# Introduction to a deformed Hermitian Yang-Mills flow

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Abstract. The deformed Hermitian Yang-Mills equation is an important fully nonlinear geometric PDE. In this survey, we sketch first some developments on the deformed Hermitian Yang-Mills equation, and then introduce a deformed Hermitian Yang-Mills flow. We present some results in our paper [9] on this flow, including the longtime existence, the convergence under the subsolution condition, and as an application, the convergence on a Kähler surface case under the semisubsolution condition.

## 1. Introduction

The deformed Hermitian Yang-Mills (dHYM) equation which was founded by Mariño-Minasian-Moore-Strominger [20] and Leung-Yau-Zaslow [17] arises from mirror symmetry in string theory. The dHYM equation is a fully nonlinear elliptic PDE and the existence problem of this equation has been extensively studied in these recent years. One can see Collins-Xie-Yau [5] for the detailed mathematical and physical introduction of the dHYM equation.

Let $(M, \omega)$ be a compact Kähler manifold of complex dimension $n$ and $\chi$ a closed real $(1,1)$-form on $M$. Jacob-Yau [16] first studied the existence and uniqueness of solutions of the dHYM equation on $(M, \omega, \chi)$, which has the following form:

$$\mathrm{Re}(\chi_u + \sqrt{-1}\omega)^n = \cot\theta_0 \, \mathrm{Im}(\chi_u + \sqrt{-1}\omega)^n, \tag{1.1}$$

where $\chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u$ for a real smooth function $u$ on $M$ and $\theta_0$ is the argument of the complex number $\int_M (\chi + \sqrt{-1}\omega)^n$.

The dHYM equation is called supercritical if $\theta_0 \in (0, \pi)$ and hypercritical if $\theta_0 \in (0, \frac{\pi}{2})$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of $\chi_u$ with respect to $\omega$. If necessary we denote $\lambda$ by $\lambda(\chi_u)$ and $\lambda_i$ by $\lambda_i(\chi_u)$ for each $1 \le i \le n$. Let

$\lambda_i = \cot\theta_i$. Then

$$\begin{aligned}(\chi_u + \sqrt{-1}\omega)^n =& \prod_{i=1}^n (\lambda_i + \sqrt{-1})\,\omega^n \\ =& \frac{\exp(\sqrt{-1}\sum_{i=1}^n \theta_i)}{\prod_{i=1}^n \sin\theta_i}\,\omega^n \\ =& \frac{\cos(\sum_{i=1}^n \theta_i)}{\prod_{i=1}^n \sin\theta_i}\omega^n + \sqrt{-1}\frac{\sin(\sum_{i=1}^n \theta_i)}{\prod_{i=1}^n \sin\theta_i}\omega^n\end{aligned}$$

So the dHYM equation becomes

$$\cos\Big(\sum_{i=1}^n \theta_i\Big) = \cot\theta_0 \sin\Big(\sum_{i=1}^n \theta_i\Big),$$

or

$$\theta(\chi_u) = \theta_0, \tag{1.2}$$

if we define

$$\theta(\chi_u) := \sum_{i=1}^n \theta_i = \sum_{i=1}^n \operatorname{arccot}\lambda_i.$$

### 1.1. Some related studies on the dHYM equation.

1.1.1. *The elliptic case.* Jacob-Yau [**16**] solved the equation for $n = 2$ by writing the dHYM equation as a complex Monge-Ampère equation which was solved by Yau [**29**]. When $n \geq 3$, Collins-Jacob-Yau [**4**] solved the dHYM equation for the supercritical case by assuming the following two conditions hold:

(i) There exists a subsolution $\underline{u}$, which means $\chi_{\underline{u}}$ satisfies the inequality

$$A_0(\underline{u}) := \max_M \max_{1\leq j\leq n} \sum_{i\neq j} \operatorname{arccot}\lambda_i(\chi_{\underline{u}}) < \theta_0; \tag{1.3}$$

(ii) $\chi_{\underline{u}}$ also satisfies the inequality

$$B_0(\underline{u}) := \max_M \theta(\chi_{\underline{u}}) < \pi. \tag{1.4}$$

To be precise, Collins, Jacob and Yau proved the following

THEOREM 1.1 (Collins-Jacob-Yau [**4**]). *Let $(M, \omega)$ be a compact Kähler manifold of dimension $n$ and $\chi$ a closed real $(1,1)$ form on $M$ with $\theta_0 \in (0, \pi)$. Suppose there exists a subsolution $\underline{u}$ of dHYM equation (1.2) in the sense of (1.3) and $\underline{u}$ also satisfies inequality (1.4). Then there exists a unique smooth solution of dHYM equation (1.2).*

Without condition (1.4), the dHYM equation in the supercritical case was solved by Pingali [**22**] when $n = 3$ and by Lin [**19**] when $n = 4$. In these two cases, the dHYM equation was written as a mixed Monge-Ampère type equation and an appropriate continuity method was used. The 4 dimensional case solved by Lin 2022 was very complicated.

For the non-Kähler case, Lin [18] generalized Collins-Jacob-Yau's result to the Hermitian case $(M,\omega)$ with $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$. Huang-Zhang-Zhang [14] considered the solution on a compact almost Hermitian manifold for the hypercritical case and the supercritical case was solved by Huang-Zhang [13].

1.1.2. *The parabolic case.* For the parabolic flow method, there are also several results.

Jacob-Yau [16] and Collins-Jacob-Yau [4] proved the existence and convergence of the line bundle mean curvature flow (LBMCF)

$$\begin{cases} u_t = \theta_0 - \theta(\chi_u) \\ u(0) = u_0 \end{cases} \tag{1.5}$$

for the hypercritical case. Here they assumed the following:

(i) There exits a subsolution $\underline{u}$;
(ii) $u_0$ satisfies: $\theta(\chi_{u_0}) \in (0, \frac{\pi}{2})$.

Han-Jin [11] considered the stability result of the above flow.

Takahashi [26] proved the existence and convergence of the tangent Lagrangian phase flow:

$$\begin{cases} u_t = \tan\big(\theta_0 - \theta(\chi_u)\big) \\ u(0) = u_0 \end{cases} \tag{1.6}$$

for the hypercritical case. Here he assumed the following:

(i) There exits a subsolution $\underline{u}$;
(ii) $u_0$ satisfies: $\theta(\chi_{u_0}) - \theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

1.1.3. *Other related works.* There are two problems raised by Collins-Jacob-Yau [4]. One is whether condition (1.4) is superfluous. The other is to find a sufficient and necessary geometric condition on the existence of a solution to the dHYM equation.

Jacob and Shen [15] solved the dHYM equation on the blowup of $\mathbb{P}^n$ under an algebraic stability condition. There are some important progresses made by Chen [1] on the $J$-equation and the dHYM equation. Based on the work of Chen [1], Song [23] proved Nakai-Moishezon criterions for the $J$-equation. Recently, motivated by Chen [1] and Song [23], Chu-Lee-Takahashi [3] established the following

THEOREM 1.2 (Chu-Lee-Takahashi [3]). *The dHYM equation (1.2) on a compact Kähler manifold $(M,\omega)$ with complex dimension $n$ is solvable for the supercritical case if and only if there exists a Kähler metric $\gamma$ on $M$ such that for any $1 \le k \le n$,*

$$\int_M \big(\mathrm{Re}(\chi + \sqrt{-1}\omega)^k - \cot\theta_0 \mathrm{Im}(\chi + \sqrt{-1}\omega)^k\big) \wedge \gamma^{n-k} \ge 0$$

*and for any proper $m$-dimensional subvariety $Y$ of $M$ and $1 \le k \le m$,*

$$\int_Y \big(\mathrm{Re}(\chi + \sqrt{-1}\omega)^k - \cot\theta_0 \mathrm{Im}(\chi + \sqrt{-1}\omega)^k\big) \wedge \gamma^{m-k} > 0.$$

**1.2. A new dHYM flow and our results.** Motivated by the concavity of $\cot\theta(\chi_u)$ by Chen [1], we consider the following dHYM flow:

$$\begin{cases} u_t = \cot\theta(\chi_u) - \cot\theta_0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.7}$$

1.2.1. *The longtime existence and the convergence.* We first prove the longtime existence.

THEOREM 1.3. *Let $(M,\omega)$ be a compact Kähler manifold and $\chi$ be a closed real $(1,1)$ form with $\theta_0 \in (0,\pi)$. If $u_0$ satisfies: $\theta(\chi_{u_0}) < \pi$, then dHYM flow (1.7) has a unique smooth longtime solution $u$.*

Then we consider the convergence of the longtime solution of flow (1.7) when there exists a subsolution of the dHYM equation and the subsolution satisfies (1.4).

THEOREM 1.4. *Let $(M,\omega)$ be a compact Kähler manifold of dimension $n$ and $\chi$ be a closed real $(1,1)$ form with $\theta_0 \in (0,\pi)$. Assume dHYM equation (1.2) has a subsolution $\underline{u}$ in the sense of $A_0(\underline{u}) < \theta_0$ which also satisfies $B_0(\underline{u}) < \pi$. Then the longtime solution $u(x,t)$ of dHYM flow (1.7) converges to a smooth solution $u^\infty$ to the dHYM equation:*

$$\theta(\chi_{u^\infty}) = \theta_0.$$

Hence we reprove the Collins-Jacob-Yau's existence theorem [4]. Our proof looks like simpler than the one by Collins-Jacob-Yau.

The advantage of this new flow is that the imaginary part of the Calabi-Yau functional is constant along the flow. However, we are still subject to the condition $\theta(\chi_{u_0}) < \pi$.

Chu-Lee [2] used the twisted version of the above dHYM flow to study the equivalence of the coerciveness, properness of the J-functional and the existence of the solution of the hypercritical dHYM equation.

1.2.2. *The Kähler surface case under the semi-subsolution condition.* A smooth function $\underline{u}$ is called a semi-subsolution of the dHYM equation if

$$A_0(\underline{u}) \le \theta_0. \tag{1.8}$$

In the 2-dimensional case, this condition is equivalent to

$$\chi_{\underline{u}} \ge \cot\theta_0 \omega. \tag{1.9}$$

Assume there exists a semi-subsolution $\underline{u}$ of the dHYM equation. For simplicity, we assume $\underline{u} = 0$.

For any $B_1 \in (0,\pi)$, we define the set

$$\mathcal{H}_{B_1} = \{v \in C^\infty(M,\mathbb{R}) : \theta(\chi_v) \in (0,B_1)\}. \tag{1.10}$$

If $\theta_0 \in (0,\frac{\pi}{2})$, we have $0 \in \mathcal{H}_{B_1}$ for any $B_1 \in (2\theta_0,\pi)$. If $\theta_0 \in [\frac{\pi}{2},\pi)$, we can show that the semi-subsolution condition implies the non-empty of $\mathcal{H}_{B_1}$ for any $B_1 \in (\theta_0,\pi)$ (see Lemma 4.1).

We take the initial function $u_0 \in \mathcal{H}_{B_1}$ with $B_1 \in (2\theta_0,\pi)$ if $\theta_0 \in (0,\frac{\pi}{2})$ or $B_1 \in (\theta_0,\pi)$ if $\theta_0 \in [\frac{\pi}{2},\pi)$. Then we have the following result.

THEOREM 1.5. *Let $(M,\omega)$ be a compact Kähler surface and $\chi$ be a closed real $(1,1)$ form. Assume $\theta_0 \in (0,\pi)$ and $\chi \geq \cot\theta_0\omega$. There exist a finite number of curves $E_i$ of negative self-intersection on $M$ such that the solution $u(x,t)$ of dHYM flow (1.7) converges to a bounded function $u^\infty$ in $C^\infty_{loc}(M \setminus \cup_i E_i)$ satisfying the following*

(i) $\chi + \sqrt{-1}\partial\bar{\partial}u^\infty - \cot B_1\omega$ *is a Kähler current which is smooth on* $M\setminus \cup_i E_i$;

(ii) $u^\infty$ *satisfies the dHYM equation on* $M\setminus \cup_i E_i$

$$\operatorname{Re}(\chi_{u^\infty} + \sqrt{-1}\omega)^n = \cot\theta_0 \operatorname{Im}(\chi_{u^\infty} + \sqrt{-1}\omega)^n; \tag{1.11}$$

(iii) $\chi_{u(x,t)}$ *coverges to* $\chi_{u^\infty}$ *and* $u^\infty$ *satisfies* (1.11) *on* $M$ *in the sense of currents.*

Fang-Lai-Song-Weinkove [8] considered such type problem for the $J$-flow on a Kähler surface. By assuming $\theta_0 \in (0,\frac{\pi}{2})$ and $B_1 \leq \frac{\pi}{2} + \theta_0$, Takahashi [27] proved the similar result for LBMCF (1.5).

## 2. Preliminaries

**2.1. The linearized operator.** Note that

$$\cot\theta(\chi_u) = \frac{\operatorname{Re}(\chi_u + \sqrt{-1}\omega)^n}{\operatorname{Im}(\chi_u + \sqrt{-1}\omega)^n}. \tag{2.1}$$

LEMMA 2.1. *The linearized operator $\mathcal{P}$ of the dHYM flow has the form:*

$$\mathcal{P}(v) = v_t - F^{i\bar{j}}v_{i\bar{j}},$$

*where*

$$F^{i\bar{j}} = \csc^2\theta(\chi_u)\big(wg^{-1}w + g\big)^{i\bar{j}},$$

*where $g = (g_{i\bar{j}})_{n\times n}$, $w = (w_{i\bar{j}})_{n\times n}$ for $w_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$, and $D^{i\bar{j}} := (D^{-1})_{i\bar{j}}$ for an invertible Hermitian symmetric matrix $D$.*

**2.2. The concavity.** Define the function

$$\theta(\lambda) := \sum_{i=1}^{n} \operatorname{arccot}\lambda_i \quad \text{for } \lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n \tag{2.2}$$

and define the set

$$\Gamma_\tau := \{\lambda \in \mathbb{R}^n \mid \theta(\lambda) < \tau\} \subset \mathbb{R}^n \quad \text{for } \tau \in (0,\pi).$$

We have the following two useful lemmas.

LEMMA 2.2 (Yuan [30], Wang-Yuan [28]). *If $\theta(\lambda) \leq \tau \in (0,\pi)$ for $\lambda = (\lambda_1,\ldots,\lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then the following inequalities hold.*

(i) $\lambda_{n-1} \geq \cot\frac{\tau}{2}\,(>0)$;

(ii) $\lambda_{n-1} \geq |\lambda_n|$;

(iii) $\lambda_1 + (n-1)\lambda_n \geq 0$.

*Moreover, $\Gamma_\tau$ is convex for any $\tau < \pi$.*

LEMMA 2.3 (Chen [1]). *For any $\tau \in (0,\pi)$, the function $\cot\theta(\lambda)$ on $\Gamma_\tau$ is concave.*

**2.3. The parabolic subsolution.** Motivated by B. Guan's definition [10] of a subsolution of fully nonlinear equations, Székelyhidi [25] gave a weaker version of a subsolution and Collins-Jacob-Yau [4] used it to the dHYM equation which is equivalent to (1.3).

On the other hand, Phong-Tô [21] modified Székelyhidi's definition to the parabolic case. We use their definition to the dHYM flow.

DEFINITION 2.4. A smooth function $\underline{u}(x,t)$ on $M \times [0,T)$ is called a subsolution of the dHYM flow if there exists a constant $\delta > 0$ such that for any $(x,t) \in M \times [0,T)$, the subset of $\mathbb{R}^{n+1}$

$$\begin{aligned} S_\delta(x,t) :=& \big\{(\mu,\tau) \in \mathbb{R}^n \times \mathbb{R} \mid \mu_i > -\delta \text{ for each } i, \tau > -\delta, \text{and} \\ & \cot\theta\big(\lambda(\chi_{\underline{u}(x,t)}) + \mu\big) - \underline{u}_t(x,t) + \tau = \cot\theta_0\big\} \end{aligned}$$

is uniformly bounded.

We have the following observation.

LEMMA 2.5. *If $\underline{u}$ is a subsolution of the dHYM equation with $B_0(\underline{u}) < \pi$, then the function $\underline{u}(x,t) = \underline{u}(x)$ on $M \times [0,\infty)$ is also a subsolution of the dHYM flow.*

**2.4. The Calabi-Yau functional.** Recall the definition of the Calabi-Yau functional by Collins-Yau [6]: for any $v \in C^2(M,\mathbb{R})$,

$$\mathrm{CY}_{\mathbb{C}}(v) := \frac{1}{n+1}\sum_{i=0}^{n}\int_M v(\chi_v + \sqrt{-1}\omega)^i \wedge (\chi + \sqrt{-1}\omega)^{n-i}.$$

Let $v(s) \in C^{2,1}(M \times [0,T],\mathbb{R})$ be a variation of the function $v$, i.e., $v(0) = v$. By integration by parts, it holds

$$\frac{d}{ds}\mathrm{CY}_{\mathbb{C}}(v(s)) = \int_M \frac{\partial v(s)}{\partial s}\big(\chi_{v(s)} + \sqrt{-1}\omega\big)^n. \tag{2.3}$$

The $\mathcal{J}$-functional is defined by Collins-Yau [6] as follows

$$\mathcal{J}(v) := \mathrm{Im}\big(e^{-\sqrt{-1}\theta_0}\mathrm{CY}_{\mathbb{C}}(v)\big).$$

The $\mathcal{J}$-functional is very important since it is the Kempf-Ness functional for Collins-Yau's infinite dimensional GIT problem by Collins-Yau [6].

LEMMA 2.6. *Let $u(x,t)$ be a solution of the dHYM flow. Then*

$$\begin{aligned} &\mathrm{Im}\big(\mathrm{CY}_{\mathbb{C}}(u(\cdot,t))\big) = \mathrm{Im}\big(\mathrm{CY}_{\mathbb{C}}(u_0)\big), \\ &\frac{d}{dt}\mathrm{Re}\big(\mathrm{CY}_{\mathbb{C}}(u(\cdot,t))\big) = \int_M \Big(\frac{\partial u(t)}{\partial t}\Big)^2 \mathrm{Im}(\chi_u + \sqrt{-1}\omega)^n, \\ &\frac{d}{dt}\mathcal{J}(u(\cdot,t))) \leq 0. \end{aligned}$$

## 3. The longtime existence and convergence

**3.1. The longtime existence under the condition: $B_0(u_0) < \pi$.** We assume that $u$ is the solution of dHYM flow (1.7) in $M \times [0, T)$, where $T$ is the maximal existence time. Assuming $B_0(u_0) < \pi$, We will prove $T = \infty$.

3.1.1. *The $u_t$-estimate.* The maximum principle implies directly

LEMMA 3.1. *For any $(x,t) \in M \times [0,T)$, we have*

$$\min_M u_t|_{t=0} \leq u_t(x,t) \leq \max_M u_t|_{t=0}; \tag{3.1}$$

*in particular,*

$$0 < \min_M \theta(\chi_{u_0(x)}) \leq \theta(\chi_{u(x,t)}) \leq B_0(u_0) < \pi. \tag{3.2}$$

As a consequence of the above lemma, we have the following

LEMMA 3.2. *Let $\lambda_n(x,t)$ be the minimum eigenvalue of $\chi_u$ with respect to the metric $\omega$ at $(x,t)$. Then*

$$\max_{M\times[0,T)} |\lambda_n| \leq C_0 \quad \text{for} \quad C_0 := |\cot B_0| + \left|\cot\Big(\frac{\min_M \theta(\chi_{u_0})}{n}\Big)\right|.$$

3.1.2. *The longtime existence.* By the $u_t$-estimate, we have

$$\sup_{M\times[0,T)} |u| \leq CT + \sup_M |u_0|.$$

By the concavity of $\cot\theta(\lambda)$, we can prove the second order estimate and the higher order estimate. Then we get the longtime existence.

3.1.3. *The estimate of the real part of the Calabi-Yau functional.* Along the dHYM flow, we can prove that the real part of the Calabi-Yau functional can be controled by $|u|_{L^\infty}$ without the subsolution condition.

PROPOSITION 3.3. *Let $u(x,t)$ be a solution of dHYM flow (1.7) with the initial data satisfying (1.4). Then there exists a uniform constant $C$ such that*

$$\mathrm{Re}\,(\,\mathrm{CY}_{\mathbb{C}}(u)) \leq C|u|_{L^\infty}. \tag{3.3}$$

**3.2. The convergence under the subsolution condition and the condition (1.4).** To prove the convergence we need the existence of the subsolution and also the additional condition on the subsolution. The key point is to prove the uniform $C^2$-estimate which are independent of the time.

3.2.1. *The $C^0$ estimate.* A Harnack type inequality along the dHYM flow can be proved.

LEMMA 3.4. *Let $u$ be the solution of the dHYM flow on $M \times [0, \infty)$. Then for any $T_0 > 0$ we have the following Harnack type inequality:*

$$\sup_{M\times[0,T_0]} u(x,t) \leq C\Big(-\inf_{M\times[0,T_0]}\big(u(x,t) - u_0(x)\big) + 1\Big).$$

Now we prove the $C^0$ estimate similar as Phong-Tô [**21**].

PROPOSITION 3.5. *Along the dHYM flow, there exists a uniform constant $M_0$ independent of $T$ such that*

$$|u|_{C^0(M\times[0,\infty))} \le C.$$

3.2.2. *The gradient estimate.* The gradient estimate follows from the argument in the elliptic case by Collins-Yau [**6**].

PROPOSITION 3.6. *Let $u$ be the solution of dHYM flow* (1.7)*. There exists a uniform constant $C$ such that*

$$\max_{M\times[0,\infty)} |\nabla u|_\omega \le C.$$

3.2.3. *The second order estimate.* In the elliptic case, Collins-Jacob-Yau [**4**] used an auxiliary function containing the gradient term which modifies the one by Hou-Ma-Wu [**12**]. Our auxiliary function does not contain the gradient term.

PROPOSITION 3.7. *There exists a uniform constant $C$ such that*

$$\sup_{M\times[0,\infty)} |\partial\bar{\partial}u|_\omega \le C.$$

3.2.4. *The convergence.* We have got the uniform a priori estimates up to the second order. By the concavity of $\theta(\chi_u(x,t))$, we obtain the uniform $C^{2,\alpha}$ estimates and then the higher order estimates hold. Then the proof of the convergence is the similar as that by Phong-Tô [**21**]. Firstly, we can prove $u(x,t)$ converges exponentially to a function $u^\infty$. Then by the uniform $C^k$ estimates of $u(x,t)$ for all $k\in\mathbb{N}$, $u(x,t)$ converges to $u^\infty$ in $C^\infty$ and $u^\infty$ satisfies

$$\theta(\chi_{u^\infty}) := \sum_{i=1}^{n} \operatorname{arccot}\lambda_i(\chi_{u^\infty}) = \theta_0.$$

## 4. An application to the Kähler surface case under the semi-subsolution condition

If $\theta_0 \in [\frac{\pi}{2},\pi)$, we can show the nonempty of the set $\mathcal{H}_{B_1}$ for any $B_1 \in (\theta_0,\pi)$.

LEMMA 4.1. *Let $(M,\omega)$ be a compact Kähler surface. Assume $\chi \ge \cot\theta_0\omega$ and $\theta_0\in[\frac{\pi}{2},\pi)$. Then for any $B_1\in(\theta_0,\pi)$, there exists a smooth function $u$ such that $u\in\mathcal{H}_{B_1}$.*

To study the dHYM flow on the Kähler surface under the semi-subsolution condition, we first need a result of Song-Weinkove [**24**].

LEMMA 4.2. *Let $M$ be a Kähler surface with a Kähler class $\beta\in H^{1,1}(M,\mathbb{R})$. If $\alpha\in H^{1,1}(M,\mathbb{R})$ satisfies $\alpha^2>0$ and $\alpha\cdot\beta>0$, then either $\alpha$ is Kähler* *or there exists a positive integer $m$, curves of negative self-intersection $E_i$, $1 \leq i \leq m$ and positive numbers $a_i$, $1 \leq i \leq m$ such that*

$$\alpha - \sum_{i=1}^{m} a_i[E_i]$$

*is a Kähler class.*

Let $\tilde{\chi} = \chi - \cot\theta_0\omega$, we can check the following

$$[\tilde{\chi}]^2 = (1 + \cot^2\theta_0)[\omega]^2 > 0,$$
$$[\tilde{\chi}] \cdot [\omega] > 0.$$

By applying the above lemma, there exists a finite number (say $m \geq 1$) curves of negative self-intersection $E_i$ and $a_i > 0$, $1 \leq i \leq m$ such that $[\tilde{\chi}] - \sum_{i=1}^{m} a_i[E_i]$ is a Kähler class.

Let $h_i$ be the hermitian metric on $[E_i]$ and $s_i$ be a holomorphic section of $[E_i]$ which vanishes along $E_i$ to order 1. Define

$$S := \sum_{i=1}^{m} a_i \log|s_i|^2_{h_i},$$

then

$$\tilde{\chi} + \sqrt{-1}\partial\bar{\partial}S > 0. \tag{4.1}$$

**4.1. The $C^0$ estimate.** To prove the $C^0$ estimate of the solution along the flow, similar as Fang-Lai-Song-Weinkove [8] and Takahashi [27], we need the following lemma which was proved by Eyssidieux-Guedj-Zeriahi [7] and Zhang [31].

LEMMA 4.3. *Let $(M, \omega)$ be a compact Kähler surface, $\tilde{\chi} := \chi - \cot\theta_0\omega \geq 0$ and $\theta_0 \in (0, \pi)$. There exists a uniquely (by adding a constant) bounded $\tilde{\chi}$-PSH function $v$ on $M$ satisfying*

$$(\tilde{\chi} + \sqrt{-1}\partial\bar{\partial}v)^2 = \csc^2\theta_0\omega^2 \tag{4.2}$$

*in the sense of currents. Moreover, $v \in C^\infty_{loc}(M \setminus \cup_i E_i)$.*

The $u_t$ estimate has been proved in Section 3 and thus along the flow $\theta(\chi_u) \in (\min_M \theta(\chi_{u_0}), B_1)$.

PROPOSITION 4.4. *There exists uniform constant $M_0$ such that*

$$\sup_{M\times[0,\infty)} |u| \leq M_0. \tag{4.3}$$

The upper bound of $u$ follows from applying maximum principal to the following auxiliary function.

$$w_\varepsilon(x,t) := u - (1+\varepsilon)v + \varepsilon S - C_0\varepsilon t,$$

Similarly the lower bound of $u$ can be proved by

$$\tilde{w}_\varepsilon := u - (1-\varepsilon)v - \varepsilon S + C_0\varepsilon t.$$

Recall along the dHYM flow, $\mathrm{Re}(\mathrm{CY}_{\mathbb{C}}(u)) \leq C|u|_{L^\infty}$. By the $C^0$ estimate, we have the following.

Corollary 4.5. *Along the dHYM flow, there exists a uniform constant $C$ such that*

$$\mathrm{Re}(\mathrm{CY}_{\mathbb{C}}(u)) \leq C. \tag{4.4}$$

This estimate is used to prove $\lim\limits_{t\to\infty} u_t = 0$ on $M \setminus \cup_i E_i$.

Proposition 4.6. *For any compact set $K \subset M \setminus \cup_i E_i$, $u_t$ uniformly converges to $0$ in $K$ as $t$ tends to $\infty$.*

**4.2. The gradient estimate and the second order estimate.** To prove the gradient and second order estimate, we prove a useful inequality.

Lemma 4.7. *There exists uniform constants $K_0 > 0$ and $c_0 > 0$ such that if $|\lambda(\chi_u)| > K_0$, then*

$$u_t - F^{i\bar{j}}(u_{i\bar{j}} - S_{i\bar{j}}) \geq c_0.$$

Based on Lemma 4.7 and the $C^0$ estimate, we can prove the following gradient estimate.

Proposition 4.8. *There exist uniform constants $C$ and $D$ such that for any $(x, t) \in M \setminus \cup_i E_i \times [0, \infty)$*

$$|\nabla u|_\omega(x, t) \leq M_1 \prod_i |s_i|_{h_i}^{-D_1 a_i}(x). \tag{4.5}$$

Based on Lemma 4.7 and the $C^1$ estimate, we can derive the second order estimate.

Proposition 4.9. *There exist uniform constants $C$ and $D$ such that for any $(x, t) \in M \setminus \cup_i E_i \times [0, \infty)$*

$$|\partial\bar{\partial} u|_\omega(x, t) \leq C \prod_i |s_i|_{h_i}^{-2Da_i}(x, t). \tag{4.6}$$

Proposition 4.10. *For any compact set $K \subset M \setminus \cup_i E_i$ and positive integer $k$, there exists a uniform constant $C_{k,K}$ such that*

$$|u|_{C^k(K)} \leq C_{k,K}. \tag{4.7}$$

Then we can get the convergence of the flow locally in $M \setminus \cup_i E_i$.

Similarly as Fang-Lai-Song-Weinkove [**8**] and Takahashi [**27**], Theorem 1.5 follows.

**4.3. The lower bound of $\mathcal{J}$-functional in $\mathcal{H}_{B_1}$.** As an application of our dHYM flow, we have the lower bound of the $\mathcal{J}$-functional in the following set.

$$\mathcal{H}_{B_1} = \{w \in C^\infty(M, \mathbb{R}) : \theta(\chi_w) \in (0, B_1)\}.$$

COROLLARY 4.11. *Let $(M,\omega)$ be a compact Kähler surface and $\chi$ be a closed real $(1,1)$ form. Assume that $\theta_0 \in (0,\pi)$ and $\chi \geq \cot\theta_0\omega$, the $\mathcal{J}$-functional is bounded from below in $\mathcal{H}_{B_1}$ for any $B_1 \in (\theta_0,\pi)$.*

If $\theta_0 \in (0,\frac{\pi}{2})$, Takahashi [**27**] proved the lower bound of the $\mathcal{J}$-functional in $\mathcal{H}_{\theta_0+\frac{\pi}{2}}$.

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