Stochastic Kähler geometry: from random zeros to random metrics

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Abstract. We provide a survey of results on the statistics of random sections of holomorphic line bundles on Kähler manifolds, with an emphasis on the resulting asymptotics when a line bundle is raised to increasing tensor powers. We conclude with a brief discussion of the 'Bergman' Kähler metrics induced by these random sections.

Contents

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Introduction

Stochastic Kähler geometry refers to the study of probabilistic problems in complex algebraic or analytic geometry in the setting of Kähler manifolds (M,ω) of any complex dimension m. It concerns random fields on a Kähler manifold which are defined in terms of the complex structure J and Kähler form ω. The basic random fields are holomorphic sections $s \in H^0(M,L^k)$ of powers of a holomorphic Hermitian line bundle $(L,h) \to (M,\omega)$. From these holomorphic fields one can construct random complex submanifolds $Z_{\vec{s}}$ (zero sets of one or several sections), random embeddings into complex projective spaces $\mathbb{CP}^N$, and random 'Bergman' or 'Fubini–Study' Kähler metrics induced by the embeddings. Zero sets and embeddings both determine positive (1,1)-forms $\omega = i\partial\bar{\partial}\varphi$, where φ is a psh (plurisubharmonic) function. Although the behavior of random zero sets in the high tensor power limit $k \to \infty$ is the heart of stochastic Kähler geometry, the same techniques often apply with little change to random Kähler metrics and other more general objects. The goal of this survey is to review some of the main results on random zero sets and also to briefly discuss these generalizations to random Kähler metrics.

Most results of stochastic Kähler geometry to date pertain to the asymptotics of probabilistic invariants such as distribution and correlation functions of zeros and of critical points as the degree $k \to \infty$. One of the main results is universality of the limit of rescaled invariants on small balls of radius $k^{-\frac{1}{2}}$. Recently such scaling limits have been used to study the local topology of random zero sets. Another focal point is on the asymptotic normality of linear statistics, showing that fluctuations of linear statistics, i.e. integrals $\int_{Z_s} \psi$ of a test form ψ over the zeros of random sections, tend to Gaussian random variables determined by the variance current. Asymptotic normality of integrals against a random positive (1,1)-form ω is equivalent to asymptotic normality of the potential u of ω, and that is the way it is often stated in the physics literature (e.g. [**CLW15a, CLW15b**]). We call attention to some natural ensembles of potentials for which asymptotic normality is as yet unknown: linear statistics for critical points and for zeros of codimension greater than 1.

As the reference to potentials indicates, the unifying theme is that of random psh functions. A Kähler metric is defined as a mixed Hessian $\omega = i\partial\bar{\partial}u$ of a local psh function u, known as the 'Kähler potential'. Zero sets are also defined as $Z_{\vec{f}} := (\frac{i}{2\pi}\partial\bar{\partial}u)^q$ where $u = \log\sum_{j=1}^{q}|f_j(z)|^2$ with $q \leq m = \dim_{\mathbb{C}} M$ and f_j are local holomorphic functions. The same formula when $q > m$ is a way to define a smooth Kähler metric, and $Z_{\vec{f}}$ can be viewed as a 'singular Kähler metric'. As this suggests, many results about random zero sets have analogues for random smooth metrics. If $q = d_k := \dim H^0(M, L^k)$ and if $\vec{f}$ is a basis of $H^0(M, L^k)$ then $i\partial\bar{\partial}\log\sum_{j=1}^{d_k}|f_j(z)|^2$ is known as a Bergman metric of degree k.

One of the themes of stochastic Kähler geometry is the response of the probabilistic results to changes in the input geometry. By 'geometry' we mean line bundles $L \to M$, Hermitian metrics h on L, curvature forms Θ_h and the 'quantization' of Hermitian metrics h (together with a choice of measure ν on M) as inner products $G(h^k, \nu)$ on spaces $H^0(M, L^k)$ of holomorphic sections of powers of L. The inner product determines a Gaussian measure $\gamma_{h^k,\nu}$ on $H^0(M, L^k)$, and this provides the notion of 'random polynomial' or more generally 'random section'. The geometric language is useful (and even necessary) to formulate generalizations of logarithmic potential theory and random polynomial theory on $\mathbb{C}$ to compact Riemann surfaces or to higher dimensional complex manifolds. Holomorphic sections of line bundles are the analogues on a compact manifold M of holomorphic functions on $\mathbb{C}^n$, and specifically $H^0(M, L^k)$ is the replacement for polynomials of degree $\leq k$.

We may contrast stochastic Kähler geometry with the much-studied one-dimensional theory of stochastic (or, random) conformal geometry. Conformal stochastic geometry is a highly developed field of probability, mathematical physics and complex analysis. It contains such subfields as SLE, the quantum Hall effect, Hele–Shale flow, and Liouville quantum gravity in mathematical physics, and probabilistic problems in one dimensional complex analysis. As the name 'conformal' suggests, it is strictly a complex one-dimensional theory. The key difference is that stochastic conformal geometry is concerned with conformally-invariant ensembles of real objects such as the Gaussian free field (GFF), SLE curves, Coulomb gas point processes, or random LQG area forms in Liouville quantum gravity [**AHM11, Du06, KN13**]. The key objects are often random fractals. In stochastic Kähler geometry, the emphasis is on holomorphic fields and the objects they induce in complex geometry.

In this survey, we only refer briefly to results in the complex one dimensional case, although it is a very rich field. Moreover, many of the recent constructions on higher dimensional Kähler manifolds use ideas that originated in the probabilistic study of real algebraic manifolds and zero sets of random real functions, in particular ideas stemming from the work of Nazarov–Sodin [**NS09**] on counting connected components of spherical harmonics and other random real functions, and their topological applications due to Sarnak–Wigman [**SW10**], Canzani–Sarnak [**CS19**] and others on

Betti numbers and combinatorial configurations. We omit these important results because they would take us too far afield.

1. Background

In this section, we introduce some background and notation pertaining to random holomorphic sections of positive Hermitian line bundles.

Let (M, L) be an m-dimensional compact complex manifold polarized with a Hermitian holomorphic line bundle (L, h). We consider a local holomorphic frame e_L over a trivializing chart U. If $s = fe_L$ is a holomorphic section of L over U, its Hermitian norm is given by $\|s(z)\|_h = e^{-\varphi_h}|f(z)|$ where

$$\varphi_h(z) := -\log \|e_L(z)\|_h \ . \tag{1}$$

The curvature form of (L, h) is given locally by $\Theta_h = 2\partial\bar\partial\varphi_h$, and the *Chern form* $c_1(L, h)$ is given by

$$c_1(L,h) = \frac{\sqrt{-1}}{2\pi}\Theta_h = \frac{\sqrt{-1}}{\pi}\,\partial\bar\partial\varphi_h \ . \tag{2}$$

We now assume that the Hermitian metric h has strictly positive curvature and we give M the Kähler form

$$\omega_h := i\partial\bar\partial\varphi_h = \pi c_1(L,h) \ . \tag{3}$$

1.1. From metrics and measures to inner products and Gaussian measures. We denote by $H^0(M, L^k)$ the space of global holomorphic sections of $L^k = L \otimes \cdots \otimes L$. The metric h induces Hermitian metrics h^k on L^k given by $\|s^{\otimes k}\|_{h^k} = \|s\|_h^k$.

We let ν denote a (finite, positive) Borel measure on M. Together, the data (h, ν) induces Hermitian inner products $G(h^k, \nu)$ on the spaces $H^0(M, L^k)$ of global holomorphic sections of powers $L^k \to M$ given by

(4)

$$\langle s_1, \overline{s_2}\rangle_k = \langle s_1, \overline{s_2}\rangle_{G(h^k,\nu)} := \int_M \langle s_1(z), \overline{s_2(z)}\rangle_{h^k}\, d\nu(z), \quad s_1, s_2 \in H^0(M, L^k)\,.$$

In turn, each inner product on $H^0(M, L^k)$ induces an orthonormal basis $\{S_1^k, \dots, S_{d_k}^k\}$ and associated Gaussian measure $\gamma_{h^k,\nu}$ given by the formula,

$$d\gamma_{h^k,\nu}(s^k) := \frac{1}{\pi^{d_k}} e^{-|c|^2} dc\,, \quad s^k = \sum_{j=1}^{d_k} c_j S_j^k\,, \quad c = (c_1, \dots, c_{d_k}) \in \mathbb{C}^{d_k}\,, \tag{5}$$

where dc denotes $2d_k$-dimensional Lebesgue measure. The measure $\gamma_{h^k,\nu}$ is characterized by the property that the $2d_k$ real variables $\Re c_j$, $\Im c_j$ ($j = 1, \dots, d_k$) are independent Gaussian random variables with mean 0 and variance 1/2; equivalently,

$$\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_l), \quad \mathbf{E}(c_j \bar c_l) = \delta_{jl}.$$

Here, $\mathbf{E}$ denotes the expectation.

The inner product $G(h^k,\nu)$ further induces an associated spherical measure on the unit sphere $SH^0(M,L^k)$ in $H^0(M,L^k)$ with respect to $G(h^k,\nu)$. In this survey, we restrict our discussion to inner products where ν is the volume form of M. For results with more general metrics and measures, see for example [**Be09, BSh07, BeBW11**].

1.2. Polynomials and holomorphic sections of line bundles. The space $Poly_k$ of univariate polynomials of degree k is a complex vector space of dimension $k+1$. The 'SU(2) inner product' on $Poly_k$ may be written in the form

$$\begin{aligned}\langle f_1,\bar f_2\rangle &= \frac{i}{2}\int_{\mathbb{C}} f_1(z)\overline{f_2(z)}e^{-k\log(1+|z|^2)}\frac{dz\wedge d\bar z}{(1+|z|^2)^2}\\ &= \frac{i}{2}\int_{\mathbb{C}} f(_1z)\overline{f_2(z)}\frac{dz\wedge d\bar z}{(1+|z|^2)^{k+2}}\,,\quad f_1,f_2\in Poly_k\,.\end{aligned}$$

This has a simple geometric interpretation: namely, we view polynomials of degree k as holomorphic sections of the line bundle $\mathcal{O}(k)=L^k$, where $L\to\mathbb{CP}^1$ is the hyperplane section bundle. We give L the Hermitian metric $\|e_L\|_h = e^{-\frac12\log(1+|z|^2)}$, where $e_L=1$. Then $\omega_h=\frac{i}{2}\frac{dz\wedge d\bar z}{(1+|z|^2)^2}$ is the usual area form on $\hat{\mathbb{C}}=\mathbb{CP}^1$, and $\langle f_1,\bar f_2\rangle=\langle f_1,\bar f_2\rangle_{G(h^k,\omega_h)}$.

For multivariable polynomials, we let $M=\mathbb{CP}^m$ with $L\to\mathbb{CP}^m$ the hyperplane section bundle $\mathcal{O}(1)$, so that the space of global sections $H^0(\mathbb{CP}^m,L)$ consists of the linear functions $f(z)=\sum_{j=0}^m c_jz_j$ on $\mathbb{C}^{m+1}$. Then $H^0(\mathbb{CP}^m,L^k)$ is the vector space $Poly_k^m$ of homogeneous polynomials of degree k on $\mathbb{C}^{m+1}$, which we identify with the space of polynomials of degree $\le k$ in the variables $z_1,\dots,z_m$ by setting $z_0=1$. If we let $w_j=z_j/z_0$, $1\le j\le m$, be local coordinates on $\mathbb{CP}^m$ and we give $L=\mathcal{O}(1)$ the Hermitian metric $\|e_L\|_h=(1+\|w\|^2)^{-1/2}$, then $\varphi_h=\frac12\log(1+\|w\|^2)$ and $\omega_h=\frac{i}{2}\partial\bar\partial\log(1+\|w\|^2)$, the Fubini–Study metric on $\mathbb{CP}^m$. Then the volume form

$$dV=\frac{1}{m!}\omega_h^m=(1+\|w\|^2)^{-m-1}\,d_{2m}w\,,$$

where $d_{2m}w$ is Euclidean volume. We then have the SU$(m+1)$-invariant inner product

$$\langle f_1,\bar f_2\rangle_{G(h^k,dV)}=\int_{\mathbb{C}^m}\frac{f_1(z)\,\overline{f_2(z)}}{(1+\|z\|^2)^{k+m+1}}\,d_{2m}z\,,\quad f_1,f_2\in Poly_k^m\,.$$

1.3. Asymptotics of Bergman kernels on positive line bundles. We let $\mathcal{L}^2(M,L^k)$ denote the $\mathcal{L}^2$ sections of $L^k\to M$ with respect to the inner product $G(h^k,dV)$, where $dV=\frac{1}{m!}\omega_h^m$.

We define the *Bergman kernel* as the orthogonal projection

$$B_k(z,w):\mathcal{L}^2(M,L^k)\to H^0(M,L^k)\,.$$

Then

$$B_k(z,w) = \sum_{j=1}^{d_k} S_j^k(z) \otimes \overline{S_j^k(w)}\,, \tag{6}$$

where $\{S_1^k, \cdots, S_{d_k}^k\}$ is an orthonormal basis of $H^0(M, L^k)$ with respect to $G(h^k, dV)$. Along the diagonal, the contraction of the Bergman kernel is

$$\|B_k(z,z)\| = \sum_{j=1}^{d_k} \|S_j^k(z)\|_{h^k}^2\,. \tag{7}$$

In the case where the curvature form Θ_h of the Hermitian line bundle (L, h) is everywhere positive, we have the following *Tian–Yau–Zelditch asymptotic expansion* [**Ca97, Z97**]:

$$\|B_k(z,z)\| \sim \frac{1}{\pi^m}[k^m + a_1(z)k^{m-1} + a_2(z)k^{m-2} + \cdots]\,, \tag{8}$$

where each coefficient $a_j(z)$ is a polynomial of the curvature and its covariant derivatives. Formulas for the first three coefficients were given by Lu [**Lu00**]. In particular, $a_1(z)$ equals one-half the scalar curvature of ω_h.

Example. In the case of $(\mathbb{CP}^m, \omega_{FS})$ with the line bundle $(\mathcal{O}(1), h_{FS})$, the Bergman kernel is easily computed to be a constant along the diagonal [**BShZ00b**]:

$$\|B_k(z,z)\| = \frac{(k+m)!}{\pi^m\, k!}\,. \tag{9}$$

Recall that the inner product $G(h^k, dV)$ induces the Gaussian field $(H^0(M, L^k), \gamma_{h^k,dV})$, where $\gamma_{h^k,dV}$ is given by (5). In fact, the Bergman kernel $B_k(z,w)$ can be interpreted as the covariance function for the Gaussian field $\big(H^0(M, L^k),\, \gamma_{h^k,dV}\big)$:

$$\mathbf{E}\big(s^k(z) \otimes \overline{s^k(w)}\big) = B_k(z,w)\,, \tag{10}$$

where $\mathbf{E}$ denotes the expected value with respect to $\gamma_{h^k,dV}$.

Proof. Apply $\mathbf{E}(c_j \bar{c}_l) = \delta_{jl}$ to

$$\mathbf{E}\big(s^k(z) \otimes \overline{s^k(w)}\big) = \mathbf{E}\left(\sum_{j=1}^{d_k} c_j S_j^k(z) \otimes \sum_{l=1}^{d_k} \overline{c_l S_l^k(w)}\right). \qquad \square$$

1.4. Off-diagonal scaling asymptotics of the Szegő kernel. To provide asymptotics for the Bergman kernel off the diagonal, it is convenient to lift the Bergman kernel to the circle bundle X of the dual bundle to L. To describe the lifted kernel, we let $L^* \to M$ denote the dual line bundle to $L \to M$ with the dual metric h^*, and we let $X := \{\lambda \in L^* : \|\lambda\|_{h^*} = 1\}$. We regard a section $s^k \in H^0(M, L^k)$ as a function on X by setting

$$s^k(\lambda) = (\lambda \otimes \cdots \otimes \lambda, s^k(z)), \quad \lambda \in L_z^*\,,$$

and we note that s^k is k-equivariant: $s^k(e^{i\theta}\lambda) = e^{ik\theta}s^k(\lambda)$. We assume that (L,h) has positive curvature; then X is the boundary of the strictly pseudoconvex disk bundle $\{\lambda \in L^* : \ell(\lambda) < 1\}$ where $\ell(\lambda) = \|\lambda\|^2_{h^*}$. We let $\Pi : \mathcal{L}^2(X) \to \mathcal{H}^2(X)$ denote the orthogonal projection to the space $\mathcal{H}^2(X)$ of square-integrable CR functions on X, where we give X the volume form

$$\frac{i}{2\pi\, m!}\bar\partial\ell \wedge (i\partial\bar\partial\ell)^m = \frac{i}{2\pi}\bar\partial\ell \wedge dV_M\,.$$

Then $\Pi = \bigoplus_{k=0}^{\infty}\Pi_k$, where $\Pi_k : \mathcal{L}^2(X) \to \mathcal{H}^2_k(X)$ is the orthogonal projection onto the space of k-equivariant functions $\mathcal{H}^2_k(X)$ in $\mathcal{H}^2(X)$. Indeed, $\mathcal{H}^2(X) = \bigoplus_{k=0}^{\infty}\mathcal{H}^2_k(X)$, where $\mathcal{H}^2_k(X) \approx H^0(M, L^k)$. We call $\Pi_k(x,y)$ the (k-th) Szegő kernel; the sum $\Pi(x,y)$ is the classical Szegő kernel for the strictly pseudoconvex boundary X.

To relate the Bergman kernel $B_k(z,w)$ to the Szegő kernel $\Pi_k(z,\theta;w,\varphi)$, we use a local frame e_L to write $S^k_j = F^k_j e_L^{\otimes k}$. Recalling (6), we have

$$\begin{aligned} B_k(z,w) &= \left(\textstyle\sum_{j=1}^{d_k} F^k_j(z)\overline{F^k_j(w)}\right)e_L(z)^{\otimes k}\otimes\overline{e_L(w)}^{\otimes k}\,,\\ \Pi_k(z,\theta_1;w,\theta_2) &= e^{ik(\theta_1-\theta_2)}e^{-k\varphi(z)-k\varphi(w)}\textstyle\sum_{j=1}^{d_k} F^k_j(z)\overline{F^k_j(w)}\,. \end{aligned}$$

Here, (z,θ) denotes the point $e^{i\theta}\|e_L(z)\|_h e^*_L(z) \in X$. Thus

$$\begin{aligned} |\Pi_k(z,\theta_1;w,\theta_2)| &= \|B_k(z,w)\|\,,\\ \Pi_k(z,z) := \Pi_k(z,0;z,0) &= \|B_k(z,z)\|\,. \end{aligned}$$

The asymptotics of the Bergman kernel are used in Section 2 to study the distributions of zeros of a random section $s^k \in H^0(M,L^k)$. In particular, the off-diagonal asymptotics of the Bergman kernel provides information on correlations and variances of random zeros. To this end, a general asymptotic expansion was given in [**ShZ02**] and further clarified in [**ShZ08**] as follows:

THEOREM 1.1. *Let $(L,h) \to (M,\omega_h)$ be a positive holomorphic line bundle over a compact Kähler manifold. Let $z_0 \in M$ and choose local coordinates $\{z^j\}$ in a neighborhood of z_0 so that $z_0 = 0$ and $\Theta_h(z_0) = \sum dz^j \wedge d\bar z^j$. Then*

$$\begin{aligned} \frac{\pi^m}{k^m}\Pi_k\left(\frac{z}{\sqrt{k}},\frac{\theta_1}{k};\frac{w}{\sqrt{k}},\frac{\theta_2}{k}\right) &= e^{i(\theta_1-\theta_2)+i\Im(z\cdot\bar w)-\frac12|z-w|^2}\\ &\quad\times\left[1+\sum_{r=1}^{n}k^{-r/2}p_r(z,w)+k^{-(n+1)/2}R_{kn}(z,w)\right]\,, \end{aligned}$$

where p_r is a polynomial in $(z,\bar z,w,\bar w)$ of the same parity as r, and

$$|\nabla^j R_{kn}(z,w)| \le C_{jn\varepsilon b}k^{\varepsilon} \quad \text{for} \quad |z|+|w| < b\sqrt{\log k}\,,$$

for $\varepsilon, b \in \mathbb{R}^+$, $j,k \ge 0$. Furthermore, the constant $C_{jk\varepsilon b}$ can be chosen independently of z_0.

Here, ∇^j stands for the j-th covariant derivative.

The theorem shows that on Kähler manifolds, there is a characteristic length scale associated to the k-th power $L^k \to M$ of a positive line bundle: the Planck scale $\frac{1}{\sqrt{k}}$. It arises in the following ways:

- The Szegő kernel $\Pi_k(z, z_0)$ is of size $\simeq k^m$ for $\mathrm{dist}(z, z_0) < \frac{b}{\sqrt{k}}$, and then decays rapidly outside the ball.
- On the length scale $\frac{1}{\sqrt{k}}$, all Kähler manifolds and positive line bundles look alike in the scaling limit: they all look like the (trivial) line bundle $\mathbb{C}^{m+1} \to \mathbb{C}^m$ with the Euclidean Kähler form on $\mathbb{C}^m$.
- Correlations become universal on this length scale.

Specific formulas for the coefficients in the off-diagonal expansion using Bochner coordinates are given by Lu–Shiffman in [**LuSh15**]. For real-analytic metrics, [**HLX20**] gives symptotics on an enlarged length scale. A detailed study of the off-diagonal asymptotics is given in the book of Ma–Marinescu [**MaMar07**] using different techniques involving normal coordinates instead of holomorphic coordinates.

Away from the diagonal, we have the following decay estimate [**ShZ08**]:

THEOREM 1.2. *Let $(L, h) \to (M, \omega_h)$ be as above. For $b > \sqrt{j + 2\alpha + 2m}$, $j, \alpha \geq 0$, we have*

$$\nabla^j \Pi_k(z, 0; w, 0) = O(k^{-\alpha}) \qquad \textit{uniformly for } \ \mathrm{dist}(z, w) \geq b\sqrt{\frac{\log k}{k}}\,.$$

In particular, our variance formulas are expressed in terms of the *normalized Bergman kernel*

$$P_k(z, w) := \frac{\|B_k(z, w)\|}{\|B_k(z, z)\|^{\frac{1}{2}}\, \|B_k(w, w)\|^{\frac{1}{2}}}\,, \tag{11}$$

which is the square root of the so-called *Berezin kernel*. Note that $0 \leq P_k(z, w) \leq 1$ by Cauchy–Schwarz, and $P_k(z, z) = 1$.

Theorems 1.1–1.2 yield the following counterparts for the normalized kernel $P_k(z, w)$:

PROPOSITION 1.3. *Let $(L, h) \to (M, \omega_h)$ be a positive holomorphic line bundle over a compact Kähler manifold. For $b > \sqrt{j + 2\alpha}$, $j, \alpha \geq 0$, the normalized Bergman kernel satisfies the asymptotic estimate*

$$\nabla^j P_k(z, w) = O(k^{-\alpha}) \qquad \textit{uniformly for } \ d(z, w) \geq b\sqrt{\frac{\log k}{k}}\,.$$

PROPOSITION 1.4. *Using the hypotheses and notation of Theorem 1.1, we have the following asymptotics for the normalized Bergman kernel near the diagonal:*

For $\varepsilon, b > 0$, there are constants $C_j = C_j(M, \varepsilon, b)$, $j \geq 2$, independent of the point z_0, such that

$$P_k\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right) = e^{-\frac{1}{2}|z-w|^2}[1 + R_k(z, w)]\,,$$

where

$$|R_k(z,w)| \le \tfrac{C_2}{2}\,|z-w|^2 k^{-1/2+\varepsilon}\,, \quad |\nabla R_k(z)| \le C_2\,|z-w|\,k^{-1/2+\varepsilon}\,,$$

$$|\nabla^j R_k(z,w)| \le C_j\,k^{-1/2+\varepsilon} \quad j \ge 2\,,$$

for $|z|+|w| < b\sqrt{\log k}$.

2. Random zero sets

We now consider zero sets

$$Z_s = \{z \in M : s(z) = 0\}$$

of Gaussian random holomorphic sections $s \in H^0(M,L)$. In the case where M is a compact Riemann surface C (complex dimension 1), the zero set Z_s is a finite set $\{\zeta_1,\dots,\zeta_k\}$ of points in C. For example, if $C = \mathbb{CP}^1$ and $L = \mathcal{O}(k)$, then s is a polynomial on $\mathbb{C} \subset \mathbb{CP}^1$ of degree $\le k$ and the zero set consists of the roots of s (and the point at infinity if $\deg s < k$).

From the probabilistic viewpoint, the zeros of a random holomorphic section define a *point process* on C, that is, a measure on the configuration space $\mathrm{Conf}(C)$ of finite subsets of C (where the points may have positive integral multiplicities). Each holomorphic section gives rise to the discrete set of its zeros, and the point process is the probability measure on $\mathrm{Conf}(C)$ induced by the probability measure on the vector space $H^0(C,L)$. A probability measure on $\mathrm{Conf}(C)$ is determined by its n-point correlations $\mathbf{K}_n(z_1,\dots,z_n)$, $n \ge 1$, which are the probability densities (in $C^n = C \times \cdots \times C$) that $z_1,\dots,z_n \in C$ are the (simultaneous) zeros of a random section. For example, the *pair correlation* $\mathbf{K}_2(z_1,z_2)$ determines whether the zeros tend to cluster or to 'repel' each other.

The zero set $s^{-1}(0) = \{\zeta_1,\dots,\zeta_k\} \in \mathrm{Conf}(C)$ of zeros of a section s yields the *normalized empirical measure*

$$\frac{1}{k} Z_s = \frac{1}{k}\sum_j \delta_{\zeta_j}\,,$$

(again, counting multiplicities), so that the point process can be considered as a measure on the space of probability measures on C with discrete support. Here, δ_z is the Dirac delta-function at z. Thus the normalized empirical measure of zeros,

$$\left(\frac{1}{k} Z_s, \psi\right) = \frac{1}{k}\sum_j \psi(\zeta_j), \quad \psi \in \mathcal{C}(C),$$

is a random probability measure on C. Its expectation is a measure called the *expected distribution of zeros.*

For $m = \dim M \ge 2$, Z_s is the *current of integration* over the zero set of s:

$$(Z_s,\psi) = \int_{Z'_s} \psi, \qquad \psi \in \mathcal{D}^{m-1,m-1}(M)\,, \tag{12}$$

where Z'_s is the set of smooth points (counted with multiplicities) of the analytic hypersurface $\{\zeta : f(\zeta) = 0\}$. In Section 4.1, we discuss point processes of simultaneous zeros of m holomorphic sections on M.

In [**ShZ99**], we showed the following:

THEOREM 2.1. *Let $(L, h) \to (M, \omega_h)$ be a positive line bundle over a compact Kähler manifold. Then*

$$\frac{1}{k}\mathbf{E}(Z_{s^k}) \to \frac{1}{\pi}\omega_h$$

weakly in the sense of measures, where $\mathbf{E}$ is the expectation with respect to the Gaussian measure $\gamma_{h^k,dV}$ on $H^0(M, L^k)$. In fact,

$$(13) \quad \frac{1}{k}\mathbf{E}(Z_{s^k}, \psi) = \frac{1}{\pi}\int_M \omega_h \wedge \psi \; + \; O\left(\frac{1}{k^2}\right), \qquad \psi \in \mathcal{D}^{m-1,m-1}(M)\,.$$

If $\{s^k \in H^0(M, L^k)\}$ is a sequence of independent random sections, then

$$\frac{1}{k}Z_{s^k} \to \frac{1}{\pi}\omega_h \quad a.s.$$

Precisely, we form the probability space $\mathcal{S} := \prod_{k=1}^{\infty} H^0(M, L^k)$ with the product measure. Its elements are sequences $\{s^k\}$ of independent random sections. (In [**ShZ99**], (13) was stated with remainder term $O(\frac{1}{k})$ in place of $O(\frac{1}{k^2})$.)

In particular, we have

$$\lim_{k\to\infty} \frac{1}{k}\mathbf{E}\mathrm{Vol}_{2m-2}(Z_{s^k} \cap U) \to \frac{m}{\pi}\mathrm{Vol}_{2m}(U)\,,$$

for U open in M. In the Riemann surface case ($m = 1$),

$$\lim_{k\to\infty} \frac{1}{k}\mathbf{E}\#\{Z_{s^k} \cap U\} = \frac{1}{\pi}\,\mathrm{Area}(U)\,.$$

We outline the proof of Theorem 2.1 in the next section.

2.1. Poincaré–Lelong formula. In complex dimension one, if $f(z)$ is a holomorphic function on a domain in $\mathbb{C}$, then the fundamental solution $\Delta \log |z|^2 = 2\pi\delta_0$ of the Laplace operator immediately yields

$$(14) \qquad Z_f = \sum_{f(\zeta)=0} \delta_\zeta = \frac{i}{2\pi}\partial\bar\partial \log |f|^2 = \frac{i}{2\pi}\frac{\partial^2 \log |f|^2}{\partial z \partial \bar z} dz \wedge d\bar z\,,$$

as a singular $(1, 1)$-current.

In higher dimensions, we similarly have (see [**Le67**])

$$(15) \qquad Z_f = \frac{i}{2\pi}\partial\bar\partial \log |f|^2\,,$$

where $Z_f \in \mathcal{D}'^{1,1}(M)$ denotes the current of integration given in (12).

For a section $s^k = fe_L^{\otimes k} \in H^0(M, L^k)$ of a Hermitian holomorphic line bundle $L \to M$, we then have by (15) the *Poincaré–Lelong formula,*

$$Z_{s^k} = \frac{i}{\pi}\partial\bar\partial \log|f| = \frac{i}{\pi}\partial\bar\partial \log\|s^k\|_{h^k} + \frac{k}{\pi}\omega_h \ . \tag{16}$$

Averaging (16), we obtain:

THEOREM 2.2. *Let $\{S_j^k\}$ be an orthonormal basis of $H^0(M, L^k)$. Write $S_j^k = f_j^k e_L^{\otimes k}$. Then,*

$$\frac{1}{k}\mathbf{E}(Z_{s^k}) = \frac{\sqrt{-1}}{2\pi k}\partial\bar\partial \log \sum_{j=1}^{d_k} |f_j^k|^2 = \frac{\sqrt{-1}}{2\pi k}\partial\bar\partial \log\|B_k(z,z)\| + \frac{1}{\pi}\omega,$$

where we recall that B_k is the Bergman kernel.

PROOF. Let $s = \sum_j a_j S_j^k$ and write it as $s = \langle \vec a, \vec S^k\rangle = \langle \vec a, \vec f\rangle e_L^k$. Let $\psi \in \mathcal{D}^{m-1,m-1}(M)$. Then

$$\mathbf{E}\langle \frac{1}{k}[Z_s^k]), \psi\rangle = \frac{\sqrt{-1}}{\pi k}\int_{\mathbb{C}^{d_k}} d\gamma_k(a)\int_M \partial\bar\partial \log|\langle \vec a, \vec f\rangle| \wedge \psi \ .$$

To compute the integral, we write $\vec f = |\vec f|\vec u$ where $|\vec u| \equiv 1$. Evidently, $\log|\langle \vec a, \vec f\rangle| = \log|\vec f| + \log|\langle \vec a, \vec u\rangle|$. The first term gives

$$\frac{\sqrt{-1}}{\pi k}\int_M \partial\bar\partial \log|\vec f| \wedge \psi = \frac{\sqrt{-1}}{2\pi k}\int_M \partial\bar\partial \log\|B_k(z,z)\| \wedge \psi + \frac{1}{\pi}\int_M \omega\wedge\psi. \tag{17}$$

We now look at the second term. We have

$$\begin{aligned}\frac{\sqrt{-1}}{\pi}\int_{\mathbb{C}^{d_k}} d\gamma_k(a)\int_M \partial\bar\partial \log|\langle \vec a, \vec u\rangle| \wedge \psi \qquad\qquad & \\ = \frac{\sqrt{-1}}{\pi}\int_M \partial\bar\partial \left[\int_{\mathbb{C}^{d_k}} \log|\langle \vec a, \vec u\rangle|\, d\gamma_k(a)\right] \wedge \psi = 0,&\end{aligned} \tag{18}$$

since the average $\int \log|\langle \vec a, \vec u\rangle| d\gamma_k(a)$ is a constant independent of $\vec u$ for $|\vec u| = 1$, and thus the operator $\partial\bar\partial$ kills it. □

Combining Lemma 2.2 with the Bergman kernel asymptotics (8) yields (13), and Theorem 2.1 then follows from a variance estimate (see also Theorem 2.8 below).

2.2. Correlation of zeros. In this section, we discuss n-point 'correlations' between zeros, or 'joint intensities', of random sections $s^k \in H^0(M, L^k)$ of powers of a positive line bundle. We first consider pair correlations ($n = 2$): the pair correlation current for random zeros is defined by

$$\mathbf{K}_2^k(z,w) := \mathbf{E}\left(Z_{s^k}(z) \otimes Z_{s^k}(w)\right); \tag{19}$$

i.e., for test forms $\psi_1,\ \psi_2 \in \mathcal{D}^{m-1,m-1}(M)$,

$$\big(\mathbf{K}_2^k(z,w), \psi_1(z)\otimes\psi_2(w)\big) := \mathbf{E}\left[(Z_{s^k}, \psi_1)\otimes(Z_{s^k}, \psi_2)\right]. \tag{20}$$

In the case of complex dimension 1, the zeros form a point process, as discussed above, and the pair correlation measures take the form

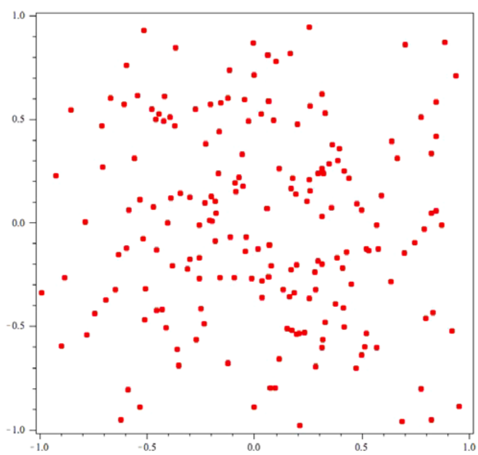
$$\mathbf{K}_2^k(z,w) = [\Delta] \wedge (\mathbf{K}_1^k(z) \otimes 1) + \tilde{K}_2^k(z,w)\, \mathbf{K}_1^k(z) \otimes \mathbf{K}_1^k(w)\,,$$

where $[\Delta]$ denotes the current of integration along the diagonal $\Delta = \{(z,z)\} \subset C \times C$, and $\mathbf{K}_1^k = \mathbf{E}(Z_{s^k}) \approx \frac{k}{\pi}\omega_h$ for large k. Then $\tilde{K}_2^k \in \mathcal{C}^\infty(C \times C)$ for k sufficiently large. The diagonal term comes from 'self-correlations' of a zero with itself. The second term is the interesting one. In [**BShZ00a**], it was shown that $\tilde{K}_2^k$ has a universal limit using the $1/\sqrt{k}$ scale of Section 1.4:

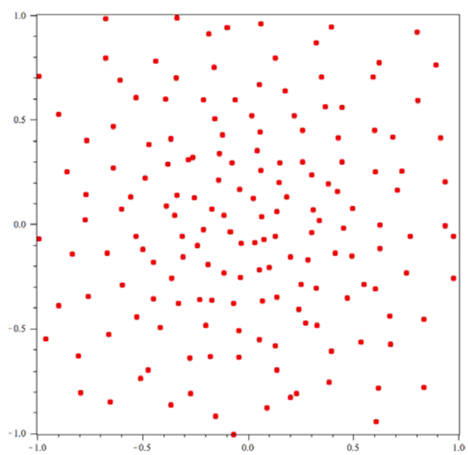
$$\begin{aligned} \tilde{K}_2^k\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right) \quad &\to \quad \frac{\left(\sinh^2(r^2/2) + r^4/4\right)\cosh(r^2/2) - r^2\sinh(r^2/2)}{\sinh^3(r^2/2)} \\ &= \frac{1}{2}r^2 - \frac{1}{36}r^6 + \frac{1}{720}r^{10} - \cdots, \qquad r = |z-w|\,, \end{aligned} \tag{21}$$

using local holomorphic coordinates about $z_0 \in M$ with $\omega(z_0) = \frac{i}{2}dz \wedge d\bar{z}$. Equation (21), which holds for all Riemann surfaces, was given in [**Ha96**] for $\mathbb{CP}^1$ and in [**NV98**] for genus $(C) = 1$.

The fact that the pair correlation $\kappa^k \to 0$ as the distance $r \to 0$ (with k fixed) tells us that the zeros 'repel' in the sense that they cluster less than independent random points cluster, as illustrated below:



Poisson point process



zeros of random polynomials

It was shown in [**BShZ00b**, Th. 3.6] that n-point correlations for random zero sets have universal scaling limits in all dimensions and codimensions of the form

$$\frac{1}{k^{np}}\mathbf{K}_{npm}^k\left(\frac{z^1}{\sqrt{k}}, \ldots, \frac{z^n}{\sqrt{k}}\right) = \mathbf{K}_{npm}^\infty(z^1, \ldots, z^n) + O\left(\frac{1}{\sqrt{k}}\right),$$

where p is the codimension of the simultaneous zero set (of p holomorphic sections of L^k). Formulas for $\mathbf{K}_{npm}^k$ are given in [**BShZ00b**] and [**BShZ01**].

In particular, for the point process case $p = m$,

$$\tilde{K}^{\infty}_{2mm}\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right) = \frac{m+1}{4}\, r^{4-2m} + O(r^{8-2m})\,, \qquad r = \|z - w\|\,.$$

Hence, random simultaneous zeros of m sections in $H^0(M, L^k)$ do not 'repel' when $\dim M \geq 2$, and in fact for $\dim M \geq 3$ they cluster more than those in Poisson processes, for large k.

2.3. A pluri-bipotential for the zero variance. In this section we give a formula for the variance $\mathrm{Var}(Z_{s^k})$ of the zero current Z_{s^k} of a Gaussian random holomorphic section $s^k \in H^0(M, L^k)$ (Theorem 2.4). Let us first describe the variance of a (general) random current:

DEFINITION 2.3. Let $X : \Omega \to \mathcal{D}'^{\,j}(M)$ be a random variable with values in the space $\mathcal{D}'^{\,j}_{\mathbb{R}}(M)$ of real currents of degree j on a manifold M. The variance of X is the current

$$\mathbf{Var}(X) := \mathbf{E}(X \boxtimes X) - \mathbf{E}(X) \boxtimes \mathbf{E}(X)\,, \tag{22}$$

where we use the notation

$$S \boxtimes T = \pi_1^* S \wedge \pi_2^* T \in \mathcal{D}'^{\,p+q}(M \times M)\,, \qquad \text{for } \; S \in \mathcal{D}'^{\,p}(M),\; T \in \mathcal{D}'^{\,q}(M)\,.$$

Here, $\pi_1, \pi_2 : M \times M \to M$ are the projections to the first and second factors, respectively. Using more intuitive notation, we shall write $(S \boxtimes T)(z, w) = S(z) \wedge T(w)$, where (z, w) denotes a point of $M \times M$.

The rationale behind Definition 2.3 is that the variance of the pairing of X with a compactly supported real test form $\psi \in \mathcal{D}^{\dim M - k}_{\mathbb{R}}(M)$ is given by

$$\mathrm{Var}(X, \psi) = \big(\mathbf{Var}(X),\, \psi \boxtimes \psi\big)\,. \tag{23}$$

We now show that the variance $\mathrm{Var}(Z_{s^k})$ of the zero current depends only on the normalized Szegő kernel P_k given in equation (11):

THEOREM 2.4 ([**ShZ08**]). *Let $(L, h) \to (M, \omega_h)$ be a positive holomorphic line bundle over a compact Kähler manifold. Then the variance of the zero current of holomorphic sections of L^k is given by*

$$\mathbf{Var}\big(Z_{s^k}\big) = -\frac{1}{4\pi^2} \partial_z \bar{\partial}_z \partial_w \bar{\partial}_w \mathrm{Li}_2[P_k(z, w)^2] \in \mathcal{D}'^{\,2,2}(M \times M)\,, \tag{24}$$

where Li_2 *is the 'di-logarithm'*

$$\mathrm{Li}_2(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^2} = -\int_0^t \frac{\log(1-x)}{x}\, dx\,.$$

Theorem 2.4 is equivalent via (23) to the variance formula:

$$\mathrm{Var}\left(\int_{Z_s} \psi\right) = \frac{1}{4\pi^2} \int_{M \times M} \mathrm{Li}_2[P_k(z, w)^2]\; i\partial\bar{\partial}\psi(z) \wedge i\partial\bar{\partial}\psi(w)\,, \tag{25}$$

for test forms $\psi \in \mathcal{D}^{m-1, m-1}(M)$. (In [**ShZ08**, Th. 3.1], $\widetilde{G}(t) = \frac{1}{4\pi^2}\mathrm{Li}_2(t^2)$.)

In the Riemann surface case, (24) becomes (in local coordinates)

$$\mathbf{Var}\big(Z_{s^k}\big) = \frac{1}{4\pi^2}\Delta_z\Delta_w \mathrm{Li}_2[P_k(z,w)^2]\;(idz\wedge d\bar z)\wedge(idw\wedge d\bar w),$$

so that $\frac{1}{4\pi^2}\mathrm{Li}_2[P_k(z,w)^2]$ is a bipotential for the variance of zeros. In higher dimensions, we say that $\frac{1}{4\pi^2}\mathrm{Li}_2[P_k(z,w)^2]$ is a *pluri-bipotential* for the variance current.

To prove Theorem 2.4, we first note that it suffices to verify the identity over a trivializing neighborhood U of (z,w). Using the notation of Section 1.1, we write $s^k = \langle \vec c, \vec{S^k}\rangle$, where $\vec{S^k} = \vec F e_L^{\otimes k}$, $s^k = f e_L^{\otimes k} = \langle \vec c, \vec F\rangle e_L^{\otimes k}$. We have by Theorem 2.2,

$$\mathbf{E}(Z_{s^k}) = \frac{i}{\pi}\partial\bar\partial \log\|\vec F\|\,. \tag{26}$$

The first step of the proof of Theorem 2.4 is the following lemma:

Lemma 2.5. *Writing $\tilde f = \|\vec F\|^{-1} f$, we have*

$$\mathbf{Var}(Z_{s^k}) = -\frac{1}{\pi^2}\partial_z\bar\partial_z\partial_w\bar\partial_w \mathbf{E}\left(\log|\tilde f(z)|\,\log|\tilde f(w)|\right)\,.$$

Proof. By the Poincaré–Lelong formula (15),

$$\mathbf{E}(Z_{s^k}\boxtimes Z_{s^k}) = -\frac{1}{\pi^2}\partial_z\bar\partial_z\partial_w\bar\partial_w \mathbf{E}\left(\log|f(z)|\,\log|f(w)|\right)\,. \tag{27}$$

Then $\tilde f = \langle \vec c, \vec u\rangle$, where $\vec u = \|\vec F\|^{-1}\vec F$, and

$$\begin{aligned}\log|f(z)|\,\log|f(w)| \;&=\; \log\|\vec F(z)\|\,\log\|\vec F(w)\| + \log\|\vec F(z)\|\,\log|\tilde f(w)|\\ &\quad + \log|\tilde f(z)|\,\log\|\vec F(w)\| + \log|\tilde f(z)|\,\log|\tilde f(w)|\,,\end{aligned}\tag{28}$$

which decomposes (27) into four terms. By (26), the first term contributes

$$-\frac{1}{\pi^2}\,\partial\bar\partial\log\|\vec F(z)\|\wedge\partial\bar\partial\log\|\vec F(w)\| = \mathbf{E}(Z_f)\boxtimes\mathbf{E}(Z_f)\,.$$

Since $\|\vec u\|\equiv 1$, $\mathbf{E}(\log|\tilde f(w)|)$ is independent of w and hence the second term vanishes when applying $\partial_w\bar\partial_w$. The third term likewise vanishes when applying $\partial_z\bar\partial_z$. Therefore, the fourth term yields the variance current $\mathrm{Var}(Z_{s^k})$. □

Next we use the following formula from [ShZ08, Lemma 3.3]:

Lemma 2.6. *Let (Y_1, Y_2) be joint complex Gaussian random variables of mean 0 and variances $\mathbf{E}(|Y_1|^2)=\mathbf{E}(|Y_2|^2)=1$. Then*

$$\mathbf{E}\big(\log|Y_1|\,\log|Y_2|\big) = \frac{1}{4}\mathrm{Li}_2\big(\big|\mathbf{E}(Y_1\overline{Y}_2)\big|^2\big) + \frac{\gamma^2}{4}\qquad (\gamma = \textit{Euler's constant})\,.$$

Completion of the proof of Theorem 2.4. Fix points $z,w\in M$, and let $Y_1=\tilde f(z)$, $Y_2 = \tilde f(w)$. Recalling (10), we have

$$|\mathbf{E}(Y_1\overline{Y}_2)| = \frac{|\sum_j F_j^k(z)\overline{F_j^k(w)}|}{\|\vec F(z)\|\,\|\vec F(w)\|} = \frac{\|B_k(z,w)\|}{\|B_k(z,z)\|^{\frac12}\,\|B_k(w,w)\|^{\frac12}} = P_k(z,w)\,.$$

Therefore, by Lemma 2.6,

$$\mathbf{E}\big(\log|\tilde f(z)|\,\log|\tilde f(w)|\big) = \mathbf{E}\big(\log|Y_1|\,\log|Y_2|\big) = \frac{1}{4}\,\mathrm{Li}_2\big(P_k(z,w)^2\big) + \frac{\gamma^2}{4}\,.$$

Equation (24) then follows from Lemma 2.5. □

2.4. Smooth linear statistics of zeros. By *linear statistics* for $H^0(M,L^k)$, we mean the random variable on the probability space $(H^0(M,L),\gamma_{h,dV})$

$$s^k \mapsto ([Z_{s^k}],\psi) := \int_{Z_{s^k}} \psi(z), \qquad s^k \in H^0(M,L^k), \tag{29}$$

for a fixed continuous test form $\psi \in \mathcal{D}^{m-1,m-1}(M)$. In particular, when M is a Riemann surface C, we have

$$([Z_{s^k}],f) = \sum_{z:s^k(z)=0} f(z), \qquad s^k \in H^0(C,L^k),$$

for a fixed continuous test function f.

Both the expectation and the variance of (29) have asymptotic expansions. To determine the asymptotic expansion of $\mathbf{E}(Z_{s^k},\psi)$, for $s^k \in H^0(M,L^k)$, we first apply (8) to obtain

$$\log\|B_k(z,z)\| \sim \log\left(\frac{k^m}{\pi^m}\right) + \frac{\rho_h}{2}k^{-1} + b_2k^{-2} + \cdots\,, \tag{30}$$

where ρ_h is the scalar curvature of ω_h. Then by Theorem 2.2 and (8), we obtain the complete asymptotic expansion of the linear statistics

$$\frac{1}{k}\,\mathbf{E}(Z_{s^k},\psi) \sim \frac{1}{\pi}\int_M \omega_h\wedge\psi + \left(\frac{i}{4\pi}\int_M \rho_h\,\partial\bar\partial\psi\right)k^{-2} + \cdots\,. \tag{31}$$

Similarly, the variance has the following complete asymptotic expansion:

THEOREM 2.7 ([**Sh21**]). *Let $(L,h)\to(M,\omega)$ be a positive holomorphic line bundle over a compact Kähler manifold, and let $\psi\in\mathcal{D}^{m-1,m-1}_{\mathbb{R}}(M)$. The variance of the linear statistics (Z_{s^k},ψ) has an asymptotic expansion of the form*

$$\mathrm{Var}\big(Z_{s^k},\psi\big) \sim A_0k^{-m} + A_1k^{-m-1} + \cdots + A_jk^{-m-j} + \cdots\,. \tag{32}$$

The leading and sub-leading coefficients are given by

$$A_0 = \frac{\pi^{m-2}\zeta(m+2)}{4}\,\|\partial\bar\partial\psi\|_2^2\,, \tag{33}$$

$$A_1 = -\pi^{m-2}\zeta(m+3)\left\{\frac{1}{8}\int_M \rho_h|\partial\bar\partial\psi|^2\frac{1}{m!}\omega_h^m + \frac{1}{4}\|\partial^*\partial\bar\partial\psi\|_2^2\right\}, \tag{34}$$

where ζ denotes the Riemann zeta function, and ρ_h is the scalar curvature of ω_h.

The expansion (32) builds on the methods of [**ShZ10**], where the asymptotic formula $\mathrm{Var}\big(Z_{s^k},\psi\big)=k^{-m}[A_0+O(k^{-1/2+\varepsilon})]$ was given.

In the complex curve case, (32) becomes

(35)
$$\mathrm{Var}\left(Z_{s^k},f\right)\sim\frac{\zeta(3)}{16\pi}\|\Delta f\|^2k^{-1}-\frac{\pi^3}{2880}\left\{\int_M\rho_h|\Delta f|^2\,\omega+\|d\Delta f\|_2^2\right\}k^{-2}+\cdots,$$

for $f\in\mathcal{C}^\infty(M)$.

Thus, smooth linear statistics are self-averaging in the sense that its fluctuations are of smaller order than its typical values. The fact that the variance involves $\|\Delta f\|_2^2$ rather than $\|\nabla f\|_2^2$ signals that the covariance kernel is not Δ^{-1} but Δ^{-2}.

2.5. Asymptotic normality of zero distributions. The following theorem was proved first by Sodin–Tsirelson [**ST04**] for certain model random analytic functions on $\mathbb{C}$, $\mathbb{CP}^1$ and the unit disc and then in [**ShZ10**] to general one-dimensional ensembles and to codimension one zero sets in higher dimensions:

THEOREM 2.8 ([**ShZ10**]). *Let $(L,h)\to(M,\omega)$ be a positive holomorphic line bundle over a compact Kähler manifold, and let ψ be a real $(m-1,m-1)$-form on M with $\mathcal{C}^3$ coefficients. Then for random sections $s^k\in H^0(M,L^k)$, the distributions of the random variables*

$$k^{m/2}\left(Z_{s^k}-\frac{k}{\pi}\,\omega,\psi\right)$$

converge weakly to the Gaussian distribution of mean 0 and variance $\frac{\pi^{m-2}\,\zeta(m+2)}{4}\,\|\partial\bar\partial\psi\|_2^2$, as $k\to\infty$.

Theorem 2.8 follows from a general result of Sodin-Tsirelson [**ST04**] on asymptotic normality of nonlinear functionals of Gaussian processes and the properties of the normalized Szegő kernel (11). To describe the result of [**ST04**], we recall that a (complex) Gaussian process on a measure space (T,μ) is a random variable (with values in the space of complex measurable functions on T) of the form

$$w(t)=\sum c_jg_j(t)\,,$$

where the c_j are i.i.d. complex Gaussian random variables of mean 0, variance 1, and the g_j are (fixed) complex-valued measurable functions. We say that $w(t)$ is *normalized* if $\sum|g_j(t)|^2=1$ for all $t\in T$; i.e., if $w(t)\sim\mathcal{N}_{\mathbb{C}}(0,1)$ for all $t\in T$.

THEOREM 2.9 ([**ST04**]). *Let $w^1,w^2,w^3,\dots$ be a sequence of normalized complex Gaussian processes on a finite measure space (T,μ). Let $f:\mathbb{R}^+\to\mathbb{R}$ be monotonically increasing such that $f(r)\in L^2(\mathbb{R}^+,e^{-r^2/2}rdr)$, and let $\eta:T\to\mathbb{R}$ be bounded measurable.*

Let $\mathcal{C}_k(s,t) := \mathbf{E}\left(w^k(s)\overline{w^k(t)}\right)$ *be the covariance function for* w^k *and suppose that*

i) $\displaystyle \liminf_{k\to\infty} \frac{\int_T \int_T |\mathcal{C}_k(s,t)|^2 \eta(s)\eta(t) d\mu(s) d\mu(t)}{\sup_{s\in T} \int_T |\mathcal{C}_k(s,t)| d\mu(t)} > 0$ *;*

ii) $\displaystyle \lim_{k\to\infty} \sup_{s\in T} \int_T |\mathcal{C}_k(s,t)| d\mu(t) = 0.$

Consider the random variables

$$Y_k = \int_T f(|w^k(t)|)\eta(t) d\mu(t).$$

Then the distributions of the random variables

$$\frac{Y_k - \mathbf{E}Y_k}{\sqrt{\mathrm{Var}(Y_k)}}$$

converge weakly to $\mathcal{N}(0,1)$ *as* $k \to \infty$*.*

To prove Theorem 2.8, we apply Theorem 2.9 with $f(r) = \log r$ and $(T,\mu) = (M, dV)$. To define the normalized Gaussian processes w^k on M, choose a measurable section $\sigma_L : M \to L$ of L with $\|\sigma_L(z)\|_h = 1$ for all $z \in M$, and let

$$S^k_j = F^k_j \sigma_L^{\otimes k}\,, \quad j = 1, \dots, d_k,$$

be an orthonormal basis for $H^0(M, L^k)$. We then let

$$w^k(z) := \sum_{j=1}^{d_k} c_j \,\frac{F^k_j(z)}{\sqrt{\Pi_k(z,z)}}\,.$$

Since $|F^k_j| = \|S^k_j\|_{h^k}$, it follows that w^k defines a normalized complex Gaussian process. In fact,

$$\sqrt{\Pi_k(z,z)}\, w^k \sigma_L^{\otimes k} = \sum c_j S^k_j = s^k\,, \tag{36}$$

where s^k is a random holomorphic section in $H^0(M, L^k)$.

We now let ψ be a fixed real $(m-1, m-1)$-form on M and we write

$$\frac{i}{\pi}\partial\bar\partial\psi = \eta\, dV\ .$$

Then by (36),

$$\begin{aligned} Y_k = \int_M \log|w^k|\eta\, dV &= \int_M \left(\log\|s^k(z)\|_{h^k} - \log\sqrt{\Pi_k(z,z)}\right)\frac{i}{\pi}\partial\bar\partial\psi(z) \\ &= \left(Z_{s^k}, \psi\right) + C_k\,, \end{aligned}$$

where each C_k is a constant independent of the random section s^k. Hence Y_k has the same variance as the linear statistic $\left(Z_{s^k}, \psi\right)$. In fact, the covariance functions $\mathcal{C}_k(z,w)$ for these Gaussian processes satisfy

$$|\mathcal{C}_k(z,w)| = P_k(z,w)\ .$$

It was shown in [**ShZ10**], using the properties of the normalized Bergman kernel P_k given by Propositions (1.3)–(1.4), that conditions (i)–(ii) of Theorem 2.9 hold. Hence, the distributions of the random variables

$$\frac{(Z_{s^k},\psi)-\mathbf{E}(Z_{s^k},\psi)}{\sqrt{\mathrm{Var}(Z_{s^k},\psi)}}=\frac{Y_k-\mathbf{E}(Y_k)}{\sqrt{\mathrm{Var}(Y_k)}}$$

converge weakly to the standard Gaussian distribution $\mathcal{N}(0,1)$ as $k\to\infty$.

The conclusion of Theorem 2.8 then follows from the leading asymptotics of the expectation $\mathbf{E}(Z_{s^k},\psi)$ and the variance $\mathrm{Var}(Z_{s^k},\psi)$ given by equations (31) and (32). □

Nazarov and Sodin [**NS11, NS12**] give results on variances and asymptotic normality for linear statistics on $\mathbb{C}$ with test functions that are not continuous. It is open whether similar results hold for line bundles on compact Kähler manifolds and whether asymptotic normality holds for linear statistics of zeros of any codimension.

One can also consider linear statistics for the point process consisting of simultaneous zeros of m independent random sections of $H^0(M,L^k)$:

$$(s_1^k,\dots,s_m^k)\mapsto\left([Z_{s_1^k,\dots,s_m^k}],f\right)=\sum_{\{s_1^k(z)=\cdots=s_m^k(z)=0\}}f(z),\tag{37}$$

for a fixed continuous test function f. Asymptotics for the expectation and variance of (37) are given in [**ShZ10**], but it is an open problem whether asymptotic normality holds for (37).

2.6. Counting random zeros in a set and hole probabilities. We say that a set $U\subset M$ is a 'hole' in the zero set Z_s of a section $s\in H^0(M,L)$ if $Z_s\cap U=\emptyset$. Hole probabilities, overcrowding and other number statistics for special ensembles of functions of one complex variable were given, for example, in [**Kr06, ST05**]. To describe the framework of these results in the case where $\dim M=1$, i.e. where M is a compact Riemann surface C, we consider the random variable $\mathcal{N}_k^U(s^k):=\#(Z_{s^k}\cap U)$ on $H^0(C,L^k)$ which counts the zeros of a section s^k in an open set U. Hence the hole probability is the probability that $\mathcal{N}_k^U(s^k)=0$. Sodin and Tsirelson [**ST05**] gave an asymptotic formula for the variance of $\mathcal{N}_k^U$ when $M=\mathbb{CP}^1$ (and for the analogous cases of holomorphic functions on $\mathbb{C}$ and on the disk). This formula was sharpened and generalized using Theorem 2.4 to arbitrary compact Riemann surfaces as well as to compact Kähler manifolds of any dimension, to obtain the following analogue of Theorem 2.7:

THEOREM 2.10 ([**ShZ08**]). *Let $(L,h)\to(M,\omega_h)$ be a positive holomorphic line bundle over a compact Kähler manifold. Let U be a domain in M with piecewise $\mathcal{C}^2$ boundary without cusps. Then for $m=\dim M$ independent Gaussian random sections $s_1^k,\dots,s_m^k$ in $H^0(M,L^k)$, the variance of the random variable*

$$\mathcal{N}_k^U(s_1^k,\dots,s_m^k):=\#\{z\in U: s_1^k(z)=\cdots=s_m^k(z)=0\}$$

has the asymptotics

$$\mathrm{Var}(\mathcal{N}_k^U) = k^{m-1/2}[\nu_m \mathrm{Vol}_{2m-1}(\partial U) + O(k^{-1/2+\varepsilon})]\,, \tag{38}$$

where ν_m is a universal positive constant depending only on m.

For the Riemann surface case, $\nu_1 = \zeta(3/2)/(8\pi^{3/2})$. For random zero sets of one section, we similarly have:

THEOREM 2.11 ([**ShZ08**]). *With the hypotheses of Theorem 2.10,*

$$\begin{aligned}&\mathrm{Var}(\mathrm{Vol}_{2m-2} Z_{s^k} \cap U)\\ &\quad = k^{3/2-m}\left[\tfrac{1}{8}\pi^{m-5/2}\zeta(m+1/2)\,\mathrm{Vol}_{2m-1}(\partial U) + O(k^{-1/2+\varepsilon})\right].\end{aligned}$$

Theorems 2.10 and 2.11 are special cases of a general result for simultaneous zeros of p holomorphic sections on M, for $1 \le p \le m$.

The volume of the zero set Z_{s^k} inside a domain also satisfies a large deviations bound of the form:

THEOREM 2.12 ([**ShZZr08**]). *Let $(L, h) \to (M, \omega_h)$ be as in Theorem 2.10, and let U be an open subset of M such that ∂U has zero measure in M. Then for all $\delta > 0$ sufficiently small, there is a constant $C_{U,\delta} > 0$ such that*

$$Prob\left\{\left|\mathrm{Vol}_{2m-2}(Z_{s^k} \cap U) - \frac{m}{\pi}\mathrm{Vol}_{2m}(U)\,k\right| > \delta\, k\right\} \le e^{-C_{U,\delta}k^{m+1}} \quad \forall\ k \gg 0\,.$$

Here, $k \gg 0$ means $k \ge k_0$ for some $k_0 \in \mathbb{Z}^+$. In particular, for the case where $\dim M = 1$, we have:

COROLLARY 2.13. *Let $(L, h) \to (C, \omega_h)$ be a positive Hermitian line bundle over a compact Riemann surface. Let $U \subset C$ be an open set in C such that ∂U has zero measure in C. Then for all $\delta > 0$, there is a constant $c_{U,\delta} > 0$ such that*

$$Prob\left\{s^k :\ \left|\mathcal{N}_k^U(s^k) - \frac{k}{\pi}\,\mathrm{Area}(U)\right| > \delta\, k\right\} \le e^{-c_{U,\delta}k^2} \quad \forall\ k \gg 0\,.$$

We also have upper and lower estimates for the 'hole probability':

THEOREM 2.14 ([**ShZZr08**]). *Let $(L, h) \to (M, \omega_h)$ and $U \subset M$ be as above, and suppose there is a section $s \in H^0(M, L)$ that does not vanish anywhere on $\overline{U}$. Then there exist constants $c'_U > c_U > 0$ such that*

$$e^{-c'_U k^{m+1}} \le Prob\{s^k : Z_{s^k} \cap U = \emptyset\} \le e^{-c_U k^{m+1}} \quad \forall\ k \gg 0\,. \tag{39}$$

The upper bound in (39) is an immediate consequence of Theorem 2.12 with $\delta < \frac{m}{\pi}\mathrm{Vol}_{2m}(U)$. The analogue of (39) for random entire functions of one variable was given in [**ST05**].

It is an open question whether

$$\log(Prob\{s^k : Z_{s^k} \cap U = \emptyset\}) \sim -\tilde{c}(U)k^{m+1}$$

for some constant $\tilde{c}(U)$. This was shown in [**Zhu14**] to hold for $M = \mathbb{CP}^m$ and $U = \Delta_r^m$, where $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$, with a specific formula for $\tilde{c}(\Delta_r^m)$. When $r \geq 1$,

$$\tilde{c}(\Delta_r^m) = \frac{2m \log r}{(m+1)!} + \frac{1}{m!} \sum_{j=1}^{m} \frac{1}{j+1} .$$

2.7. Expected local topology of random zero sets. An active topic of recent research in random *real* algebraic geometry is the random topology of random real algebraic varieties defined by the zero locus of one or several independent real polynomials of a fixed degree k: the number of connected components, the betti numbers, and the combinatorics of the components. Works of Gayet and Welschinger [**GaW14, GaW15, GaW17**] and others have resolved many problems in this area. For this survey, the question is whether there exist problems of this nature in the complex case. Globally, the answer is no: all of these topological invariants are deterministic. Recently, D. Gayet [**Ga22b**] has studied the local analogues of these problems in open sets $U \subset M$, and the local topology of $Z_{s^k} \cap U$ is very random. The main result of [**Ga22b**], stated here in the rank 1 case, is the following:

THEOREM 2.15. *Let $(L, h) \to (M, \omega_h)$ be a positive holomorphic line bundle over a compact Kähler manifold. Let U be an open set in M with smooth boundary. Then for random holomorphic sections $s^k \in H^0(M, L^k)$,*

$$\lim_{k\to\infty} \frac{1}{k^m} \mathbf{E}\, b_j(Z_{s^k} \cap U) = \begin{cases} 0, & \text{for } \ 0 \leq j \leq 2m-2, \ j \neq m-1 \\ \frac{m!}{\pi^m} \mathrm{Vol}_{2m}(U)\,, & \text{for } \ j = m-1 \end{cases} .$$

By contrast, in the real domain, all Betti numbers grow like λ^m where $1/\lambda$ is the natural scale of the model. In the complex setting, $\lambda = \frac{1}{\sqrt{k}}$. Yet the $(m-1)$-th Betti number (and only that Betti number) grows like k^m.

In [**Au97**], Auroux proved that the (deterministic) quantitatively transversal Donaldson hypersurfaces, which are zeros of sections that vanish transversally with a controlled derivative, satisfy this local topology estimate for the $(m-1)$-th Betti number. The Donaldson hypersurfaces have common features with the random ones of Theorem 2.15 in the complex setting. For instance, the current of integration Z_s fills out M uniformly for large degrees k in both contexts.

Gayet explains the intuition behind Theorem 2.15 when $m = 2$, as follows: By the maximum principle, if a complex curve $C \subset \mathbb{C}^2$ contacts a real hyperplane H, and is locally on one side of H, then $C \subset H$. Let $u : \mathbb{C}^2 \to \mathbb{R}$ be a Morse function and for $r > 0$ let $u_r(z) = u(r^{-1}z)$. For increasing r, the level sets of u_r locally become closer and closer to being planar so that there are fewer random curves touching them from the interior; i.e., there are fewer critical points of $u|Z_{s^k}$ of index 0 compared to critical points of index 1. The result then follows from Morse theory.

As this intuition suggests, the proof of Theorem 2.15 involves the strong Morse inequalities and a statistical study of critical points via the Kac–Rice formula.

3. Critical points and values of random holomorphic sections

In this section, we review some results on random critical points, critical values, and excursion sets of Gaussian random holomorphic sections of $H^0(M, L^k)$ and their asymptotics.

3.1. Critical points. The critical point set $\mathrm{Crit}(s, h)$ of a holomorphic section $s \in H^0(M, L)$ is defined by

$$\begin{aligned}\mathrm{Crit}(s,h) &:= \{z \in M : d(\|s\|_h^2)\big|_z = 0,\ s(z) \neq 0\} \\ &= \{z \in M : (\nabla_h s)(z) = 0,\ s(z) \neq 0\},\end{aligned} \tag{40}$$

where ∇_h is the Chern connection of the Hermitian holomorphic line bundle (L, h).

Along with the expected distribution of zeros (of a random section of a line bundle on a Riemann surface, or of simultaneous zeros in higher dimensions), we are also interested in the expected distributions of critical points

$$\mathbf{K}_1^{k\,\mathrm{Crit}} := \mathbf{E}\left[\textstyle\sum_{z\in\mathrm{Crit}(s^k,h^k)} \delta_z\right] \tag{41}$$

of random holomorphic sections $s^k \in H^0(M, L^k)$.

Additionally, the total number of critical points, $\#\mathrm{Crit}(s, h)$, is a (non-constant) random variable, unlike the total number of zeros of m holomorphic sections of $L \to M$, which equals the topological invariant $c_1(L)^m$. Although the alternating sum of critical points of each Morse index is a topological invariant, the sum $\#\mathrm{Crit}(s, h)$ as well as the number of critical points of a given Morse index is non-constant. Hence, we are interested in the average number of critical points,

$$\mathcal{N}_k^{\mathrm{Crit}}(L,h) := \mathbf{E}\big(\#\mathrm{Crit}(s^k,h^k)\big) = \int_M d\mathbf{K}_1^{k\,\mathrm{Crit}}, \tag{42}$$

of a random section $s^k \in H^0(M, L^k)$. We also consider the average number of critical points of Morse index q, which we denote by $\mathcal{N}_{k,q}^{\mathrm{Crit}}(L, h)$, for $m \le q \le 2m$. (It was observed in [**DShZ04**] that the critical points are all of Morse index $\ge m$.)

The expected number $\mathcal{N}_k^{\mathrm{Crit}}(L, h)$ and expected distribution of critical points can be expressed as formulas involving the Bergman kernel (see [**DShZ04**, Theorems 1 & 6]). These formulas can be used to obtain explicit expressions for the hyperplane section bundle $\mathcal{O}(1) \to \mathbb{CP}^m$ with the Fubini–Study metric. In fact, the averages $\mathcal{N}_k^{\mathrm{Crit}}(\mathcal{O}(1) \to \mathbb{CP}^m)$ are rational functions of k. In dimensions 1 and 2, we have

$$\begin{aligned}\mathcal{N}_k^{\mathrm{Crit}}\left(\mathcal{O}(1)\to\mathbb{CP}^1\right) &= \frac{5k^2-8k+4}{3k-2}\,,\\ \mathcal{N}_k^{\mathrm{Crit}}\left(\mathcal{O}(1)\to\mathbb{CP}^2\right) &= \frac{59\,k^5-231\,k^4+375\,k^3-310\,k^2+132\,k-24}{(3\,k-2)^3}\,.\end{aligned}$$

The average numbers of critical points of sections of $\mathcal{O}(k)\to\mathbb{CP}^m$ of each Morse index q ($m\le q\le 2m$) are likewise rational functions of k, for $m\ge 1$. See [**DShZ06**, Appendix 1] for some explicit formulas.

The expected number of critical points depends on the metric h, but surprisingly its leading asymptotics are independent of h. This can be seen from the following asymptotic formulas for $\mathcal{N}_k^{\mathrm{Crit}}(L,h)$ and $\mathcal{N}_{k,q}^{\mathrm{Crit}}(L,h)$:

Theorem 3.1 ([**DShZ06**, Cor. 1.4]). *Let $(L,h)\to(M,\omega_h)$ be a positive holomorphic line bundle on a compact Kähler manifold. Then for $m\le q\le 2m$,*

$$\begin{aligned}\mathcal{N}_{k,q}^{\mathrm{Crit}}(L,h)\sim{}&[\beta_{0qm}\,c_1(L)^m]\,k^m+\left[\beta_{1qm}\,c_1(M)\cdot c_1(L)^{m-1}\right]k^{m-1}\\ &+\left[\beta_{2qm}\int_M\rho_h^2\,dV_h+\beta'_{2qm}\,c_1(M)^2\cdot c_1(L)^{m-2}\right.\\ &\left.+\beta''_{2qm}\,c_2(M)\cdot c_1(L)^{m-2}\right]k^{m-2}+\cdots\,,\end{aligned}\tag{43}$$

where ρ_h is the scalar curvature of ω_h, and $\beta_{0qm},\beta_{1qm},\beta_{2qm},\beta'_{2qm},\beta''_{2qm}$ are universal constants depending only on q and m.

Taking the sum over q from m to $2m$, one obtains a similar asymptotic expansion for $\mathcal{N}_k^{\mathrm{Crit}}(L,h)$.

In fact, both the leading term and the subleading term of the expansion do not depend on the choice of metric on L. Note that the non-topological part of the third term is β_{2qm} times the Calabi functional

$$Cal(\omega_h):=\int_M\rho_h^2\,dV_h\,.$$

It was shown in [**Bau10**] that $\sum_{q=m}^{2m}\beta_{2qm}$ is positive for all $m\ge 1$, and hence we have:

Corollary 3.2. *Let h and h' be metrics on L with positive curvature. If $Cal(h')>Cal(h)$, then there exists k_0 such that $\mathcal{N}_k^{\mathrm{Crit}}(L,h')>\mathcal{N}_k^{\mathrm{Crit}}(L,h)$ for all $k\ge k_0$.*

It is known that Kähler metrics of constant scalar curvature are critical metrics of the functional Cal on the space of Kähler metrics in $c_1(L)$ and that all critical metrics are global minimums for Cal on this space [**Ca85, Hw95**]. Furthermore, if $c_1(L)$ contains a constant scalar curvature Kähler metric, then every critical metric has constant scalar curvature [**Ca85**]. Thus, constant scalar curvature metrics have “asymptotically minimal critical numbers” in the sense that if h and h' are two positive metrics on L such that ω_h

but not $\omega_{h'}$ has constant scalar curvature, then the conclusion of Corollary 3.2 holds.

It is known that $\beta_{2mm} > 0$ for all m [**Bau10**] and that $\beta_{2qm} > 0$ in low dimensions [**DShZ06**], but it remains an open question whether $\beta_{2qm} > 0$ for all q, m.

Like the zeros of a random section on a Riemann surface (or the simultaneous zeros of m-sections on M), the critical points of random sections on M form a point process. Its one-point function (or expected density) has an asymptotic expansion [**DShZ06**]

$$\frac{1}{k^m}\mathbf{K}_1^{k\,\mathrm{Crit}}(z) \sim \left[b_0 + b_1(z)k^{-1} + b_2(z)k^{-2} + \cdots\right]\frac{1}{m!}\omega_h^m\,. \tag{44}$$

The constant b_0 depends only on the dimension m; see [**DShZ06**] for its values. To our knowledge, the only result on the pair correlation of critical points of sections in $H^0(M, L^k)$ is the asymptotic formula for the case of Riemann surfaces in [**Bab12**]:

THEOREM 3.3. *Let $(L, h) \to (C, \omega_h)$ be a positive line bundle over a compact Riemann surface C. Then the pair correlation of critical points $\tilde{K}_2^{k\,\mathrm{Crit}}(z, w)$ has the scaling limit*

$$\lim_{k\to\infty}\frac{1}{k^2}\,\tilde{K}_2^{k\,\mathrm{Crit}}\left(\frac{z}{\sqrt{k}}, \frac{w}{\sqrt{k}}\right) = \frac{2}{3\pi^2} + O(r^2)\,,$$

where $r = \mathrm{dist}(z, w)$.

Thus the clustering of critical points on Riemann surfaces is similar to that of Poisson point processes. It is an open problem to determine critical point pair correlation formulas in complex dimension greater than one.

3.2. Sup norms and random excursion sets. In this section, we discuss the excursion sets

$$\{z \in M : \|s(z)\|_h > r\}$$

and sup norms of random holomorphic sections s of positive holomorphic line bundles (L, h). Random excursion sets have been much studied in the case of real Gaussian fields for over three decades; we refer to [**Ga22a, TA03**] for background and references to that subject.

It is most useful to study the sup norms and excursion sets of random sections of unit $\mathcal{L}^2$ norm with respect to Haar probability measure ν_k on the unit $(2d_k - 1)$-sphere,

$$SH^0(M, L^k) = \{s^k \in H^0(M, L^k) : \|s^k\|_{G(h^k, dV)} = 1\}, \tag{45}$$

which probability space we call the *spherical ensemble*. One well-known problem is to determine the expected Euler characteristics of the excursion sets,

$$\mathbf{E}_{\nu_k}\left[\chi\{\|s^k(z)\|_{h^k} > r\}\right] = \int_{H^0(M,L^k)} \chi\{\|s^k(z)\|_{h^k} > r\}\, d\nu_k(s),$$

and also the probability that the excursion set is non-empty. (Here and in the following, χ denotes the Euler characteristic.)

We note that for all $s^k = \sum c_j S_j^k \in SH^0(M, L^k)$, we have by (6)–(8),

(46)
$$|s^k(z)|^2_{h^k} \leq \sum_{l=1}^{d_k} |c_l|^2 \sum_{j=1}^{d_k} \|S_j^k(z)\|^2_{h^k} = 1 \cdot \|B_k(z,z)\| = \left(\frac{1}{\pi^m} + o(1)\right) k^m,$$

and thus the excursion sets $\{z : \|s^k(z)\|_{h^k} > (\pi^{m/2} + \varepsilon)k^{m/2}\}$ are empty for all $s^k \in H^0(M, L^k)$, for k sufficiently large. Furthermore, a much sharper asymptotic upper bound usually holds:

THEOREM 3.4 ([**ShZ03**, Th. 1.1]). *For all $n > 0$, there exists a positive constant C_n such that the probability*

$$Prob\left\{\sup_{z\in M} \|s^k(z)\|_{h^k} > C_n\sqrt{\log k}\right\} < O\left(\frac{1}{k^n}\right), \qquad s^k \in H^0(M, L^k).$$

Thus for almost all random sequences $\{s^k\} \in \prod_{k=1}^\infty SH^0(M, L^k)$,

$$\|s^k\|_\infty = O\left(\sqrt{\log k}\right). \tag{47}$$

To our knowledge, the only article studying excursion sets in the holomorphic setting is a paper of Jingzhou Sun [**Sun12**], and we summarize Sun's results below.

THEOREM 3.5. *Let $(L, h) \to (M, \omega_h)$ be a positive holomorphic line bundle on a compact Kähler manifold, and let k_0 be sufficiently large so that L^k is very ample for all $k \geq k_0$. Then there exists $u_0 < 1$ independent of k, such that for $1 \geq u > u_0$, and $k \geq k_0$,*

i) *the excursion set*

$$\mathcal{E}_u^k(s^k) := \left\{z \in M : \|s^k(z)\|_{h^k} > u\,\|B_k(z,z)\|^{1/2}\right\}$$

is either empty or contractible, and

ii) $$Prob\big[\mathcal{E}_u^k(s^k) \neq \emptyset\big] = Prob\left\{\sup_{z\in M} \|s^k(z)\|_{h^k} > u\,\|B_k(z,z)\|^{1/2}\right\}$$
$$= \int_M c(M)(1 - kc_1(L)) \wedge (ku^2\, c_1(L) - u^2 + 1)^{d_k - 1},$$

where $c(M)(1 - kc_1(L))$ is the Chern polynomial evaluated at $1 - kc_1(L)$.

COROLLARY 3.6. *With the hypotheses and notation of Theorem 3.5, for $1 \geq u > u_0$, the expected Euler characteristic*

$$\begin{aligned}\mathbf{E}\,\chi\big[\mathcal{E}_u^k(s^k)\big] &= \int_M c(M)(1 - kc_1(L)) \wedge (ku^2\, c_1(L) - u^2 + 1)^{d_k-1} \\ &= (1 + o(1))d_k^{m+1}(1 - u^2)^{d_k - m - 1} u^{2m}\,.\end{aligned}$$

Recall that

$$d_k = \dim H^0(M, L^k) = \frac{k^m}{m!} \int_M c_1^m(L) + O(k^{m-1}),$$

by the Hirzebruch–Riemann–Roch formula and Kodaira vanishing theorem (or see e.g., [**ShS85**, Lemma 7.6]).

When $\dim M = 1$, Theorem 3.5 yields the following:

COROLLARY 3.7. *Let (L, h) be a positive line bundle of degree δ over a compact Riemann surface (M, ω_h) of genus g. Then with the notation of Theorem 3.5, there exists $u_0 < 1$ independent of k, such that for $1 \geq u > u_0$, the expected Euler characteristic*

$$\mathbf{E}\,\chi\big[\mathcal{E}_u^k(s^k)\big] = (1-u^2)^{k\delta-g-1}\left[k^2\delta^2u^2 - k\delta(gu^2 - 1 + u^2) + (2-2g)(1-u^2)\right],$$

for $k\delta > 2g - 2$.

To prove Theorem 3.5, J. Sun proves an embedding theorem of independent interest:

THEOREM 3.8. *Let $(L, h) \to (M, \omega_h)$ be a positive holomorphic line bundle on a compact Kähler manifold, and let $k_0 \in \mathbb{Z}^+$ so that L^k is very ample for all $k \geq k_0$. Let $\Phi_k : M \to \mathbb{CP}^{d_k-1}$ be an embedding given by an orthonormal basis of $H^0(M, L^k)$ with respect to the Hermitian inner product $G(h^k, dV)$, for $k \geq k_0$. Let r_k be the critical radius of $\Phi_k(M) \subset \mathbb{CP}^{d_k-1}$, where $\mathbb{CP}^{d_k-1}$ is given the Fubini–Study metric. Then $\inf_{k\geq k_0} r_k > 0$.*

The proof of Theorem 3.5 uses the volume of tubes formula in [**Gr85**], setting $u_0 = \cos r_0$, where $r_0 = \inf_{k\geq k_0} r_k$.

3.3. Critical values. We now turn to the distribution of critical values of random holomorphic sections of powers of $L \to M$. By the "value" of $s^k(z) \in H^0(M, L^k)$, we mean the norm $\|s^k(z)\|_{h^k} \in \mathbb{R}^+$. We study the norms, since the values $s^k(z)$ lie in different fibers L_z^k of L^k. Thus we let

$$\begin{aligned} \mathrm{CV}(s^k) &:= \{\|s^k(z)\|_{h^k} : z \in \mathrm{Crit}(s^k, h^k)\} \\ &= \{\|s^k(z)\|_{h^k} : z \in M\,, \nabla s^k(z) = 0\,, s^k(z) \neq 0\} \end{aligned} \tag{48}$$

denote the set of critical values of a section $s^k \in H^0(M, L^k)$.

Since $\mathrm{CV}(\lambda s^k) = |\lambda|\,\mathrm{CV}(s^k)$, it is most useful to study the distribution of critical values of random sections of unit $\mathcal{L}^2$ norm with respect to Haar probability measure ν_k on the unit $(2d_k - 1)$-sphere

$$SH^0(M, L^k) = \{s^k \in H^0(M, L^k) : \|s^k\|_{\mathcal{L}^2} = 1\}, \tag{49}$$

which probability space we call the *spherical ensemble*. (Clearly, the expected distribution of critical points of s^k in the spherical ensemble is identical to that in the Gaussian ensemble and both ensembles have the same expected numbers $\mathcal{N}_{k,q}^{\mathrm{Crit}}(L, h)$.)

We recall that by (46), we have the deterministic bound $\mathrm{CV}(s^k) \subset (0, Ck^{m/2})$, and furthermore by (47), $\mathrm{CV}(s^k) \subset \big(0, C\sqrt{\log k}\big)$ almost surely.

We let

$$[\mathrm{CV}(s^k)] = \textstyle\sum_{z \in \mathrm{Crit}(s^k, h^k)} \delta_{|s^k(z)|_{h^k}} \tag{50}$$

denote the critical value distribution of a section $s^k \in SH^0(M, L^k)$. To describe the asymptotics of the spherical averages $\mathbf{E}_{\nu_k}[\mathrm{CV}(s^k)]$, we use the following notation: denote by $\mathrm{Sym}(m, \mathbb{C}) \cong \mathbb{C}^{\frac{m^2+m}{2}}$ the space of $m \times m$ complex symmetric matrices, and define the special (positive definite) operator

(51)

$$Q = (Q_{jq}^{j'q'}) := \left(\delta_{jj'}\delta_{qq'} + \delta_{jq'}\delta_{qj'}\right), \quad 1 \le j \le q \le m, \ 1 \le j' \le q' \le m.$$

We define the universal function (depending only on the dimension m)

$$f_m(t) = \frac{2}{\pi^{m(m+3)}} \int_{\mathrm{Sym}(m,\mathbb{C})} e^{-|\xi|^2} \left| \det \left(\left| \sqrt{Q}\, \Xi \right|^2 - t^2 I \right) \right| d\Xi,$$

where $d\Xi$ denotes Lebesgue measure on $\mathrm{Sym}(m, \mathbb{C}) \cong \mathbb{R}^{m^2+m}$. We then have:

THEOREM 3.9 ([**FZ14**]). *Let $(L, h) \to (M, \omega_h)$ be a positive holomorphic line bundle on a compact Kähler manifold. The normalized expected density of critical values in the spherical ensemble $SH^0(M, L^k)$ has the asymptotics*

$$\frac{1}{k^m} \mathbf{E}_{\nu_k}[\mathrm{CV}(s^k)] \to \mathrm{Vol}(M)\, f_m(t)\, t e^{-t^2}\, dt\,.$$

In the case of complex curves, $f_1(t) = \frac{1}{\pi}(2t^2 - 4 + 8e^{-t^2/2})$.

4. Point processes and Kähler metrics

In this section we give two examples showing how Kähler metrics can be constructed using point processes.

4.1. Zero point processes. Given a positive line bundle (L, h) over a compact Riemann surface C, we can form the point processes Z_{s^k} of zeros of random sections of powers L^k of the line bundle. Recall from Theorem 2.1 that the expected measure $\frac{1}{k}\mathbf{E}Z_{s^k}$ converges to $\frac{1}{\pi}\omega_h$. In higher dimensions, for holomorphic sections $s_1, \cdots, s_m$ of a line bundle $L \to M$, chosen so that their common zero set

$$Z_{s_1,\ldots,s_m} = \{z \in M : s_1(z) = \cdots = s_m(z) = 0\} = \{\zeta_1, \cdots, \zeta_p\}$$

is finite, we define the empirical probability measure

$$\frac{1}{p}[Z_{s_1,\ldots,s_m}] = \frac{1}{p}\sum_{j=1}^{p} \delta_{\zeta_j}\,.$$

Given random holomorphic sections of a positive line bundle $L^k \to M$, the probability measures $\gamma_{h^k,dV}$ on $H^0(M, L^k)$ induce point processes $Z_{s_1^k,\ldots,s_m^k}$ on M (for k sufficiently large so the common zero set is 0-dimensional a.s.). We then have

THEOREM 4.1 ([**ShZ99**, Prop. 4.4]). *Let $(L,h)\to(M,\omega_h)$ be a positive holomorphic line bundle over a compact Kähler manifold. Then*

$$\frac{1}{k^m}\,\mathbf{E}[Z_{s_1^k,\dots,s_m^k}]\to\frac{1}{\pi^m}\omega_h^m=\frac{m!}{\pi^m}\,d\mathrm{Vol}_M\,.$$

4.2. Berman's canonical Kähler point process. In a series of articles, [**Be11, Be14, Be17, Be18, Be20, Be21**], R. Berman investigated determinantal point processes on Kähler manifolds defined in terms of Bergman kernels and related geometric invariants.

Let M be a compact Kähler manifold and suppose that the canonical line bundle $K_M=\bigwedge_m T_M^*$ is ample. It was shown by Aubin [**Aub78**] and by Yau [**Yau78**] that M carries a Kähler–Einstein metric ω_{KE}; i.e., the metric has constant scalar curvature: $\mathrm{Ric}\,\omega_{KE}=-\omega_{KE}$. Berman constructs determinantal point processes $[\mathbf{z}_k]$ and uses their expected empirical measures to obtain Kähler metrics on M converging to ω_{KE} as $k\to\infty$.

Whereas the zero point process above depends on the choice of the metric (i.e., on the measure $\gamma_{h^k,dV}$), Berman's canonical point processes are independent of the choice of metric. As before (with $L=K_M$), let $\{S_1^k,\dots,S_{d_k}^k\}$ be a basis for the pluricanonical system $H^0(M,K_M^k)$, where $d_k=\dim H^0(M,K_M^k)$. Berman's canonical probability measure (or point process) on the configuration space M^{d_k} of d_k points in M is the probability measure μ_k^B defined by

$$\mu_k^B:=\frac{1}{\lambda_k}\left|\det S^{(k)}(z_1,\dots,z_{d_k})\right|^{\frac{2}{k}},$$

where λ_k is a normalizing constant and

$$\det S^{(k)}(z_1,\dots,z_{d_k}):=\det\left(S_j^k(z_l)\right)_{1\le j,l\le d_k}$$

is a holomorphic section of the pluricanonical bundle $K^k_{M^{d_k}}\to M^{d_k}=M\times\cdots\times M$. Thus $|\det S^{(k)}|^{2/k}$ is a semi-positive section of $K_{M^{d_k}}\wedge\overline{K_{M^{d_k}}}$, i.e. a positive measure on M^{d_k}. Changing the basis changes $|\det S^{(k)}|^{2/k}$ by a constant factor, and we divide by λ_k so that μ_k^B is a well-defined probability measure.

As mentioned at the beginning of Section 2, a point $\mathbf{z}_k=(z_1,\dots,z_{d_k})\in M^{d_k}$ gives rise to the empirical probability measure on M,

$$[\mathbf{z}_k]:=\frac{1}{d_k}\sum_{j=1}^{d_k}\delta_{z_j}\,.$$

Thus (after dividing out by the symmetric group) we can consider μ_k^B to be a probability measure on the space of discrete probability measures on M.

Berman then obtains canonical sequences of Kähler forms and volume forms on M converging to the Kähler-Einstein metric and volume, respectively:

Theorem 4.2 ([**Be17**]). *Let M be a compact Kähler manifold such that K_M is ample.*

i)
$$\mathbf{E}_{\mu_k^B}[\mathbf{z}_k] \to c\,\omega_{KE}^m\,, \quad as \quad k\to\infty,$$
where c is a normalizing constant;

ii) *Writing $\mathbf{E}_{\mu_k^B}[\mathbf{z}_k] = f_k\,(i^{m^2}\eta\wedge\bar\eta)$ over an open set U, where η is a nonvanishing holomorphic m-form, the Kähler form*
$$\omega_k := \frac{i}{2\pi}\partial\bar\partial \log f_k \to \omega_{KE}\,, \quad as \quad k\to\infty.$$

Note that the Kähler forms ω_k are globally defined and independent of the choice of η. Part (i) is an analog of Theorem 4.1. The constant c is chosen so that $\int_M c\,\omega_{KE}^m = c\,m!\mathrm{Vol}_{KE}(M) = 1$.

A link between determinantal point processes, Coulomb gas point processes and the Liouville functional in LQG is given in [**K14**].

5. Random Bergman metrics

In this section, we discuss a recent direction to stochastic Kähler geometry: the study of random Kähler metrics in a fixed class $\mathcal{K}_{[\omega_0]}$ and their approximations by random Bergman metrics, as given by Ferrari, Klevtsov, and Zelditch [**FKZ12, FKZ13, KZ16**]. Here, $\omega_0 = \omega_h = \pi c_1(L,h)$ is the Kähler metric of a positive line bundle $(L,h)\to(M,\omega_h)$ and $\mathcal{K}_{[\omega_0]}$ is the infinite dimensional space of Kähler metrics $\omega\in[\omega_0]$, the cohomology class of ω_0. The space of all Kähler metrics $\mathcal{K}_{[\omega_0]}$ on M in the Kähler class $[\omega_0]$ is parametrized as

$$\mathcal{K}_{[\omega_0]} = \{\varphi\in C^\infty(M)/\mathbb{R} : \omega_0 + i\partial\bar\partial\varphi > 0\}. \tag{52}$$

The motivation to study rather general types of random Kähler metrics originates in some sense in Polyakov's approach to quantum gravity. In complex dimension one, it has led to an explosion of articles on LQG (Liouville quantum gravity). In keeping with our emphasis on higher dimensional Kähler manifolds, we do not review the voluminous literature on LQG but only the random Kähler metrics studied in [**FKZ12, FKZ13, KZ16**]. To endow (52) with an interesting probability measure is very difficult because of its infinite dimensionality. In LQG, one specific measure is studied and it is induced by a well-studied Gaussian field, the Gaussian free field. More precisely, it is a renormalized version of the exponential of the GFF and is known as the Gaussian multiplicative chaos. In higher dimensions, there is no parallel construction and one has to start from scratch. The main idea is to define a sequence μ_k of probability measures on finite dimensional spaces $\mathcal{B}_k$ of Bergman metrics and then to study their limits.

We begin by describing the spaces $\mathcal{B}_k$ and then focus on one specific choice of probability measure induced by Brownian motion on $\mathcal{B}_k$ with respect to its symmetric space Riemannian metric. The space $\mathcal{B}_k$ of Bergman metrics of degree k is the space of metrics given by the pullbacks of Fubini–Study metrics by the Kodaira map for $H^0(M,L^k)$. I.e., let $\{\sigma_1,\dots,\sigma_{d_k}\}$ be a basis for $H^0(M, L^k)$, and let

$$\iota_\sigma = [\sigma_1, \ldots, \sigma_{d_k}] : M \to \mathbb{CP}^{d_k - 1} . \tag{53}$$

Since positive line bundles are ample, we can choose k sufficiently large so that (53) is an imbedding (for all bases $\{\sigma_j\}$ of $H^0(M, L^k)$). The associated Bergman metric is

$$\frac{1}{k} \iota_\sigma^* \omega_{FS} = \frac{i}{2k} \partial\bar{\partial} \log \sum_{j=1}^{d_k} |f_j|^2 , \tag{54}$$

where $\sigma_j = f_j e_L^{\otimes k}$ for a local frame e_L. The space $\mathcal{B}_k$ of Bergman metrics of degree k then consists of all metrics of the form (54).

The space $\mathcal{B}_k$ can be parametrized by the symmetric space $\mathrm{SU}(d_k)\backslash \mathrm{SL}(d_k, \mathbb{C})$ as follows: let $\{S_1^k, \ldots, S_{d_k}^k\}$ be a fixed orthonormal basis with respect to the inner product (4) induced by h and ω_0, and write $S_j^k = F_j^k e_L^{\otimes k}$ as above. For matrices $A = (A_{jl}) \in \mathrm{SL}(d_k, \mathbb{C})$, we let $\sigma_j^A = \sum_l A_{jl} S_l^k$. Then $\sigma^A = \{\sigma_1^A, \ldots \sigma_{d_k}^A\}$ is a basis for $H^0(M, L^k)$, and the associated Bergman metric is

$$\frac{1}{k} (\iota_{\sigma^A})^* \omega_{FS} = \frac{i}{2k} \partial\bar{\partial} \log \sum_{jl} \bar{F}_j^k P_{jl} F_l^k = \omega_0 + \frac{i}{2k} \partial\bar{\partial} \log \sum_{jl} \left\| \bar{S}_j^k P_{jl} S_l^k \right\|_{h^k} , \tag{55}$$

where $P = A^* A$ is in the space of positive definite Hermitian $d_k \times d_k$ matrices with determinant one. We denote this space by $\mathcal{P}_{d_k}$ and note that $\mathrm{SU}(d_k)\backslash \mathrm{SL}(d_k, \mathbb{C}) \cong \mathcal{P}_{d_k}$ via the map $A \mapsto A^* A$.

For matrices $P = A^* A \in \mathcal{P}_{d_k}$, we define the *Bergman potential*

$$\varphi_P := \frac{1}{2k} \log \sum_{jl} \bar{F}_j^k P_{jl} F_l^k , \tag{56}$$

and we let

$$\omega_P := \frac{1}{2k} (\iota_{\sigma^A})^* \omega_{FS} = i\, \partial\bar{\partial} \varphi_P \tag{57}$$

denote the corresponding Bergman metric.[1] In particular, $\varphi_{\mathbf{I}_k} = \varphi_h + \frac{1}{k} \log \|B_k(z, z)\|$, and

$$\omega_{\mathbf{I}_k} = \omega_0 + \frac{i}{2k} \partial\bar{\partial} \log \|B_k(z, z)\| = \omega_0 + O\left(\frac{1}{k^2}\right) , \tag{58}$$

by (8), where $\mathbf{I}_k$ is the $d_k \times d_k$ identity matrix.

[1] Here, we are using the convention that $\pi\omega_{\mathrm{FS}}$ is in the Chern class of the hyperplane section bundle $\mathcal{O}(1) \to \mathbb{CP}^m$, and thus $[\pi\omega_P] = c_1(M, L)$ for $\omega_P \in \mathcal{B}_k$. Hence in this article, ω_P and φ_P equal $\frac{1}{2}$ the corresponding terms in [**FKZ12, KZ16**].

5.1. Heat kernel measures. Given an orthonormal basis $\{S_j^k\}$ of $H^0(M, L^k)$ with respect to h and $\omega_0 = \omega_h$, the space $\mathcal{B}_k$ can be identified with the symmetric space $\mathcal{P}_{d_k}$ via equations (55)–(57). The general question is to find sequences $\{d\mu_k\}$ of measures on $\mathcal{B}_k$ that are independent of the choices of the basis $\{S_j^k\}$ and which vary in a simple way under the change of the reference point $\omega_0 \in \mathcal{K}_{[\omega_0]}$ and have good asymptotic properties as $k \to \infty$. Such measures can be given as the heat kernel measures

$$d\mu_k^t(P) := p_k(t, \mathbf{I}_k, P)\, dP, \tag{59}$$

where dP is Haar measure on $\mathcal{P}_{d_k}$, and $p_k(t, P_1, P_2)$ is the heat kernel of the symmetric space $\mathcal{P}_{d_k}$. The measure is invariant under the action of the unitary group $\mathrm{U}(d_k)$ and thus is independent of the choice of the orthonormal basis of sections $\{S_j^k\}$ used for the matrix-metric identification in (55). Then (59) is the probability measure on $\mathcal{B}_k$ induced by Brownian motion on $\mathcal{P}_{d_k}$ starting at the identity $\mathbf{I}_k$ at time $t = 0$.

In this section we review results of [**KZ16**] on the behavior of the heat kernel measure (59) on $\mathcal{B}_k$ as $k \to \infty$. The heat kernel measure is only one among many possible measures to study; we choose it because it has a simple geometric and probabilistic interpretation and because we obtain surprisingly explicit formulae for its correlation function. However, it is so closely tied to the symmetric space geometry of positive Hermitian matrices that it does not reflect the deeper geometric aspects of $\mathcal{B}_k$. At the end of this section, we propose a model which does go deeper, namely the Calabi metric measure on $\mathcal{B}_k$. However it is difficult to obtain analytic expressions for the key probabilistic objects for this Calabi model.

It was shown in [**FKZ12**], that for all probability measures ν on $\mathcal{P}_{d_k}$ on $\mathcal{P}_{d_k}$ invariant under the $\mathrm{U}(d_k)$ action $P \mapsto U^* P U$, one has

$$\mathbf{E}_\nu \varphi_P = \varphi_{\mathbf{I}_k} = \frac{1}{2k} \log \|B_k(z,z)\| - \log \|e_L(z)\|\,, \tag{60}$$

and thus by (8)

$$\mathbf{E}_\nu \omega_P = \omega_{\mathbf{I}_k} = \omega_0 + O\left(\frac{1}{k^2}\right). \tag{61}$$

In particular (61) holds for the heat kernel measures.

However, the two-point correlations depend on the choice of invariant measure. The two-point correlations for the heat kernel measures (59) were given in [**KZ16**], where it was shown that the correlations have the form

$$\mathbf{E}_{\mu_k^t} \varphi_P(z) \varphi_P(w) = \varphi_{\mathbf{I}_k}(z) \varphi_{\mathbf{I}_k}(w) + \frac{1}{4k^2} I_2(t, \beta_k(z,w)), \tag{62}$$

where

$$\beta_k(z,w) = \frac{\|B_k(z,w)\|^2}{\|B_k(z,z)\|\,\|B_k(w,w)\|} = P_k(z,w)^2 \tag{63}$$

is the *Berezin kernel*, and

$$\frac{\partial}{\partial x} I_2(t,x) = \frac{2t}{x} - \frac{e^{-t/2}}{\sqrt{2\pi t}} \frac{\sqrt{1-x}}{x} \int_{-\infty}^{\infty} d\lambda \, \frac{e^{-\frac{1}{2t}\lambda^2} \cosh \lambda}{\sqrt{\coth^2 \lambda - x}} \times \log \frac{\sqrt{\coth^2 \lambda - x} + \sqrt{1-x}}{\sqrt{\coth^2 \lambda - x} - \sqrt{1-x}}. \tag{64}$$

It follows from (60) and (62) that

$$\mathbf{Var}(\varphi_P) = \frac{1}{4k^2} I_2(t, \beta_k)\,, \tag{65}$$

where $\mathbf{Var} = \mathbf{Var}_{\mu_k^t}$ is given by Definition 2.3. Note that

$$\big(\mathbf{Var}(\varphi_P)\big)(z,w) = \mathrm{Cov}\big(\varphi_P(z), \varphi_P(w)\big)\,.$$

Furthermore, differentiating (65), we have

$$\mathbf{Var}\big(\omega_P\big) = \frac{1}{4k^2} (i\partial\bar\partial)_z \, (i\partial\bar\partial)_w \, I_2(t, \beta_k(z,w))\,. \tag{66}$$

Formula (66) says that $I_2(t,\beta_k)$ is the pluri-bipotential of the variance of the Kähler metric for the heat kernel measure at time t. In the Riemann surface case ($\dim M = 1$), the variance of the area of a domain $U \subset M$ is given by

$$\mathrm{Var}\left(\int_U \omega_P\right) = \int_{U\times U} \mathbf{Var}(\omega_P) = -\frac{1}{4k^2} \int_{\partial U \times \partial U} \partial_z \, \partial_w \, I_2(t, \beta_k(z,w))\,.$$

If we fix k and let $t \to \infty$. Then

$$\frac{\partial}{\partial x} I_2(\infty, x) := \lim_{k\to\infty} \frac{\partial}{\partial x} I_2(t,x) = -\frac{\log(1-x)}{x}\,, \tag{67}$$

(see [**KZ16**]) and thus

$$I_2(\infty, x) = \mathrm{Li}_2(x)\,.$$

Therefore for fixed k, the variance of random Kähler metrics with respect to the heat kernel measure μ_k^t on the space $\mathcal{B}_k \cong \mathcal{P}_{d_k}$ converges to the variance of the scaled zero current $\frac{\pi}{k} Z_{s^k} = \frac{i}{k}\partial\bar\partial \log|f|$ of a random section $s^k = f e_L^{\otimes k} \in H^0(M, L^k)$ (given in Theorem 2.4) as $t \to \infty$. In fact, in the $t \to \infty$ limit, the random metrics converge to random zero divisors regarded as singular metrics, in the sense described in [**KZ16**, §5.1].

Since (66) gives an exact formula for any t, k, one may also consider a variety of limits as $t \to \infty$, $k \to \infty$ in some relation. There is a natural choice of coupled limit motivated by the metric asymptotics of $\mathcal{B}_k$. If we rescale the Cartan–Killing (CK) metric $g_{CK,k}$ on $\mathcal{P}_{d_k} \approx \mathcal{B}_k \subset \mathcal{K}_{[\omega_0]}$ as $g_k = \epsilon_k^2 g_{CK,k}$, with $\epsilon_k = k^{-1} d_k^{-1/2}$, then $g_k \to g_M$ on $T\mathcal{B}_k$. Here g_M is the Mabuchi metric on $\mathcal{K}_{[\omega_0]}$, i.e. the Riemannian metric on $\mathcal{K}_{[\omega_0]}$ defined by $||\delta\varphi||^2_{\varphi_0} = \int_M (\delta\varphi)^2 \omega_\varphi^m / m!$, where $\omega_\varphi = \omega_0 + i\partial\bar\partial\varphi$. Thus, a ball of radius one with respect to the usual CK metric $g_{CK,k}$ has radius approximately ϵ_k with respect to the Mabuchi distance. With the rescaling $g_k = \epsilon_k^2 g_{CK,k}$, the corresponding Laplacian scales as $\Delta_{g_k} \mapsto \epsilon_k^{-2}\Delta_{g_{CK,k}}$. It follows that the heat operator scales as

$$\exp t\Delta_{g_k} = \exp t\epsilon_k^{-2}\Delta_{g_{CK,k}}.$$

In effect, it is only the time that is rescaled, and the rescaled heat kernel is $p_k(\epsilon_k^{-2}t, \mathbf{I}_k, P)$.

We therefore study the metric scaling limit with $t \mapsto t\epsilon_k^{-2}$ and evaluate $I_2(\epsilon_k^{-2}t, \beta_k)$ asymptotically as $k \to \infty$. This scaling keeps the d_k-balls of uniform size as $k \to \infty$ with respect to the limit Mabuchi metric. Thus, as k changes, the Brownian motion with respect to g_k probes distances of size t from the initial metric ω_0 for all k. It turns out that

$$\lim_{k\to\infty} I_2\left(\epsilon_k^{-2}t, \beta_k(z, z + u/\sqrt{k})\right) = \mathrm{Li}_2(e^{-|u|^2})\,. \tag{68}$$

5.2. The Calabi model. The Calabi metric is the natural (background independent) $\mathcal{L}^2$ metric on either $\mathcal{K}_{[\omega_0]}$ or $\mathcal{B}_k$. If $\dot{\omega} = \delta\omega \in T_\omega\mathcal{K}_{[\omega_0]}$, the Calabi metric

$$||\delta\omega||_C^2 = \int_M ||\delta\omega(z)||_\omega^2 dV_\omega.$$

It is the restriction to a Kähler class of the deWitt–Ebin metric on metric tensors [**E70, DeW67**]. In terms of relative Kähler potentials, $\dot{\omega} = i\partial\bar{\partial}\dot{\varphi}$, the Calabi metric inner product is,

$$||\Delta_\omega\dot{\varphi}||_{C,\omega}^2 = \int_M |\Delta_\omega\dot{\varphi}|^2\, dV_\omega. \tag{69}$$

It is known that the sectional curvatures of $(\mathcal{K}_{[\omega_0]}, g_C)$ are all equal to 1, i.e. this Riemannian manifold is an open subset of the infinite dimensional sphere of constant curvature 1 (see [**Cal12**]). The finite dimensional approximations to the Calabi metric should approximate domains in finite dimensional spheres. Hence,

CONJECTURE 5.1. *The Calabi volume* $\mathrm{Vol_k}(\mathcal{B}_\mathrm{k})$ *with respect to* $G|_{\mathcal{B}_k}$ *is finite for each* k*.*

If the conjecture is true, one obtains a purely geometric sequence of probability measures. This could give a rigorous definition of the Polyakov path integral over metrics, which uses the Calabi metric to define its volume form, as observed in [**FKZ13**].

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