Local scalar invariants of Kähler metric

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Dedicated to the memory of Professor Shiing-Shen Chern

ABSTRACT. On Kähler manifolds, the asymptotic coefficients of Bergman kernel, heat kernel and deformation quantization are local scalar invariants which are universal polynomials of jets of the Kähler metric. We show that they could be canonically expressed as a summation over directed graphs and the coefficients of these graphs are explicit graph invariants. The method should work for all geometrically meaningful local Kähler invariants. We survey the related works and give applications to heat coefficients of Kähler manifold. In particular, we show an explicit formula of the heat coefficients of $\mathbb{C}P^d$ as polynomials in d and present a heuristic approach to Chern-Gauss-Bonnet formula of Kähler manifold.

1. Introduction

The asymptotic expansion of the heat kernel and the Bergman kernel has found many applications in geometry, topology, analysis and mathematical physics. The coefficients in these asymptotic expansion are local invariants, i.e., universal polynomials of contractions of curvature tensors and their covariant derivatives, which encode important geometric information. For example, the inverse spectral problem aims at recovering geometry of a manifold from the spectrum of Laplacian, or equivalently the integrals of the heat coefficients. It was rephrased by M. Kac in 1966 as the famous question: "Can one hear the shape of a drum?"

It is well-known that in a normal coordinate system, the coefficients in the Taylor expansion of g_{ij} about the origin are polynomials in the curvature R_{ijkl} and its covariant derivatives. Conversely, R_{ijkl} and its covariant derivatives can be expressed as polynomials of partial derivatives of g_{ij}. In the case of Kähler manifold, when local scalar invariants are written in terms of partial derivatives of $g_{i\bar{j}}$, they could be neatly expressed as summations over directed graphs.

The computations for the first coefficients of the asymptotic expansion are usually quite involved. Although we have recursive ways of doing the calculation, it would be useful to have closed formulas and structural description of local Kähler invariants. These are important questions in subjects like CR invariant theory of Chern-Moser and Fefferman on strictly pseudoconvex domains, the asymptotic expansion of Bergman kernel, Yau's program on balanced conditions and Kähler-Einstein metrics, and local index theorem.

Asymptotic expansions are also ubiquitous in quantum theory. Deformation quantization provides rules for deforming the commutative algebra of classical observables to a noncommutative algebra of quantum observables. The input is a Poisson manifold M and the output is a noncommutative associative algebra, which is a formal deformation of the algebra of smooth functions on M. The quantization of Chern-Simons theory, which not only led to Witten's reinterpretation of Jones polynomial and Kontsevich integral for knot invariant, but also it describes quantum Hall effect which is the cornerstone of modern condensed matter physics. Asymptotic expansions play important roles in these theories.

Prof. Chern has done many foundational works in complex and Kähler geometry, as can be seen from the following description of the paper, which covers only a small part of Chern's important works.

The paper is organized as follows: In §2, we study asymptotic expansions of weighted Bergman kernels on a polarized compact Kähler manifold or a strictly pseudoconvext domain equipped with a Kähler potential. For the definition, we need Chern connection of holomorphic Hermitian vector bundles.

In §3, we describe how to use graph to represent local Kähler invariants and give closed formulas for the asymptotic coefficients of the weighted Bergman kernel.

In §4, we discuss CR invariant theory of Chern-Moser and Fefferman for a strictly pseudoconvex domain. It could be used to express the coefficients of the expansion of the Bergman kernel. We give a graph-theoretic formula to do the calculation.

In §5, we discuss deformation quantization on Kähler manifolds. In particular, we present a graph theoretic formula for the Berezin star product.

In §6, we study the structure of heat coefficients of Kähler manifold and present a heuristic approach to the Chern-Gauss-Bonnet formula via graph-theoretic point of view, Patodi's formula and the local index theorem.

2. Local and global weighted Bergman kernels

Weighted Bergman kernel can be defined either on a polarized compact Kähler manifold or a complex domain equipped with a Kähler potential.

Let E be a holomorphic vector bundle on a complex manifold M and h a Hermitian metric on E. Then a connection ∇ on E is said to be compatible with h if for any two smooth sections ξ, η of E and a smooth vector field X

on M, we have

$$Xh(\xi,\eta) = h(\nabla_X \xi,\eta) + h(\xi,\nabla_X \eta).$$

A connection ∇ is said to be compatible with the complex structure of E if $\nabla'' = \bar\partial$, or equivalently if the connection matrix ω of ∇ consists of $(1,0)$-forms with respect to any local holomorphic frame.

LEMMA 2.1. *Let E be a holomorphic vector bundle on a complex manifold M and h a Hermitian metric on E. Then there is a unique connection that is compatible with both the metric h and the complex structure of E. Such a connection is called Chern connection or Hermitian connection. Moreover its connection and curvature matrices are given by*

$$\omega = \partial H \cdot H^{-1}, \qquad \Omega = \bar\partial(\partial H \cdot H^{-1})$$

where $H = (H_{i\bar j}) = (H(e_i, e_j))$ is the matrix of the metric h under a local holomorphic frame $\{e_1, \ldots, e_r\}$.

PROOF. See [78] for a proof. We only give a proof of the formula of Ω which was missing in [78].

$$\begin{aligned}
\Omega &= d\omega - \omega\wedge\omega \\
&= d(\partial H\cdot H^{-1}) - (\partial H\cdot H^{-1})\wedge(\partial H\cdot H^{-1}) \\
&= \bar\partial\partial H\cdot H^{-1} + \partial H\cdot H^{-1}\wedge dH\cdot H^{-1} - \partial H\cdot H^{-1}\wedge \partial H\cdot H^{-1} \\
&= \bar\partial\partial H\cdot H^{-1} + \partial H\cdot H^{-1}\wedge \bar\partial H\cdot H^{-1} \\
&= \bar\partial(\partial H\cdot H^{-1})
\end{aligned}$$

where in the second equation we used $d(H^{-1}) = -H^{-1}dH\cdot H^{-1}$. □

REMARK 2.2. Yang-Zheng [72] proved that on a Hermitian manifold (M, g), the Chern connection and the Levi-Civita connection are equal if and only (M, g) is Kähler.

Let M be a projective algebraic manifold. A polarization on M is the assignment of an ample line bundle L on M. A Kähler metric g is called a polarized metric, if the corresponding Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi}\sum_{i,j=1}^{n} g_{i\bar j}\, dz_i\wedge dz_{\bar j}$$

represents the first Chern class $c_1(L)$ of L in $H^2(M,\mathbb{Z})$. Given any polarized Kähler metric g, there is a Hermitian metric h on L whose curvature form is equal to ω_g. This last assertion could be proved as follows [78]. By Lemma 2.1, for any Hermitian metric h on L, its curvature form is given by

$$\Omega^h = -\partial\bar\partial \log |s|_h^2,$$

where s is a local nowhere vanishing holomorphic section of L. Then $\frac{\sqrt{-1}}{2\pi}\Omega^h$ represents $c_1(L)$ in $H^2(M,\mathbb{Z})$. Since ω_g also represents $c_1(L)$, by $\partial\bar\partial$-Lemma, there exists a real-valued function f on M such that $\frac{\sqrt{-1}}{2\pi}\Omega^h - \omega_g = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f$. Hence

$$\omega_g = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(e^f|s|_h^2) = \frac{\sqrt{-1}}{2\pi}\Omega^{\tilde{h}},$$

where $\tilde{h}$ is the Hermitian metric on L defined by $\tilde{h}(s,s) = e^f h(s,s)$.

For each $m \in \mathbb{N}$, h induces a Hermitian metric h_m on L^m. Let $\{S_1, \dots, S_d\}$ be an orthonormal basis of $H^0(M, L^m)$ with respect to the inner product

$$(S_i, S_j)_{h_m} = \int_M h_m(S_i(x), S_j(x))\frac{1}{n!}\omega_g^n.$$

The Bergman kernel is defined by

$$K_m(x,y) = \sum_{j=1}^{d} h_m(S_j(x), S_j(y)). \tag{1}$$

For any given $m \geq 1$, one can define a holomorphic map

$$\phi_m : M \to \mathbb{C}P^d, \qquad z \mapsto [S_1(z), \dots, S_d(z)]. \tag{2}$$

Theorem 2.3. *Let g_{FS} be the Fubini-Study metric on $\mathbb{C}P^d$. Then for any k,*

$$\| \phi_m^*(g_{FS}) - g \|_{C^k} = O\left(\frac{1}{m}\right), \quad m \to \infty. \tag{3}$$

The pull-back metric $\phi_m^*(g_{FS})$ is called the Bergman metric induced by L. The above theorem arose out of a question of Yau [74]: whether a Kähler-Einstein metric on M can be the limit of a sequence of Bergman metrics induced by pluricanonical line bundles K_M^m. The C^2 convergence was proved by Tian [64]. The C^∞ convergence was proved by Ruan [58].

Theorem 2.3 is a corollary of the following Tian-Yau-Zelditch asymptotic expansion of $K_m(x) := K_m(x,x)$ proved independently by Zelditch [75] and Catlin [10].

Theorem 2.4. *When $m \to \infty$,*

$$K_m(x) = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots, \tag{4}$$

where $a_0(x) = 1$. More precisely, for any $k, \mu \geq 0$,

$$\| K_m(x) - \sum_{j=0}^{k} a_j(x)m^{n-j} \|_{C^\mu} \leq C_{k,\mu} m^{n-k-1},$$

where $C_{k,\mu}$ depends on k, μ and the manifold M.

Different proofs and generalizations can be found in [7, 16, 36, 44, 48, 63]. The coefficient a_k for $k \leq 3$ were computed by Lu [47] using peak section method. The above theorem and the coefficient $a_1 = \frac{1}{2}\rho$ have important applications in Donaldson's breakthrough work [17] on Yau's conjecture that the existence of extremal metrics is equivalent to the stability of manifolds.

Some recent work around Theorem 2.4 include: Arezzo-Loi-Zudda [3] studied balance metrics in relation to the coefficients a_n. Feng-Tu [25] studied geometry of Cartan-Hartogs domains when a_2 is constant. Alexakis-Hirachi [1] proved a structure theorem of a_n as an application of their work on local Kähler invariants. Shiffman [62] derived an asymptotic expansion for the variance of the zero sets of random holomorphic sections on a compact Kähler manifold.

A physical derivation of Theorem 2.4 was given by Douglas and Klevtsov [18] using path integral and perturbation theory. Klevtsov [41] and Can-Laskin-Wiegmann [9] studied the relations between Bergman kernel expansion and quantum Hall effect.

Now we consider the weighted Bergman kernel in a local domain. Let Ω be a domain in $\mathbb{C}^n$ with a Kähler potential $\Phi(x)$, i.e., $g_{i\bar{j}} := \partial_i \partial_{\bar{j}} \Phi$ is a Kähler metric. Let $\Phi(x,y)$ be an almost analytic extension of $\Phi(x)$ to a neighborhood of the diagonal, i.e., $\bar{\partial}_x \Phi$ and $\partial_y \Phi$ vanish to infinite order for $x = y$. We may assume $\overline{\Phi(x,y)} = \Phi(y,x)$.

For $\alpha > 0$, consider the weighted Bergman space of all holomorphic function on Ω square-integrable with respect to the measure $e^{-\alpha\Phi} \frac{w_g^n}{n!}$ and denote by $K_\alpha(x,y)$ the reproducing kernel. Locally, it is often the case that $K_\alpha(x,y)$ has an asymptotic expansion in a small neighborhood of the diagonal when $\alpha \to \infty$,

$$K_\alpha(x,y) = e^{\alpha\Phi(x,y)} \sum_{k=0}^{\infty} B_k(x,y) \alpha^{n-k} \tag{5}$$

uniformly on compact subsets. For instance, the asymptotic expansion has been established by Berezin [6] for bounded symmetric domains and by Engliš [20] for bounded strictly pseudoconvex domains with real analytic boundary. The coefficients B_k were computed by Engliš [19] for $k \le 3$ by a recursive formula of B_k derived from the asymptotics of Laplace integrals. It was proved by Loi [46] (cf. [67, §3]) that coefficients of the asymptotic expansion of local and global Bergman kernels are equal, i.e., $B_k = a_k$, $k \ge 0$. Recall Loi's formula [46] (which is a refinement of Engliš' formula),

$$B_k(x) = -\sum_{\substack{i+j=k \\ i,j\ge 1}} B_i(x) B_j(x) - \sum_{\substack{\ell+i+j=k \\ 1\le \ell \le k}} R_\ell(B_i(x,y) B_j(y,x))|_{y=x}, \tag{6}$$

where $R_j : C^\infty(\Omega) \to C^\infty(\Omega)$ are explicit differential operators defined by

$$R_j f(x) = \frac{1}{\det g} \sum_{k=j}^{3j} \frac{1}{k!(k-j)!} L^k (f \det g S^{k-j})|_{y=x}, \tag{7}$$

where L is the (constant-coefficient) differential operator

$$Lf(y) = g^{i\bar{j}}(x) \partial_i \partial_{\bar{j}} f(y)$$

and the function $S(x,y)$ satisfies

$$S = \partial_\alpha S = \partial_{\alpha\beta} S = \partial_{i_1 i_2 \dots i_m} S = \partial_{\bar{i}_1 \bar{i}_2 \dots \bar{i}_m} S = 0 \quad \text{at } y = x,$$
$$\partial_{i\bar{j}\alpha_1\alpha_2\dots\alpha_m} S|_{y=x} = -\partial_{\alpha_1\alpha_2\dots\alpha_m} g_{i\bar{j}}(x).$$

Here indices $i, j, k, \dots$ run from 1 to n and Greek indices α, β, γ may represent either i or $\bar{i}$.

3. Graph theoretic formulas of the Bergman kernel asymptotics

First we introduce some notions of graph theory. A *digraph* $G = (V, E)$ is defined to be a finite directed multigraph which is permitted to have multi-edges and loops. Here V and E are the sets of vertices and edges respectively. The weight $w(G)$ of G is defined to be $|E| - |V|$. The adjacency matrix $A = A(G)$ of a digraph G with n vertices is a square matrix of order n whose entry A_{ij} is the number of directed edges from vertex i to vertex j. The *outdegree* $\deg^+(v)$ and *indgree* $\deg^-(v)$ of a vertex v are defined to be the number of outward and inward edges at v respectively.

A vertex v of G is called *stable* if $\deg^-(v) \geq 2$, $\deg^+(v) \geq 2$. We call G *stable* if each vertex of G is stable.

A vertex v of G is called *semistable* if $\deg^-(v) \geq 1$, $\deg^+(v) \geq 1$ and $\deg^-(v) + \deg^+(v) \geq 3$. We call G *semistable* if each vertex of G is semistable.

A digraph G is *strongly connected* if there is a directed path from each vertex in G to every other vertex. We call G *quasi-strong* if each connected component of G is strongly connected.

Thanks to the Kähler condition $\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}$ and $\partial_{\bar{l}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{l}}$, we can canonically associate a polynomial in the variables $\{g_{i\bar{j}\,\alpha}\}_{|\alpha|\geq 1}$ to a stable digraph G, such that each vertex represents a partial derivative of $g_{i\bar{j}}$ and each edge represents the contraction of a pair of barred and unbarred indices.

In [**67**], we proved the following closed formula for asymptotic coefficients of the weighted Bergman kernel.

THEOREM 3.1 ([**67**]). *For $k \geq 0$,*

$$B_k = \sum_{G:\, w(G)=k}^{\text{quasi-strong stable}} \frac{(-1)^{n(G)} \det(A(G) - I)}{|\mathrm{Aut}(G)|}, \tag{8}$$

where G runs over quasi-strong stable digraphs of weight k and $n(G)$ is the number of components of G.

The above theorem was inspired by Feynman diagram formulas in deformation quantization (cf. §5). The proof began with a graph-theoretic interpretation of Engliš' work on asymptotic expansion of Laplace integrals and Loi's formula (6). A key observation is that a graph with a non-strongly-connected component will cancel out. Finally we used the coefficient theorem from spectral graph theory to get coefficients for quasi-strong graphs.

Recall that at each point x on a Kähler manifold, there exists a normal coordinate system such that at x the Kähler metric satisfies

$$g_{i\bar{j}}(x) = \delta_{ij}, \qquad g_{i\bar{j}k_1\ldots k_r}(x) = g_{i\bar{j}\bar{l}_1\ldots\bar{l}_r}(x) = 0$$

for all $r \leq N \in \mathbb{N}$, where N can be chosen arbitrary large.

The curvature tensor is given by

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}k\bar{l}} + g^{m\bar{p}} g_{m\bar{j}\bar{l}} g_{i\bar{p}k}. \tag{9}$$

The covariant derivative of a covariant tensor field $T_{\beta_1\ldots\beta_p}$ is defined by

$$T_{\beta_1\ldots\beta_p/\gamma} = \partial_\gamma T_{\beta_1\ldots\beta_p} - \sum_{i=1}^{p} \Gamma^{\delta}_{\gamma\beta_i} T_{\beta_1\ldots\beta_{i-1}\delta\beta_{i+1}\ldots\beta_p}, \tag{10}$$

where the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma} = 0$ except for

$$\Gamma^{i}_{jk} = g^{i\bar{l}} g_{j\bar{l}k}, \quad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = g^{l\bar{i}} g_{l\bar{j}\bar{k}}.$$

The identities (9) and (10) can be used to convert partial derivatives of metrics in (8) to covariant derivatives of curvature tensors and vice versa. See [**70, 71**] for more discussions, where some closed formulas were derived.

EXAMPLE 3.2. We represent a digraph as a weighted graph. The weight of a directed edge is the number of multi-edges. The number attached to a vertex denotes the number of its self-loops. A vertex without loops will be denoted by a small circle $\circ$.

There is only one quasi-strong digraph with weight 1. By (8) and (9),

$$B_1 = -\frac{1}{2}\left[\textcircled{2}\right] = -\frac{1}{2} g_{i\bar{i}j\bar{j}} = \frac{1}{2} R_{i\bar{i}j\bar{j}} = \frac{1}{2}\rho, \tag{11}$$

where $(i, \bar{i})$, $(j, \bar{j})$ are paired indices to be contracted.

There are four quasi-strong digraphs with weight 2. By (8) we have

$$\begin{aligned} B_2 =& -\frac{1}{3}\left[\textcircled{3}\right] + \frac{1}{2}\left[\textcircled{1} \underset{1}{\overset{1}{\rightleftarrows}} \textcircled{1}\right] + \frac{3}{8}\left[\circ \underset{2}{\overset{2}{\rightleftarrows}} \circ\right] + \frac{1}{8}\left[\textcircled{2} \mid \textcircled{2}\right] \\ =& -\frac{1}{3} g_{i\bar{i}j\bar{j}k\bar{k}} + \frac{1}{2} g_{i\bar{i}k\bar{l}} g_{j\bar{j}l\bar{k}} + \frac{3}{8} g_{i\bar{j}k\bar{l}} g_{j\bar{i}l\bar{k}} + \frac{1}{8} g_{i\bar{i}j\bar{j}} g_{k\bar{k}l\bar{l}} \\ =& \frac{1}{3}\Delta\rho + \frac{1}{24}|R|^2 - \frac{1}{6}|Ric|^2 + \frac{1}{8}\rho^2. \end{aligned}$$

The last equation used the identities $g_{i\bar{j}k\bar{l}} = -R_{i\bar{j}k\bar{l}}$ and

$$\begin{aligned} g_{i\bar{i}j\bar{j}k\bar{k}} &= -R_{i\bar{i}j\bar{j};k\bar{k}} + R_{k\bar{i}s\bar{j}} R_{i\bar{k}j\bar{s}} + R_{j\bar{j}s\bar{i}} R_{k\bar{k}i\bar{s}} + R_{i\bar{i}s\bar{j}} R_{k\bar{k}j\bar{s}} \\ &= -\Delta\rho + |R|^2 + 2|Ric|^2, \end{aligned}$$

Note that the tour de force computations of B_3 (containing 13 terms) by Lu [**47**] and Engliš [**19**] occupy more than ten pages in both papers. The computation of B_3 using the graph-theoretic formula (8) is much shorter (see [**67**]).

4. CR invariant and Bergman kernel

Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. If $r \in C^\infty(\overline{\Omega})$ is a defining function in the sense $\Omega = \{r > 0\}$ with $dr \neq 0$ on $\partial\Omega$, Fefferman [22] showed that the boundary singularity of the Bergman kernel has the form

$$K(z) = \frac{n!}{\pi^n}\left(\frac{\varphi(z)}{r(z)^{n+1}} + \psi(z)\log r(z)\right), \qquad \varphi, \psi \in C^\infty(\overline{\Omega}). \tag{12}$$

A weighted analogue of Fefferman's expansion was obtained by Engliš [20].

Hörmander [35] proved that $r(z)^{n+1}K(z) \to \frac{n!}{\pi^n}J[r](z_0)$ as $z \to z_0 \in \partial\Omega$, where $J[r]$ denotes the complex Monge-Ampère operator defined by

$$J[r] = (-1)^n \det \begin{pmatrix} r & \partial r/\partial \bar{z}_j \\ \partial r/\partial z_i & \partial^2 r/\partial z_i \partial \bar{z}_j \end{pmatrix}_{1\leq i,j\leq n}. \tag{13}$$

By (12), we have $\varphi = J[r]$ on $\partial\Omega$. Starting from an arbitrary smooth defining function of Ω, Fefferman [23] devised a recursive algorithm to explicitly construct another defining function $r^F \in C^\infty(\overline{\Omega})$ satisfying

$$J[r^F] = 1 + O^{n+1}(r^F), \quad r^F > 0 \text{ in } \Omega, \quad r^F|_{\partial\Omega} = 0, \tag{14}$$

where $O^{n+1}(r^F)$ denotes a term of the form $(r^F)^{n+1}f$ with $f \in C^\infty(\overline{\Omega})$.

Let r^F be a Fefferman's defining function of Ω. Define a Lorentz-Kähler metric

$$g = \sum_{0\leq i,j\leq n} \frac{\partial^2 r_\#}{\partial z_i \partial \bar{z}_j}\, dz_i\, d\bar{z}_j, \quad r_\#(z_0, z) = |z_0|^2 r^F(z) \text{ on } \mathbb{C}^* \times \overline{\Omega},$$

called Fefferman's ambient metric associated with $\partial\Omega$. From the curvature tensor R of g, Fefferman [24] constructed Weyl invariants by complete contractions of covariant derivatives $R^{(p,q)} := R_{a_1\bar{b}_1 a_2 \bar{b}_2/a_3\cdots a_p \bar{b}_3 \cdots \bar{b}_q}$, e.g. the following Weyl invariant

$$W_\# = \mathrm{contr}(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}), \tag{15}$$

is defined to be of weight $\sum_{j=1}^s (p_j + q_j)/2 - s$ and gives rise to a function $W = W_\#|_{z_0=1}$ on $\overline{\Omega}$. Fefferman proposed a program [24] to express φ, ψ in (12) as linear combinations of Weyl invariants W_k of weight k such that

$$\varphi = \sum_{k=0}^{n} W_k\, r^k + O^{n+1}(r), \quad \psi = \sum_{k=0}^{\infty} W_{k+n+1}\, r^k + O^\infty(r), \tag{16}$$

where $O^\infty(r)$ means that $\psi = \sum_{k=0}^m W_{k+n+1}[r]\, r^k + O^{m+1}(r)$ for any $m \geq 0$. The expansion of φ was proved by Fefferman [24] and Bailey et al. [4] for any Fefferman's defining function $r = r^F$. The expansion of ψ was proved by Hirachi [32] for more refined defining functions than (14).

The restriction of W_k to $\partial\Omega$ gives a CR invariant of weight k, which can be defined using Moser normal form in analogy to the normal coordinate system in Riemannian geometry. Let $(z', z_n) = (z_1, \ldots, z_n) \in \mathbb{C}^n$. A hypersurface $0 \in \partial\Omega \subset \mathbb{C}^n$ with local equation

$$2u = |z'|^2 + \sum_{|\alpha|,|\beta|\geq 2,\, k\geq 0} A^k_{\alpha\bar\beta}(v) v^k z'_\alpha \bar z'_\beta, \qquad z_n = u + iv \tag{17}$$

is said to be in Moser normal form if the coefficients $A^k_{\alpha\bar\beta}$ satisfy:

(i) $A^k_{\alpha\bar\beta} = \overline{A^k_{\beta\bar\alpha}}$;
(ii) $\operatorname{tr}(A_{2\bar 2}) = 0$, i.e. $\sum_{p=1}^{n-1} A^k_{pi\bar p\bar j} = 0$ for all k, i, j;
(iii) $\operatorname{tr}(A_{2\bar 3}) = 0$, i.e. $\sum_{p,q=1}^{n-1} A^k_{pq\bar p\bar q\bar j} = 0$ for all k, j;
(iv) $\operatorname{tr}(A_{3\bar 3}) = 0$, i.e. $\sum_{p,q,r=1}^{n-1} A^k_{pqr\bar p\bar q\bar r} = 0$ for all k.

A celebrated theorem of Chern and Moser [**15**] says that any real analytic hypersurface may be placed in Moser normal form through a biholomorphic map in a neighborhood of 0.

Definition 4.1 ([**24, 29, 34**]). Denote by $N(A^k_{\alpha\bar\beta})$ a real hypersurface in normal form (17). A polynomial P in variables $A^k_{\alpha\bar\beta}$ is said to be a CR invariant of weight $w \in \mathbb{N}_{\geq 0}$ if it satisfies the transformation law $P(A^k_{\alpha\bar\beta}) = |\det \Phi'(0)|^{2w/(n+1)} P(B^k_{\alpha\bar\beta})$ for any biholomorphic mapping $\Phi : N(A^k_{\alpha\bar\beta}) \to N(B^k_{\alpha\bar\beta})$ preserving the origin.

Let I_w denote the set of CR invariants of weight w. Then every $P \in I_w$ is a homogeneous polynomial of weight w if we define the weight of $A^k_{\alpha\bar\beta}$ to be $(|\alpha| + |\beta|)/2 + k - 1$. Graham [**29**] proved the following:

Theorem 4.2 ([**29**]). *(i) Let $n = 2$. Then $I_1 = I_2 = \{0\}$ and $\dim I_3 = \dim I_4 = 1$. Moreover, I_3 and I_4 are respectively spanned by $A^0_{4\bar 4}$ and $|A^0_{2\bar 4}|^2$.*

(ii) Let $n \geq 3$. Then $I_1 = \{0\}$ and $\dim I_2 = 1$. Moreover, I_2 is spanned by $\|A^0_{2\bar 2}\|^2 = \sum |A^0_{\alpha\bar\beta}|^2$, where the summation runs over $|\alpha| = |\beta| = 2$.

When $n = 2$, a basis of $\dim I_5 = 2$ has been determined in [**29, 34**] and a basis of $\dim I_6 = 3$ has been determined by Hirachi [**33**].

For the Dirichlet problem of the complex Monge-Ampère equation

$$J[u] = 1, \quad u > 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{18}$$

Cheng-Yau [**13**] proved that there exists a unique solution $u \in C^\infty(\Omega) \cap C^{n+3/2-\epsilon}(\overline{\Omega})$ for any $\epsilon > 0$ which implies that Ω admits a unique complete Einstein-Kahler metric with scalar curvature -1. Cheng-Yau's original proof requires that the boundary $\partial\Omega$ is C^2, which was later dropped by Mok-Yau [**51**]. Lee-Melrose [**43**] proved that for any smooth defining function r, Cheng-Yau's solution has an asymptotic expansion

$$u \sim r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k, \quad \eta_k \in C^\infty(\overline{\Omega}), \tag{19}$$

which implies that $u \in C^{n+2-\epsilon}(\overline{\Omega})$ for any $\epsilon > 0$ improving Cheng-Yau's estimate. However, the solution to (18) is not C^∞ smooth up to the boundary, so we have to use Fefferman's defining function r^F when studying the invariant expansions (16).

Let us fix $r = r^{\mathrm{F}}$ and $a \in C^\infty(\partial\Omega)$ locally near $0 \in \partial\Omega$. Then Graham [**30**] proved that there exists a unique formal series u of the form (19) satisfying

$$J[u] = 1 + O(r^\infty), \qquad \eta_0 = 1 + ar^{n+1} + O(r^{n+2}) \tag{20}$$

near $0 \in \partial\Omega$. For any $k \geq 1$, η_k modulo $O(r^{n+1})$ is independent of $r = r^{\mathrm{F}}$ and a. Each $\eta_k|_{\partial\Omega}$ modulo $O(r^{n+1})$ is a CR invariant of weight $k(n+1)$. From Theorem 4.2, Graham [**29**] proved that:

Theorem 4.3 ([**29**]). *(i) Let $n = 2$. Then $\eta_1 = 4A^0_{4\bar{4}}$ and the singularity of the Bergman kernel* (12) *has the expansions*

$$\varphi = 1 + O(r^3), \qquad \psi = -3\eta_1 + c|A^0_{2\bar{4}}|^2 r + O(r^2), \tag{21}$$

where c is a constant independent on Ω.

(ii) Let $n \geq 3$. There is a constant c_n depending only on n such that

$$\varphi = 1 + c_n \|A^0_{2\bar{2}}\|^2 r^2 + O(r^3). \tag{22}$$

Note that Theorem 4.3 was used in Huang and Xiao's proof [**37**] of Cheng's conjecture that the Bergman metric of a smoothly bounded strictly pseudoconvex domain is Kähler-Einstein if and only if the domain is biholomorphic to the ball. Hirachi, Komatsu and Nakazawa [**34**] gave two different methods of identifying the above universal constants.

Theorem 4.4 ([**34**]). *The constants in* (21) *and* (22) *are given by $c = 24/5$ and $n(n-1)c_n = 2/3$.*

One of their proofs used an explicit asymptotic expansion for Reinhardt domains. In the rest of the section, assume that $\Omega \subset \mathbb{C}^n$ is a bounded strictly pseudoconvex complete Reinhardt domain. Its logarithmic real representation domain is given by

$$-\log|\Omega| = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid (e^{-x_1}, \ldots, e^{-x_{n-1}}, e^{-y}) \in \Omega\}.$$

First we assume $n = 2$. Let $f(x) := \inf\{y \in \mathbb{R} \mid (x, y) \in -\log|\Omega|\}$. Then $\lambda = y - f(x) (> 0)$ is a defining function of $\partial\Omega \cap \{z_1 z_2 \neq 0\}$. We make change of variables $(x, y) \to (\lambda, v)$ with $v = f'(x)$ and set $p(v) = f''(x)$, the hodograph transformation. We have the following asymptotic expansion in dimension 2 due to Nakazawa [**52**].

Theorem 4.5 ([**52**]). *Let $n = 2$. Near $\partial\Omega \cap \{z_1 z_2 \neq 0\}$, we have*

$$K(z) = \frac{2}{\pi^2} J[\lambda] \left(\frac{\widetilde{\varphi}(v, \lambda)}{\lambda^3} + \widetilde{\psi}(v, \lambda) \log \lambda \right), \tag{23}$$

where $J[\lambda] = \frac{p}{|4z_1z_2|^2}$. *Let* $e_1 = p''$, $e_2 = (pp^{(3)})'$, $e_3 = (p^2p^{(4)})''$, $e_{41} = e_1e_3$, $e_{42} = (pe_3')'$ *and* $e_{43} = (pp^{(4)})^2$. *Then*

$$\widetilde{\varphi}(v,\lambda) = 1 + \frac{\lambda}{4}e_1 + \frac{\lambda^2}{12}e_2,$$
$$\widetilde{\psi}(v,\lambda) = \frac{-e_3}{48} + \frac{\lambda}{480}(2e_{42} + e_{43} - e_{41}) + O(\lambda^2).$$

Theorem 4.5 and the following lemma immediately implies $c = 24/5$ in (21).

LEMMA 4.6 ([**34**]). *Under the notation of the above theorem, we have* $|A^0_{2\bar{4}}|^2 = J[\lambda]^{4/3}e_{43}/48^2$, $r^F = J[\lambda]^{-1/3}(\widetilde{r} + O(\lambda^4))$ *and* $\eta_1 = J[\lambda](\widetilde{\eta}_1 + O(\lambda^2))$, *where*

$$\widetilde{r} = \lambda - \frac{\lambda^2}{12}e_1 - \frac{\lambda^3}{36}\left(e_2 - \frac{e_1^2}{2}\right), \quad \widetilde{\eta}_1 = \frac{e_3}{144} - \frac{\lambda}{720}\left(e_{42} - \frac{e_{41}}{2}\right).$$

Write

$$K = \frac{p}{8\pi^2|z_1z_2|^2}\left(\frac{L_0}{\lambda^3} + \frac{L_1}{\lambda^2} + \frac{L_2}{\lambda} + \sum_{k=3}^{\infty} L_k\lambda^{k-3}\log\lambda\right). \tag{24}$$

Engliš [**19**] studied the asymptotic expansion of a Laplace integral and proved a recursion relation for its coefficients. As an application, he derived a formula of Fefferman's invariants for the Bergman kernel of strictly pseudoconvex Hartogs domains using the Forelli-Rudin construction. We observed that some key quantities in Engliš' formula can be expressed as explicit summations over strongly connected graphs when the domain is complete Reinhardt and proved a graph-theoretic formula of L_k.

THEOREM 4.7 ([**70**]). *Let* $k \geq 0$. *Define a function* $W_k(p)$ *by*

$$W_k(p) = \frac{1}{p^k}\sum_{G:\, w(G)=k}^{\textit{quasi-strong semistable}} \frac{(-1)^{|V(G)|+n(G)}}{|\mathrm{Aut}(G)|}\prod_{v\in V(G)} h(\deg(v)-2), \tag{25}$$

where G *runs over all quasi-strong semistable graphs of weight* k *and* $n(G)$ *is the number of components of* G; *the function* h *is defined recursively by*

$$h(1) = p', \quad h(k) = [p\cdot h(k-1)]', \; k \geq 2.$$

Then the coefficients of (24) *are given by*

$$L_k = \begin{cases} \dfrac{(2-k)!}{2}W_k(p), & 0 \leq k \leq 2, \\ \dfrac{(-1)^k}{2(k-3)!}W_k(p), & k \geq 3. \end{cases} \tag{26}$$

Example 4.8. Obviously $L_0 = W_0(p) = 1$. Note that

$$\begin{aligned}
&h(1) = p', \qquad h(2) = (p')^2 + pp'',\\
&h(3) = (p')^3 + 4pp'p'' + p^2p^{(3)},\\
&h(4) = (p')^4 + 11p(p')^2p'' + 7p^2p'p^{(3)} + 4p^2(p'')^2 + p^3p^{(4)}.
\end{aligned}$$

We now compute L_1, L_2, L_3 by using the above theorem. There are two quasi-strong semistable graphs of weight 1,

$$[2] \qquad \left[\begin{tikzcd} \circ \arrow[r, bend left, "2"] & \circ \arrow[l, bend left, "1"] \end{tikzcd}\right]$$

So $W_1(p) = \frac{1}{p}(\frac{h(2)}{2} - \frac{h(1)^2}{2}) = \frac{1}{2}p''$, which implies $L_1 = \frac{1}{4}p''$.

There are 19 quasi-strong semistable graphs of weight 2, among which 4 are stable. It is a routine calculation that $W_2(p) = \frac{1}{6}(pp^{(3)})'$, which implies $L_2 = \frac{1}{12}(pp^{(3)})'$.

There are 300 quasi-strong semistable graphs of weight 3, among which 14 are stable. With the help of a computer program, we get $W_3(p) = \frac{1}{24}(p^2p^{(4)})''$, which implies $L_3 = -\frac{1}{48}(p^2p^{(4)})''$.

Next we assume $n \geq 3$. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex complete Reinhardt domain satisfying $-\log|\Omega| = \{\lambda := y - (f_1(x) + \cdots + f_{n-1}(x)) > 0\}$ with hodograph variables $v_j = f_j'(x_j)$ and $p_j(v_j) = f_j''(x_j)$. We introduce

$$e_1 = \sum_{j=1}^{n-1} p_j'', \quad e_{21} = \sum_{j=1}^{n-1}(p_jp_j''')', \quad e_{22} = \sum_{j=1}^{n-1}(p_j'')^2, \quad e_{23} = \sum_{j\neq k} p_j''p_k''.$$

Theorem 4.9 ([34]). *Under the above notation, we have*

$$\begin{aligned}
\|A^0_{2\bar{2}}\|^2 &= \frac{J[\lambda]^{2/(n+1)}}{16n(n+1)}\big((n-2)(n-1)e_{22} + 2e_{23}\big),\\
r^F &= J[\lambda]^{\frac{-1}{n+1}}\Bigg(\lambda - \frac{e_1\lambda^2}{2n(n+1)}\\
&\qquad + \frac{-n(n+1)e_{21} + (n^2-1)e_{22} - e_{23}}{6(n-1)n^2(n+1)^2}\lambda^3 + O(\lambda^4)\Bigg).
\end{aligned}$$

The Bergman kernel has the expansion

$$K(z) = \frac{n!}{\pi^n}J[\lambda]\left(\frac{\widetilde{\varphi}(v,\lambda)}{\lambda^{n+1}} + \widetilde{\psi}(v,\lambda)\log\lambda\right), \tag{27}$$

where $J[\lambda] = \frac{p}{4^n|z_1\cdots z_n|^2}$ *and*

$$\widetilde{\varphi}(v,\lambda) = 1 + \frac{\lambda}{2n}e_1 + \frac{\lambda^2}{n(n-1)}\left(\frac{1}{6}e_{21} + \frac{1}{8}e_{23}\right) + O(\lambda^3). \tag{28}$$

Theorem 4.9 immediately implies $c_n = \frac{2}{3n(n-1)}$ in (22). Explicit graph theoretic formulae for the coefficients of (27) similar to (26) can be found in [**70**].

5. Deformation quantization

Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [**5**] introduced quantization as a deformation of the usual commutative product into a non-commutative associative star product. Let M be a smooth manifold and $A = C^\infty(M)$ the algebra over $\mathbb{R}$ of smooth functions on M endowed with the pointwise product. A *star product* on M is an associative $\mathbb{R}[[\hbar]]$-bilinear product on $A[[\hbar]]$ given by the following formula for $f, g \in A$,

$$f * g = fg + \sum_{k=1}^{\infty} \hbar^k C_k(f,g), \tag{29}$$

where $\hbar$ is the formal variable and C_k are bidifferential operators.

Two star products $\star$ and $\star'$ are called *gauge equivalent* if there is an $\mathbb{R}[[\hbar]]$-module automorphism D of $A[[\hbar]]$:

$$D\left(\sum_{n\geq 0} f_n \hbar^n\right) = \sum_{n\geq 0} f_n \hbar^n + \sum_{n\geq 0, m\geq 1}^{\infty} \hbar^{n+m} D_m(f_n),$$

where $D_i : A \to A$ are differential operators, such that $f \star' g = D((D^{-1}f) \star (D^{-1})g)$.

It is not difficult to check that a star product $\star$ gives a Poisson bracket on A:

$$\{f,g\} = \frac{f \star g - g \star g}{\hbar} = C_1(f,g) - C_1(g,f), \tag{30}$$

which depends only on the gauge equivalence class of $\star$.

Kontsevich proved a a long-standing conjecture that every Poisson structure arises from the first term of a star product. Before that it was known only for symplectic manifolds.

THEOREM 5.1 (Kontsevich [**42**]). *The set of gauge equivalence classes of star products on a smooth manifold M can be naturally identified with the set of equivalence classes of formal Poisson structures on M.*

In fact, it follows from the more general Kontsevich's formality theorem which establishes an L_∞-algebra quasi-isomorphism between the DGLAs of polyvector fields and Hochschild complex. Moreover, Kontsevich [**42**] gave an explicit graph-theoretic formula whose coefficients are integrals over the space of configurations.

If M is a Kähler manifold, a canonical Poisson bracket is locally given by

$$\{f_1, f_2\} = -g^{k\bar{l}} \left(\frac{\partial f_1}{\partial z^k} \frac{\partial f_2}{\partial \bar{z}^l} - \frac{\partial f_2}{\partial z^k} \frac{\partial f_1}{\partial \bar{z}^l} \right). \tag{31}$$

A star product $\star$ has the property of *separation of variables*, if it satisfies $f \star h = fh$ and $h \star g = hg$ for any locally defined antiholomorphic function f, holomorphic function g and an arbitrary function h.

In fact, all star products with separation of variables on Kähler manifolds are equivalent. We give two examples.

EXAMPLE 5.2. The *Berezin transform*

$$I_\alpha f(x) = \int_\Omega f(y) \frac{|K_\alpha(x,y)|^2}{K_\alpha(x,x)} e^{-\alpha\Phi(y)} \frac{w_g^n(y)}{n!} \tag{32}$$

has an asymptotic expansion (cf. [**19, 40**]),

$$I_\alpha f(x) = \sum_{k=0}^{\infty} Q_k f(x) \alpha^{-k}, \quad \alpha \to \infty \tag{33}$$

where Q_k are linear differential operators. Denote by $c_{j\alpha\beta}\partial^\alpha\bar\partial^\beta$ the coefficients in

$$Q_j f = \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta}\partial^\alpha\bar\partial^\beta f. \tag{34}$$

Then the coefficients of *Berezin star product* are given by bidifferential operators

$$C_j(f_1, f_2) := \sum_{\alpha,\beta} c_{j\alpha\beta}(\bar\partial^\beta f_1)(\partial^\alpha f_2). \tag{35}$$

The Berezin transform has important applications in linear operator theory.

EXAMPLE 5.3. The *Berezin-Toeplitz star product* $\star_{BT}$ is equivalent to Berezin star product $\star_B$ via the Berezin transform (cf. [**40**])

$$f_1 \star_{BT} f_2 = I^{-1}(If_1 \star_B If_2), \tag{36}$$

where $I := I_{1/\hbar}$ is obtained by substituting α by $1/\hbar$ in the Berezin transform I_α.

Recall that the Toeplitz operator $T_f^{(m)}$ for $f \in C^\infty(M)$ is defined by

$$T_f^{(m)} := \Pi^{(m)}(f\cdot): \quad H^0(M, L^m) \to H^0(M, L^m), \tag{37}$$

where $\Pi^{(m)} : L^2(M, L^m) \to H^0(M, L^m)$ is the projection.

Schlichenmaier [**60**] proved that $\star_{BT}$ is the unique star product

$$f_1 \star_{BT} f_2 := \sum_{j=0}^{\infty} h^j C_j^{BT}(f_1, f_2), \tag{38}$$

such that the following asymptotic expansion holds

$$T_{f_1}^{(m)} T_{f_2}^{(m)} = \sum_{j=0}^{\infty} m^{-j} T^{(m)}_{C_j^{BT}(f_1,f_2)}, \quad m \to \infty. \tag{39}$$

Let (M, ω_{-1}) be a Kähler manifold with Kähler form ω_{-1}. A formal deformation of the form $(1/\nu)\omega_{-1}$ is a formal $(1,1)$-form,

$$\hat{\omega} = \frac{1}{\nu}\omega_{-1} + \omega_0 + \nu\omega_1 + \nu^2\omega_2 + \cdots, \tag{40}$$

where each ω_k is a closed $(1,1)$-form.

THEOREM 5.4 (Karabegov [38]). *Each star product with separation of variables on a Kähler manifold M bijectively corresponds to a formal deformation of the Kähler metric ω_{-1} on M.*

The Karabegov forms of Berezin and Berezin-Toeplitz quantizations were identified by Karabegov and Schlichenmaier [40].

THEOREM 5.5 ([40]). (i) *The Karabegov form of Berezin star product is*

$$\frac{1}{\nu}\omega_{-1} + \sqrt{-1}\,\partial\bar{\partial}\log\left(\sum_{k=0}^{\infty}\nu^k B_k(x)\right),$$

where $B_k(x)$ are the coefficients in the asymptotic expansion (5).

(ii) *The Karabegov form of Berezin-Toeplitz star product is*

$$-\frac{1}{\nu}\omega_{-1} + Ric,$$

where $Ric = \sqrt{-1}\partial\bar{\partial}\log\det g$ is the Ricci curvature. The negative sign indicates that the role of holomorphic and antiholomorphic variables are swapped.

Feynman diagram formulas for star products on Kähler manifolds was first studied by Reshetikhin and Takhtajan [57], who proved a Feynman diagram formula for the non-normalized Berezin star product. Gammelgaard [26] obtained a universal formula as a summation over weighted acyclic graphs for any star product with separation of variables corresponding to a given classifying Karabegov form. Karabegov [39] gave an algebraic proof of Gammelgaard's formula and clarified why acyclic graphs play a role.

Berezin-Toeplitz quantization was extensively studied in the literature [11, 12, 21, 49, 61]. Zelditch [76] studied connections between Berezin-Toeplitz quantization and quantum chaos. Andersen [2] applied Berezin-Toeplitz technique to prove asymptotic faithfulness of the mapping class groups action on Verlinde bundles.

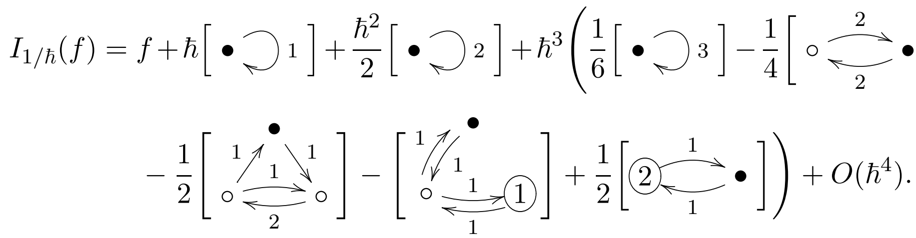
In [68], we proved the following graph theoretic formula for Berezin star product. The proof follows similar lines as that of Theorem 3.1. We also discussed its relation to Gammelgaard's formula in [69].

THEOREM 5.6 ([68]). *Fix a normal coordinate system around $x \in M$, at x*

$$f_1 \star_B f_2(x) = \sum_{\Gamma=(V\cup\{\bullet\},E)}^{strong} \frac{\det(A(\Gamma_-) - I)}{|\dot{\mathrm{Aut}}(\Gamma)|}\hbar^{|E|-|V|} D_\Gamma(f_1, f_2)\Big|_x. \tag{41}$$

Let us explain notations in (41). Γ runs over all strongly connected stable pointed graphs; Γ_- is the graph obtained from Γ by removing the distinguished vertex $\bullet$ and all adjacent edges of $\bullet$; $A(\Gamma_-)$ is the adjacency matrix of Γ_-. The partition function $D_\Gamma(f_1, f_2)$ is defined to be a Weyl invariant generated from Γ by replacing the vertex $\bullet$ with two vertices f_1 and f_2, such that all inward edges of $\bullet$ are connected to f_1 and all outward edges of f are connected to f_2.

Example 5.7. The first terms of the Berezin transform are



By (35), we can get Berezin star product up to order $\hbar^3$.

$$\begin{aligned}
C_0(f_1, f_2) &= f_1 f_2,\\
C_1(f_1, f_2) &= f_{1;\bar{i}} f_{2;i},\\
C_2(f_1, f_2) &= \frac{1}{2} f_{1;\bar{i}\bar{j}} f_{2;ij},\\
C_3(f_1, f_2) &= \frac{1}{6} f_{1;\bar{i}\bar{j}\bar{k}} f_{2;ijk} + \frac{1}{4} R_{i\bar{j}k\bar{l}} f_{1;\bar{i}\bar{k}} f_{2;jl} - \frac{1}{2}\rho_{;i\bar{j}} f_{1;\bar{i}} f_{2;j}.
\end{aligned}$$

6. Heat kernel

The Laplace operator Δ on a Riemannian manifold (M, g) of dimension d is given by

$$\Delta = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{d} \partial_i (g^{ij} \sqrt{\det g}\, \partial_j).$$

The heat kernel is a smooth function $e(t, x, y) \in C^\infty(M \times M \times \mathbb{R}^+)$ that solves the heat equation $\frac{\partial e}{\partial t} + \Delta_x e = 0$ and satisfies $e(t, x, y, t) = e(t, y, x)$ and

$$\lim_{t\to 0} \int_M e(t, x, y) f(y) dV = f(x)$$

for any smooth function f of compact support.

For example, the heat kernel of $\mathbb{R}^d$ is

$$e(t, x, y) = (4\pi t)^{-d/2} e^{-|x-y|^2/4t}.$$

If M is compact, there is a unique heat kernel $H(x, y, t)$ on M with the on-diagonal asymptotic expansion as $t \to 0^+$,

$$e(t, x, x) = (4\pi t)^{-d/2}(a_0 + a_1 t + a_2 t^2 + \cdots). \tag{42}$$

The first few terms can be computed by using, e.g., the Minakshiundaram-Pleijel recursive formula.

$$a_0 = 1, \quad a_1 = \frac{1}{6}\rho, \quad a_2 = -\frac{1}{30}\Delta\rho + \frac{1}{72}\rho^2 - \frac{1}{180}|Ric|^2 + \frac{1}{180}|R|^2, \tag{43}$$

where ρ is the scalar curvature and $|Ric|^2 = R_{ij}R^{ij}$, $|R|^2 = R_{ijkl}R^{ijkl}$.

There exists a complete orthonormal basis $\{\phi_0, \phi_1, \phi_2, \dots\}$ of $L^2(M)$, consisting of eigenfunctions of Δ, with corresponding eigenvalues $\lambda_0, \lambda_1, \lambda_2, \dots$ arranged in increasing order $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$, we have

$$e(t,x,y) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\phi_k(x)\phi_k(y)$$

with uniformly convergence for any fixed t. Therefore by (42), we get

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} = \int_M e(t,x,x) = (4\pi t)^{-d/2}\left(\mathrm{Vol}(M) + t\int_M \frac{1}{6}\rho\, dV + \cdots\right).$$

The integrals $\int_M a_n$ are called heat traces. This equation implies that if two Riemannian manifolds are isospectral (i.e., have the same spectrum of Laplacian), then they have the same heat traces. By (42) and (43), one sees that two isospectral Riemannian manifolds have the same dimensions, volumes and integrals of scalar curvatures.

Polterovich [**55**] proved a closed formula for all heat coefficients using a generalization of the Agmon-Kannai asymptotic expansion on the resolvent kernels of elliptic operators.

THEOREM 6.1 ([**55**]). *Let $w \ge 3n$. Then the heat coefficients $a_n(x)$ are equal to*

$$a_n(x) = (-1)^n \sum_{j=0}^{w} \binom{w+\frac{d}{2}}{j+\frac{d}{2}} \frac{1}{4^j j!(j+n)!}\Delta^{j+n}\big(f(dist(y,x)^2)^j\big)|_{y=x}, \tag{44}$$

where $dist(y,x)$ is the distance function and f is an arbitrary smooth function with $f(s) = s + O(s^2)$ for $s \in [0,\epsilon]$.

It was proved by Weingart [**66**] that when $f(s) = s$, Polterovich's formula (44) holds for $w \ge n$. However, in general, it is highly nontrivial to convert powers of the Laplacian and the distance function to curvature tensors and their covariant derivatives, which also seems hopeless to have a nice compact form.

In [**45**], we proved that on a Kähler manifold, Polterovich's formula implies a graph-theoretic formula of heat coefficients.

THEOREM 6.2. *On a Kähler manifold, the heat coefficients of the (real) Laplacian are given by*

$$a_n = \sum_{G:\, w(G)=n}^{stable} \frac{(-1)^{|V(G)|}2^n}{|\mathrm{Aut}(G)|} \sum_{C\subset E(G)} \frac{(-1)^{|C|}\varphi(\Gamma_C)}{(|C|+n)!}\, G. \tag{45}$$

We explain the notations in (45). Here G runs over all stable digraphs of weight n and C runs over all subsets of edges of G. Here Γ_C is a pointed graph obtained by cutting each edge in C in the middle and connecting all loose ends to a new distinguished vertex $\bullet$. Finally $\varphi(\Gamma_C)$ is a graph invariant that can be calculated as follows:

(i) if Γ is not strongly connected, then $\varphi(\Gamma) = 0$,
(ii) if Γ has only one vertex with l loops, then $\varphi(\Gamma) = l!$,
(iii) a vertex $v \in V(\Gamma_-)$ is called removable if it satisfies $\deg^+(v) = \deg^-(v) = 1$. If v is removable, denote by $\Gamma/\{v\}$ the graph obtained by removing v and connecting its two neighboring vertices, then $\varphi(\Gamma) = \varphi(\Gamma/\{v\})$,
(iv) if Γ is a strongly connected pointed graph and has no removable vertices, then

$$\varphi(\Gamma) = \sum_{e \in E(\Gamma)} \varphi(\Gamma - \{e\}),$$

where $\Gamma - \{e\}$ denotes deleting an edge e from Γ while keeping the endpoints.

Example 6.3. There is only one stable graph with weight 1.

$$a_1 = -\frac{1}{3}\left[\textcircled{2}\right] = \frac{1}{3}\rho. \tag{46}$$

There are four stable graphs with weight 2.

$$\begin{aligned} a_2 = &-\frac{2}{15}\left[\textcircled{3}\right] + \frac{1}{18}\left[\textcircled{2} \mid \textcircled{2}\right] + \frac{23}{90}\left[\textcircled{1} \underset{1}{\overset{1}{\rightleftarrows}} \textcircled{1}\right] + \frac{7}{45}\left[\circ \underset{2}{\overset{2}{\rightleftarrows}} \circ\right] \\ =&\frac{2}{15}\Box\rho + \frac{1}{18}\rho^2 - \frac{1}{90}|Ric|^2 + \frac{1}{45}|R|^2. \end{aligned} \tag{47}$$

Here $\Box f = f_{;i\bar{i}}$ for any function f. We computed a_3 in the appendix. a_3 for Riemannian metric was obtain in [27].

Definition 6.4. Let G be a stable digraph. Define

$$h(G) = \sum_{C \subset E(G)} \frac{(-1)^{|C|}\varphi(\Gamma_C)}{(|C| + w(G))!}, \qquad a(G) = \frac{(-1)^{|V(G)|}2^n}{|\mathrm{Aut}(G)|}h(G).$$

Then by (45), we have

$$a_n(z) = \sum_{G:\, w(G)=n}^{\text{stable}} a(G)G.$$

LEMMA 6.5. *Let $G = \cup_{i=1}^{k} G_i$ be a disjoint union of connected subgraphs. Then*

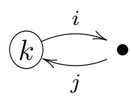
$$h(G) = \prod_{j=1}^{k} h(G_j), \tag{48}$$

$$a(G) = \frac{1}{|Sym(G_1, \ldots, G_k)|} \prod_{j=1}^{k} a(G_j), \tag{49}$$

where $Sym(G_1, \ldots, G_k)$ is the permutation group of the connected subgraphs.

PROOF. The equation (49) follows from the fact that the heat kernel of a product manifold $M \times N$ is equal to the product of the heat kernels of M and N. Obviously (48) follows from (49). $\square$

EXAMPLE 6.6. Let k, i, j be nonnegative integers. Consider a pointed digraph $\Gamma_{k,i,j}$



Here k in the circle means k loops. One may recursively show that

$$\varphi(\Gamma_{k,i,j}) = ij \cdot (k + i + j - 2)!.$$

Let G_k be the digraph with a single vertex and $k + 1$ loops. Then

$$\begin{aligned} h(G_k) &= \sum_{i=1}^{k+1} \binom{k+1}{i} \frac{(-1)^i i^2 \cdot (k+i-1)!}{(i+k)!} \\ &= \frac{k}{\binom{2k+1}{k+1}} = \frac{k \cdot k!(k+1)!}{(2k+1)!} \end{aligned}$$

and $z(G_k) = -2^k k \cdot k!/(2k+1)!$.

On a Riemannian manifold of constant sectional curvature or a Kähler manifold of constant holomorphic sectional curvature, the heat kernel coefficients are constants.

LEMMA 6.7. *Let $g_{i\bar{j}}$ be the Fubini-Study metric on $\mathbb{C}P^d$. Then $g_{i\bar{j}\alpha_1\alpha_2\ldots\alpha_r}(0)$ is nonzero only if the number of barred and unbarred indices in $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ are equal. In this case, we have*

$$g_{i_1\bar{j}_1 i_2\bar{j}_2\ldots i_k\bar{j}_k}(0) = (-1)^{k-1}(k-1)! \sum_{\sigma\in S_k} g_{i_1\bar{j}_{\sigma(1)}} g_{i_2\bar{j}_{\sigma(2)}} \cdots g_{i_k\bar{j}_{\sigma(k)}}(0), \tag{50}$$

where 0 is the center of Kähler normal coordinates.

PROOF. The Fubini-Study metric has constant holomorphic sectional curvature, namely $R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$, which implies

$$g_{i_1\bar{j}_1 i_2\bar{j}_2} = g^{m\bar{n}} g_{m\bar{j}_1\bar{j}_2} g_{i_1\bar{n}i_2} - g_{i_1\bar{j}_1}g_{i_2\bar{j}_2} - g_{i_1\bar{j}_2}g_{i_2\bar{j}_1}. \tag{51}$$

The remaining argument is similar to that of [67, Lemma 7.4]. $\square$

EXAMPLE 6.8. For the 2-sphere $S^2 = \mathbb{C}P^1$, the heat coefficients of the Fubini-Study metric can be calculated by using (45) and (50),

$$a_n = \sum_{G:w(G)=n}^{balanced} (-1)^n a(G) \prod_{v\in V(G)} (\deg^+(v)-1)!\deg^+(v)!,$$

where G runs over all stable balanced digraphs of weight n and $a(G)$ is defined in Definition 6.4. Recall that a balanced digraph means $\deg^+(v) = \deg^-(v)$ for each vertex $v \in V(G)$.

On the other hand, Polterovich [56] obtained a combinatorial formula of heat coefficients of the unit sphere S^2 from his formula (44),

$$c_n = \frac{1}{n!2^{2n}} \sum_{r=0}^{n} \binom{n}{r} (2-2^{2r}) B_{2r},$$

where B_{2r} are Bernoulli numbers. See [65] for a different calculation of heat coefficients of $\mathbb{C}P^1$.

Since the Fubini-Study metric on $\mathbb{C}P^1$ gives a round ball of radius $\frac{1}{2}$, we have $a_n = 2^n c_n$ for any $n \geq 0$. Namely,

(52)
$$c_n = \sum_{G:w(G)=n}^{balanced} \frac{(-1)^{|E(G)|}}{|\mathrm{Aut}(G)|} \sum_{C\subset E(G)} \frac{(-1)^{|C|}\varphi(\Gamma_C)}{(|C|+n)!} \prod_{v\in V(G)} (\deg^+(v)-1)!\deg^+(v)!$$

We have checked (52) for $n \leq 4$, but it seems not easy to give a direct proof.

THEOREM 6.9. *On the complex projective space $\mathbb{C}P^d$ with Fubini-Study metric, the heat coefficients are given by*

$$a_n = \sum_{G:w(G)=n}^{balanced} (-1)^n a(G) \prod_{v\in V(G)} (\deg^+(v)-1)! \sum_{H\in\mathscr{C}_G} d^{p(H)}, \tag{53}$$

where $\mathscr{C}_G$ denote the set of all cycle decomposition of G and $p(H)$ is the number of cycles in the cycle decomposition $H \in \mathscr{C}_G$.

PROOF. It follows from (45) and (50). □

COROLLARY 6.10. *Each heat coefficient a_n of $\mathbb{C}P^d$ is a polynomial in d. The leading term of a_n is $\frac{1}{3^n n!}d^{2n}$. In particular,*

$$a_1 = \frac{1}{3}d^2 + \frac{1}{3}d, \qquad a_2 = \frac{1}{18}d^4 + \frac{1}{10}d^3 + \frac{7}{90}d^2 + \frac{1}{30}d.$$

PROOF. The polynomiality of a_n follows from (45) and (50). The contribution to the top degree of a_n comes from a single graph $G = \underbrace{[(2) \mid \cdots \mid (2)]}_{n}$ with $a(G) = \frac{1}{n!}(-\frac{1}{3})^n$. a_1 and a_2 could be calculated using Example 6.3 and Theorem 6.9. □

Note that Cahn-Wolf [8] computed heat coefficients of compact symmetric spaces of rank one (e.g., S^d, $\mathbb{R}P^d$ and $\mathbb{C}P^d$) by using representation theory of Lie groups. Their formula for heat coefficients of $\mathbb{C}P^d$ need to treat separately cases of odd and even d as well as $n < d-1$ and $n \geq d-1$. It is totally unclear from their formula that a_n should be polynomials in d.

LEMMA 6.11. *Let $T_{\alpha_1 \ldots \alpha_p}$ be a covariant tensor of type $(0,p)$. When we interchange the order of its covariant derivative, the difference is*

$$T_{\alpha_1\ldots\alpha_p;i\bar{j}} - T_{\alpha_1\ldots\alpha_p;\bar{j}i} = \sum_{k=1}^{p} R^{\beta}_{\alpha_k i\bar{j}} T_{\alpha_1\ldots\alpha_{k-1}\beta\alpha_{k+1}\ldots\alpha_p}, \tag{54}$$

where $R^{\bar{k}}_{\bar{l}i\bar{j}} = -g^{m\bar{k}} R_{m\bar{l}i\bar{j}}$, $R^{k}_{li\bar{j}} = g^{k\bar{m}} R_{l\bar{m}i\bar{j}}$ *and* $R^{k}_{\bar{l}i\bar{j}} = R^{\bar{k}}_{li\bar{j}} = 0$.

PROPOSITION 6.12. *The heat traces $u_i = \int a_i$ of compact Kähler manifold are*

$$\begin{aligned}
u_1 =& \frac{1}{3}\int \rho,\\
u_2 =& \frac{1}{90}\int (5\rho^2 - 2|Ric|^2 + 4|R|^2),\\
u_3 =& \int \Big(\frac{1}{162}\rho^3 - \frac{1}{270}\rho|Ric|^2 + \frac{1}{135}\rho|R|^2 - \frac{2}{2835} R_{i\bar{j}} R_{k\bar{i}l\bar{m}} R_{j\bar{k}m\bar{l}}\\
&\qquad - \frac{2}{2835} R_{i\bar{j}} R_{j\bar{k}} R_{k\bar{i}} + \frac{4}{945} R_{i\bar{j}k\bar{l}} R_{j\bar{i}m\bar{n}} R_{l\bar{k}n\bar{m}} - \frac{7}{270}|\nabla\rho|^2\\
&\qquad - \frac{4}{2835}|\nabla Ric|^2 - \frac{1}{810}|\nabla R|^2 \Big).
\end{aligned}$$

PROOF. The formulas of u_1 and u_2 follow immediately from Example 6.3. For u_3, recall Green's theorem that the integral of the divergence of a tangent vector field on a closed manifold is zero. The Laplacian of a function is a divergence, for example,

$$\begin{aligned}
(\rho^2)_{;i\bar{i}} &= 2\rho_{;i}\rho_{;\bar{i}} + 2\rho\rho_{i\bar{i}} = 2\sigma_{11} + 2\sigma_8,\\
(R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k}})_{m\bar{m}} &= 2R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k};m\bar{m}} + 2R_{i\bar{j}k\bar{l};m} R_{j\bar{i}l\bar{k};\bar{m}} = 2\sigma_{10} + 2\sigma_{13},\\
(R_{i\bar{j}} R_{j\bar{i}})_{;k\bar{k}} &= 2R_{i\bar{j}} R_{j\bar{i};k\bar{k}} + 2R_{i\bar{j};k} R_{j\bar{i};\bar{k}} = 2\sigma_9 + 2\sigma_{12},
\end{aligned}$$

where the σ notations are defined in the Appendix A.

We compute three more divergences. The first is

$$\begin{aligned}
(R_{i\bar{j}k\bar{l}} R_{j\bar{m}l\bar{k}})_{;m\bar{i}} =& R_{i\bar{j}k\bar{l};m\bar{i}} R_{j\bar{m}l\bar{k}} + R_{i\bar{j}k\bar{l};m} R_{j\bar{m}l\bar{k};\bar{i}} + R_{i\bar{j}k\bar{l};\bar{i}} R_{j\bar{m}l\bar{k};m}\\
&+ R_{i\bar{j}k\bar{l}} R_{j\bar{m}l\bar{k};m\bar{i}}\\
=& \sigma_{10} + \sigma_{13} + \sigma_{12} + R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k};\bar{m}m} + R_{i\bar{j}k\bar{l}} R_{j\bar{n}m\bar{i}} R_{n\bar{m}l\bar{k}}\\
&- R_{i\bar{j}k\bar{l}} R_{n\bar{m}m\bar{i}} R_{j\bar{n}l\bar{k}} + R_{i\bar{j}k\bar{l}} R_{l\bar{n}m\bar{i}} R_{j\bar{m}n\bar{k}}\\
&- R_{i\bar{j}k\bar{l}} R_{n\bar{k}m\bar{i}} R_{j\bar{m}l\bar{n}}\\
=& \sigma_{10} + \sigma_{13} + \sigma_{12} + \sigma_{10} + \sigma_7 - \sigma_5 + \sigma_7 - \sigma_{15}\\
=& 2\sigma_{10} + \sigma_{13} + \sigma_{12} + 2\sigma_7 - \sigma_5 - \sigma_{15},
\end{aligned}$$

where we used

$R_{j\bar{m}l\bar{k};m\bar{i}}-R_{j\bar{m}l\bar{k};\bar{i}m}=R_{j\bar{n}m\bar{i}}R_{n\bar{m}l\bar{k}}-R_{n\bar{m}m\bar{i}}R_{j\bar{n}l\bar{k}}+R_{l\bar{n}m\bar{i}}R_{j\bar{m}n\bar{k}}-R_{n\bar{k}m\bar{i}}R_{j\bar{m}l\bar{n}}$ and $R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k};\bar{m}m}=R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k};m\bar{m}}$.

The second divergence is

$$\begin{aligned}(R_{i\bar{j}}R_{j\bar{m}})_{;m\bar{i}} &= R_{i\bar{j};m\bar{i}}R_{j\bar{m}}+R_{i\bar{j};m}R_{j\bar{m};\bar{i}}+R_{i\bar{j};\bar{i}}R_{j\bar{m};m}+R_{i\bar{j}}R_{j\bar{m};m\bar{i}}\\ &=\sigma_9+\sigma_{12}+\sigma_{11}+R_{i\bar{j}}R_{j\bar{m};\bar{i}m}+R_{i\bar{j}}R_{j\bar{k}m\bar{i}}R_{k\bar{m}}-R_{i\bar{j}}R_{k\bar{i}}R_{j\bar{k}}\\ &=\sigma_9+\sigma_{12}+\sigma_{11}+\sigma_9+\sigma_4-\sigma_6\\ &=2\sigma_9+\sigma_{12}+\sigma_{11}+\sigma_4-\sigma_6\end{aligned}$$

The third divergence is

$$(R_{i\bar{j}}R_{j\bar{i}m\bar{k}})_{;k\bar{m}}=\sigma_{10}+2\sigma_7-\sigma_{15}-\sigma_5+\sigma_{12}.$$

Finally we make substitutions

$$\int\sigma_8=-\int\sigma_{11},\qquad \int\sigma_9=-\int\sigma_{12},\qquad \int\sigma_{10}=-\int\sigma_{13}$$

$$\int\sigma_4=\int(\sigma_6-\sigma_{11}+\sigma_{12}),\qquad \int\sigma_{15}=\int(-\sigma_{13}+\sigma_{12}+2\sigma_7-\sigma_5)$$

to the integral of (61) and get

$$\begin{aligned}u_3=\int\Big(&\frac{1}{162}\sigma_1-\frac{1}{270}\sigma_2+\frac{1}{135}\sigma_3-\frac{2}{2835}\sigma_5-\frac{2}{2835}\sigma_6\\ &+\frac{4}{945}\sigma_7-\frac{7}{270}\sigma_{11}-\frac{4}{2835}\sigma_{12}-\frac{1}{810}\sigma_{13}\Big)\end{aligned}$$

as desired. □

Corollary 6.13. *Let M and M' be two isospectral Kähler-Einstein manifolds. If their curvature tensors satisfy*

$$\int_M R_{i\bar{j}k\bar{l}}R_{j\bar{i}m\bar{n}}R_{l\bar{k}n\bar{m}}=\int_{M'}R'_{i\bar{j}k\bar{l}}R'_{j\bar{i}m\bar{n}}R'_{l\bar{k}n\bar{m}},\tag{55}$$

then M is locally symmetric if and only if M' is locally symmetric.

Proof. On Kähler-Einstein manifold, we have $|Ric|^2=\rho^2/d$ where both sides are constants. Then the heat traces are

$$\begin{aligned}u_0&=\mathrm{vol}(M),\\ u_1&=\frac{1}{3}\rho\mathrm{vol}(M),\\ u_2&=\left(\frac{1}{18}-\frac{1}{90d}\right)\rho^2\mathrm{vol}(M)+\frac{1}{45}\int|R|^2,\\ u_3&=\left(\frac{1}{162}-\frac{1}{270d}-\frac{2}{2835d^2}\right)\rho^3\mathrm{vol}(M)+\left(\frac{1}{135}-\frac{2}{2835d}\right)\rho\int|R|^2\\ &\quad+\frac{4}{945}\int R_{i\bar{j}k\bar{l}}R_{j\bar{i}m\bar{n}}R_{l\bar{k}n\bar{m}}-\frac{1}{810}\int|\nabla R|^2.\end{aligned}$$

The conclusion follows easily. □

Remark 6.14. Proposition 6.12 for Riemannian metric and Corollary 6.13 for (real) Einstein manifold were proved by Sakai [59]. The condition (55) in Proposition 6.12 could be replaced by

$$\int R_{i\bar{j}k\bar{l}} R_{j\bar{m}l\bar{n}} R_{m\bar{i}n\bar{k}} = \int R'_{i\bar{j}k\bar{l}} R'_{j\bar{m}l\bar{n}} R'_{m\bar{i}n\bar{k}}.$$

Consider the Laplacian acting on p-forms on a compact Riemannian manifold of dimension d, the corresponding heat kernel $e^p x, y, t$ has asymptotic expansion as $t \to 0^+$,

$$\operatorname{tr} e^p(t,x,x) = (4\pi t)^{-d/2}\left(\binom{d}{p} + a_{1,p}t + a_{2,p}t^2 + \cdots\right). \tag{56}$$

Patodi [53] computed $a_{1,p}$ and $a_{2,p}$.

The local index theorem (conjectured by McKean and Singer [50]) for Laplacian says

$$\sum_{p=0}^{d} (-1)^p a_{n,p} = \begin{cases} 0 & \text{if } 2n < d; \\ (-2\pi)^n \mathrm{Pf}(R) & \text{if } 2n = d, \end{cases} \tag{57}$$

where $\mathrm{Pf}(R)$ is the Pfaffian of the Riemann curvature tensor. The local index theorem was first proved by Patodi [54]. It implies the Chern-Gauss-Bonnet theorem [14] by integration when $d = 2n$. See [77] for a comprehensive survey on the local index theorem. In fact, Chern-Gauss-Bonnet theorem is the only formula such that the integral of local invariants gives a topological invariant. This is a conjecture of Singer, a proof was given in [28].

On Kähler manifold (of complex dimension d), Patodi's formulas in terms of digraphs are given by

$$a_{1,p} = \left(-\frac{1}{3}\binom{2d}{p} + 2\binom{2d-2}{p-1}\right)[2],$$

$$a_{2,p} = C_1(d,p)\,[2 \mid 2] + C_2(d,p)\left[\begin{tikzcd} 1 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "1"] \end{tikzcd}\right] + C_3(d,p)\left[\begin{tikzcd} \circ \arrow[r, bend left, "2"] & \circ \arrow[l, bend left, "2"] \end{tikzcd}\right] + C_4(d,p)\,[3],$$

where the coefficients $C_i(d,p)$ are given by

$$\begin{cases} C_1(d,p) = \frac{1}{18}\binom{2d}{p} - \frac{2}{3}\binom{2d-2}{p-1} + 2\binom{2d-4}{p-2}, \\ C_2(d,p) = \frac{23}{90}\binom{2d}{p} - \frac{1}{3}\binom{2d-2}{p-1} - 4\binom{2d-4}{p-2}, \\ C_3(d,p) = \frac{7}{45}\binom{2d}{p} - \binom{2d-2}{p-1} + 2\binom{2d-4}{p-2}, \\ C_4(d,p) = -\frac{2}{15}\binom{2d}{p} + \frac{2}{3}\binom{2d-2}{p-1}. \end{cases}$$

From the local index theorem and Patodi's formula, we expect that on Kähler manifold (of complex dimension d)

$$a_{n,p} = \sum_{G\colon w(G)=n}^{\text{stable}} \sum_{k=0}^{|V(G)|} c_{n,k}(G) \binom{2d-2k}{p-k} G, \tag{58}$$

where $c_{n,k}(G)$ are rational numbers depending only on n, k and G. In particular, setting $p = 0$, we get $c_{n,0}(G) = a(G)$.

Assuming (58), it is not difficult to see that if $n < d$,

$$\sum_{p=0}^{d} (-1)^p a_{n,p} = 0 \tag{59}$$

and if $n = d$,

$$\sum_{p=0}^{d} (-1)^p a_{n,p} = \sum_{G\colon w(G)=|V(G)|=n}^{\text{stable}} (-1)^n c_{n,n}(G) G, \tag{60}$$

where G runs over digraphs with weight n and n vertices. Note that a stable digraph of weight n has at most n vertices.

The Chern-Gauss-Bonnet formula for Kähler manifold reads $\chi(M) = \int_M c_d(M)$. The Chern classes c_k are defined by

$$\det\left(I + \frac{\sqrt{-1}}{2\pi}\Omega\right) = 1 + c_1 + \cdots + c_d.$$

In other words,

$$c_k = \frac{1}{k!}\left(\frac{\sqrt{-1}}{2\pi}\right)^k \sum \delta^{j_1\cdots j_k}_{i_1\cdots i_k} \Omega^{i_1}_{j_1} \wedge \cdots \wedge \Omega^{i_k}_{j_k},$$

where $\Omega^i_j = R_{j\bar{i}k\bar{l}} dz^k \wedge d\bar{z}^l$

Let us compute top Chern class c_d on M for $d = 1$ and 2.

$$\begin{aligned}
c_1 =& \frac{\sqrt{-1}}{2\pi}\Omega^i_i = \frac{\sqrt{-1}}{2\pi}\rho dz^1 \wedge d\bar{z}^1, \\
c_2 =& -\frac{1}{8\pi^2}(\Omega^i_i \wedge \Omega^j_j - \Omega^j_i \wedge \Omega^i_j) \\
=& -\frac{1}{8\pi^2}(R_{i\bar{i}k\bar{l}} dz^k \wedge d\bar{z}^l \wedge R_{j\bar{j}p\bar{q}} dz^p \wedge d\bar{z}^q - R_{i\bar{j}k\bar{l}} dz^k \wedge d\bar{z}^l \wedge R_{j\bar{i}p\bar{q}} dz^p \wedge d\bar{z}^q) \\
=& -\frac{1}{8\pi^2}(\rho^2 - 2|Ric|^2 + |R|^2) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2.
\end{aligned}$$

The last equation needs some reorganizations of terms. We omit the details. So we verified that for $n = 1$ and 2, c_n is equal to the right-hand side of (60) up to constant factors. Of course this equality for all n will prove Chern-Gauss-Bonnet formula for Kähler manifold.

Appendix A. Heat coefficient a_3 on Kähler manifold

We express a_3 in terms of the following basis as used by Engliš [19].

$$\sigma_1 = \rho^3, \quad \sigma_2 = \rho R_{i\bar{j}} R_{j\bar{i}}, \quad \sigma_3 = \rho R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k}},$$
$$\sigma_4 = R_{i\bar{j}} R_{k\bar{l}} R_{j\bar{i}l\bar{k}}, \quad \sigma_5 = R_{i\bar{j}} R_{k\bar{i}l\bar{m}} R_{j\bar{k}m\bar{l}}, \quad \sigma_6 = R_{i\bar{j}} R_{j\bar{k}} R_{k\bar{i}},$$
$$\sigma_7 = R_{i\bar{j}k\bar{l}} R_{j\bar{i}m\bar{n}} R_{l\bar{k}n\bar{m}}, \quad \sigma_8 = \rho \Box \rho, \quad \sigma_9 = R_{i\bar{j}} R_{j\bar{i};k\bar{k}},$$
$$\sigma_{10} = R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k};m\bar{m}}, \quad \sigma_{11} = \rho_{;i}\rho_{;\bar{i}}, \quad \sigma_{12} = R_{i\bar{j};k} R_{j\bar{i};\bar{k}},$$
$$\sigma_{13} = R_{i\bar{j}k\bar{l};m} R_{j\bar{i}l\bar{k};\bar{m}}, \quad \sigma_{14} = \Box^2 \rho, \quad \sigma_{15} = R_{i\bar{j}k\bar{l}} R_{j\bar{m}l\bar{n}} R_{m\bar{i}n\bar{k}}.$$

Note that our convention of curvatures $R_{i\bar{j}k\bar{l}}$, $R_{i\bar{j}}$, ρ all differ by a minus sign with that of [19].

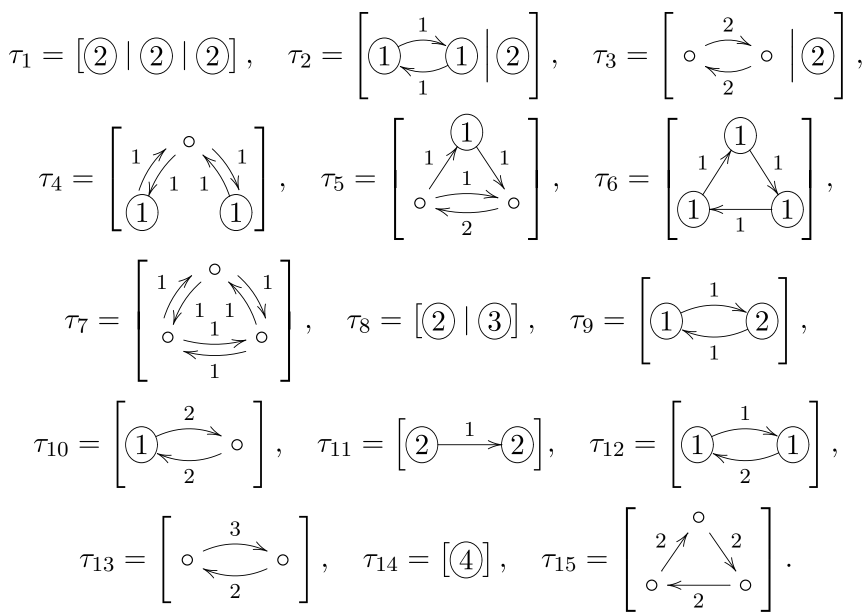
THEOREM A.1. *On a Kähler manifold, the heat coefficient a_3 is given by*

$$a_3 = \sum_{i=1}^{15} c_i \sigma_i \tag{61}$$

where c_i are given by

$$c_1 = 1/162, \quad c_2 = -1/270, \quad c_3 = 1/135, \quad c_4 = 8/945, \quad c_5 = -4/945,$$
$$c_6 = -26/2835, \quad c_7 = 32/2835, \quad c_8 = 2/45, \quad c_9 = 1/315, \quad c_{10} = 2/105,$$
$$c_{11} = 17/630, \quad c_{12} = -1/315, \quad c_{13} = 1/70, \quad c_{14} = 1/35, \quad c_{15} = -2/567.$$

Below we outline the derivation. There are 15 stable graphs of weight 3.



By (45), we wrote a computer program to get $a_3 = \sum_{i=1}^{15} z_i \tau_i$ with

$$z_1 = -1/162, \quad z_2 = -23/270, \quad z_3 = -7/135, \quad z_4 = -17/135,$$
$$z_5 = -332/945, \quad z_6 = -307/2835, \quad z_7 = -74/405, \quad z_8 = 2/45,$$
$$z_9 = 64/315, \quad z_{10} = 26/105, \quad z_{11} = 17/630, \quad z_{12} = 89/315,$$
$$z_{13} = 1/10, \quad z_{14} = -1/35, \quad z_{15} = -206/2835.$$

We express each τ_i as a linear combination of σ_i, $1 \le i \le 15$.

$$\tau_i = -\sigma_i, \quad 1 \le i \le 7,$$
$$\tau_8 = -2\sigma_2 - \sigma_3 + \sigma_8, \quad \tau_9 = -\sigma_4 - \sigma_5 - \sigma_6 + \sigma_9, \quad \tau_{10} = -2\sigma_5 + \sigma_{10} - \sigma_{15},$$
$$\tau_{11} = \sigma_{11}, \quad \tau_{12} = \sigma_{12}, \quad \tau_{13} = \sigma_{13},$$
$$\tau_{14} = -3\sigma_4 - 12\sigma_5 - 3\sigma_6 + 6\sigma_7 + 7\sigma_9 + 8\sigma_{10} + 10\sigma_{12} + 3\sigma_{13} - \sigma_{14} - 6\sigma_{15},$$
$$\tau_{15} = -\sigma_{15}.$$

The only nontrivial computation is for τ_{14}. Then we get (61).

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