Periodic and open classical spin Calogero-Moser chains

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Abstract. We construct classical Hamiltonian system similar to spin Calogero-Moser type systems. They can be regarded as a many particle system with composite spin degrees of freedom and also as an integrable spin chain of Gaudin type. We prove that these Hamiltonian systems are superintegrable.

Introduction

1. Classical Calogero-Moser (CM) systems were among the first integrable N-particle systems of one dimensional particles [3][25] with the potential $1/(q_i-q_j)^2$. This model was generalized to the potential $1/sh^2(q_i-q_j)$ in [36]. Then it was extended to other root systems and to elliptic potentials in [28], to a model involving spin degrees of freedom in [16].

There is an extensive literature on spin versions of CM systems. For example in [21][22] solutions to equations of motion to the elliptic spin Calogero-Moser system were related to special elliptic solutions to the matrix KP hierarchy. The relation to gauge theories were explored in many papers, see for example [27][15]. A variety of spin CM systems were obtained by L. Feher, see for example [10][11][12], in particular he derived important examples related to homogeneous spaces. Two spin CM systems were studied in [18][19]. Integrable chains of relativistic spin CM type systems were studied in [5][2].

Superintegrability of spin CM systems and of spin Ruijsenaars systems was established in [29]. In [31] the superintegrability of spin CM systems on homogeneous spaces was established. A family of superintegrable systems on moduli spaces of flat connections was constructed in [1]. This family includes systems studied in [5][2]. In these particular case the system is also Liouville integrable.

In this paper we will describe classical superintegrable system which we call spin Calogero-Moser (CM) chains. We call them spin CM chains because they combine features of many particle systems (as in CM systems) and of spin chains. We distinguish two cases: a *periodic chain* and an *open chain*. The periodic case is the classical version of a quantum integrable system where joint eigenfunctions of quantum commuting Hamiltonians are trace functions, see [7]. In this case the spin part of the system reminds a spin chain with periodic boundary conditions. In case of rank 1 orbits for $\mathfrak{sl}_n$ these systems are linearized versions of [5] and [2]. In the open case they are a classical version of quantum integrable systems constructed in [35][33]. For these systems the spin part of the system is similar to an open spin chain.

In both cases, i.e. in the periodic and in the open spin Calogero-Moser chains, the phases space is a stratified symplectic space [23], which, in some cases have only one stratum and becomes a symplectic manifold.

2. Recall that a *superintegrable system* is the structure on a symplectic manifold $\mathcal{M}$ that consists of a Poisson manifold $\mathcal{P}$, a Poisson manifold $\mathcal{B}$ with the trivial Poisson structure (i.e. zero Poisson tensor) and two surjective Poisson projections

$$\mathcal{M} \xrightarrow{p_1} \mathcal{P} \xrightarrow{p_2} \mathcal{B} \tag{1}$$

such that $\dim(\mathcal{M}) = \dim(\mathcal{P}) + \dim(\mathcal{B})$. For a superintegrable system a generic fiber of p_1 is an isotropic submanifolds of dimension $\dim(\mathcal{B})$ and connected components of a generic fiber of p_2 is a disjoint union of symplectic leaves of $\mathcal{P}$. For details see [26][30] and references therein. Here we adopt this notion to the case of stratified symplectic and Poisson spaces in which case p_1 and p_2 are Poisson mapping between stratifies spaces. In this paper the superintegrability means the balance of dimensions for the big stratum. How the system behave at smaller strata will be a subject of a separate publication. In the algebraic case, the appropriate setting is symplectic and Poisson stacks.

Let I be a Poisson commutative subalgebra of $A = C^\infty(\mathcal{M})$ that consists of functions which are constant on fibers of $p_2 \circ p_1$ (the pull-back of functions on $\mathcal{B}$ to functions on $\mathcal{M}$) and J be the Poisson algebra of functions which are constant on fibers of p_1 (the pull-back of functions on $\mathcal{P}$). The condition on $(\mathcal{M}, \mathcal{P}, \mathcal{B})$ for being a superintegrable system is equivalent to the following condition on $I \subset J \subset A$. The Poisson algebra A has trivial center, $I \subset A$ is a Poisson commutative subalgebra, such that J, its centralizer in A maximal possible Gelfand-Kirillov dimension for the given Gelfand-Kirillov dimension of I.

The Hamiltonian dynamics generated by a function $H \in I$ is called *superintegrable*. Any function from J is constant along flow lines of the vector field generated by H and thus, it is an integral of motion for the Hamiltonian dynamics generated by H. This is why we call elements of the Poisson commutative subalgebra I Hamiltonians and elements of J conservation laws.

3. Throughout this paper G is a split real connected semisimple Lie algebra with finite center which admits a complexification, and $\Theta \in \mathrm{Aut}(G)$

is a Cartan involution. We denote by $K = G^{\Theta}$ the closed subgroup of fixed points of Θ, which is connected and maximal compact. Let θ the corresponding Cartan involution[1] of $\mathfrak{g}$, and $\mathfrak{k}$ the Lie algebra of K. The associated Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{p}$ the (-1)-eigenspace of θ.

Let $\mathfrak{a} \subset \mathfrak{g}$ be maximally noncompact θ-stable Cartan subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}$ is split we have $\mathfrak{a} \subseteq \mathfrak{p}$. On the Lie group level, $A = \exp(\mathfrak{a}) \subset G$ is a maximal real split torus in G and $H := Z_G(A)$, the centraliser of A in G, is a Cartan subgroup in G containing A. The exponential map provides an isomorphism $\mathfrak{a} \xrightarrow{\sim} A$, whose inverse we denote by $\log : A \to \mathfrak{a}$.

Consider the root decomposition of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{a}$,

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$$

where $R \subset \mathfrak{a}^*$ is the root system of $\mathfrak{g}$ relative to $\mathfrak{a}$. Choose $e_\alpha \in \mathfrak{g}_\alpha$ such that

$$\theta(e_\alpha) = -e_{-\alpha} \tag{2}$$

and $(e_\alpha, e_{-\alpha}) = 1$ for each $\alpha \in R$, and choose a subset $R_+ \subset R$ of positive roots. Let $W \subset \mathrm{GL}(\mathfrak{a}^*)$ be the Weyl group of R.

The Weyl group W is isomorphic to $N_G(A)/H$, where $N_G(A)$ is the normaliser of A in G. Denote by A_{reg} the set of regular elements in $a \in A$,

$$A_{reg} := \{a \in A \mid a_\alpha := e^{\alpha(\log(a))} \neq 1 \text{ for all } \alpha \in R\}.$$

It is the union of all the regular W-orbits in A. A fundamental domain for the W-action on A_{reg}[2] is the positive Weyl chamber

$$A_+ = \{a \in A \mid a_\alpha := e^{\alpha(\log(a))} > 1 \text{ for any } \alpha \in R_+\}.$$

Let $G' \subset G$ be the set of elements $g \in G$ which are G-conjugate to some element in A_{reg}. The inclusion $A_{reg} \hookrightarrow G'$ induces a bijection $A_{reg}/W \xrightarrow{\sim} G'/G$, with A_{reg}/W the set of W-orbits in A_{reg} and G'/G the set of G-conjugacy classes in G'.

The Weyl group W is also isomorphic to $N_K(A)/M$, where $N_K(A) = N_G(A) \cap K$ and $M = H \cap K$ (note that M is a finite group since G is split). The inclusion map $A \hookrightarrow G$ induces an isomorphism $A/W \xrightarrow{\sim} K\backslash G/K$. We write $G_{reg} = KA_+K$ for the union of the double (K,K)-cosets intersecting A_{reg}.

4. The phase space of a periodic spin Calogero-Moser chain corresponding to a collection $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_n\}$ of coadjoint orbits $\mathcal{O}_i \subset \mathfrak{g}^*$ is the regular part of the symplectic leaf $\mathcal{S}(\mathcal{O})$ of the stratified Poisson space $T^*(G^{\times n})/G_n$, with the action of the gauge group $G_n = G^{\times n}$ the lift of a twisted conjugation action on $G^{\times n}$ (see section 1.1).[3] Here we assume that

[1] Recall that an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution when the bilinear from $(-\theta(x), y)$ on $\mathfrak{g}$ is positive definite. Here $(\cdot, \cdot)$ is the Killing form.

[2] In case of $SL_n(\mathbb{R})$ one can take A to be the diagonal unimodular matrices with positive real entries, and A_{reg} consists of those with distinct diagonal entries.

[3] In this paper we assume that all quotients X/H are GIT quotients.

each of $\mathcal{O}_i$ is non-trivial, i.e. $\mathcal{O}_i \neq \{0\}$ These symplectic leaves are obtained by the Hamiltonian reduction, as it is described in section 1.1. As a stratified symplectic space

$$\mathcal{S}(\mathcal{O}) \simeq \{(x_1, \dots, x_n, g) \in \mathfrak{g}^{*\times n} \times G \mid x_1 - \mathrm{Ad}^*_{g^{-1}} x_n \in \mathcal{O}_1, \\ x_i - x_{i-1} \in \mathcal{O}_i,\ i = 2, \dots, n\}/G,$$

see section 1.3.[4] Its regular part is defined as the intersection $\mathcal{S}(\mathcal{O})_{reg} = \mathcal{S}(\mathcal{O}) \cap (\mathfrak{g}^{*\times n} \times G')/G$.[5] The regular part has the following structure as a symplectic manifold, $\mathcal{S}(\mathcal{O})_{reg} \simeq \big(\nu_{\mathcal{O}}^{-1}(0)/H \times T^*A_{reg}\big)/W$, where $\nu_{\mathcal{O}} : \mathcal{O}_1 \times \cdots \mathcal{O}_n \to \mathfrak{a}^*$ is the moment map for the diagonal coadjoint action of H on $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ (see section 1.3).

Trivialization of T^*G by right translations gives an isomorphism $T^*(G^{\times n}) \simeq \mathfrak{g}^{*\times n} \times G^{\times n}$ and the Poisson projection $T^*(G^{\times n})/G_n \to (\mathfrak{g}^*/G)^{\times n}$, which is the projection to the cotangent directions followed by the quotienting with respect to the coadjoint action of $G^{\times n}$. Poisson commuting Hamiltonians of the periodic spin Calogero-Moser system are functions on $T^*(G^{\times n})/G_n$ which are constant on fibers of this Poisson projection. More precisely, tPoisson commuting Hamiltonians are such functions restricted to $\mathcal{S}(\mathcal{O})_{reg}$.

Consider the increasing set of natural numbers $2 = d_1 \leq \cdots \leq d_r$ with $r = \mathrm{rank}(\mathfrak{g})$ and $d_k - 1$ the exponents of $\mathfrak{g}$. Let c_{d_k} be the nonzero coadjoint invariant functions on $\mathfrak{g}^*$ of degree d_k, known as Casimir functions. The function c_2 is the quadratic Casimir of $\mathfrak{g}$. Let $H^{(l)}_{d_k}$ be the function on $(\mathfrak{g}^*/G)^{\times n}$ which is c_{d_k} on the l-th factor and constant on all other factors.

Let us denote vectors in $\mathcal{O}_k \subset \mathfrak{g}^*$ by $\mu^{(k)}$, its Cartan component by $\mu_0^{(k)}$ and set $\mu_\alpha^{(k)} = \mu^{(k)}(e_{-\alpha})$ for $\alpha \in R$. Denote (p, a) points on $T^*A \simeq \mathfrak{a}^* \times A$. Now let us describe quadratic Hamiltonians in terms of these variables.

The n-th quadratic Hamiltonian is the spin Calogero-Moser Hamiltonian. It has particularly simple form:

$$H_2^{(n)} = \frac{1}{2}(p, p) - \sum_{\alpha > 0} \frac{\mu_\alpha \mu_{-\alpha}}{2\mathrm{sh}^2(q_\alpha)}$$

where we used the parametrization $a_\alpha = e^{q_\alpha}$ (so $q_\alpha = \alpha(\log(a))$) and $\mu_\alpha = \mu_\alpha^{(1)} + \cdots + \mu_\alpha^{(n)}$, and $(\cdot, \cdot)$ is the Euclidean form on $\mathfrak{a}^*$ obtained by dualising the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{a}$.

[4]There are n such natural isomorphisms φ_j $(1 \leq j \leq n)$, see section 1.2. In the introduction we use φ_n.

[5]A better way to think about the periodic spin Calogero-Moser system for a real split simple Lie group G is to define $\mathcal{S}(\mathcal{O})_{\mathbb{C}}$ is complex algebraic setting and then to take the corresponding real slice. This will be addressed in another publication.

The differences $D_k = H_2^{(k)} - H_2^{(k-1)}$ for $1 < k \le n$ are classical analogs of topological Knizhnik-Zamolodchikov-Bernard differential operators,

$$D_k = (\mu_0^{(k)}, p) - \sum_{l=1}^{k-1} r_{lk} + \sum_{l=k+1}^{n} r_{kl} \tag{3}$$

where r_{kl} for $k \neq l$ is a classical version of the Felder's dynamical r-matrix [**13**],

$$r_{kl} = -\frac{1}{2}(\mu_0^{(k)}, \mu_0^{(l)}) + \sum_\alpha \frac{\mu_{-\alpha}^{(k)}\mu_\alpha^{(l)}}{a_\alpha - 1} \tag{4}$$

and $\sum_\alpha$ stands for the sum over all the roots $\alpha \in R$. This explicit form of D_k is derived in section 1.3.

The superintegrability of this system is described in section 1.5. The projection method for constructing solutions of equations of motion and angle variables are described in section 1.6.

One can choose G to be a the maximal compact real form of the complexification $G_\mathbb{C}$. In this case the integrable system is similar, but hyperbolic functions gets replaces by the trigonometric ones. The structure of the phase space is again a stratified symplectic space. The superintegrability of the quantum counterpart of such compact case is proven in [**32**].

5. The phase space of an open Calogero-Moser spin chain is the regular part of a symplectic leaf of the Poisson manifold $T^*(G^{\times n+1})/(K \times G^{\times n} \times K)$ where the action of the gauge group $K \times G^{\times n} \times K$ is described in section 2.3, and $K \subset G$ is as above. Such symplectic leaves are given by the Hamiltonian reduction. They are parametrized by collections of coadjoint orbits $\mathcal{O} = \{\mathcal{O}_\ell^K, \mathcal{O}_1, \ldots, \mathcal{O}_n, \mathcal{O}_r^K\}$ where $\mathcal{O}_i \subset \mathfrak{g}^*$ and $\mathcal{O}_{\ell,r}^K \subset \mathfrak{k}^* \subset \mathfrak{g}^*$ are coadjoint orbits. We assume that none of $\mathcal{O}_i$ is trivial, i.e. $\mathcal{O}_i \neq \{0\}$.

We denote the corresponding symplectic leaf by $\mathcal{S}(\mathcal{O})$. It is a stratified symplectic space. Using Cartan decomposition $G = KAK$ and a "gauge fixing fixing", we define the regular part $\mathcal{S}(\mathcal{O})_{reg}$ of $\mathcal{S}(\mathcal{O})$ as the stratum

$$S(\mathcal{O})_{reg} \simeq (T^*A_{reg} \times \mathcal{O}_1^K \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \times \mathcal{O}_2^K))/N_K(A),$$

where on the right we have a natural product symplectic structure.

Similarly to the periodic case, quadratic Hamiltonians can be computed explicitly in terms of Cartan components $\mu_0^{(k)}$ and root coordinates $\mu_\alpha^{(k)}$ of vectors $\mu^{(k)} \in \mathcal{O}_k$, coordinates $\mu'_{[\alpha]}$, $\mu''_{[\alpha]}$ on $\mathcal{O}_\ell^K$ and $\mathcal{O}_r^K$ respectively (in the basis elements $e_{[\alpha]} = e_{-\alpha} - e_\alpha \in \mathfrak{k} \subset \mathfrak{g}$ for $\alpha \in R_+$), and $(p, a) \in T^*A_{reg}$.

Assuming the gauge fixing ϕ_n (see section 2.3) we have

$$\begin{aligned} H_2^{(n)} &= \frac{1}{2}(p,p) \\ &\quad + \sum_{\alpha>0} \frac{(a_\alpha\mu'_{[\alpha]} + \mu''_{[\alpha]} + a_\alpha(\mu_\alpha - \mu_{-\alpha}))(a_\alpha^{-1}\mu'_{[\alpha]} + \mu''_{[\alpha]} + a_\alpha^{-1}(\mu_\alpha - \mu_{-\alpha}))}{(a_\alpha - a_{-\alpha})^2} \end{aligned}$$

For other quadratic Hamiltonians the differences

$$D_k = H_2^{(k)} - H_2^{(k-1)} \qquad (1 \le k \le n)$$

are more interesting. They are classical analogs of boundary Knizhnik-Zamolodchikov-Bernard differential operators [**35**][**33**]. We have the following formula for D_k:

$$D_k = (\mu_0^{(k)}, p) - \sum_{l=1}^{k-1}(r_{lk} + r_{lk}^{\theta_l}) + (\sum_\alpha K_\alpha \mu_{-\alpha}^{(k)} - \kappa_k) + \sum_{l=k+1}^{n}(r_{kl} - r_{kl}^{\theta_k}).$$

Here r_{kl} for $k \neq l$ now is Felder dynamical r-matrix rescaled in $a \in A_{reg}$,

$$r_{kl} = -\frac{1}{2}(\mu_0^{(k)}, \mu_0^{(l)}) + \sum_\alpha \frac{\mu_{-\alpha}^{(k)}\mu_\alpha^{(l)}}{a_\alpha^2 - 1}, \tag{5}$$

θ_k is the transpose of the Cartan involution acting on $\mu^{(k)}$,

$$\kappa_k = \frac{1}{2}(\mu_0^{(k)}, \mu_0^{(k)}) + \sum_\alpha \frac{(\mu_\alpha^{(k)})^2}{1 - a_\alpha^2}$$

and

$$K_\alpha = \frac{a_\alpha \mu'_{[\alpha]} + \mu''_{[\alpha]}}{a_\alpha - a_\alpha^{-1}}. \tag{6}$$

The differences $D_k = H_2^{(k)} - H_2^{(k-1)}$ are classical analogs of boundary KZB operators derived in [**35**][**33**]. The superintegrability of open spin CM chains is proven in section 2.6. The projection method for solving equations of motion and angle coordinates are described in section 2.7.

6. The structure of the paper is as follows. In section 1 we construct periodic spin CM chains by the Hamiltonian reduction and prove the superintegrability. In section 1.1 we describe the phase space of such a system. In sections 1.2, 1.3 we describe the regular part of the phase space. Hamiltonians of a periodic spin CM chain, restricted to the regular part of the phase space are described in section 1.4. The superintegrability of a periodic spin CM chain is proven in section 1.5. In section 1.6 solutions to equations of motion are described algebraically by the projection method, and angle variables are described. In section 2 we focus on open spin CM chains. In section 2.1 we describe phase spaces. In section 2.3, 2.4 we describe the regular part of the phase space. Hamiltonians of an open spin CM chain, restricted to the regular part of the phase space are described in section 2.5. The superintegrability of an open spin CM chain is proven in section 2.6. In section 2.7 solutions to equations of motion are described algebraically by the projection method, and angle variables are described. In the conclusion (section 3) we discuss some open problems and describe in details periodic CM spin chain for SL_N with orbits of rank 1. In Appendix A we compare our symplectic leaves with the ones from [**31**].

Throughout this paper we will focus on split real semisimple Lie groups. However, since all constructions are algebraic they extend (with appropriate modifications) to the complex algebraic case. The non-split real case will be the subject of a separate publication (see [33] for the quantum case). Another important real case is when G is compact, which can be deduced from the complex algebraic case by restriction to a compact real form. The structure of phase spaces as stratified symplectic spaces will be explored further in [6].

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1. Periodic spin Calogero-Moser chains

1.1. The phase space as the Hamiltonian reduction. Here we will describe the phase space of a periodic spin Calogero-Moser chain as a Hamiltonian reduction of $T^*(G^{\times n})$. Let us start with the description of these symplectic spaces.

Consider the manifold $T^*(G^{\times n})$ with the standard symplectic structure. The cotangent bundle over a Lie group can be trivialized by right translations, which gives an isomorphism of vector bundles

$$T^*(G^{\times n}) \simeq (T^*G)^{\times n} \simeq \mathfrak{g}^{*\times n} \times G^{\times n}$$

We will choose this trivialization throughout the paper.

The Lie group $G_n := G^{\times n}$ acts naturally on itself by left and right translations. Lifting these actions to $T^*(G^{\times n})$, after the trivialization of the cotangent bundle, we can write the action by left translations as:

$$h_L(x,g) = (Ad^*_{h_1}(x_1), Ad^*_{h_2}(x_2)\dots, Ad^*_{h_n}(x_n), h_1g_1, h_2g_2, \dots, h_ng_n)$$

and the action by right translations as

$$h_R(x,g) = (x_1,\dots,x_n, g_1h_1^{-1},\dots,g_nh_n^{-1})$$

Both these actions are Hamiltonian with moment maps

$$\mu_L(x,g) = (x_1,x_2,\dots,x_n)$$

and

$$\mu_R(x,g) = (-Ad^*_{g_1^{-1}}(x_1),\dots,-Ad^*_{g_n^{-1}}(x_n))$$

respectively.

Actions by left and right translations can be twisted by permutations. In particular, we can twist the action by left translations by a cyclic permutation. Combining the twisted left action with the non-twisted right action we obtain the "gauge action" of G_n on $G^{\times n}$[6]

$$h(g_1,\dots,g_n)=(h_1g_1h_2^{-1},h_2g_2h_3^{-1},\dots,h_ng_nh_1^{-1})$$

Lifting the twisted conjugation action of G_n on $G^{\times n}$ to $T^*(G^{\times n})$ we obtain the "gauge action" on the cotangent bundle:

$$\begin{aligned}h(x,g)=(&Ad^*_{h_1}(x_1),Ad^*_{h_2}(x_2),\dots,Ad^*_{h_n}(x_n),\\ &h_1g_1h_2^{-1},h_2g_2h_3^{-1},\dots,h_ng_nh_1^{-1})\end{aligned} \tag{7}$$

Because this is the diagonal action for two Hamiltonian actions, the gauge action is also Hamiltonian with the moment map $\mu: T^*(G^{\times n})\to\mathfrak{g}^{*\times n}$:

$$\begin{aligned}\mu(x,g)&=\mu_L(x,g)+\mu_R^{tw}(x,g)\\ &=(x_1-Ad^*_{g_n^{-1}}(x_n),x_2-Ad^*_{g_1^{-1}}(x_1),\dots,x_n-Ad^*_{g_{n-1}^{-1}}(x_{n-1}))\end{aligned} \tag{8}$$

where μ_R^{tw} is the right moment map, twisted by cyclic permutation.

Because the gauge action (7) of G_n is Hamiltonian, the quotient space $T^*(G^{\times n})/G_n$ is a Poisson space.[7] Symplectic leaves of $T^*(G^{\times n})/G_n$ are given by the Hamiltonian reduction with respect to the moment map (8). Let $\mathcal{O}_1,\dots,\mathcal{O}_n$ be coadjoint orbits in $\mathfrak{g}^*$, then the corresponding symplectic leaf in $T^*(G^{\times n})/G_n$ is

$$\begin{aligned}\mathcal{S}(\mathcal{O})&=\mu^{-1}(\mathcal{O}_1\times\cdots\times\mathcal{O}_n)/G_n\\ &=\{(x,g)\in\mathfrak{g}^{*\times n}\times G^{\times n}|x_i-Ad^*_{g_{i-1}^{-1}}(x_{i-1})\in\mathcal{O}_i\}/G_n\end{aligned} \tag{9}$$

where G_n acts by the gauge transformations (7) and the indices i should be taken modulo n.

On each of these symplectic leaves we will construct a superintegrable system which we will call a *periodic spin Calogero-Moser chain*.

1.2. The gauge fixing. Let us fix $i\in 1,\dots,n$ and $g=(g_1,\dots,g_n)\in G^{\times n}$. Let $h\in G_n$ such that

$$h_j=\begin{cases}h_ig_{i-1}^{-1}\cdots g_{j+1}^{-1}g_j^{-1} & \text{for } 1\le j<i,\\ h_ig_{i-1}^{-1}\cdots g_2^{-1}g_1^{-1}g_n^{-1}\cdots g_{j+1}^{-1}g_j^{-1} & \text{for } i<j\le n.\end{cases}$$

Denote such element of G_n by h_g (we suppress the dependence on i).

It is easy to check that the gauge transformation of $g=(g_1,\dots,g_n)$ by the element h_g brings it to $(1,\dots,1,h_i(g_ig_{i+1}\cdots g_ng_1g_2,\dots g_{i-1})h_i^{-1},1,\dots,1)$,

[6]One can twist both left and right actions by a permutation. This leads to other superintegrable systems.

[7]This space is singular. Having in mind classical-quantum correspondence we need the algebra of functions on $T^*G^{\times n}/G_n$. Thus, by the quotient space we will always mean the GIT quotient. By definition, functions on $T^*(G^{\times n})/G_n$ are G_n-invariant functions on $T^*(G^{\times n})$.

with the $i^{\rm th}$-entry being the nontrivial entry. This identifies the G_n gauge orbit through $g=(g_1,\ldots,g_n)$ with the G-conjugation orbit through $g_1\cdots g_n$. It thus gives an (i-independent) isomorphism

$$G^{\times n}/G_n \xrightarrow{\sim} G/G,$$

where G/G denotes the set of conjugacy classes in G. On the cotangent bundles the gauge fixing with gives the isomorphism $\varphi_i : \big(\mathfrak{g}^{*\times n}\times G^{\times n}\big)/G_n \xrightarrow{\sim} \big(\mathfrak{g}^{*\times n}\times G\big)/G$ mapping the G_n-orbit $G_n(x,g)$ through $(x,g)\in\mathfrak{g}^{*\times n}\times G^{\times n}$ to the G-orbit through

$$\begin{gathered}\big(Ad^*_{g_i^{-1}\cdots g_1^{-1}}(x_1),\ldots,Ad^*_{g_i^{-1}}(x_i),Ad^*_{g_i^{-1}\cdots g_1^{-1}g_n^{-1}\cdots g_{i+1}^{-1}}(x_{i+1}),\ldots,\\ Ad^*_{g_i^{-1}\cdots g_1^{-1}g_n^{-1}}(x_n),g_{i+1}\cdots g_ng_1\cdots g_i\big)\end{gathered}$$

(for $i=n$ this should be read as $\big(Ad^*_{g_n^{-1}\cdots g_1^{-1}}(x_1),\ldots,Ad^*_{g_n^{-1}}(x_n),g_1\cdots g_n\big)$). Here G is acting diagonally on $\mathfrak{g}^{*\times n}\times G$ via the coadjoint action on $\mathfrak{g}^*$ and the conjugation action on G. From now on we will work with the isomorphism φ_n.

1.3. The regular part of the phase space. The image of the symplectic leaf $\mathcal{S}(\mathcal{O})$ under the isomorphism φ_n is

$$\begin{aligned}\mathcal{S}(\mathcal{O})=\{(z_1,\ldots,z_n,g)\in\mathfrak{g}^{*\times n}\times G\mid z_1-Ad^*_{g^{-1}}z_n\in\mathcal{O}_1,&\\ z_i-z_{i-1}\in\mathcal{O}_i,\ \ i=2,\ldots,n\}/G.&\end{aligned}$$

Define the *regular* part $\mathcal{S}(\mathcal{O})_{reg}\subset\mathcal{S}(\mathcal{O})$ of the phase space as $\mathcal{S}(\mathcal{O})\cap(\mathfrak{g}^{*\times n}\times G')/G$. On $\mathcal{S}(\mathcal{O})_{reg}$ we can choose a representative where g is in the regular part A_{reg} of the real split torus A in G: $g=bab^{-1}$, $z_i=\mathrm{Ad}^*_bx^{(i)}$ with $a\in A_{reg}$. Then we have

$$\begin{aligned}\mathcal{S}(\mathcal{O})_{reg}=\{(x^{(1)},\ldots,x^{(n)},a)\in\mathfrak{g}^{*\times n}\times A_{reg}\mid x^{(1)}-Ad^*_{a^{-1}}x^{(n)}&\in\mathcal{O}_1,\\ x^{(i)}-x^{(i-1)}&\in\mathcal{O}_i,\\ i=2,\ldots,n\}/N_G(A).&\end{aligned}$$

Identify $\mathfrak{g}^*\simeq\mathfrak{g}$ and $\mathfrak{a}^*\simeq\mathfrak{a}$ via the Killing form of $\mathfrak{g}$. The element $y\in\mathfrak{g}^*$ then corresponds to $y_0+\sum_\alpha y_\alpha e_\alpha$, where y_0 is the element in $\mathfrak{a}$ corresponding to $y|_{\mathfrak{a}}$ and $y_\alpha=y(e_{-\alpha})$. Let $\mu^{(j)}\in\mathcal{O}_j$ be vectors $\mu^{(1)}=x^{(1)}-Ad^*_{a^{-1}}x^{(n)}$ and $\mu^{(i)}=x^{(i)}-x^{(i-1)}$ for $i=2,\ldots,n$. For coordinates $x^{(i)}_\alpha$ and $\mu^{(i)}_\alpha$ of vectors $x^{(i)}$ and $\mu^{(i)}$ we then have

$$x^{(1)}_\alpha-a_\alpha^{-1}x^{(n)}_\alpha=\mu^{(1)}_\alpha,\qquad x^{(i)}_\alpha-x^{(i-1)}_\alpha=\mu^{(i)}_\alpha,\qquad i=2,\ldots,n.$$

For the Cartan components we have

$$x^{(i)}_0-x^{(i-1)}_0=\mu^{(i)}_0,\qquad i=1,\ldots,n,$$

with the index i taken to be modulo n.

Solving these equations for $x^{(i)}$ we have

$$x^{(i)}_\alpha = \frac{a_\alpha(\mu^{(1)}_\alpha + \mu^{(2)}_\alpha + \cdots + \mu^{(i)}_\alpha) + \mu^{(i+1)}_\alpha + \mu^{(i+2)} + \cdots + \mu^{(n)}_\alpha}{a_\alpha - 1},$$
$$x^{(i)}_0 = x^{(1)}_0 + \mu^{(2)}_0 + \cdots + \mu^{(i)}_0 = x^{(n)}_0 - \mu^{(n)}_0 - \cdots - \mu^{(i+1)}_0 \tag{10}$$

and we have the constraint

$$\mu^{(1)}_0 + \cdots + \mu^{(n)}_0 = 0. \tag{11}$$

This gives an isomorphism

$$\mathcal{S}(\mathcal{O})_{reg} \xrightarrow{\sim} \big(\nu_{\mathcal{O}}^{-1}(0)/H \times T^*A_{reg}\big)/W \tag{12}$$

which preserves the natural symplectic structures, where $\nu_{\mathcal{O}} : \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \to \mathfrak{a}^*$ is the moment map $(\mu^{(1)}, \ldots, \mu^{(n)}) \mapsto (\mu^{(1)} + \cdots + \mu^{(n)})|_{\mathfrak{a}}$ for the diagonal action of H on the product $\mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ of coadjoint orbits, and $W = N_G(A)/H$ acts diagonally on $\nu_{\mathcal{O}}^{-1}(0)/H \times T^*A_{reg}$. The isomorphism (12) maps the $N_G(A)$-orbit through $\big(x^{(1)}, \ldots, x^{(n)}, a\big)$ to the W-orbit through $\big(H(x^{(1)} - Ad^*_{a^{-1}}x^{(n)}, x^{(2)} - x^{(1)}, \ldots, x^{(n)} - x^{(n-1)}), x^{(n)}_0, a\big)$, where we used the trivialisation $T^*A_{reg} \simeq \mathfrak{a} \times A_{reg}$. The inverse maps the W-orbit through $\big(H(\mu^{(1)}, \ldots, \mu^{(n)}), p, a\big)$ to the $N_G(A)$-orbit through $(x^{(1)}, \ldots, x^{(n)}, a)$, with

$$x^{(i)} = p - \mu^{(n)}_0 - \cdots - \mu^{(i+1)}_0$$
$$+ \sum_\alpha \left(\frac{a_\alpha(\mu^{(1)}_\alpha + \cdots + \mu^{(i)}_\alpha) + \mu^{(i+1)}_\alpha + \cdots + \mu^{(n)}_\alpha}{a_\alpha - 1} \right) e_\alpha, \tag{13}$$

where we use the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ via the Killing form.

1.4. Hamiltonians of a periodic spin CM chain. After the trivialization of the cotangent bundle by translations, we have a natural projection:

$$T^*(G^{\times n}) \simeq \mathfrak{g}^{*\times n} \times G^{\times n} \to \mathfrak{g}^{*\times n} \tag{14}$$

which is simply the projection to the first factor. This projection depends on the trivialization. In this paper we alway assume that we use the trivialization by right translations. However, the corresponding projection of quotient spaces

$$T^*(G^{\times n})/G_n \to (\mathfrak{g}^*/G)^{\times n} \tag{15}$$

does not depend on the trivialization and in this sense is canonical.

The projection (15) is Poisson[8] with the trivial Poisson structure on $(\mathfrak{g}^*/G)^{\times n}$. Thus the $G^{\times n}$-invariant functions on $\mathfrak{g}^{*\times n}$ give rise to a Poisson commutative subalgebra in the algebra of functions on $T^*(G^{\times n})/G_n$. The restriction of these functions to the symplectic leaf $\mathcal{S}(\mathcal{O})$ gives the algebra of

[8]One of the reasons for this is that the equation (14) is the moment map for the left diagonal action of $G^{\times n}$ on the cotangent bundle.

Poisson commuting functions on it. This is the subalgebra of *Hamiltonians of the periodic spin Calogero-Moser chain.*

Now let us describe the restriction of the Hamiltonians corresponding to quadratic Casimir functions

$$H_2^{(k)}(x,g)=\frac{1}{2}\big(x^{(k)},x^{(k)}\big)=\frac{1}{2}\big(x_0^{(k)},x_0^{(k)}\big)+\sum_{\alpha>0}x_\alpha^{(k)}x_{-\alpha}^{(k)} \qquad (1\le k\le n)$$

to the regular part of $\mathcal{S}(\mathcal{O})$, where $x=(x^{(1)},\dots,x^{(n)})\in\mathfrak{g}^{*\times n}$ and $g\in G^{\times n}$. Consider the functions

$$D_k=H_2^{(k)}-H_2^{(k-1)} \qquad (1<k\le n)$$

which we call *Knizhnik-Zamolodchikov-Bernard (KZB) Hamiltonians*[9].

THEOREM 1. *The restriction of the KZB Hamiltonians to $\mathcal{S}(\mathcal{O})_{reg}$ can be written as*

$$D_k=(\mu_0^{(k)},p)-\sum_{l=1}^{k-1}r_{lk}+\sum_{l=k+1}^{n}r_{kl} \tag{16}$$

where r_{kl} for $k\neq l$ is the classical version of the Felder's dynamical r-matrix [**13**]*:*

$$r_{kl}=-\frac{1}{2}(\mu_0^{(k)},\mu_0^{(l)})+\sum_{\alpha}\frac{\mu_{-\alpha}^{(k)}\mu_\alpha^{(l)}}{a_\alpha-1}. \tag{17}$$

REMARK 1. Note that (17) can also be written as

$$r_{kl}=-\frac{1}{2}(\mu_0^{(k)},\mu_0^{(l)})+\sum_{\alpha>0}\frac{\mu_{-\alpha}^{(k)}\mu_\alpha^{(l)}}{a_\alpha-1}-\sum_{\alpha>0}\frac{a_\alpha\mu_\alpha^{(k)}\mu_{-\alpha}^{(l)}}{a_\alpha-1}.$$

PROOF. We need to show that formula (16) gives the expression of D_k in terms of the coordinates on $\mathcal{S}(\mathcal{O})_{reg}$ of $\big(\nu_{\mathcal{O}}^{-1}(0)/H\times T^*A_{reg}\big)/W$, obtained from the isomorphism (12). In particular, let $\big(H(\mu^{(1)},\dots,\mu^{(n)}),p,a\big)\in\nu_{\mathcal{O}}^{-1}(0)/H\times\mathfrak{a}^*\times A_{reg}$ and let $\big((x^{(1)},\dots,x^{(n)}),(1,\dots,1,a)\big)$ be the corresponding point in $\mathfrak{g}^{*\times n}\times G^{\times n}$, with $x^{(i)}$ given by (13). Taking into account the relation $x^{(k)}-x^{(k-1)}=\mu^{(k)}$ between the $x^{(i)}$ and the $\mu^{(j)}$ we have

$$\begin{aligned}D_k&=\big(\mu^{(k)},x^{(k-1)}\big)+\frac{1}{2}\big(\mu^{(k)},\mu^{(k)}\big)\\&=\big(\mu_0^{(k)},x_0^{(k-1)}+\frac{1}{2}\mu_0^{(k)}\big)+\sum_{\alpha}x_\alpha^{(k-1)}\mu_{-\alpha}^{(k)}+\sum_{\alpha>0}\mu_\alpha^{(k)}\mu_{-\alpha}^{(k)}.\end{aligned}$$

[9]The proper name would be *constant Knizhnik-Zamolodchikov-Bernard Hamiltonians* emphasizing the fact that they are related to finite dimensional simple Lie algebras, not to the affine Kac-Moody algebras. See for example references [**13**][**8**][**9**][**34**].

Substitute here the expression (10) for $x_\alpha^{(k-1)}$ in terms of the $\mu_\alpha^{(j)}$:

$$D_k = (\mu_0^{(k)}, x_0^{(k-1)} + \frac{1}{2}\mu_0^{(k)}) + \sum_{l=1}^{k-1}\sum_\alpha \frac{a_\alpha \mu_\alpha^{(l)}\mu_{-\alpha}^{(k)}}{a_\alpha - 1} + \sum_{l=k+1}^{n}\sum_\alpha \frac{\mu_\alpha^{(l)}\mu_{-\alpha}^{(k)}}{a_\alpha - 1}.$$

From here using the identities

$$\sum_\alpha \frac{a_\alpha \mu_\alpha^{(l)}\mu_{-\alpha}^{(k)}}{a_\alpha - 1} = -r_{lk} - \frac{1}{2}(\mu_0^{(k)}, \mu_0^{(l)})$$

and

$$\sum_\alpha \frac{\mu_\alpha^{(l)}\mu_{-\alpha}^{(k)}}{a_\alpha - 1} = r_{kl} + \frac{1}{2}(\mu_0^{(k)}, \mu_0^{(l)})$$

we conclude

$$D_k = \left(\mu_0^{(k)}, x_0^{(k-1)} + \frac{1}{2}\sum_{l=k}^{n}\mu_0^{(l)} - \frac{1}{2}\sum_{l=1}^{k-1}\mu_0^{(l)}\right) - \sum_{l=1}^{k-1} r_{lk} + \sum_{l=k+1}^{n} r_{kl}.$$

Using $x_0^{(k-1)} = p - \mu_0^{(n)} - \cdots - \mu_0^{(k)}$ (see (13)) and the constraint (11) we obtain (16). □

A particularly simple expression has the quadratic Hamiltonian $H_2^{(n)}$ on $\mathcal{S}(\mathcal{O})_{reg}$,

$$H_2^{(n)} = \frac{1}{2}(p,p) + \sum_\alpha \frac{\mu_\alpha \mu_{-\alpha}}{(1-a_\alpha)(1-a_\alpha^{-1})}.$$

Here $\mu_\alpha = \mu_\alpha^{(1)} + \cdots + \mu_\alpha^{(n)}$. Setting $q_\alpha = \alpha(\log(a))$ this formula becomes a familiar formula for the spin Calogero-Moser Hamiltonian,

$$H_2^{(n)} = \frac{1}{2}(p,p) - \sum_{\alpha>0} \frac{\mu_\alpha \mu_{-\alpha}}{2sh^2(q_\alpha)}. \tag{18}$$

Note that the periodic spin CM chain is the classical version of the dynamical Knizhnik-Zamolodchikov equation from [8][9].

1.5. Periodic spin Calogero-Moser chain as a superintegrable system. Now let us establish the superintegrability of the periodic spin CM chain. For this we should construct an intermediate Poisson manifold and projections as in [26][30].

Observe that we have natural Poisson projections:

$$T^*(G^{\times n})/G_n \stackrel{p_1}{\to} \mathcal{P}_n \stackrel{p_2}{\to} \mathcal{B}_n \tag{19}$$

Firstly, $\mathcal{P}_n = (\mathfrak{g}^{*\times n} \times_{(\mathfrak{g}^*/G)^{\times n}} \mathfrak{g}^{*\times n})/G_n$ with

$$\mathfrak{g}^{*\times n} \times_{(\mathfrak{g}^*/G)^{\times n}} \mathfrak{g}^{*\times n} := \{(x,y) \in \mathfrak{g}^{*\times n} \times \mathfrak{g}^{*\times n} | Gy_i = -Gx_{i-1}\},$$

where Gz is the coadjoint orbit through $z \in \mathfrak{g}^*$ and the indices i are taken modulo n, and G_n is acting by

$$g(x,y) := (Ad^*_{g_1}(x_1), \ldots, Ad^*_{g_n}(x_n), Ad^*_{g_1}(y_1), \ldots, Ad^*_{g_n}(x_n)). \tag{20}$$

The map p_1 is the map induced from the G_n-equivariant map $\mu_L \times \mu_R^{tw}$. Explicitly, the mapping p_1 acts as

$$\begin{aligned} p_1 : G_n(x, g) \mapsto & \; G_n(\mu_L(x, g), \mu_R^{tw}(x, g)) \\ & = G_n(x_1, x_2, \dots, x_n, -Ad^*_{g_n^{-1}}(x_n), -Ad_{g_1^{-1}}(x_1), \dots, \\ & \quad - Ad_{g_{n-1}^{-1}}(x_{n-1})). \end{aligned} \tag{21}$$

Secondly,

$$\mathcal{B}_n = (\mathfrak{g}^*/G)^{\times n}$$

and the map p_2 is the projection to the first factor.

Restricting projection p_1 to the symplectic leaf $\mathcal{S}(\mathcal{O})$ (see (9)), we obtain the surjective Poisson projection

$$p_{1,\mathcal{O}} : \mathcal{S}(\mathcal{O}) \to \mathcal{P}(\mathcal{O})$$

where

$$\mathcal{P}(\mathcal{O}) = \{(x, y) \in \mathfrak{g}^{*\times n} \times_{(\mathfrak{g}^*/G)^{\times n}} \mathfrak{g}^{*\times n} \mid x_i + y_i \in \mathcal{O}_i\}/G_n \subset \mathcal{P}_n$$

with the G_n-action described by (20).

Restricting the second projection p_2 to $\mathcal{P}(\mathcal{O})$ we have the Poisson projection

$$p_{2,\mathcal{O}} : \mathcal{P}(\mathcal{O}) \to \mathcal{B}(\mathcal{O}) \subset \mathcal{B}_n, \qquad G_n(x, y) \mapsto (Gx_1, \dots, Gx_n)$$

where $\mathcal{B}(\mathcal{O})$ is the image of $p_{2,\mathcal{O}}$. It can be explicitly described as

$$\mathcal{B}(\mathcal{O}) = \{(\mathcal{O}^{(1)}, \dots, \mathcal{O}^{(n)}) \in (\mathfrak{g}^*/G)^{\times n} \mid \mathcal{O}_i \subseteq \mathcal{O}^{(i)} - \mathcal{O}^{(i-1)}\},$$

with the indices i taken modulo n.

LEMMA 1. *The dimension of $\mathcal{B}(\mathcal{O})$ is nr where r is the rank of the Lie algebra $\mathfrak{g}$.*

PROOF. Let $\mathfrak{h}^*_{\geq 0}$ be the positive Weyl chamber in the dual space to Cartan subalgebra of $\mathfrak{g}$. For each generic orbit $\mathcal{O}$ there is a unique representative $y \in \mathcal{O} \cup \mathfrak{h}^* > 0$. Let x_1 be such representative of $\mathcal{O}^{(1)}$. Let us describe orbits $\mathcal{O}^{(2)}$ such that $x_1 + y_1 \in \mathcal{O}^{(2)}$ for some $y_1 \in \mathcal{O}_1$. Assume that $\mathcal{O}^{(1)}$ is very large, i.e. $||x_1|| >> 1$. Because $\mathcal{O}_1$ is compact $||y_1|| < C_1$ for some constant C_1 determined by the orbit $\mathcal{O}_1$. Let $c_k^{(i)}$ be the value of k-th Casimir function on the orbit $\mathcal{O}^{(i)}$.

For k-th Casimir function $c_k^{(2)}$ we have:

$$c_k^{(2)} = c_k(x_1 + y_1) = c_k^{(1)} + \sum_{i=1}^{r} \frac{\partial c_k(h)}{\partial h_i}|_{h=x_1}(y_1)_i + O(y^2)$$

Because the matrix $\frac{\partial c_k(h)}{\partial h_i}$ is nondegerate for generic h, possible values of the Euclidean vector with components $c_k^{(2)}$ span an r-dimensional neighborhood of $\{c_k^{(1)}\}$.

Repeating this argument for each $\mathcal{O}^{(i)}$ we conclude that each of $\mathcal{O}_i$ is non-zero, $\dim(\mathcal{B}(\mathcal{O})) = nr$. $\square$

Now let us describe the fiber $\mathcal{P}(\mathcal{O};\mathcal{O}^{(1)},\dots,\mathcal{O}^{(n)})$ of $p_{2,\mathcal{O}}$ over $(\mathcal{O}^{(1)},\dots,\mathcal{O}^{(n)})\in\mathcal{B}(\mathcal{O})$:

$$\begin{aligned}&\mathcal{P}(\mathcal{O};\mathcal{O}^{(1)},\dots,\mathcal{O}^{(n)})\\ &\quad=\{(x,y)\in\mathfrak{g}^{*\times n}\times\mathfrak{g}^{*\times n}|x_i+y_i\in\mathcal{O}_i,\quad x_i\in\mathcal{O}^{(i)},y_i\in-\mathcal{O}^{(i-1)}\}/G_n\\ &\quad=\prod_{i=1}^n\big\{(x_i,y_i)\in\mathcal{O}^{(i)}\times-\mathcal{O}^{(i-1)}\mid x_i+y_i\in\mathcal{O}_i\big\}/G\end{aligned}$$

with the index i taken modulo n and with G acting by the diagonal coadjoint action on $\mathcal{O}^{(i)}\times-\mathcal{O}^{(i-1)}$. Set

$$\mathcal{M}(\mathcal{O}^{(1)},\mathcal{O}^{(2)},\mathcal{O}^{(3)})=\{(x,y,z)\in\mathcal{O}^{(1)}\times\mathcal{O}^{(2)}\times\mathcal{O}^{(3)}\mid x+y+z=0\}/G,\tag{22}$$

with G acting by the diagonal coadjoint action. Then

$$\begin{aligned}\big\{(x_i,y_i)\in\mathcal{O}^{(i)}\times-\mathcal{O}^{(i-1)}\mid x_i+y_i\in\mathcal{O}_i\big\}/G&\xrightarrow{\sim}\mathcal{M}(-\mathcal{O}^{(i)},\mathcal{O}^{(i-1)},\mathcal{O}_i)\\ G(x_i,y_i)&\mapsto G(-x_i,-y_i,x_i+y_i),\end{aligned}\tag{23}$$

and hence we conclude that

$$\mathcal{P}(\mathcal{O};\mathcal{O}^{(1)},\dots,\mathcal{O}^{(n)})\simeq\prod_{i=1}^n\mathcal{M}(-\mathcal{O}^{(i)},\mathcal{O}^{(i-1)},\mathcal{O}_i),\tag{24}$$

with the index i taken modulo n.

LEMMA 2. *Let $\mathcal{O}^{(1)}$, $\mathcal{O}^{(2)}$ be generic, sufficiently large coadjoint orbits and $\mathcal{O}^{(3)}\neq 0$, then*

$$\dim(\mathcal{M}(\mathcal{O}^{(1)},\mathcal{O}^{(2)},\mathcal{O}^{(3)}))=\dim(\mathcal{O}^{(3)})-2r.$$

PROOF. Let $x\in\mathcal{O}^{(1)}$ be the unique representative which lies in the positive Weyl chamber. Assume this orbit is “big”, i.e. $||x||>>1$. The condition $x+y+z=0$ for $y\in\mathcal{O}^{(2)}$ and $z\in\mathcal{O}^{(3)}$ for a large orbit $\mathcal{O}^{(2)}$ means that we have r constraints $c_k(-x-y)=c_k^{(3)}$ on y. For large orbits $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$ these constraints are independent. Taking into account that we are quotienting by H we have $\dim(\mathcal{M}(\mathcal{O}^{(1)},\mathcal{O}^{(2)},\mathcal{O}^{(3)}))=\dim(\mathcal{O}^{(3)})-2r$. □

COROLLARY 1. *Thus the dimension of the fiber is* $\dim(\mathcal{P}(\mathcal{O};\mathcal{O}^{(1)},\dots,\mathcal{O}^{(n)}))=\sum_{i=1}^n\dim(\mathcal{O}_i)-2nr$.

Each of the factors in (24) is the Hamiltonian reduction of the product of the three coadjoint orbits relative to the moment map of the diagonal coadjoint G-action, and therefore carries a natural symplectic structure. Moduli spaces $\mathcal{M}(\mathcal{O}^{(1)},\mathcal{O}^{(2)},\mathcal{O}^{(3)})$ and therefore fibers of $p_{2,\mathcal{O}}$ are stratified symplectic spaces.

THEOREM 2. *The Hamiltonian system generated by any Hamiltonian for the periodic spin CM chain described in section 1.4 is superintegrable with* *the superintegrable structure described by the Poisson maps*

$$\mathcal{S}(\mathcal{O}) \xrightarrow{p_{1,\mathcal{O}}} \mathcal{P}(\mathcal{O}) \xrightarrow{p_{2,\mathcal{O}}} \mathcal{B}(\mathcal{O})$$

as introduced earlier in this section.

Here, as everywhere above, we assume that $\mathcal{O}_i \neq \{0\}$ for each $i = 1, \ldots, n$.

Proof. For $G_n(x, y) \in \mathcal{P}(\mathcal{O})$ let $\widetilde{g}_i \in G$ such that $y_{i+1} = -Ad^*_{\widetilde{g}_i^{-1}}(x_i)$. Then

$$p_{1,\mathcal{O}}^{-1}\big(G_n(x, y)\big) = \{G_n(x, g) \in \mathcal{S}(\mathcal{O}) \mid g_i \in \widetilde{g}_i Z_G(y_{i+1})\}$$

(index i taken modulo n) which, generically, is isotropic and of dimension $nr = \dim(\mathcal{B}(\mathcal{O}))$.

It remains to check the balance of dimensions. It follows from (12) that

$$\dim(\mathcal{S}(\mathcal{O})) = \sum_{i=1}^{n} \dim(\mathcal{O}_i).$$

By Remark 2 we have, for generic $(\mathcal{O}^{(1)}, \ldots, \mathcal{O}^{(n)}) \in (\mathfrak{g}^*/G)^{\times n}$,

$$\begin{aligned}\dim(\mathcal{P}(\mathcal{O})) &= \dim(\mathcal{B}(\mathcal{O})) + \dim(\mathcal{P}(\mathcal{O}; \mathcal{O}^{(1)}, \ldots, \mathcal{O}^{(n)})) \\ &= \dim(\mathcal{B}(\mathcal{O})) + \sum_{i=1}^{n} \dim(\mathcal{O}_i) - 2nr.\end{aligned}$$

Then

$$\dim(\mathcal{P}(\mathcal{O})) + \dim(\mathcal{B}(\mathcal{O})) = \sum_{i=1}^{n} \dim(\mathcal{O}_i) = \dim(\mathcal{S}(\mathcal{O})),$$

as desired. □

Remark 2. In the compact case, the quantum version of functions on $\mathcal{M}(\mathcal{O}^{(1)}, \mathcal{O}^{(2)}, \mathcal{O}^{(3)})$ is the algebra of endomorphisms $End((V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3})^G)$ of the subspace of G-invariant vectors in the tensor product $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$ with V_{λ_i} the representation corresponding to $\mathcal{O}^{(i)}$.

The quantum version of the algebra of functions on the fiber $\mathcal{P}(\mathcal{O}; \mathcal{O}^{(1)}, \ldots, \mathcal{O}^{(n)})$ is the algebra of endomorphisms of the vector space

$$\mathrm{Hom}_G(V_{\lambda_1}, V_{\lambda_n} \otimes V_1) \otimes \mathrm{Hom}_G(V_{\lambda_2}, V_{\lambda_1} \otimes V_2) \otimes \cdots \otimes \mathrm{Hom}_G(V_{\lambda_n}, V_{\lambda_{n-1}} \otimes V_n).$$

Here the orbits $\mathcal{O}_i$ correspond to V_i, and $\mathrm{Hom}_G(V_{\lambda_i}, V_{\lambda_{i-1}} \otimes V_i)$ is the space of G-linear intertwiners $V_{\lambda_i} \to V_{\lambda_{i-1}} \otimes V_i$. For details see [**35**].

1.6. Constructing solutions by the projection method and angle variables. For $\mathcal{H}$ a G-invariant function on $\mathfrak{g}^*$, write $\mathcal{H}^{(i)}$ for the G_n-invariant function on $T^*(G^{\times n})$ defined by $\mathcal{H}^{(i)}(x, g) := \mathcal{H}(x_i)$. The Hamiltonian flow through $(x, g) \in \mathfrak{g}^{*\times n} \times G^{\times n}$ generated by $\mathcal{H}^{(i)}$ is

$$(x(t_i), g(t_i)) = (x_1, \ldots, x_n, g_1, \ldots, g_{i-1}, e^{\nabla\mathcal{H}(x_i)t_i} g_i, g_{i+1} \ldots, g_n) \tag{25}$$

where $\nabla\mathcal{H}(x) \in \mathfrak{g}$ is the gradient of $\mathcal{H}$ at $x \in \mathfrak{g}^*$, i.e.,

$$y(\nabla\mathcal{H}(x)) = \frac{d}{dt}\mathcal{H}(x+ty)|_{t=0}$$

for all $y \in \mathfrak{g}^*$. The projection of such flow line to $T^*(G^{\times n})/G^{\times n}$ and further restricted to $\mathcal{S}(\mathcal{O}) \subset T^*(G^{\times n})/G_n$ is a flow line of the Hamiltonian vector field generated by the restriction of $\mathcal{H}^{(i)}$ to $\mathcal{S}(\mathcal{O})$.

Now let us construct angle variables, i.e., functions on $S(\mathcal{O})$ which evolve linearly with respect to the evolution (25) for each $i = 1, \ldots, n$. Write $\mathfrak{a}_+^* \subset \mathfrak{g}^*$ for the elements $x \in \mathfrak{g}^*$ which vanish on root spaces and satisfy $(x, \alpha) > 0$ for $\alpha \in R_+$, where $(\cdot, \cdot)$ is the bilinear form on $\mathfrak{g}^*$ induced by the Killing form. Write $\mathfrak{g}'^*$ for the elements in $\mathfrak{g}^*$ which are G-conjugate to an element in $\mathfrak{a}_+^*$, relative to the coadjoint action.

For $(x, g) \in \mathfrak{g}'^{*\times n} \times G^{\times n}$ define $s_i \in G$ by the property $Ad^*_{s_i}(x_i) \in \mathfrak{a}_+^*$. These elements are defined only up to $s_i \mapsto a_i s_i$ where $a_i \in H$. Gauge transformations $h \in G_n$ act by $(s_1, \ldots, s_n) \mapsto (s_1 h_1^{-1}, \ldots, s_n h_n^{-1})$.

Let $G_{\mathbb{C}}$ be a complexification of G, which we take to be connected. Let $H_{\mathbb{C}} \subset G_{\mathbb{C}}$ be the Cartan subgroup $Z_{G_{\mathbb{C}}}(\mathfrak{h})$, where $\mathfrak{h}$ is the Cartan subalgebra $\mathfrak{a} \oplus i\mathfrak{a}$ of the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ of $G_{\mathbb{C}}$. Then $A \subseteq H \subset H_{\mathbb{C}}$. We identify $\mathfrak{g}^*$ with the real subspace of $\mathfrak{g}_{\mathbb{C}}^*$ consisting of the complex linear functionals that take real values on $\mathfrak{g}$. For finite dimensional $G_{\mathbb{C}}$-representations $V_1, \ldots, V_n$ choose vector $v_i \in V_i$ of $H_{\mathbb{C}}$-weight λ_{i+1} and linear functionals $u_i^* \in V_i^*$ of $H_{\mathbb{C}}$-weight $-\lambda_i$ (indices i taken modulo n).[10] Define

$$f_{u,v}(x, g) = u_1^*(s_1 g_1 s_2^{-1} v_1) u_2^*(s_2 g_2 s_3^{-1} v_2) \ldots u_n^*(s_n g_n s_1^{-1} v_n) \tag{26}$$

for $(x, g) \in (\mathfrak{g}'^* \times G)^{\times n}$. This expression is well defined (i.e., invariant with respect to transformations $s_i \to a_i s_i$ with $a_i \in H$), and invariant with respect to gauge transformations. Thus, it defines a function on the subset $(\mathfrak{g}'^{*\times n} \times G^{\times n})/G_n$ of $T^*(G^{\times n})/G_n$.

From the G-invariance of $\mathcal{H}$ we have the identity

$$u_i^*(s_i e^{t_i \nabla\mathcal{H}(x_i)} g_i s_{i+1}^{-1} v_i) = e^{t_i \lambda_i(\nabla\mathcal{H}(y_i))} u_i^*(s_i g_i s_{i+1}^{-1} v_i)$$

where $y_i = Ad^*_{s_i}(x_i) \in \mathfrak{a}_+^*$, and consequently

$$f_{u,v}(x(t_i), g(t_i)) = e^{t_i \lambda_i(\nabla\mathcal{H}(y_i))} f_{u,v}(x, g). \tag{27}$$

Logarithms of these functions evolve linearly, and hence they produce angle variables for the Hamiltonians $\mathcal{H}^{(i)}$ on $S(\mathcal{O}) \cap (\mathfrak{g}'^{*\times n} \times G^{\times n})/G_n$.

2. Open spin Calogero-Moser chains

Recall from the introduction that G is a split real connected Lie group with finite center which admits a complexification, and $K \subset G$ is the subgroup of fixed points of a fixed Cartan involution Θ of G. Recall furthermore the root space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha>0}(\mathbb{R}e_\alpha \oplus \mathbb{R}e_{-\alpha})$ with the Cartan

[10] Recall that G acts on dual vectors as $(gu^*)(v) = u^*(g^{-1}v)$.

subalgebra $\mathfrak{a} \subset \mathfrak{g}$ and the root vectors $e_\alpha \in \mathfrak{g}_\alpha$ such that the infinitesimal Cartan involution θ acts as

$$\theta(h) = -h, \quad \theta(e_\alpha) = -e_{-\alpha}$$

for $h \in \mathfrak{a}$ and $\alpha \in R$. We will furthermore normalise the root vectors in such a way that $(e_\alpha, e_{-\alpha}) = 1$, with $(\cdot, \cdot)$ the Killing form of $\mathfrak{g}$.

To avoid cumbersome notations, we will not always indicate in notations that we are in the open case. This leads to an overlap of some of the notations with the ones for the periodic case. For instance, the moment maps, Poisson spaces and Poisson projections will be denoted in the same way.

2.1. The phase space as the Hamiltonian reduction. Consider for $n \geq 0$ the manifold $T^*(G^{\times n+1})$ with the standard symplectic structure. We trivialize the cotangent bundle $T^*(G^{\times n+1})$ by right translations:

$$T^*(G^{\times n+1}) \simeq (T^*G)^{\times n+1} \simeq \mathfrak{g}^{*\times n+1} \times G^{\times n+1} \tag{28}$$

We have a natural action of $K \times G^{\times n}$ on $G^{\times n+1}$ by left translations:

$$(k_\ell, h_1, \ldots, h_n)_L(g_0, g_1, \ldots, g_n) = (k_\ell g_0, h_1 g_1, \ldots, h_n g_n)$$

This action lifts to the following Hamiltonian action on $T^*G^{\times n+1}$,

$$\begin{aligned}(k_\ell, h_1, \ldots, &h_n)_L(x_0, \ldots, x_n, g_0, g_1, \ldots, g_n) = \\ &= (Ad^*_{k_\ell}(x_0), Ad^*_{h_1}(x_1), \ldots, Ad^*_{h_n}(x_n), k_\ell g_0, h_1 g_1, \ldots, h_n g_n)\end{aligned}$$

with the moment map

$$\mu_L(x, g) = (\pi(x_0), x_1, \ldots, x_n)$$

where the projection $\pi : \mathfrak{g}^* \to \mathfrak{k}^*$ is the dual map dual to the embedding $\mathfrak{k} \hookrightarrow \mathfrak{g}$.

Similarly, the action of $G^{\times n} \times K$ on $G^{\times n+1}$ by right translations

$$(h_1, \ldots, h_n, k_r)_R(g_0, g_1, \ldots, g_n) = (g_0 h_1^{-1}, g_1 h_2^{-1}, \ldots, g_{n-1} h_n^{-1}, g_n k_r^{-1})$$

lifts to the following Hamiltonian action on $T^*G^{\times n+1}$,

$$\begin{aligned}(h_1, \ldots, h_n, k_r)_R&(x_0, \ldots, x_n, g_0, g_1, \ldots, g_n) \\ &= (x_0, x_1, \ldots, x_n, g_0 h_1^{-1}, g_1 h_2^{-1}, \ldots, g_{n-1} h_n^{-1}, g_n k_r^{-1}),\end{aligned}$$

with the moment map

$$\mu_R(x, g) = (-Ad^*_{g_0^{-1}}(x_0), -Ad^*_{g_1^{-1}}(x_1), \ldots, -Ad^*_{g_{n-1}^{-1}}(x_{n-1}), -\pi(Ad^*_{g_n^{-1}}(x_n))).$$

As a result, the group $G_{n,K} := K \times G^{\times n} \times K$ acts on $T^*(G^{\times n+1})$ as

$$\begin{aligned}&(k_\ell, h_1, \ldots, h_n, k_r)(x_0, \ldots, x_n, g_0, \ldots, g_n) = \\ &\quad =(Ad^*_{k_\ell}(x_0), Ad^*_{h_1}(x_1), \ldots, Ad^*_{h_n}(x_n), k_\ell g_0 h_1^{-1}, h_1 g_1 h_2^{-1}, \ldots, h_n g_n k_r^{-1})\end{aligned} \tag{29}$$

with $k_\ell, k_r \in K$ and $h_i \in G$. This action is Hamiltonian with the moment map $\mu : T^*(G^{\times n}) \to \mathfrak{k}^* \times \mathfrak{g}^{*\times n} \times \mathfrak{k}^*$ given by

(30)
$$\begin{aligned}
&\mu((x_0, \dots, x_n, g_0, \dots, g_n) = (\mu_L(x, g), 0) + (0, \mu_R(x, g)) = \\
&\quad = (\pi(x_0), x_1 - Ad^*_{g_0^{-1}}(x_0), x_2 - Ad^*_{g_1^{-1}}(x_1), \dots, x_n - Ad^*_{g_{n-1}^{-1}}(x_{n-1}), \\
&\qquad -\pi(Ad^*_{g_n^{-1}}(x_n))).
\end{aligned}$$

For $n = 0$ this is the $K \times K$ action $(k_\ell, k_r)(x, g) = (Ad^*_{k_\ell}(x), k_\ell g k_r^{-1})$ on T^*G, with the moment map $(x, g) \mapsto (\pi(x), -\pi(Ad_{g^{-1}}(x)))$. It is easy to check explicitly that this moment map intertwines the action of $G_{n,K}$ on $T^*(G^{\times n+1})$ given by (29) with its diagonal coadjoint action on $\mathfrak{k}^* \times \mathfrak{g}^{*\times(n-1)} \times \mathfrak{k}^*$.

Because the action of $G_{n,K}$ on $T^*(G^{\times n+1})$ is Hamiltonian, the space $T^*(G^{\times n+1})/G_{n,K}$[11] is Poisson with symplectic leaves being given by the Hamiltonian reduction with respect to the moment map (30). Let $\mathcal{O} = (\mathcal{O}^K_\ell \times \mathcal{O}_1 \times \dots, \mathcal{O}_n \times \mathcal{O}^K_r)$ with $\mathcal{O}_i \subset \mathfrak{g}^*$ coadjoint G-orbits and $\mathcal{O}^K_\ell, \mathcal{O}^K_r \subset \mathfrak{k}^*$ coadjoint K-orbits, then the corresponding symplectic leaf in $T^*(G^{\times n+1})/G_{n,K}$ is

(31)
$$\begin{aligned}
\mathcal{S}(\mathcal{O}) &= \mu^{-1}(\mathcal{O})/G_{n,K} \\
&= \big\{(x_0, \dots, x_n, g_0, \dots, g_n) \in \mathfrak{g}^{*\times n+1} \times G^{\times n+1} \mid \pi(x_0) \in \mathcal{O}^K_\ell, \\
&\qquad - \pi(Ad^*_{g_n^{-1}}(x_n)) \in \mathcal{O}^K_r, \\
&\qquad x_1 - Ad^*_{g_0^{-1}}(x_0) \in \mathcal{O}_1, \dots, x_n - Ad^*_{g_{n-1}^{-1}}(x_{n-1}) \in \mathcal{O}_n\big\}/G_{n,K}.
\end{aligned}$$

Each symplectic leaf $\S(\mathcal{O})$ is a stratified symplectic space and is the phase space for the corresponding open spin Calogero-Moser chain. We will describe the largest stratum of $\mathcal{S}(\mathcal{O})$ later.

2.2. The Hamiltonians of the open spin Calogero-Moser chain. After the trivialization (28) of $T^*(G^{\times n+1})$ by right translations we have a natural Poisson projection $T^*(G^{\times n+1}) \to \mathfrak{g}^{*\times n+1}$ to the first factor. It is $G_{n,K}$-invariant with the following action of $G_{n,K}$ on $\mathfrak{g}^{*\times n+1}$

$$(k_\ell, h_1, \dots, h_n, k_r) : (x_0, x_1, \dots, x_n) \mapsto (Ad^*_{k_\ell}(x_0), Ad^*_{h_1}(x_1), \dots, Ad^*_{h_n}(x_n)).$$

This gives rise to the projection

$$p : T^*(G^{\times n+1})/G_{n,K} \to (\mathfrak{g}^*/G)^{\times n+1}$$

which is Poisson because it is the composition of natural Poisson projections

(32)
$$T^*(G^{\times n+1})/G_{n,K} \to (\mathfrak{g}^{*\times n+1})/G_{n,K} = \mathfrak{g}^*/K \times (\mathfrak{g}^*/G)^{\times n} \to (\mathfrak{g}^*/G)^{\times n+1}.$$

[11] Recall that here and in everywhere else in this paper X/H means the GIT quotient for a Lie group H action on a manifold X.

Here the Poisson structure on the right is trivial (the Poisson tensor is zero). The last projection is a consequence of the embedding $K \hookrightarrow G$.

Restricting this projection to the symplectic leaf $\mathcal{S}(\mathcal{O})$ we have the Poisson projection

$$p_{\mathcal{O}} : \mathcal{S}(\mathcal{O}) \to \mathcal{B}(\mathcal{O}), \quad G_{n,K}(x_0, \dots, x_n, g_0, \dots, g_n) \mapsto (Gx_0, \dots, Gx_n) \tag{33}$$

where $\mathcal{B}(\mathcal{O}) \subset (\mathfrak{g}^*/G)^{\times n+1}$ is, by definition, the image of $\mathcal{S}(\mathcal{O})$. It can be described explicitly from the description of $\mathcal{S}(\mathcal{O})$ as

$$\begin{aligned}\mathcal{B}(\mathcal{O}) = \big\{(\mathcal{O}^{(0)}, \dots, \mathcal{O}^{(n)}) \in (\mathfrak{g}^*/G)^{\times n+1} \mid \mathcal{O}_\ell^K \subseteq \pi(\mathcal{O}^{(0)}), \mathcal{O}_r^K \subseteq -\pi(\mathcal{O}^{(n)}),\\ \mathcal{O}_1 \subseteq \mathcal{O}^{(1)} - \mathcal{O}^{(0)}, \dots, \mathcal{O}_n \subseteq \mathcal{O}^{(n)} - \mathcal{O}^{(n-1)}\big\}.\end{aligned} \tag{34}$$

Hamiltonians of the open spin Calogero-Moser system are pull back p^* of functions on $(\mathfrak{g}^*/G)^{\times n+1}$ restricted to $\mathcal{S}(\mathcal{O})$.

The subalgebra of Hamiltonians is a Poisson commuting subalgebra. Quadratic Hamiltonians are given by Casimir functions. We will compute the radial components of the quadratic Casimirs explicitly in section 2.5. Hamiltonians are constant on fibers of the projection $p_{\mathcal{O}}$.

2.3. The gauge fixing. Fix $i \in \{0, \dots, n\}$. For $g = (g_0, \dots, g_n) \in G^{\times n+1}$ and $k_\ell, k_r \in K$ define $h \in G^{\times n}$ (depending on i, g, k_ℓ, k_r) by

$$h_j = \begin{cases} k_\ell g_0 g_1 \cdots g_{j-1} & \text{if } \ j \le i, \\ k_r g_n^{-1} g_{n-1}^{-1} \cdots g_j^{-1} & \text{if } \ j > i. \end{cases}$$

Then $(k_\ell, h_1, \dots, h_n, k_r)$ acts on $g \in G^{\times n+1}$ as

$$g \mapsto (1, \dots, 1, k_\ell g_0 g_1 \dots g_n k_r^{-1}, 1, \dots, 1).$$

Here the nontrivial entry is at the position i. This gives an i-independent isomorphism

$$G^{\times n+1}/G_{n,K} \to K\backslash G/K, \qquad G_{n,K}(g_0, \dots, g_n) \mapsto K g_0 g_1 \dots g_n K. \tag{35}$$

For the cotangent bundles the gauge fixing gives an isomorphism

$$\phi_i : \big(\mathfrak{g}^{*\times n+1} \times G^{\times n+1}\big)/G_{n,K} \xrightarrow{\sim} K\backslash(\mathfrak{g}^{*\times n+1} \times G)/K \tag{36}$$

mapping the $G_{n,K}$-orbit through $(x, g) \in \mathfrak{g}^{*\times n+1} \times G^{\times n+1}$ to the double K-coset through

$$\begin{aligned}&\big(x_0, Ad^*_{g_0}(x_1), \dots, Ad^*_{g_0 \cdots g_{i-1}}(x_i), Ad^*_{g_n^{-1} \cdots g_{i+1}^{-1}}(x_{i+1}), \dots,\\ &\quad Ad^*_{g_n^{-1}}(x_n), g_0 g_1 \cdots g_n\big)\end{aligned}$$

For example for $i = n$ this expression is $\big(x_0, Ad^*_{g_0}(x_1), \dots, Ad^*_{g_0 \cdots g_{n-1}}(x_n), g_0 g_1 \cdots g_n\big)$. In (36) the double K-cosets in the codomain of ϕ_i are taken relative to the i-dependent $K \times K$-action

$$\begin{aligned}(k_\ell, k_r)(x_0, \dots, x_n, g) = (Ad^*_{k_\ell}(x_0), \cdots, Ad^*_{k_\ell}(x_i), Ad^*_{k_r}(x_{i+1}), \dots,\\ Ad^*_{k_r}(x_n), k_\ell g k_r^{-1})\end{aligned}$$

on $\mathfrak{g}^{*\times n+1}\times G$.

Now we can describe the symplectic leaf $\mathcal{S}(\mathcal{O})$ as a subvariety in $K\backslash(\mathfrak{g}^{*\times n+1}\times G)/K$ though the isomorphism φ_n as

(37)
$$\begin{aligned}\mathcal{S}(\mathcal{O})=K\backslash\big\{(y_0,y_1,\ldots,y_n,g)\in\mathfrak{g}^{*\times n+1}\times G\mid \pi(y_0)\in\mathcal{O}_\ell^K,\\ -\pi\big(Ad^*_{g^{-1}}(y_n)\big)\in\mathcal{O}_r^K,\\ y_1-y_0\in\mathcal{O}_1,\ldots,y_n-y_{n-1}\in\mathcal{O}_n\big\}/K.\end{aligned}$$

Note that, as in the periodic case, $\mathcal{S}(\mathcal{O})$ is a symplectic stratified space. From now on we will focus mostly on the largest stratum $\mathcal{S}(\mathcal{O})_{reg}$.

2.4. The regular part of the symplectic leaf $\mathcal{S}(\mathcal{O})$. We use the gauge fixing isomorphism φ_n in the remainder of the text. We now use $K\backslash G/K\simeq A/W$ with $W=N_K(A)/M$ the Weyl group of G (see subsection §3 of the introduction) to describe the regular part of the symplectic leaf $\mathcal{S}(\mathcal{O})$ in radial coordinates.

Define the regular part of the phase space $\mathcal{S}(\mathcal{O})$ (see (37)) as
$$\mathcal{S}(\mathcal{O})_{reg}=\mathcal{S}(\mathcal{O})\cap K\backslash(\mathfrak{g}^{*\times n+1}\times G_{reg})/K.$$
The regular part $\mathcal{S}(\mathcal{O})_{reg}\subset\mathcal{S}(\mathcal{O})$ is its largest stratum of the stratified symplectic space $\mathcal{S}(\mathcal{O})$.

We can then choose a representative of $K(y_0,\ldots,y_n,g)K\in\mathcal{S}(\mathcal{O})_{reg}$ with the G-component in A_{reg} by writing $g=k_\ell a k_r^{-1}$ and $y_i=Ad^*_{k_\ell}(x^{(i)})$ with $k_\ell,k_r\in K$ and $a\in A_{reg}$. It follows that
$$\begin{aligned}\mathcal{S}(\mathcal{O})_{reg}\simeq\big\{(x^{(0)},\ldots,x^{(n)},a)\in\mathfrak{g}^{*\times n+1}\times A_{reg}\mid\pi(x^{(0)})\in\mathcal{O}_\ell^K,\\ -\pi(Ad^*_{a^{-1}}(x^{(n)}))\in\mathcal{O}_r^K,\\ x^{(1)}-x^{(0)}\in\mathcal{O}_1,\ldots,x^{(n)}-x^{(n-1)}\in\mathcal{O}_n\big\}/N_K(A).\end{aligned}$$
Here $N_K(A)$ acts diagonally on $\mathfrak{g}^{*\times n+1}\times A_{reg}$ via the coadjoint action on $\mathfrak{g}^*$ and the conjugation action on A_{reg}.[12] We can now also divide out the action of $M=Z_K(A)$, to obtain the isomorphism
$$\begin{aligned}\mathcal{S}(\mathcal{O})_{reg}\simeq\big\{(M(x^{(0)},\ldots,x^{(n)}),a)\in\mathfrak{g}^{*\times n+1}/M\times A_{reg}\mid\pi(x^{(0)})\in\mathcal{O}_\ell^K,\\ -\pi(Ad^*_{a^{-1}}(x^{(n)}))\in\mathcal{O}_r^K,\\ x^{(1)}-x^{(0)}\in\mathcal{O}_1,\ldots,x^{(n)}-x^{(n-1)}\in\mathcal{O}_n\big\}/W,\end{aligned}$$
where M acts by the diagonal coadjoint action on $\mathfrak{g}^{*\times n+1}$ and W acts diagonally on the space $\mathfrak{g}^{*\times n+1}/M\times A_{reg}$ in the natural way.

Recall that we identified $\mathfrak{g}\simeq\mathfrak{g}^*$ and $\mathfrak{a}\simeq\mathfrak{a}^*$ via the Killing form, so that $x\in\mathfrak{g}^*$ corresponds to $x_0+\sum_\alpha x_\alpha e_\alpha$ with x_0 the element in $\mathfrak{a}$ corresponding

[12]We use here the fact that $k_\ell a k_r^{-1}=k'_\ell a' k_r'^{-1}$ for $k_\ell,k'_\ell,k_r,k'_r\in K$ and $a,a'\in A_{reg}$ is implying that $k_\ell^{-1}k'_\ell=k_r^{-1}k'_r\in N_K(A)$, cf., e.g., [20, §VII.3]. This essentially follows from the global Cartan decomposition of G.

to $x|_{\mathfrak{a}} \in \mathfrak{a}^*$ and $x_\alpha = x(e_{-\alpha})$. Denote by $x_0^{(k)}$, $x_\alpha^{(k)}$ the components of vectors $x \in \mathfrak{g}^{*\times n+1}$ from the k-th factor of $\mathfrak{g}^{*n+1}$, and $\mu_0^{(k)}$, $\mu_\alpha^{(k)}$ the components of $\mu \in \mathcal{O}_k$. For $y \in \mathfrak{k}^*$ we write $y_{[\alpha]} = y(e_{-\alpha} - e_\alpha)$, so that $y_{[\alpha]} = -y_{[-\alpha]}$.

Consider

$$T(\mathcal{O})_{reg} = \{(x^{(0)}, \dots, x^{(n)}, a) \in \mathfrak{g}^{*\times n+1} \times A_{reg} \mid \pi(x^{(0)}) \in \mathcal{O}_\ell^K, \\ -\pi(Ad^*_{a^{-1}}(x^{(n)})) \in \mathcal{O}_r^K, \\ x^{(1)} - x^{(0)} \in \mathcal{O}_1, \dots, x^{(n)} - x^{(n-1)} \in \mathcal{O}_n\}.$$

Clearly $S(\mathcal{O})_{reg} = T(\mathcal{O})_{reg}/N_K(A)$.

For $(x^{(0)}, \dots, x^{(n)}, a) \in T(\mathcal{O})_{reg}$ write $\mu' = \pi(x^{(0)}) \in \mathcal{O}_\ell^K$, $\mu'' = -\pi(Ad^*_{a^{-1}}(x^{(n)})) \in \mathcal{O}_r^K$ and $\mu^{(i)} = x^{(i)} - x^{(i-1)} \in \mathcal{O}_i$ for $i = 1, \dots, n$. The Cartan components of $x^{(k)}$ and their root coordinates then satisfy

$$\begin{aligned} x_\alpha^{(0)} - x_{-\alpha}^{(0)} &= \mu'_{[\alpha]}, & a_\alpha x_{-\alpha}^{(n)} - a_\alpha^{-1} x_\alpha^{(n)} &= \mu''_{[\alpha]}, \\ x_\alpha^{(i)} - x_\alpha^{(i-1)} &= \mu_\alpha^{(i)}, & x_0^{(i)} - x_0^{(i-1)} &= \mu_0^{(i)} \end{aligned} \tag{38}$$

for $i = 1, \dots, n$.

It is easy to solve the equations for Cartan parts $x_0^{(i)}$ $(0 < i < n)$ in terms of Catran components of $x^{(0)}$, $x^{(n)}$ and $\mu^{(j)}$,

$$x_0^{(i)} = \mu_0^{(i)} + \dots + \mu_0^{(1)} + x_0^{(0)} = x_0^{(n)} - \mu_0^{(n)} - \dots - \mu_0^{(i+1)} \tag{39}$$

Proposition 1. *The following identities hold for $\alpha \in R$ and $k = 0, 1, \dots, n$:*

$$x_\alpha^{(k)} = K_\alpha + \sum_{l=1}^{k} \frac{a_\alpha \mu_\alpha^{(l)} - a_\alpha \mu_{-\alpha}^{(l)}}{a_\alpha - a_\alpha^{-1}} + \sum_{l=k+1}^{n} \frac{a_\alpha^{-1} \mu_\alpha^{(l)} - a_\alpha \mu_{-\alpha}^{(l)}}{a_\alpha - a_\alpha^{-1}} \tag{40}$$

where

$$K_\alpha = \frac{a_\alpha \mu'_{[\alpha]} + \mu''_{[\alpha]}}{a_\alpha - a_\alpha^{-1}}. \tag{41}$$

Proof. Denote

$$\mu = \mu^{(1)} + \dots + \mu^{(n)}. \tag{42}$$

Note that $x^{(n)} - x^{(0)} = \mu$.

Fix $\beta \in R_+$. For $x_{\pm\beta}^{(1)}$ and $x_{\pm\beta}^{(n)}$ the formula $x^{(n)} - x^{(0)} = \mu$ implies

$$x_\beta^{(n)} - x_\beta^{(0)} = \mu_\beta, \quad x_{-\beta}^{(n)} - x_{-\beta}^{(0)} = \mu_{-\beta}.$$

Combined with the first line of (38) we end up with four linear equations in $x_\beta^{(0)}$, $x_{-\beta}^{(0)}$, $x_\beta^{(n)}$, $x_{-\beta}^{(n)}$ which, by the assumption that a is regular, are uniquely solved by

$$
(43)\qquad \begin{aligned} x_\alpha^{(0)} &= \frac{a_\alpha \mu'_{[\alpha]} + \mu''_{[\alpha]} + (a_\alpha^{-1}\mu_\alpha - a_\alpha \mu_{-\alpha})}{a_\alpha - a_\alpha^{-1}}, \\ x_\alpha^{(n)} &= \frac{a_\alpha \mu'_{[\alpha]} + \mu''_{[\alpha]} + (a_\alpha \mu_\alpha - a_\alpha \mu_{-\alpha})}{a_\alpha - a_\alpha^{-1}} \end{aligned}
$$

for $\alpha = \beta, -\beta$ (here we used that $a_{-\beta} = a_\beta^{-1}$, $\mu'_{[-\beta]} = -\mu'_{[\beta]}$ and $\mu''_{[-\beta]} = -\mu''_{[\beta]}$).

By the second line of (38) we then obtain

$$
x_\alpha^{(k)} = \frac{a_\alpha \mu'_{[\alpha]} + \mu''_{[\alpha]} + (a_\alpha^{-1}\mu_\alpha - a_\alpha \mu_{-\alpha}) + (a_\alpha - a_\alpha^{-1})(\mu_\alpha^{(1)} + \cdots + \mu_\alpha^{(k)})}{a_\alpha - a_\alpha^{-1}}
$$

for $k = 0, 1, \ldots, n$. Substituting (42) it is now easy to see that this is exactly what we wanted to prove. □

The proposition and (39) give an isomorphism

$$
(44)\qquad S(\mathcal{O})_{reg} \simeq \big((\mathcal{O}_\ell^K \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \times \mathcal{O}_r^K)/M \times T^*A_{reg}\big)/W,
$$

mapping $N_K(A)(x^{(0)}, \ldots, x^{(n)}, a)$ to the W-orbit of $(M(\mu', \mu^{(1)}, \ldots, \mu^{(n)}, \mu''), x_0^{(n)}, a)$, which preserves the natural symplectic structures. Here the finite discrete group $M = Z_K(A) \subset K$ acts diagonally via the coadjoint action, and $W = N_K(A)/M$ acts diagonally. The quantum version of this isomorphism is described in [**35**].

2.5. Quadratic Hamiltonians of open spin Calogero-Moser chain on the regular part of the phase space. In this section we compute the restriction of the Hamiltonian corresponding to the quadratic Casimir function on $\mathfrak{g}^*$,

$$
H_2^{(k)}(x,g) = \frac{1}{2}(x^{(k)}, x^{(k)}) = \frac{1}{2}(x_0^{(k)}, x_0^{(k)}) + \sum_{\alpha>0} x_\alpha^{(k)} x_{-\alpha}^{(k)}
$$

to the regular part of $\mathcal{S}(\mathcal{O})$ (see (31)) for $k = 0, \ldots, n$, where $(x, g) \in \mathfrak{g}^{*\times n+1} \times G^{\times n+1}$. Here $(\cdot,\cdot)$ is the Killing form and $x_\alpha^{(i)}$, $x_0^{(i)}$ are the components of $x^{(i)}$ which were computed in the previous section on the regular part of the phase space.

We first consider the differences, which we will call the *boundary Knizhnik-Zamolodchikov-Bernard (bKZB) Hamiltonians*,

$$
D_k = H_2^{(k)} - H_2^{(k-1)} \qquad (1 \le k \le n).
$$

THEOREM 3. *For the bKZB Hamiltonians we have the following formula:*

$$
(45)\quad D_k = (\mu_0^{(k)}, x_0^{(n)}) - \sum_{l=1}^{k-1}(r_{lk} + r_{lk}^{\theta_l}) + (\sum_\alpha K_\alpha \mu_{-\alpha}^{(k)} - \kappa_k) + \sum_{l=k+1}^{n}(r_{kl} - r_{kl}^{\theta_k}).
$$

Here r_{kl} for $k \neq l$ is Felder's rescaled dynamical r-matrix

$$r_{kl} = -\frac{1}{2}(\mu_0^{(k)}, \mu_0^{(l)}) + \sum_\alpha \frac{\mu_{-\alpha}^{(k)}\mu_\alpha^{(l)}}{a_\alpha^2 - 1}, \tag{46}$$

θ_k is the transpose of the Chevalley involution θ acting on $\mu^{(k)}$,

$$\kappa_k = \frac{1}{2}(\mu_0^{(k)}, \mu_0^{(k)}) + \sum_\alpha \frac{(\mu_\alpha^{(k)})^2}{1 - a_\alpha^2}$$

is the core quadratic classical dynamical k-matrix and K_α is given by (41).

Proof. The first step of the proof is the same as in the proof of Theorem 1, resulting in the expression

$$D_k = (\mu_0^{(k)}, x_0^{(k-1)} + \frac{1}{2}\mu_0^{(k)}) + \sum_\alpha x_\alpha^{(k-1)}\mu_{-\alpha}^{(k)} + \sum_{\alpha>0} \mu_\alpha^{(k)}\mu_{-\alpha}^{(k)}. \tag{47}$$

Now let us the formula (40) for $x_\alpha^{(k-1)}$,

$$\begin{aligned}\sum_\alpha x_\alpha^{(k-1)}\mu_{-\alpha}^{(k)} &= \sum_{l=1}^{k-1}\sum_\alpha \frac{a_\alpha\mu_\alpha^{(l)}\mu_{-\alpha}^{(k)} - a_\alpha\mu_{-\alpha}^{(l)}\mu_{-\alpha}^{(k)}}{a_\alpha - a_\alpha^{-1}} \\ &+ \sum_\alpha K_\alpha\mu_{-\alpha}^{(k)} + \sum_\alpha \frac{a_\alpha^{-1}\mu_\alpha^{(k)}\mu_{-\alpha}^{(k)} - a_\alpha\mu_{-\alpha}^{(k)}\mu_{-\alpha}^{(k)}}{a_\alpha - a_\alpha^{-1}} \\ &+ \sum_{l=k+1}^{n}\sum_\alpha \frac{a_\alpha^{-1}\mu_{-\alpha}^{(k)}\mu_\alpha^{(l)} - a_\alpha\mu_{-\alpha}^{(k)}\mu_{-\alpha}^{(l)}}{a_\alpha - a_\alpha^{-1}}.\end{aligned} \tag{48}$$

We express the different terms in the right hand side of (48) in terms of the dynamical r-matrix and k-matrix.

Note first that

$$r_{kl}^{\theta_k} = \frac{1}{2}(\mu_0^{(k)}, \mu_0^{(l)}) - \sum_\alpha \frac{\mu_\alpha^{(k)}\mu_\alpha^{(l)}}{a_\alpha^2 - 1} = r_{lk}^{\theta_l}.$$

Then the terms in the right hand side of (48) with l strictly smaller than k can be rewritten as

$$\sum_\alpha \frac{a_\alpha\mu_\alpha^{(l)}\mu_{-\alpha}^{(k)} - a_\alpha\mu_{-\alpha}^{(l)}\mu_{-\alpha}^{(k)}}{a_\alpha - a_\alpha^{-1}} = -(r_{lk} + r_{lk}^{\theta_l})$$

while the terms in the right hand side of (48) with l strictly larger than k reduce to

$$\sum_\alpha \frac{a_\alpha^{-1}\mu_{-\alpha}^{(k)}\mu_\alpha^{(l)} - a_\alpha\mu_{-\alpha}^{(k)}\mu_{-\alpha}^{(l)}}{a_\alpha - a_\alpha^{-1}} = (\mu_0^{(k)}, \mu_0^{(l)}) + (r_{kl} - r_{kl}^{\theta_k}).$$

Finally, for the middle term in (48) a direct computation shows that

$$\sum_{\alpha} \frac{a_\alpha^{-1}\mu_\alpha^{(k)}\mu_{-\alpha}^{(k)} - a_\alpha\mu_{-\alpha}^{(k)}\mu_{-\alpha}^{(k)}}{a_\alpha - a_\alpha^{-1}} = \frac{1}{2}(\mu_0^{(k)}, \mu_0^{(k)}) - \sum_{\alpha>0} \mu_\alpha^{(k)}\mu_{-\alpha}^{(k)} - \kappa_k.$$

Substitute these formulas in (48), then the resulting formula (47) for D_k becomes

$$\begin{aligned} D_k &= (\mu_0^{(k)}, x_0^{(k-1)} + \mu_0^{(k)} + \mu_0^{(k+1)} + \cdots + \mu_0^{(n)}) \\ &\quad - \sum_{l=1}^{k-1}(r_{lk} + r_{lk}^{\theta_l}) + (\sum_\alpha K_\alpha \mu_{-\alpha}^{(k)} - \kappa_k) + \sum_{l=k+1}^{n}(r_{kl} - r_{kl}^{\theta_k}). \end{aligned}$$

By (39) this reduces to the formula (45). □

The quantum versions of the boundary KZB Hamiltonians in the present context were obtained in [**35**, §6]. It was extended to the case of non-split real semisimple Lie groups G in [**33**].

For the Hamiltonian $H_2^{(n)}$ we obtain by (43) the expression

$$\begin{aligned} H_2^{(n)} &= \frac{1}{2}(p,p) \\ &\quad + \sum_{\alpha>0} \frac{(a_\alpha \mu'_{[\alpha]} + \mu''_{[\alpha]} + a_\alpha(\mu_\alpha - \mu_{-\alpha}))(a_\alpha^{-1}\mu'_{[\alpha]} + \mu''_{[\alpha]} + a_\alpha^{-1}(\mu_\alpha - \mu_{-\alpha}))}{(a_\alpha - a_{-\alpha})^2} \end{aligned}$$

on $\mathcal{S}(\mathcal{O})_{reg}$, where $\mu = \mu^{(1)} + \cdots + \mu^{(n)}$. Here we use notation $p = x_0^{(n)}$ for the cotangent vectors to A_{reg} in formula (44). Note that the potential term only depends on the restrictions $\pi(\mu^{(i)})$ of $\mu^{(i)} \in \mathfrak{g}^*$ to $\mathfrak{k}$, since $\mu_\alpha - \mu_{-\alpha} = -(\pi(\mu))_{[\alpha]}$. The radial component of the quantum quadratic Hamiltonian in the current open context was obtained in [**35**, §6].

2.6. The superintegrability of the open spin CM chain. In this section we will prove that Poisson commutative subalgebra of Hamiltonians constructed in section 2.2 defines a superintegrable system. Fix $\mathcal{O} = (\mathcal{O}_\ell^K, \mathcal{O}_1, \ldots, \mathcal{O}_n, \mathcal{O}_r^K) \in \mathfrak{k}^*/K \times (\mathfrak{g}^*/G)^{\times n} \times \mathfrak{k}^*/K$. We will construct Poisson projections

$$S(\mathcal{O}) \xrightarrow{p_{1,\mathcal{O}}} \mathcal{P}(\mathcal{O}) \xrightarrow{p_{2,\mathcal{O}}} \mathcal{B}(\mathcal{O}) \tag{49}$$

such that $p_{\mathcal{O}} = p_{2,\mathcal{O}} \circ p_{1,\mathcal{O}}$ (see (33)), satisfying the desired properties.

Let $(\mathfrak{k}^* \times \mathfrak{g}^{*\times n}) \times_{(\mathfrak{g}^*/G)^{\times n+1}} (\mathfrak{g}^{*\times n} \times \mathfrak{k}^*)$ be the subset of $(\mathfrak{k}^* \times \mathfrak{g}^{*\times n}) \times (\mathfrak{g}^{*\times n} \times \mathfrak{k}^*)$ consisting of elements $(z_\ell, x_1, \ldots, x_n, y_1, \ldots, y_n, z_r)$ satisfying

$$z_\ell \in -\pi(Gy_1), \quad x_i \in -Gy_{i+1}\ (1 \le i < n), \quad z_r \in -\pi(Gx_n).$$

The gauge group $G_{n,K}$ acts on $(\mathfrak{k}^* \times \mathfrak{g}^{*\times n}) \times_{(\mathfrak{g}^*/G)^{\times n+1}} (\mathfrak{g}^{*\times n} \times \mathfrak{k}^*)$ by

(50)

$$\begin{aligned} &(k_\ell, h_1, \ldots, h_n, k_r)(z_\ell, x_1, \ldots, x_n, y_1, \ldots, y_n, z_r) = \\ &\qquad = (Ad^*_{k_\ell} z_\ell, Ad^*_{h_1} x_1, \ldots, Ad^*_{h_n} x_n, Ad^*_{h_1} y_1, \ldots, Ad^*_{h_n} y_n, Ad^*_{k_r} z_r). \end{aligned}$$

Consider the resulting Poisson space

$$\mathcal{P} = \big((\mathfrak{k}^* \times \mathfrak{g}^{*\times n}) \times_{(\mathfrak{g}^*/G)^{\times n+1}} (\mathfrak{g}^{*\times n} \times \mathfrak{k}^*)\big)/G_{n,K}$$

and define the Poisson map

$$p_1 : T^*(G^{\times n+1})/G_{n,K} \to \mathcal{P}$$

by

$$\begin{aligned} p_1\big(G_{n,K}(x,g)\big) &= G_{n,K}\big(\mu_L(x,g), \mu_R(x,g)\big) \\ &= G_{n,K}\big(\pi(x_0), x_1, \dots, x_n, -Ad^*_{g_0^{-1}}(x_0), \dots, \\ &\qquad -Ad^*_{g_{n-1}^{-1}}(x_{n-1}), -\pi(Ad^*_{g_n^{-1}}(x_n))\big). \end{aligned}$$

Here $(x,g) = (x_0, \dots, x_n, g_0, \dots, g_n) \in \mathfrak{g}^{*\times n+1} \times G^{\times n+1} \simeq T^*(G^{\times n+1})$. Define the Poisson projection

$$p_2 : \mathcal{P} \to (\mathfrak{g}^*/G)^{\times n+1}$$

by

$$p_2\big(G_{n,K}(z_\ell, x_1, \dots, x_n, y_1, \dots, y_n, z_r)\big) = (-Gy_1, Gx_1, \dots, Gx_n),$$

with the trivial Poisson structure on the target space.

The restriction of the Poisson projection p_1 to the symplectic leaf $S(\mathcal{O}) \subset T^*(G^{\times n+1})/G_{n,K}$ (see (31)) gives the Poisson projection

$$p_{1,\mathcal{O}} : S(\mathcal{O}) \to \mathcal{P}(\mathcal{O})$$

where

$$\mathcal{P}(\mathcal{O}) = (\mu_L \times \mu_R)(\mu^{-1}(\mathcal{O}))/G_{n,K},$$

or, more explicitly,

(51)
$$\begin{aligned} \mathcal{P}(\mathcal{O}) = \big\{&(z_\ell,\, x_1, \dots, x_n, y_1, \dots, y_n, z_r) \in (\mathfrak{k}^* \times \mathfrak{g}^{*\times n}) \times_{(\mathfrak{g}^*/G)^{\times n+1}} (\mathfrak{g}^{*\times n} \times \mathfrak{k}^*) \mid \\ &z_\ell \in \mathcal{O}_\ell^K,\, x_1 + y_1 \in \mathcal{O}_1, \dots, x_n + y_n \in \mathcal{O}_n,\, z_r \in \mathcal{O}_r^K\big\}/G_{n,K}. \end{aligned}$$

The generic fibers of this mapping are isotropic submanifolds in $\mathcal{S}(\mathcal{O})$.

The restriction of the Poisson projection p_2 to $\mathcal{P}(\mathcal{O})$ gives a surjective Poisson projection

$$p_{2,\mathcal{O}} : \mathcal{P}(\mathcal{O}) \to \mathcal{B}(\mathcal{O}),$$

with $\mathcal{B}(\mathcal{O})$ given by (34). Clearly, the composition of $p_{2,\mathcal{O}} \circ p_{1,\mathcal{O}} : \mathcal{S}(\mathcal{O}) \to \mathcal{B}(\mathcal{O})$ is the projection $p_\mathcal{O}$ as given by (33).

Now let us describe fibers of $p_{2,\mathcal{O}}$ are symplectic leaves of $\mathcal{P}(\mathcal{O})$.

Lemma 3. *We have he following symplectomorphism*

(52)
$$\begin{aligned} p_{2,\mathcal{O}}^{-1}((\mathcal{O}^{(0)}, \dots, \mathcal{O}^{(n)})) &\xrightarrow{\sim} \mathcal{M}(-\mathcal{O}^{(0)}, \mathcal{O}_\ell^K) \\ &\qquad \times \prod_{i=1}^{n} \mathcal{M}(-\mathcal{O}^{(i)}, \mathcal{O}^{(i-1)}, \mathcal{O}_i) \times \mathcal{M}(\mathcal{O}^{(n)}, \mathcal{O}_r^K) \end{aligned}$$

where symplectic spaces $\mathcal{M}(-\mathcal{O}^{(i)}, \mathcal{O}^{(i-1)}, \mathcal{O}_i)$ *are defined in (22) and*

$$\mathcal{M}(\mathcal{O}', \mathcal{O}^K) = \{(x, z) \in \mathcal{O}' \times \mathcal{O}^K \mid \pi(x) + z = 0\}/K$$

Here $\mathcal{O} \subset \mathfrak{g}^*$ *is a* G*-coadjoint orbit and* $\mathcal{O}^K \subset \mathfrak{k}^*$ *is a* K*-coadjoint orbit. It has a natural symplectic structure because it is the Hamiltonian reduction of* $\mathcal{O} \times \mathcal{O}^K$ *with respect to the Hamiltonian diagonal action of* K*.*

Proof. Let $(\mathcal{O}^{(0)}, \dots, \mathcal{O}^{(n)}) \in \mathcal{B}(\mathcal{O})$. By a direct computation, the fiber $p_{2,\mathcal{O}}^{-1}((\mathcal{O}^{(0)}, \dots, \mathcal{O}^{(n)}))$ consists of the $G_{n,K}$-orbits in $\mathcal{P}$ with representatives

$$(z_\ell, x_1, \dots, x_n, y_1, \dots, y_n, z_r) \in (\mathfrak{k}^* \times \mathfrak{g}^{*\times n}) \times (\mathfrak{g}^{*\times n} \times \mathfrak{k}^*)$$

satisfying the following conditions

$$z_\ell \in \mathcal{O}_\ell^K \cap \pi(\mathcal{O}^{(0)}), \qquad x_i + y_i \in \mathcal{O}_i \quad (1 \le i \le n), \qquad z_r \in \mathcal{O}_r^K \cap -\pi(\mathcal{O}^{(n)}),$$
$$-y_1 \in \mathcal{O}^{(0)}, \qquad x_i \in \mathcal{O}^{(i)}, -y_{i+1}\mathcal{O}^{(i)} \in (1 \le i \le n-1), \qquad x_n \in \mathcal{O}^{(n)}.$$

Using this explicit description of the fiber, we can write it as a direct product of symplectic spaces. The isomorphism (23) for symplectic spaces $\mathcal{M}(\mathcal{O}^{(1)}, \mathcal{O}^{(2)}, \mathcal{O}^{(3)})$ defined by (22) gives factors $\mathcal{M}(-\mathcal{O}^{(i)}, \mathcal{O}^{(i-1)}, \mathcal{O}_i)$. The space

$$\mathcal{M}(\mathcal{O}', \mathcal{O}^K) = \{(x, z) \in \mathcal{O}' \times \mathcal{O}^K \mid \pi(x) + z = 0\}/K$$

with K acting by the diagonal coadjoint action is symplectic because, see above. The isomorphism

$$\mathcal{M}(\mathcal{O}', \mathcal{O}^K) \xrightarrow{\sim} (\mathcal{O}' \cap \pi^{-1}(\mathcal{O}^K))/K,$$

completes the proof. The isomorphism maps $G_{n,K}(z_\ell, x_1, \dots, x_n, y_1, \dots, y_n, z_r) \in p_{2,\mathcal{O}}^{-1}((\mathcal{O}^{(0)}, \dots, \mathcal{O}^{(n)}))$ to

$$(K(\widetilde{y}_1, z_\ell), G(-x_1, -y_1, x_1 + y_1), \dots, G(-x_n, -y_n, x_n + y_n), K(\widetilde{x}_n, z_r)),$$

with $\widetilde{y}_1 \in Gy_1 = -\mathcal{O}^{(0)}$ such that $-\pi(\widetilde{y}_1) = z_\ell$ and $\widetilde{x}_n \in Gx_n = \mathcal{O}^{(n)}$ such that $-\pi(\widetilde{x}_n) = z_r$. $\square$

Note that for generic $\mathcal{O}'$, the symplectic space $\mathcal{M}(\mathcal{O}', \mathcal{O}^K)$ is of dimension $\dim(\mathcal{O}^K)$.

Remark 3. In the compact case, the algebra of function on the fiber of $p_{2,\mathcal{O}}$ has the algebra of endomorphisms of the vector space

$$\mathrm{Hom}_K(V_{\lambda_0}, U_{\nu_\ell}) \otimes \mathrm{Hom}_G(V_{\lambda_1}, V_{\lambda_0} \otimes V_{\mu_1}) \otimes \cdots \otimes \mathrm{Hom}_G(V_{\lambda_n}, V_{\lambda_{n-1}} \otimes V_{\mu_n}) \otimes \mathrm{Hom}_K(U_{\nu_r}, V_{\lambda_n})$$

as natural quantization, where the finite dimensional G-representation V_{λ_i} (resp. V_{μ_i}) corresponds to $\mathcal{O}^{(i)}$ (resp. $\mathcal{O}_i$) and the K-representation U_{ν_ℓ} (resp. U_{ν_r}) corresponds to $\mathcal{O}_\ell^K$ (resp. $\mathcal{O}_r^K$), compare with Remark 2 in the cyclic case. For details see [**35**] (which treats the noncompact case) and [**31**].

Lemma 4. *Dimensions of spaces* $\mathcal{B}(\mathcal{O})$ *and* $\mathcal{P}(\mathcal{O})$ *are*

$$\dim(\mathcal{B}(\mathcal{O})) = (n+1)r, \quad \dim(\mathcal{P}(\mathcal{O})) = \dim(\mathcal{O}) - 2nr$$

where we define $\dim(\mathcal{O})$ *as* $\dim(\mathcal{O}_\ell^K) + \sum_{i=1}^n \dim(\mathcal{O}_i) + \dim(\mathcal{O}_r^K)$*.*

PROOF. The proof of the dimension formula for $\mathcal{B}(\mathcal{O})$ is completely similar to the periodic case. It is enough to consider large orbits. For the dimension of $\mathcal{P}(\mathcal{O})$ we have:

$$\dim(\mathcal{P}(\mathcal{O})) = \dim(\mathcal{B}(\mathcal{O})) + \dim\big(p_{2,\mathcal{O}}^{-1}(\mathcal{O}^{(0)}, \ldots, \mathcal{O}^{(n)})\big)$$

and by (52) the dimension of $p_{2,\mathcal{O}}^{-1}((\mathcal{O}^{(0)}, \ldots, \mathcal{O}^{(n)}))$ is equal to

$$\begin{aligned} \dim(\mathcal{M}(-\mathcal{O}^{(0)}, \mathcal{O}_\ell^K)) &+ \sum_{i=1}^{n} \dim(\mathcal{M}(-\mathcal{O}^{(i)}, \mathcal{O}^{(i-1)}, \mathcal{O}_i)) \\ &+ \dim(\mathcal{M}(\mathcal{O}^{(n)}, \mathcal{O}_r^K)) = \\ &= \dim(\mathcal{O}_\ell^K) + \sum_{i=1}^{n}(\dim(\mathcal{O}_i) - 2r) + \dim(\mathcal{O}_r^K) = \dim(\mathcal{O}) - 2nr. \end{aligned}$$

This finishes the proof. □

We now have the following main result of this section.

THEOREM 4. *The Hamiltonian system generated by any Hamiltonian for the open spin CM chain described in section 2.2 is superintegrable with the superintegrable structure described by the surjective Poisson maps*

$$\mathcal{S}(\mathcal{O}) \xrightarrow{p_{1,\mathcal{O}}} \mathcal{P}(\mathcal{O}) \xrightarrow{p_{2,\mathcal{O}}} \mathcal{B}(\mathcal{O})$$

as introduced earlier in this section.

Recall that $\mathcal{O}_i \neq \{0\}$ for all $i = 0, 1, \ldots, n$.

PROOF. We already verified most of the conditions. What remains to show is the matching of dimensions,

$$\dim(\mathcal{S}(\mathcal{O})) = \dim(\mathcal{P}(\mathcal{O})) + \dim(\mathcal{B}(\mathcal{O})). \tag{53}$$

For the collection $\mathcal{O} = (\mathcal{O}_\ell^K, \mathcal{O}_1, \ldots, \mathcal{O}_n, \mathcal{O}_r^K)$ of coadjoint orbits we write $\dim(\mathcal{O})$ for the sum of the dimensions of the coadjoint orbits.

For the dimension of the symplectic leaf $\mathcal{S}(\mathcal{O})$ we have, using (44),

$$\dim(\mathcal{S}(\mathcal{O})) = 2r + \dim(\mathcal{O}),$$

with r the rank of $\mathfrak{g}$. For $\mathcal{P}(\mathcal{O})$ we obtain for generic $(\mathcal{O}^{(0)}, \ldots, \mathcal{O}^{(n)}) \in \mathcal{B}(\mathcal{O})$.

Because

$$\dim(\mathcal{B}(\mathcal{O})) = (n+1)r,$$

we have

$$\begin{aligned} \dim(\mathcal{P}(\mathcal{O})) + \dim(\mathcal{B}(\mathcal{O})) &= \dim(\mathcal{O}) - 2nr + 2\dim(\mathcal{B}(\mathcal{O})) \\ &= \dim(\mathcal{O}) + 2r \\ &= \dim(\mathcal{S}(\mathcal{O})), \end{aligned}$$

as desired. □

2.7. Constructing solutions by the projection method and angle variables. Let $\mathcal{H}$ be a G-invariant function on $\mathfrak{g}^*$ and $\mathcal{H}^{(i)}$ for $i = 0, \dots, n$ the associated G_n-invariant function $(x, g) \mapsto \mathcal{H}(x_i)$ on $T^*(G^{\times n+1})$ (cf. section 1.6). The Hamiltonian flow generated by $\mathcal{H}^{(i)}$ on $T^*(G^{\times n+1}) \simeq \mathfrak{g}^{*\times n+1} \times G^{\times n+1}$ was already described in section 1.6. The flow line passing through (x, g) at $t = 0$ is

$$(x(t_i), g(t_i)) = (x_0, \dots, x_n, g_0, \dots, g_{i-1}, e^{\nabla\mathcal{H}(x_i)t_i} g_i, g_{i+1} \dots, g_n). \tag{54}$$

The corresponding Hamiltonian flow on the symplectic leaf

$$\mathcal{S}(\mathcal{O}) \subset T^*(G^{\times n+1})/G_{n,K}$$

is obtained by projecting the flow (54) to $T^*(G^{\times n+1})/G_{n,K}$ and restricting it to $\mathcal{S}(\mathcal{O})$. Thus, we reformulated the problem of solving nonlinear differential equations of motion for open spin Calogero-Moser chains to a problem of linear algebra. This is a version of the original projection method which goes back to earlier papers on Calogero-Moser type systems [**28**].

Now let us describe angle variables for this integrable dynamics. We use the notations from section 1.6. Fix $(x, g) \in \mathfrak{g}'^{*\times n+1} \times G^{\times n+1}$. For $i = 1, \dots, n$ define elements $s_i \in G$ by the condition $Ad^*_{s_i}(x_i) \in \mathfrak{a}^*_+$. As in the periodic case (see section 1.6) it is defines up to $s_i \mapsto a_i s_i$ with $a_i \in H \subset G$, where $H \subset G$ is the Cartan subgroup containing A. Define $s_0 \in K$ such that $Ad^*_{s_0}(x_0)|_{\mathfrak{p}} \in \mathfrak{a}^*_+$ (where we view $\mathfrak{a}^*_+$ now as subset of $\mathfrak{p}^*$ in the natural manner). The element s_0 is defined up to $s_0 \mapsto m s_0$ with $m \in M = Z_K(A)$. Similarly, we define $s_{n+1} \in K$ such that $Ad^*_{s_{n+1}}(x_n)|_{\mathfrak{p}} \in \mathfrak{a}^*_+$.

Choose finite dimensional representations $V_0, V_1, \dots, V_n$ of $G_{\mathbb{C}}$, $H_{\mathbb{C}}$-weight vectors $v_i \in V_i$ of weight λ_{i+1} for $0 \le i < n$ and $H_{\mathbb{C}}$-weight vectors $u^*_j \in V^*_j$ of weight $-\lambda_j$ for $0 \le j \le n$. Finally, we choose M-invariant vectors $u^*_0 \in V^*_0$ and $v_n \in V_n$ (i.e., $m u^*_0 = u^*_0$ and $m v_n = v_n$ for all $m \in M$). Define

$$\begin{aligned} f_{u,v}(x, g) = u^*_0(s_0 g_0 s_1^{-1} v_0) u^*_1(s_1 g_1 s_2^{-1} v_1) \cdots \\ u^*_{n-1}(s_{n-1} g_{n-1} s_n^{-1} v_{n-1}) u^*_n(s_n g_n s_{n+1}^{-1} v_n). \end{aligned} \tag{55}$$

It is an easy check that $f_{u,v}(x, g)$ is a well defined $G_{n,K}$-invariant function on $\mathfrak{g}'^{*\times n+1} \times G^{\times n+1}$.

Similarly as in the periodic case (see section 1.6) we then have for $i = 1, \dots, n$,

$$f_{u,v}(x(t_i), g(t_i)) = e^{t_i \lambda_i(\nabla\mathcal{H}(y_i))} f_{u,v}(x, g) \tag{56}$$

with $y_i = Ad^*_{s_i}(x_i) \in \mathfrak{a}^*_+$. Logarithms of these functions thus evolve linearly, and hence give rise to angle variables for the Hamiltonians $\mathcal{H}^{(i)}$ on $\mathcal{S}(\mathcal{O}) \cap (\mathfrak{g}'^{*\times n+1} \times G^{\times n+1})/G_{n,K}$.

For $i = 0$ we need to restrict further to $(x, g) \in \mathfrak{g}'^{*\times n+1} \times G^{\times n+1}$ with $x_0 \in \mathfrak{p}$, and assume that $u^*_0 \in V^*_0$ is not only M-invariant but also a $H_{\mathbb{C}}$-weight vector, say of weight $-\lambda_0$. In this case $\mathrm{Ad}^*_{s_0}(x_0) = y_0 \in \mathfrak{a}^*_+$ and hence

$$u^*_0(s_0 e^{t_0 \nabla\mathcal{H}(x_0)} g_0 s_1^{-1} v_0) = e^{t_0 \lambda_0(\mathcal{H}(y_0))} u^*_0(s_0 g_0 s_1^{-1} v_0).$$

As a consequence (56) then also holds true for $i=0$, and the logarithm of $f_{u,v}(x,g)$ becomes a linear functions of time t_0.

3. A Liouville integrable example of a periodic spin Calogero-Moser example for orbits of rank 1

3.1. Let us briefly discuss a particular case of periodic spin CM chain corresponding to $G = SL_N(\mathbb{R})$ with rank one orbits $\mathcal{O}_k$. This case is related to the original paper [**16**] where spin CM systems were first introduced.

Take $\mathfrak{a} \subset \mathfrak{sl}_N$ the Cartan subalgebra consisting of diagonal matrices, and denote the roots by $\{\epsilon_i - \epsilon_j\}_{i\neq j} \subset \mathfrak{a}^*$ with $\epsilon_i \in \mathfrak{a}^*$ the linear functional picking out the i^{th} diagonal entry. We identify $\mathfrak{sl}_N$ with its dual via the Killing form $(x,y) = \mathrm{Tr}(xy)$. Then for $p \in \mathfrak{a}^* \simeq \mathfrak{a}$ we have $(p,p) = \sum_{i=1}^N p_i^2$, with p_i the i^{th} diagonal entry of the diagonal matrix p. For $y \in \mathfrak{sl}_N^* \simeq \mathfrak{sl}_N$ and $i \neq j$ we have $y_{\epsilon_i - \epsilon_j} = y_{ij}$, with y_{ij} the $(i,j)^{th}$ entry of the matrix y.

For $\xi \in \mathbb{R}$ set

$$\mathcal{O}^{(\xi)} = \Big\{ z - \frac{\xi}{N}\mathrm{id}_N \;\mid\; z \text{ is a rank one } N \times N \text{ matrix with } \; \mathrm{Tr}(z) = \xi \Big\}.$$

Then $\mathcal{O}^{(\xi)}$ is a coadjoint orbit in $\mathfrak{sl}_N \simeq \mathfrak{sl}_N^*$ of dimension $2(N-1)$.

Viewing elements in $\mathbb{R}^N$ as column vectors, we have a natural mapping

$$\begin{aligned} &\big\{(a,b) \in \mathbb{R}^N \times \mathbb{R}^N \mid a^t b = \sum_{i=1}^N a_i b_i = \xi\big\}/\mathbb{R}^\times \xrightarrow{\sim} \mathcal{O}^{(\xi)}, \\ &\mathbb{R}^\times(a,b) \mapsto \mu = b a^t - \frac{\xi}{N}\mathrm{id}_N \end{aligned} \tag{57}$$

where $\lambda \in \mathbb{R}^\times$ acts by $(a,b) \mapsto (\lambda a, \lambda^{-1} b)$ and a^t is the transpose of $a \in \mathbb{R}^N$, so that $\mu_{ij} = a_i b_j - \delta_{ij}\frac{\xi}{N}$. Because of the rank one condition, this is an isomorphism. It is easy to check that this is symplomorphism, with the Poisson brackets of the coordinate functions a_i and b_j of $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$ given by

$$\{b_i, a_j\} = \delta_{ij}, \quad \{a_i, a_j\} = 0 = \{b_i, b_j\} = 0.$$

The value of the quadratic Casimir function $(y,y) = \sum_{i,j=1}^N y_{ij} y_{ji}$ being restricted to $\mathcal{O}^{(\xi)}$ can easily computed using (57):

$$\sum_{i,j=1}^N \mu_{ij}\mu_{ji} = \xi^2\Big(1 - \frac{1}{N}\Big), \qquad\qquad \mu \in \mathcal{O}^{(\xi)}.$$

Note that the action of $\mathbb{R}^\times$ in (57) is Hamiltonian and that $\mathcal{O}^{(\xi)}$ can be regarded as the Hamiltonian reduction of $T^*\mathbb{R}^N$ with coordinates a_i on $\mathbb{R}^N$ and b_i on cotangent spaces with respect the action and the moment map

$$(a,b) \mapsto H_\xi(a,b) = \sum_{i=1}^N a_i b_i - \xi$$

3.2. Now consider n-spin Calogero-Moser system for $G = SL_N$ with the phase space $\mathcal{S}(\mathcal{O})$ corresponding to $\mathcal{O} = \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_n}$. The regular part of this phase space is an open dense subset $\mathcal{S}(\mathcal{O})_{reg} \subset \mathcal{S}(\mathcal{O})$ that we described earlier by an isomorphism

$$\mathcal{S}(\mathcal{O})_{reg} \simeq (T^* A_{N,reg} \times \mathcal{O}//A)/W$$

where $\mathcal{O}//A$ is the Hamiltonian reduction of $\mathcal{O}$ with respect to the diagonal action of the Cartan subgroup $A_N \subset SL_N$ and $A_{N,reg} \subset A_N$ are diagonal matrices with positive pairwise distinct entries.

Quadratic Hamiltonians were described in section 1.4. The quadratic n-th Hamiltonian $H_2^{(n)}$ has a particularly simple form

$$H_2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \sum_{i<j} \frac{\mu_{ij}\mu_{ji}}{2\mathrm{sh}^2(q_i - q_j)} \tag{58}$$

where $q_i = \epsilon_i(\log(a))$ are natural logarithmic coordinates on the Cartan subgroup $A_N \in SL_N(\mathbb{R})$, p_i are corresponding coordinates on the cotagent spaces in $T^*A_{N,reg}$ and

$$\mu_{ij} = \sum_{k=1}^{n} \mu_{ij}^{(k)}.$$

Here for each $k = 1, \ldots, n$, $\mu_{ij}^{(k)}$ is the restriction of natural coordinates on sl_N^* to $\mathcal{O}_{\xi_k}$ regarded as a function of the k-th factor of $\mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_n}$.

Now let us describe the "spin" part of the phase space as the Hamiltonian reduction with respect to the action of $A_N \times A_n$ where $A_N \subset SL_N$ and $A_n \subset SL_n$ are Cartan subgroups (groups of unimodular diagonal matrices). For $n = 1$ we already did this in section 3.1.

As it is explained above we can write $\mu_{ij}^{(k)} = b_i^{(k)} a_j^{(k)} - \frac{\xi_k}{N}\delta_{ij}$ where for variables $a_i^{(k)}$, $b_j^{(l)}$ we have the standard Poisson structure:

$$\{a_i^{(k)}, a_j^{(l)}\} = \{b_i^{(k)}, b_j^{(l)}\} = 0, \quad \{b_i^{(k)}, a_j^{(l)}\} = \delta_{kl}\delta_{ij}$$

Define the Hamiltonians

$$H_{\xi,*}^{(k)} = \sum_{i=1}^{N} a_i^{(k)}, b_i^{(k)} - \xi, \quad H_{\eta,i}^{(*)} = \sum_{k=1}^{n} a_i^{(k)}, b_i^{(k)} - \eta$$

A function on $(\mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_n})//A_N$ is called *polynomial* if it is the restriction of a polynomial A_N-invariant function on $(sl_N^*)^n$ to $\nu^{-1}(0) \subset \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_n}$ where ν is the moment map for the diagonal Hamiltonian action of A_N.

It is easy to see that the space of polynomial functions on $(\mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_n})//A_N$ can be described as

$$\begin{aligned} Pol(\mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_n}//A_N) \simeq &< a_i^{(k)}, b_j^{(l)} | \#(a_*^{(k)}) = \#(b_*^{(k)}), \\ &\#(a_i^{(*)}) = \#(b_i^{(*)}), i = 1, \ldots N, k = 1, \ldots, n > / < H_{\xi_k,*}^{(k)}, H_{\eta,i}^{(*)} >, \end{aligned} \tag{59}$$

where $\eta = (\xi_1 + \cdots + \xi_n)/N$. We can simplify the right side if we consider the action of the Cartan subgroup $H_N \times H_n \subset GL_N \times GL_n$ (here H_n are non-degenerate diagonal matrices) on the Poisson Heisenberg algebra generated by $a_i^{(k)}$ and $b_j^{(l)}$:

$$(h, h') : a_i^{(k)} \mapsto h_i h'_k a_i^{(k)}, \quad b_j^{(l)} \mapsto h_j^{-1} h'^{-1}_l b_j^{(l)}$$

Then we can write (59) as

$$Pol(\mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_n} // A_N) \simeq < a_i^{(k)}, b_j^{(l)} >^{H_N \times H_n} / < H^{(k)}_{\xi_k, *}, H^{(*)}_{\eta, i} > .$$

The following theorem is an immediate consequence of this isomorphism.

Theorem 5. *We have the following isomorphism of symplectic varieties:*

$$(\mathcal{O}^{SL_N}_{\xi_1} \times \cdots \times \mathcal{O}^{SL_N}_{\xi_n}) // A_N \simeq (\mathcal{O}^{SL_n \times N}_{\eta})(\xi_1, \ldots, \xi_n) // A_n \tag{60}$$

Here $\eta = (\xi_1 + \cdots + \xi_n)/N$.

In terms of the variables $a^{(k)}$ and $b^{(k)}$ the Hamiltonian H_2 on $\mathcal{S}(\mathcal{O})_{reg}$ can be rewritten as

$$H_2 = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \sum_{i<j} \frac{\sum_{k,\ell=1}^{n} b_i^{(k)} a_j^{(k)} b_j^{(\ell)} a_i^{(\ell)}}{2\mathrm{sh}^2(q_i - q_j)}. \tag{61}$$

This Hamiltonian has a more natural "spin" interpretation. Now we have N particles and each of them have "internal degrees of freedom" or "spins" described by variables $\sum_{k=1}^{n} b_i^{(k)} a_j^{(k)}$ with $i \neq j$. In the next section we will give a better description of these variables in terms of coadjoint orbits.

3.3. Here we will rewrite the Hamiltonian (61) in terms of "spain" variables attached to one dimensional particles with the positions $q_i, \quad i = 1, \ldots, N$. They are defined as follows.

Denote by g^i_{kl} the restriction of standard coordinate functions on sl_n^* to $\mathcal{O}_\eta$ regraded as a function on k-th factor. We have:

$$g^{(i)}_{k\ell} = b_i^{(k)} a_i^{(\ell)} - \delta_{k\ell} \frac{\eta}{n}.$$

It is easy to check that if $i \neq j$ the following identity holds:

$$\sum_{k,\ell=1}^{n} g^{(i)}_{k\ell} g^{(j)}_{\ell k} = \mu_{ij} \mu_{ji} - \frac{\eta^2}{n}. \tag{62}$$

Thus, we can rewrite the Hamiltonian (58) in terms of spin variables from $\widetilde{\nu}^{-1}_{\widetilde{\mathcal{O}}}(t_{\underline{\xi}})/\widetilde{H} \times T^* A_{reg}$ as

$$H_2 = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \sum_{i<j} \frac{\mathrm{Tr}(g^{(i)} g^{(j)}) + \frac{\eta^2}{n}}{2\mathrm{sh}^2(\frac{q_i - q_j}{2})}. \tag{63}$$

This Hamiltonian describes N classical particles each carrying a "spin" variable which lies in the rank one coadjoint orbit in $\mathfrak{sl}_n^*$ with the Casimir value given by

$$\sum_{k,l=1}^{n} g_{k,l}^{(i)} g_{l,k}^{(i)} = \eta^2 (1 - \frac{1}{n}) \tag{64}$$

The system is Liouville integrable since we constructed $n(N-1)$ integrals for the periodic spin chain earlier (see the proof of Theorem 2).

Integrable system described above are closely related [16] and [21].

3.4. This project, together with results of [1], is the first step towards constructing superintegrable systems on moduli spaces of flat connections on a surface where on part of the boundary the gauge group G is constrained to K. When the boundary gauge group is not constrained, corresponding integrable systems are described in [1]. We expect that such moduli spaces have the structure of a cluster variety similar to the one described in [14]. It would be interesting to extend the construction of spin CM chains to the elliptic case as it was done for $N = 1$ in [21].

Appendix A. Comparison with the $n = 2$ case from [31]

Consider the periodic spin CM chain from section 1 for $n = 2$. Let $G_2 = G \times G$ be the gauge group acting on $G \times G$ as it in section 1.1. Symplectic leaves of $T^*(G^{\times 2})/G_2$ are then

$$\begin{aligned} \mathcal{S}(\mathcal{O}_1, \mathcal{O}_2) = \{(x_1, x_2, g_1, g_2) \mid \ & x_1 - Ad^*_{g_2^{-1}}(x_2) \in \mathcal{O}_1, \\ & x_2 - Ad^*_{g_1^{-1}}(x_1) \in \mathcal{O}_2\}/G_2 \end{aligned} \tag{65}$$

where $\mathcal{O}_1$, $\mathcal{O}_2$ are coadjoint orbits in $\mathfrak{g}^*$, relative to the gauge action

$$(h_1, h_2)(x_1, x_2, g_1, g_2) = (Ad^*_{h_1}(x_1), Ad^*_{h_2}(x_2), h_1 g_1 h_2^{-1}, h_2 g_2 h_1^{-1}).$$

In [31, §3 & App. C] the following Hamiltonian action of G_2 on $T^*(G^{\times 2})$ is considered,

$$(h_1, h_2)_*(x_1, x_2, g_1, g_2) = (Ad^*_{h_1}(x_1), Ad^*_{h_1}(x_2), h_1 g_1 h_2^{-1}, h_1 g_2 h_2^{-1}), \tag{66}$$

with corresponding moment map $\mu_* : T^*(G^{\times 2}) \to \mathfrak{g}^{*\times 2}$ given by

$$\mu_*(x_1, x_2, g_1, g_2) = (x_1 + x_2, -Ad^*_{g_1^{-1}}(x_1) - Ad^*_{g_2^{-1}}(x_2)).$$

The corresponding symplectic leaves are

$$\begin{aligned} \mathcal{S}_*(\mathcal{O}_1, \mathcal{O}_2) &= \mu_*^{-1}(\mathcal{O}_1, \mathcal{O}_2)/G_2 \\ &= \big\{(x_1, x_2, g_1, g_2) \mid x_1 + x_2 \in \mathcal{O}_1, \\ &\qquad -Ad^*_{g_1^{-1}}(x_1) - Ad_{g_2^{-1}}(x_2) \in \mathcal{O}_2\big\}/G_2, \end{aligned}$$

with the gauge group G_2 now acting by (66). These symplectic leaves were used in [31]. They are related to the symplectic leaves $\mathcal{S}(\mathcal{O}_1, \mathcal{O}_2)$ in the following way.

Consider the map $\psi : T^*(G^{\times 2}) \to T^*(G^{\times 2})$, defined by

$$\psi(x_1, x_2, g_1, g_2) = (-x_1, Ad^*_{g_1}(x_2), g_1, g_1 g_2 g_1).$$

Then ψ is G_2-equivariant,

$$\psi((h_1, h_2)(x_1, x_2, g_1, g_2)) = (h_1, h_2)_* \psi(x_1, x_2, g_1, g_2),$$

and the resulting map on the G_2-orbits restricts to an isomorphism

$$\mathcal{S}(\mathcal{O}_1, \mathcal{O}_2) \xrightarrow{\sim} \mathcal{S}_*(\mathcal{O}_2, \mathcal{O}_1).$$

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