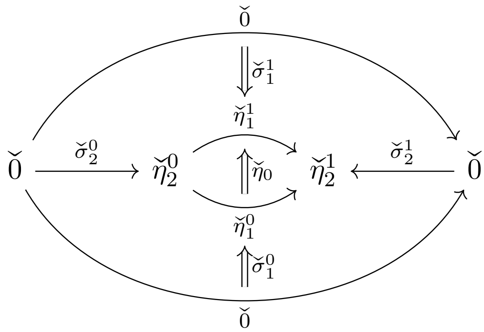
(A.28)
$$\begin{aligned} \breve{\sigma}^{\delta}_j &= \int_{M^{\delta}_{-j}/S} \breve{\tau} \qquad \in \check{C}^{q-k+j-1}(S), \\ \sigma^{\delta}_j &= \int_{M^{\delta}_{-j}/S} \tau \qquad \in \Omega^{q-k+j-1}(S). \end{aligned}$$

THEOREM A.29.

(1) *$\breve{\sigma}^{\delta}_2$ is a nonflat trivialization of $\breve{\eta}^{\delta}_2$ with covariant derivative σ^{δ}_2.*
(2) *$\breve{\sigma}^{\delta}_1\colon \breve{\sigma}^{1}_2 - \breve{\sigma}^{0}_2 \to \breve{\eta}^{\delta}_1$ is a nonflat isomorphism with covariant derivative σ^{δ}_1.*
(3) *$\breve{\sigma}_0\colon \breve{\sigma}^{1}_1 - \breve{\sigma}^{0}_1 \to \breve{\eta}_0$ is a nonflat isomorphism with covariant derivative σ_0.*

The data of $\breve{\eta}_2^\delta, \breve{\eta}_1^\delta, \breve{\eta}_0, \breve{\sigma}_2^\delta, \breve{\sigma}_1^\delta, \breve{\sigma}_0$ assemble to a 3-morphism in $\breve{\mathcal{G}}_{(q-k+2)}(S)$. We depict all but $\breve{\sigma}_0$ in (A.30).

(A.30)



Again, we leave to the reader generalizations of Theorem A.29 to greater depths.

Remark A.31. Suppose $q = k$, so that (A.30) is a diagram in $\breve{\mathcal{G}}_{(2)}(S)$. Then as in Remark A.21 we interpret $\breve{\eta}_2^\delta$ as a principal $\mathbb{R}/\mathbb{Z}$-bundle $\pi^\delta\colon P^\delta \to S$ with connection, $\breve{\eta}_1^\delta$ as a nonflat isomorphism $\pi^0 \to \pi^1$, and $\breve{\eta}_0$ as a function $S \to \mathbb{R}$ whose reduction mod $\mathbb{Z}$ maps the isomorphism $\breve{\eta}_1^0$ to the isomorphism $\breve{\eta}_1^1$. Now $\breve{\sigma}_2^\delta$ is a nonflat section of π^δ. The isomorphism $\breve{\eta}_1^\delta\colon \pi^0 \to \pi^1$ maps $\breve{\sigma}_2^0$ to a section of π^1, and $\breve{\sigma}_1^\delta$ is a function $S \to \mathbb{R}$ whose mod $\mathbb{Z}$ reduction maps the section $\breve{\sigma}_2^1$ to the section $\breve{\eta}_1^\delta(\breve{\sigma}_2^0)$. Finally, $\breve{\sigma}_0 = 0$ trivially—there are no nonzero 3-morphisms in $\breve{\mathcal{G}}_{(2)}(S)$—which means that the difference of the functions $\breve{\sigma}_1^1 - \breve{\sigma}_1^0$ equals the function $\breve{\eta}_0$, as is evident from the foregoing.

Appendix B. Invertible field theories

We briefly outline some general facts about invertible field theories, including those which are not flat, hence not topological. For simplicity we confine our exposition to a setting which applies to the Chern-Simons theory we encounter in §5.3.5. Invertible field theories which are topological in a *restricted* sense (which applies to evaluation on a single manifold, as for example is true for Chern-Simons theory (3.14) on flat connections) or are topological in a *strong* sense (which applies to evaluation in families) are modeled as maps of spectra in topology; see [**FHT, FH1, F5**] and the references therein. Here we use *generalized* differential cohomology, for which we need a model of "cocycles" which generalize the singular cocycles used in Appendix A. For background and detailed development of topics in this appendix, see [**FH2, HS, BNV, ADH**] and the references therein.

Let G be a Lie group. Let Man denote the category of smooth manifolds and smooth maps between them, and let sSet be the category of simplicial sets. Then

$$B_\nabla G\colon \mathrm{Man}^{\mathrm{op}} \longrightarrow \mathrm{sSet} \tag{B.1}$$

is the simplicial sheaf which assigns to a test manifold S the nerve of the groupoid of G-connections on S; see [**FH2**, Example 5.11]. Similarly,

$$E_\nabla G\colon \mathrm{Man}^{\mathrm{op}} \longrightarrow \mathrm{sSet} \tag{B.2}$$

is the simplicial sheaf which assigns to a test manifold S the nerve of the groupoid of G-connections on S and a trivialization of the underlying G-bundle. Then $E_\nabla G$ is equivalent to the set-valued sheaf $\Omega^1 \otimes \mathfrak{g}$ which assigns to a test manifold S the vector space $\Omega^1_S(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G; see [**FH2**, Example 5.14]. A smooth manifold M defines a representable set-valued sheaf on Man; its value on a test manifold S is the set of smooth maps $S \to M$.

Let $h^\bullet$ be a generalized cohomology theory. Define the $\mathbb{Z}$-graded real vector space

$$V_h^\bullet = h^\bullet(\mathrm{pt}) \otimes \mathbb{R}, \tag{B.3}$$

and suppose given an isomorphism of cohomology theories $h^\bullet \otimes \mathbb{R} \xrightarrow{\cong} HV_h^\bullet$, where the codomain is the Eilenberg-MacLane theory with $HV_h^\bullet(\mathrm{pt}) = V_h^\bullet$. There is then a differential cohomology theory $\check{h}^\bullet$ (of "Hopkins-Singer type") which refines the topological theory $h^\bullet$. Furthermore, these theories—as well as the de Rham complex—can be evaluated on simplicial sheaves, in particular on $B_\nabla G$. The Anderson dual $I\mathbb{Z}^\bullet$ to the sphere is a universal choice for the codomain of an invertible field theory. Nonetheless, for Chern-Simons theory in this context it is more convenient to use a truncation $E^\bullet$, the cohomology theory introduced in §5.2.1, as the codomain.

Remark B.4. Chern-Simons theory has not only a G-connection as a background field, but also a spin structure. In the formalism sketched here we do not treat them symmetrically: we use the spin structure to integrate differential E-cohomology classes (and cochains). By contrast, in *topological* invertible theories we usually do treat them symmetrically and use the universal codomain $I\mathbb{Z}^\bullet$.

An n-dimensional invertible field theory on G-connections is modeled by a map

$$\alpha\colon B_\nabla G \longrightarrow \check{h}^{n+1}. \tag{B.5}$$

The theory α has an underlying *cohomology class*

$$B_\nabla G \longrightarrow h^{n+1} \tag{B.6}$$

and *curvature*

$$B_\nabla G \longrightarrow (\Omega \otimes V_h)^{n+1} \tag{B.7}$$

of total degree $n+1$. (For E-cohomology[40] the $\mathbb{Z}$-graded vector space $V_E^\bullet = \mathbb{R}$ is supported in degree 0, hence the codomain of the curvature (B.7) is Ω^{n+1}.)

[40]In the main text we use a *complexified* version of differential E-cohomology.

The theory α is *flat* if its curvature (B.7) vanishes. In that case α factors through a topological theory

$$\hat{\alpha}\colon B_\nabla G \longrightarrow h^n_{\mathbb{R}/\mathbb{Z}}, \tag{B.8}$$

where $h^\bullet_{\mathbb{R}/\mathbb{Z}}$ is the cofiber of $h^\bullet \to h^\bullet \otimes \mathbb{R}$. The theory is *topologically trivial* if its underlying cohomology class (B.6) vanishes. A trivialization of the underlying cohomology class lifts α to a theory defined by the differential form[41]

$$\eta\colon B_\nabla G \longrightarrow (\Omega \otimes V_h)^n. \tag{B.9}$$

(Compare with Definition A.11 and the following paragraph.) Conversely, a differential form (B.9) determines an invertible theory (B.5), and the isomorphism class of the latter does not change if the form is shifted by an exact form (or, more generally, by a closed form whose periods are integral in a sense defined by the cohomology theory $h^\bullet$).

EXAMPLE B.10. The 3-dimensional spin $\mathbb{C}^\times$ Chern-Simons theory in §5.3.5 is a map

$$B_\nabla \mathbb{C}^\times \longrightarrow \check{E}^4_{\mathbb{C}} \tag{B.11}$$

Recall [**FH2**] that the complexified de Rham complex of $B_\nabla \mathbb{C}^\times$ is a complex polynomial algebra generated by a 2-form ω which is $\sqrt{-1}/2\pi$ times the curvature of the universal $\mathbb{C}^\times$-connection. Then the curvature of the theory (B.11) is the 4-form

$$\frac{1}{2}\,\omega \wedge \omega. \tag{B.12}$$

The curvature, and indeed the entire theory (B.11), should be evaluated on families

$$P \xrightarrow{\;p\;} X \xrightarrow{\;\pi\;} S \tag{B.13}$$

in which π is a proper fiber bundle equipped with a relative spin structure and p is a principal $\mathbb{C}^\times$-bundle equipped with a connection. Then (B.12) pulls back to an element of $\Omega^4_{\mathbb{C}}(X)$.

The discussions in §A.2 and §A.3 have analogs for invertible field theories. Thus there is a notion of a flat (or nonflat) isomorphism of invertible theories. Furthermore, the equivalence classes of flat isomorphisms of two n-dimensional invertible theories form a torsor over the abelian group of flat $(n-1)$-dimensional invertible theories. The latter are *topological* invertible theories in a strong sense, hence may be treated via methods of stable homotopy theory.

[41]Our convention is that the partition function for a theory defined by a form η is obtained by integrating $2\pi\sqrt{-1}\eta$.

Appendix C. $\mathbb{Z}/2\mathbb{Z}$ gradings

The invertible spin $\mathbb{C}^\times$ Chern-Simons theory is obtained by integration in the cohomology theory E with two nonzero homotopy groups; see §5.2.1. (As explained in §5.3, one integrates complex *differential* "cochains" in the differential theory $\check{E}_{\mathbb{C}}$ or, for flat connections, "cochains" in the secondary theory $E_{\mathbb{C}/\mathbb{Z}}$.) The homotopy group $\mathbb{Z}/2\mathbb{Z}$, which appears along with the standard $\mathbb{Z}$ or $\mathbb{C}/\mathbb{Z}$, introduces an additional $\mathbb{Z}/2\mathbb{Z}$-grading in the values of the field theory. In this appendix we collect some remarks and results about this grading. In particular, we complete the proof of Proposition 6.29. The bottom line is: with a suitable universal choice we can and do ignore the $\mathbb{Z}/2\mathbb{Z}$-gradings in the main text.

C.1. $s\mathbb{V}$-lines. Let $s\mathbb{V}$ be the linear category of finite-dimensional complex $\mathbb{Z}/2\mathbb{Z}$-graded[42] vector spaces; for $V_1, V_2 \in s\mathbb{V}$ the hom space $s\mathbb{V}(V_1, V_2)$ consists of *even* linear maps $V_1 \to V_2$. Impose the symmetric monoidal structure of tensor product with the Koszul sign rule. A *super category* is a complex linear module category over the tensor category $s\mathbb{V}$. Super categories form a 2-category. The multiplicative units in this 2-category are called $s\mathbb{V}$-*lines*, and there are two such up to isomorphism: the Bose line $s\mathbb{V}$ and the Fermi line $\mathbb{\Gamma}$; the latter is the category of modules over the complex Clifford algebra Cliff_1. An automorphism of an $s\mathbb{V}$-line is a functor defined by tensoring with a super line L; the complex line L can be even or odd. We refer to [**FT2**] for more details about super categories.

Remark C.1. The integral of the level of spin Chern-Simons theory $\mathscr{S}_{\mathbb{C}^\times}$ over a spin 2-manifold with flat $\mathbb{C}^\times$-connection lies in $E^1_{\mathbb{C}/\mathbb{Z}}$. The nonzero homotopy groups of $E^1_{\mathbb{C}/\mathbb{Z}}$ are $\pi_0 E^1_{\mathbb{C}/\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_1 E^1_{\mathbb{C}/\mathbb{Z}} \cong \mathbb{C}/\mathbb{Z}$; there is a nonzero k-invariant which connects them. An equivalent linear Picard groupoid is that of complex super lines, and so the value of $\mathscr{S}_{\mathbb{C}^\times}$ on a closed 2-manifold is a complex super line. Similarly, on a spin 1-manifold the relevant space is $E^2_{\mathbb{C}/\mathbb{Z}}$, whose nonzero homotopy groups are $\pi_1 E^2_{\mathbb{C}/\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_2 E^2_{\mathbb{C}/\mathbb{Z}} \cong \mathbb{C}/\mathbb{Z}$. An equivalent linear Picard 2-groupoid is that of *Bose* $s\mathbb{V}$-lines. In particular, we do not encounter Fermi $s\mathbb{V}$-lines. (We would have met Fermi $s\mathbb{V}$-lines had the level lain in the cohomology theory with an additional homotopy group $\mathbb{Z}/2\mathbb{Z}$.)

C.2. Spin flip. A Riemannian spin m-manifold is a Riemannian manifold M equipped with a principal Spin_m-bundle $\mathcal{B}_{\mathrm{Spin}}(M) \to M$ which lifts the orthonormal frame bundle $\mathcal{B}_{\mathrm{O}}(M) \to M$ under the homomorphism $\mathrm{Spin}_m \to \mathrm{O}_m$. A diffeomorphism $f\colon M' \to M$ of spin m-manifolds is an isometry of the underlying Riemannian manifolds together with a lift to the Spin_m-bundles. The *spin flip* Φ_M is the automorphism id_M with lift given as multiplication by the central element of Spin_m.

[42]As usual, we use 'super' as a synonym for '$\mathbb{Z}/2\mathbb{Z}$-graded'.

Let F be an n-dimensional field theory of spin manifolds—say nonextended—so defined on a bordism category of $(n-1)$- and n-manifolds. If Y is a closed spin $(n-1)$-manifold, then $F(Y)$ is a super vector space. We say that F *satisfies spin-statistics* if the spin flip Φ_Y acts on the super vector space $F(Y) = F(Y)^0 \oplus F(Y)^1$ as the grading automorphism $\mathrm{id}_{F(Y)^0} \oplus -\mathrm{id}_{F(Y)^1}$. Any F which is topological, invertible, and reflection positive satisfies spin-statistics [**FH1**, §11]. *Complex* Chern-Simons theories are not reflection positive, and in any case what we need is spin-statistics for the difference line of a triangle, which is defined in §6.5 via a combination of the $\mathbb{C}^\times$ spin Chern-Simons theory $\mathscr{S}_{\mathbb{C}^\times}$, the $\mathrm{SL}_2\mathbb{C}$ Chern-Simons theory $\mathscr{F}_{\mathrm{SL}_2\mathbb{C}}$, and the abelianization isomorphism (6.21). Although we cannot simply quote [**FH1**, §11] for what we need, the basic setup pertains, and so we review it briefly.

Let $\mathcal{C}$ be a symmetric monoidal category, and suppose $x \in \mathcal{C}$ is a dualizable object. Then

$$1 \xrightarrow{\text{coevaluation}} x \otimes x^* \xrightarrow{\text{symmetry}} x^* \otimes x \xrightarrow{\text{evaluation}} 1 \tag{C.2}$$

is by definition the dimension $\dim x$. For example, if $\mathcal{C} = s\mathbb{V}$ and $x = V^0 \oplus V^1$ is a finite dimensional super vector space, then $\dim x = \dim V^0 - \dim V^1$. If $f\colon x \to x$ is a morphism, then the composition

$$1 \longrightarrow x \otimes x^* \longrightarrow x^* \otimes x \xrightarrow{1\otimes f} x^* \otimes x \longrightarrow 1 \tag{C.3}$$

is by definition the trace $\mathrm{tr}\, f$. For $\mathcal{C} = s\mathbb{V}$ and f an (even) endomorphism of $V^0 \oplus V^1$, this categorical trace $\mathrm{tr}\, f = \mathrm{tr}\, f\big|_{V^0} - \mathrm{tr}\, f\big|_{V^1}$ is usually called the *supertrace*. For $\mathcal{C} = \mathrm{Bord}_{\langle n-1,n\rangle}(\mathrm{Spin})$ the spin bordism category and $x = Y$ a closed spin $(n-1)$-manifold with spin flip Φ_Y, we find

$$\begin{aligned} \dim Y &= S^1_{\text{nonbounding}} \times Y \\ \mathrm{tr}\, \Phi_Y &= S^1_{\text{bounding}} \quad\;\; \times Y \end{aligned} \tag{C.4}$$

where '(non)bounding' identifies the spin structure on S^1. If

$$F\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\mathrm{Spin}) \to s\mathbb{V}$$

is a topological field theory, then since F is a symmetric monoidal functor it maps traces to traces:

$$\mathrm{tr}\, F(\Phi_Y) = F(S^1_{\text{bounding}} \times Y). \tag{C.5}$$

If F is invertible, then spin-statistics holds for Y iff this equals $+1$; spin statistics holds for F iff this equals $+1$ for all Y.

C.3. Spin-statistics for the difference line $\mathcal{L}$. Next, recall the construction of the complex line $\mathcal{L} = \mathcal{L}(D, \epsilon, \sigma, \mathcal{A})$, defined in (6.28) in an equivalent description to what we give here. Let $D = D^2$ be the standard spin 2-disk; the spin structure is denoted σ. The edge orientations ϵ which appear on the triangle Δ in (6.28) give rise to a universal cover of $\partial D = S^1$; see 6.1.

The 2-disk D is equipped with its standard spectral network (Figure 25), and $\mathcal{A} = (P, Q, \mu, \theta)$ is stratified abelianization data (Definition 4.24).

The line $\mathcal{L}$ derives from three ingredients. First $G = \mathrm{SL}_2\mathbb{C}$ Chern-Simons theory $\mathscr{F}_G$ on D produces an $s\mathbb{V}$-line[43] $\mathscr{F}_G(\partial D)$ and an isomorphism of $s\mathbb{V}$-lines

$$s\mathbb{V} \xrightarrow[\cong]{\mathscr{F}_G(D)} \mathscr{F}_G(\partial D). \tag{C.6}$$

Let $c = D_{-2b}$ be the center point of D and let $\widetilde{D}' \to D \setminus \{c\}$ the double cover constructed from the spectral network on D. The spin structure σ lifts to a spin structure on the deleted 2-disk $\widetilde{D}'$; σ does *not* extend over the deleted point. The principal $\mathbb{Z}$-bundle derived from ϵ is used to twist this spin structure and the flat $\mathbb{C}^\times$-bundle over $\widetilde{D}'$—see §6.3—to obtain a spin structure $\tilde{\sigma}$ and a flat $\mathbb{C}^\times$-bundle over the filled-in 2-disk $\widetilde{D}$. The second ingredient in $\mathcal{L}$ is then the $\mathbb{C}^\times$-spin Chern-Simons invariant

$$s\mathbb{V} \xrightarrow[\cong]{\mathscr{S}_{\mathbb{C}^\times}(\widetilde{D})} \mathscr{S}_{\mathbb{C}^\times}(\partial\widetilde{D}). \tag{C.7}$$

The third ingredient is the isomorphism of $s\mathbb{V}$-lines

$$\mathscr{F}_G(\partial D) \xrightarrow[\cong]{\chi(\partial D)} \mathscr{S}_{\mathbb{C}^\times}(\partial\widetilde{D}) \tag{C.8}$$

constructed in §6.4. For now we simply note that $\chi(\partial D)$ is the composition of three isomorphisms; we dig into the details in §C.4 below. Finally, the line $\mathcal{L}$ is the composition

$$\mathcal{L} = \mathscr{S}_{\mathbb{C}^\times}(\widetilde{D})^{-1} \circ \chi(\partial D) \circ \mathscr{F}_G(D). \tag{C.9}$$

That is, the right hand side of (C.9) is an $s\mathbb{V}$-linear automorphism of $s\mathbb{V}$, hence it is tensoring with a line $\mathcal{L}$. Note that $\mathcal{L}$ is $\mathbb{Z}/2\mathbb{Z}$-graded; it may be even or odd.

Let Φ_D denote the spin flip of the spin 2-disk (D, σ). It induces the spin flip $\Phi_{\widetilde{D}}$ of $(\widetilde{D}, \tilde{\sigma})$, and these spin flips of 2-disks restrict to the spin flip of the boundary circles. In this way Φ_D induces an involution on each of (C.6), (C.7), and (C.8), hence too an involution $(\Phi_D)_*\colon \mathcal{L} \to \mathcal{L}$ on the line $\mathcal{L}$ which is their composition.

Proposition C.10. *$(\Phi_D)_* = \mathrm{id}_{\mathcal{L}}$ if $\mathcal{L}$ is even, and $(\Phi_D)_* = -\mathrm{id}_{\mathcal{L}}$ if $\mathcal{L}$ is odd.*

Proof. Define an invertible 1-dimensional field theory f of spin manifolds as follows. If M is a compact 0-dimensional spin manifold or a 1-dimensional spin bordism, then form the spin manifold $M \times D$ and equip it

[43]The $\mathrm{SL}_2\mathbb{C}$ Chern-Simons theory does not use the spin structure and it factors through ungraded linear objects, so $\mathscr{F}_G(\partial D)$ is canonically a $\mathbb{V}$-line. Here we base extend to $s\mathbb{V}$ for consistency with $\mathscr{S}_{\mathbb{C}^\times}$.

with the structure pulled back from $(\epsilon, \mathcal{A})$ on D. Then $f(M)$ is computed as the composition in (C.9) applied to $M \times -$:

$$f(M) = \mathscr{S}_{\mathbb{C}^\times}(M \times \widetilde{D})^{-1} \circ \chi(M \times \partial D) \circ \mathscr{F}_G(M \times D). \tag{C.11}$$

Note that $f(\mathrm{pt}) = \mathcal{L}$. By (C.5) the supertrace of $(\Phi_D)_*$ is

$$\operatorname{tr}(\Phi_D)_* = \operatorname{tr} f(\Phi_{\mathrm{pt}}) = f(S^1_{\mathrm{bounding}}). \tag{C.12}$$

Hence the proposition follows if we prove that f satisfies spin-statistics.

Compute $f(S^1_{\mathrm{bounding}})$ as the composition (C.11). Write $S^1_{\mathrm{bounding}} = \partial D^2$. The complex lines $\mathscr{F}_G(S^1_{\mathrm{bounding}} \times \partial D)$ and $\mathscr{S}_{\mathbb{C}^\times}(S^1_{\mathrm{bounding}} \times \partial \widetilde{D})$ have nonzero (basis) elements $\mathscr{F}_G(D^2 \times \partial D)$ and $\mathscr{S}_{\mathbb{C}^\times}(D^2 \times \partial \widetilde{D})$. Furthermore, since χ is a map of theories, the linear isomorphism

$$\chi(S^1_{\mathrm{bounding}} \times \partial D) \colon \mathscr{F}_G(S^1_{\mathrm{bounding}} \times \partial D) \longrightarrow \mathscr{S}_{\mathbb{C}^\times}(S^1_{\mathrm{bounding}} \times \partial \widetilde{D}) \tag{C.13}$$

maps $\mathscr{F}_G(D^2 \times \partial D)$ to $\mathscr{S}_{\mathbb{C}^\times}(D^2 \times \partial \widetilde{D})$. It remains to compute the ratio of each of the vectors $\mathscr{F}_G(S^1_{\mathrm{bounding}} \times D)$ and $\mathscr{S}_{\mathbb{C}^\times}(S^1_{\mathrm{bounding}} \times \widetilde{D})$ and these basis elements. The first ratio is $\mathscr{F}_G$ evaluated on

$$(S^1_{\mathrm{bounding}} \times D) \cup -(D^2 \times \partial D), \tag{C.14}$$

and the second ratio is $\mathscr{S}_{\mathbb{C}^\times}$ evaluated on

$$(S^1_{\mathrm{bounding}} \times \widetilde{D}) \cup -(D^2 \times \partial \widetilde{D}), \tag{C.15}$$

where the minus sign denotes the opposite spin structure (and reverse orientation). But each of these is a 3-sphere with the trivial connection, and so each partition function is $+1$. It follows that $f(S^1_{\mathrm{bounding}}) = +1$, as claimed. ☐

REMARK C.16. In the proof we encounter the 3-sphere S^3 in (C.14) with the following spectral network. The branch locus is an embedded $S^1 \subset S^3$. The walls are three disjoint embedded open 2-disks B^2, each of which has boundary $S^1 \subset S^3$.

C.4. The freedom to eliminate odd lines. The isomorphism χ is the composition of three isomorphisms: see Theorem 6.20. One of them, (6.22), is given by a canonical construction in Theorem 5.61. We fixed another, (6.24), in Corollary 5.103. The remaining isomorphism ν in (6.23) derives from Corollary 5.109, which we did not fix completely. Namely, since ν is an isomorphism of invertible 3-dimensional topological theories, we can tensor it with an invertible 2-dimensional theory to obtain a new isomorphism. The background fields of ν—listed at the beginning of §5.3.6, are: a spin structure σ, a flat H-connection, and a principal $\mathbb{Z}$-bundle which lifts the associated principal μ_2-bundle δ. Observe that the 2-dimensional invertible theory whose partition function is[44] $(-1)^{\mathrm{Arf}(\sigma+\delta)}$ takes value the *odd* line on S^1_{bounding}. Hence, possibly after tensoring ν with this shifted

[44]This is the Arf invariant (0 or 1) of the shifted spin structure $\sigma + \delta$.

Arf theory, we can arrange that $\mathcal{L}$ be an *even* complex line. It then follows from Proposition 4.32 that the spin flip acts as $\mathrm{id}_{\mathcal{L}}$, which is the claim in Proposition 6.29.

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