Shiing-Shen Chern: as a great geometer of 20-th century

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Contents

1. Introduction

In 1675, Isaac Newton (1643–1721) said that

> *If I have seen further, it is by standing on the shoulders of giants.*

Chern is a giant in geometry in the twentieth century whose shoulder most of later geometers stood on. On the other hand, he also stood on the shoulders of several great geometers before him. According to himself, mathematicians who are most influential on him are Blaschke, Kähler, Cartan and Weil. Blaschke, Kähler, Cartan taught him projective differential geometry, integral geometry, Kähler geometry, Cartan–Kähler system, theory of connections, and Schubert calculus while Weil was his friend who propose him to find an intrinsic proof of the Gauss–Bonnet formula and to study characteristic classes.

I believe it is instructive to find out who were the giants in the nineteenth century whose ideas inspired Chern and those great geometers in twentieth century.

The study of differential invariants can be traced back to Riemann, Christoffel, Ricci, Levi-Civita and Weyl. The theory of Cartan–Kähler has direct bearing on most works of Chern. His works on Gauss–Bonnet formula, constructions of Chern forms, Chern–Bott forms, Chern–Moser invariants and Chern–Simons invariants are good examples. Integral geometry and geometry of line complexes and Grassmannians also played very important roles for Chern's construction of Chern classes in understanding the cohomology of the classifying space of vector bundles.

I shall therefore begin my talk by explaining some of these works done by those giants before Chern.

2. Foundation of modern geometry in nineteenth century

2.1. Intrinsic geometry developed by Riemann based on equivalence principle. After Newton introduced Calculus to study mechanics, it was soon used by Leonhard Euler (1707–1783) to study geometry. Euler confined his study to surfaces in Euclidean spaces. But his view was similar to what Newton thought:

> *The universe is static and we have a global Cartesian coordinate to measure everything.*

This view was changed drastically by Riemann who was influenced by his teacher Gauss who noted that Gauss curvature is intrinsic.

I summarized here three important developments of geometry in addition to the classical geometry developed by Euler and others.

The purpose of Bernhard Riemann (1826–1866) was to explore the foundation of physics by studying geometry of space through equivalence principle via metric tensor and their curvature tensor, and to understand global meaning of space by linking metric geometry with topology.

Modern differential geometry was found by Riemann in 1854. His purpose was to understand the physical world through geometry. His grand picture was based on the equivalence principle that essential properties of law of physics or geometry should be independent of the choice of coordinate system (observer). The essence of geometric properties should be independent of whether we use Cartesian coordinates or polar coordinates to calculate them!

The principle of equivalence is also the same principle that built the foundation of general relativity discovered by Einstein 60 years later. The great work of Riemann was reported in his thesis: "*The hypothesis on which geometry is based*". The great desire to keep this equivalence principle in geometry drove Riemann to develop methods to decide under what conditions that two differential quadratic forms can be shown to be equivalent to each other by coordinate transformations.

Riemann introduced the curvature tensor, which was published in an essay, written to give an answer to the prize question on heat distribution posed by the Paris academy (Riemann did not receive the prize nor did any people). It was submitted on 1, July, 1861. The motto, written in Latin, was: *these principles pave the way to higher things*. In this essay, Riemann wrote down the curvature tensor of the differential quadratic form to be a necessary condition for two differential quadratic forms to be equivalent. Both the concepts of tensor and the intrinsic curvature tensor were brand new. Riemann had intended to develop it further, but was ill.

Heinrich Martin Weber (1842–1913) explained it in more detail based on an unpublished paper of Richard Dedekind (1831–1916) in 1867. In 1869 to 1870, Elwin Bruno Christoffel (1829–1900) and Rudolf Lipschitz (1832–1903) (Crelle journals) discussed the curvature tensor further and noted that they provided sufficient condition for the equivalence of two differential quadratic forms.

In 1901, Tullio Levi-Civita (1873–1941) and Ricci-Curbastro (1853–1925) published the theory of tensors in "*Methods de calcul differential absolute et Keyes applications*", where they wrote down the Ricci tensor and in fact what we call the Einstein tensor, and the conservation law associated to it.

This tensor appeared in the paper of Levi-Civita and Ricci was the same tensor used by Einstein, Grossman and David Hilbert (1862–1943) from 1912 to 1915 to describe gravity of spacetime. The discovery of general relativity due to Einstein–Hilbert can be considered as a major triumph for human beings understanding of spacetime. The major contributions by geometers should not have been ignored.

In the other direction, Riemann initiated the point of view using topology in complex analysis by the concept of Riemann surfaces. He realized there is a deep relationship between analysis and global topology of the Riemann surface, e.g., the Riemann–Roch formula showed how to calculate dimension of meromorphic functions with prescribed poles in terms of some topological data. Riemann started to develop the concept of "handle body decomposition" which led to the works of Poincaré on topology and global analysis on the manifold.

In his course of his investigations, Riemann mentioned:

> *In the course of our presentation, we have taken care to separate the topological relations from the metric relations. We found that different measurement systems are conceivable for one and the same topological structure and we have sought to find a simple system of measurements which allows all the metric relations in this space to be fully determined and all metric theorems applying to this space to be deduced as a necessary conditions.*

Riemann was puzzled by geometry of immeasurably small versus geometry of immeasurably large. The measurements of the former case will become less and less precise, but not in the later case.

> *When we extend constructions in space to the immeasurably large, a construction has to be made between the unlimited and the infinite; the first applies to relations of a topological nature, the second to metric nature.*

From the discussions of Riemann at the beginning of the development of modern geometry, we see the importance of the relationship of metric geometry with topology. This indeed the central theme of the development of geometry in the 20th century.

2.2. Geometry based on the study of linear subspaces. In 1865, Julius Plücker (1801–1868) studied line geometry which is the study of space of projective lines in a three dimensional projective space. He introduced the Plücker coordinates. This was soon generalized by Hermann Graßmann (1809–1877) to study the space of all linear subspaces of a fixed vector space. This space is later called Grassmannian. It is a universal space for the study of bundles over a manifold. The global topology of Grassmanians played a fundamental role in differential topology.

In 1879, Hermann Schubert (1848–1911) introduced a cell structure on the Grassmanian spaces which gave the basic homology structures of the Grassmanian manifold of space of linear spaces. The structure of their interactions give rise to the product structures of the homology.

The important concept of exterior algebra was introduced by Hermann Graßmann until 1844. But it was largely ignored until Henri Poincaré (1854–1912) and Élie Cartan (1869–1951) introduced the concept of differential form and its calculus with exterior differentiation. In 1928, Cartan suggested that the differential forms should be linked to topology of the manifold. Georges de Rham (1903–1990) was inspired and proved in 1931 in his thesis (under the guidance of Cartan) that the cohomology defined by differential forms is isomorphic to singular cohomology by integrating differential forms over singular chains.

In 1930's, W.V.D. Hodge (1903–1975) then discovered the star operator acting on forms which can be used to define concept of duality in de Rham theory. Hodge then generalized the work of Hermann Weyl (1885–1955) in 1913 on Riemann surfaces to higher dimensional manifold using the star operator.

He also found the (p, q) decomposition of the cohomology for algebraic manifolds where he made the famous conjecture that algebraic cycles represent exactly those (p, p) classes of the manifold. This is perhaps the most important conjecture in algebraic geometry that is still unsolved.

2.3. Symmetries in geometry. Motivated by the works of Niels H. Abel (1802–1829) and Evariste Galois (1811–1832) in group theory, and Sophus Lie (1842–1899) on contact transformations, Lie and others developed theory of Lie group in late 1860's.

In 1872, Felix Klein (1849–1925) initiated the Erlangen program of clarifying geometry based on the continuous group of global symmetries. Examples include **(a)** Projective Geometry, **(b)** Affine Geometry and **(c)** Möbius Geometry.

2.3.1. *Symmetries in geometry - projective geometry.* Projective geometry is one of the most classical and yet most influential subject in geometry. The group of projective collineations is the most encompassing group, which can transform "points at infinity" to finite points. The subject studies geometric properties that are invariant of such transformations. They include incidence relations between linear subspaces and the important concept of duality that come out from such considerations. Such concepts form the foundation of modern development of topology, geometry and algebraic geometry.

Major important contributors include: Papas of Alexandria (third century), Gerard Desargues (1591–1661), Blaise Pascal (1623–1662), Joseph Diez Gergonne (1771–1859), Jean Victor Poncelet (1788–1867), August Ferdinand Möbius (1790–1868), Jakob Steiner (1796–1863).

The subject of projective geometry was gradually developed into two different directions:

- One is the rich theory of algebraic curves which was developed by Abel (1802–1829), Riemann (1826–1866), Max Noether (1844–1921) and others.
- Invariant theory was used extensively. Italian algebraic geometers including Gino Fano (1871–1952), Federico Enriques (1871–1946), Benimano Segre (1903–1971), and Francesco Severi (1879–1961)

extended the subject of algebraic curves to higher dimensional algebraic varieties.

In the other direction, projective differential geometers is developed as a mixture of the approaches from Riemannian geometry to study local invariants with the Erlangen program of characterizing geometries according to their group symmetries. Contributors include Ernest Julius Wilczynski (1876–1932), Eduard Čech (1893–1960), Wilhelm Blaschke (1885–1962).

Many Japanese and Chinese geometers studied the subject of projective differential geometry. This includes Shiing-Shen Chern and Bu-chin Su (1902–2003).

2.3.2. *Symmetries in geometry - Möbius geometry.* Möbius geometry is also called conformal geometry which studies manifold properties invariant under conformal group. The subject is very powerful in two dimension and led to study discrete groups of the conformal group and conformally flat manifolds of higher dimension.

Joseph Liouville (1802–1889) and Poincaré studied the equation that transforms a metric conformally to one with constant scalar curvature.

Hermann Weyl identified the Weyl tensor that is part of the curvature tensor that is responsible for conformal change of the metric.

2.3.3. *Symmetries in geometry - affine geometry.* Affine geometry was studied by Guido Fubini (1879–1943), Wilhelm Blaschke (1885–1962), Eugenio Calabi (1923–) and is related to study differential invariants of hypersurfaces that are invariant under the affine transformations of the ambient linear space.

The invariants of the affine transformation group has given an important tool to solve the Monge–Ampère equations.

3. The birth of modern differential geometry

3.1. Levi-Civita, Weyl, Cartan and Hodge. André Weil and Hermann Weyl were two giants in twentieth century mathematics. Weil said:

> *The psychological aspects of true geometric intuition will perhaps never be cleared up* $\cdots$.

Whatever the truth of the matter, mathematics in our century would not have made such impressive progress without the geometric sense of Cartan, Hopf, Chern and a very few more. It seems safe to predict that such men will always be needed if mathematics is to go on as before.

Besides the above mentioned three great geometers, we should mention the great contributions due to Levi-Civita, Weyl, Weil, Whitney, Morse and Hodge.

- Levi-Civita was the first one (1917) who introduced the concept of parallel transport in Riemannian geometry.
- Soon afterwards, Weyl attempted to use connections to understand electro magnetism similar to Einstein theory of gravity. He succeeded to do so in 1928 where he introduced the gauge principle.

Weyl proposed that while equivalence principle dictates the law of gravity, gauge principle dictates the law of matter. The natural simplest action principle in general relativity is the Hilbert action which is the integral of scalar curvature and the one in gauge theory is the Weyl action integral which is the square integral of the curvature tensor.

- Cartan completed the foundational works since Gauss–Riemann. In the beginning of last century, he combined the Lie group theory and invariant theory of differential system, to develop the concept of generalized spaces which includes both Klein's homogeneous spaces and Riemann's local geometry.

  In modern terminology, he introduced the concept of principle bundles and a connection in a fiber bundle. This is so called non-Abelian gauge theory. It generalized the theory of parallelism due to Levi-Civita.

  In general, we have a fiber bundle $\pi\colon E \to M$, whose fibers $\pi^{-1}(x)$, $x \in M$, are homogeneous spaces acted on by a Lie group G. A connection is an infinitesimal transport of the fibers compatible with the group action by G.

  While Graßmann introduced exterior forms, Cartan and Poincaré introduced the operation of exterior differentiation. Cartan's theory of Pfaffian system and theory of prolongation created invariants for solving equivalence problem in geometry. Cartan's view of building invariants by moving frame had deep influence on Chern.

  The works of Weyl on differential forms on Riemann surfaces was generalized to high-dimensional manifolds by Hodge based on potential theory. It builds a very important relationship between linear analysis and topology.

3.2. Poincaré, Whitney and Hopf. Heinz Hopf (1894–1971) and Poincaré initiated the study of differential topology by proving that sum of indices of a vector field on a manifold can be used to calculate the Euler number of the manifold.

Hopf did the hypersurface case of Gauss–Bonnet in 1925 in his thesis. In 1932, Hopf emphasized that the integrand can be written as a polynomial of components of Riemann curvature tensor.

In 1935, Hopf's student Eduard Stiefel (1909–1978) generalized this work on vector fields to multi-vector fields of tangent bundle and defined Stiefel–Whitney classes for tangent bundles. At around the same time, Hassler Whitney (1907–1989) obtained the same characteristic class for a general sphere bundle.

3.3. Chern: father of global intrinsic geometry. These works of Hopf have deep influence on Chern's later work.

- Chern: Riemannian geometry and its generalization in differential geometry are local in character. It seems a mystery to me that we do need a whole space to piece the neighborhood together. This is achieved by topology.
- Both Cartan and Chern saw the importance of fiber bundle on problems in differential geometry.
- In 1934, Charles Ehresmann (1905–1979), a student of Cartan, wrote a thesis on the cell structure of complex Grassmanian showing that its cohomology has no torsion. This paper has deep influence of Chern's later paper on Chern class. Ehresmann went on to formulate the concept of connections in more modern terminology initiated by Cartan.

4. Shiing-Shen Chern: a great geometer

4.1. Chern's education. Shiing-Shen Chern was born on Oct. 26, 1911 in Jiaxing, and died on Dec. 3, 2004 in Tianjin, China. He studied at home for elementary education and four years in high school.

4.1.1. *Chern's education: Tsinghua University.* At age fifteen, he entered Nankai University and then spent another four years (1930–1934) in Tsinghua University. In undergraduate days, he studied:

- Coolidge's non-Euclidean geometry: "*Geometry of the circle and sphere*".
- Salmon's book: "*Conic sections and analytic geometry of three dimensions*".
- Castelnuovo's book: "*Analytic and projective geometry*".
- Otto Stande's book: "*Fadenkonstruktionen*".

His teacher Professor Dan Sun (1900–1979) studied projective differential geometry (found by E.J. Wilczynski in 1901 and followed by Fubini, Čech).

Chern's master thesis was on projective line geometry which studies hypersurface in the space of all lines in three dimensional projective space. He studied line congruences: two dimensional submanifold of lines and their oscillation by quadratic line complex. At the end of his study, he wrote four papers in projective differential geometry.

4.1.2. *Chern's education: with Blaschke.* In 1932, Wilhelm Blaschke (1885–1962) visited Peking. He lectured on "*topological questions in differential geometry*". He discussed pseudo-group of diffeomorphism and their local invariants.

Chern started to think about global differential geometry and realized the importance of algebraic topology. He read Veblen's book "*Analysis Situs*"(1922).

The chairman of math dept in Tsinghua University was Prof. Cheng (1887–1963) who later became father in law of Chern. He helped Chern to get a fellowship to follow Blaschke to study in Hamburg in 1934.

Chern wrote a doctoral thesis on web geometry under Blaschke. Emil Artin (1898–1962), Erich Hecke (1887–1947) and Erich Kähler (1906–2000) were also there.

Blaschke worked on web geometry and integral geometry at that time. Chern started to read Seifert–Threlfall (1934) and Alexandroff–Hopf (1935). He also started to learn integral geometry started by the formula of Morgan Crofton (1826–1915) on calculating length of a plane curve by counting the measure of a needle intersecting this curve. The other founder of integral geometry was Johann Radon (1887–1956) who invented Radon transform which is now used extensively in medical imaging: reconstruct a geometric figure by slicing the figure by moving planes.

Chern was very much fond of integral geometry partially because of the tradition created by Radon who was in Hamburg many years ago and had created a tradition that inspired Blaschke. Both Chern and Luis Santaló (1911–2001) were students of Blaschke around the same time. Santaló was a major leader on the subject after Blaschke. Perhaps this education influenced Chern's famous paper in 1939 on integral geometry.

4.1.3. *Chern's education: with Kähler.* In Hamburg, Erich Kähler (1906–2000) lectured on Cartan–Kähler theory:"*Einf ührung in die Theorie der systeme von Differentialeichungen*".

In 1933, Kähler published the first paper where Kähler geometry was introduced. It is a remarkable paper as some very important concepts were introduced. He has computed the Ricci tensor of a Kähler metric to be the complex Hessian of the log of the volume form. Kähler observed the condition of the metric to be Kähler–Einstein metric comes from solution of a complex Monge–Ampère equation, where he gave many examples. He also proved that the Ricci form defines a closed form which gives rise to a de Rham homology class which is independent of the choice of the Kähler metric. This is the first Chern form of the Kähler manifold.

Chern certainly was influenced by this paper as he was a student there. In the last thirty years of his life, Chern told many students that he would like to spend his time to teach them the powerful concept of moving frames invented by Cartan.

He probably learned Cartan–Kähler from Kähler in Hamburg in 1934 when he was taking a class from Kähler where he ended up to be the only student in that class.

4.1.4. *Chern's education: with Cartan.* When Chern graduated, he earned a postdoctoral fellowship in 1936 to pursue further study in Europe. Blaschke advised him either stayed in Hamburg to study with Artin or to go to Paris to study with Cartan. He took the latter choice. In 1936 to 1937, Chern went to Paris, to study with Cartan on moving frames (principle bundles, in modern terminology), the method of equivalence and more on Cartan–Kähler theory. He spent ten months in Paris and met Cartan every two weeks.

Chern went back to China in the summer of 1937. He spent a few years to study Cartan's work. He said that Cartan wrote more than six thousand pages in his whole life. Chern has read at least seventy to eighty percent of these works. Some of the works he read it over and over again. During the War, it is great to spend full time to read and think in isolation.

Chern mentioned the influence of Cartan on him in the following way:

- Undoubtedly one of the greatest mathematician of this century, his career was characterized by a rare harmony of genius and modesty.
- In 1940, I was struggling in learning Cartan. I realized the central role to be played by the notion of a connection and wrote several papers associating a connection to a given geometrical structures.

Hermann Weyl (1885–1955) was a great mathematician of all time who studied with Hilbert. His comment on Cartan was:

- Cartan is undoubtedly the greatest living master in differential geometry. Nevertheless, I must admit that I found the book, like most of Cartan's papers, hard reading.

It was Cartan, around 1901, who first formulated many local geometric problem as a generalization of the Pfaff problem (which was about describing the Lagrangian submanifolds associated to a contact 1-form).

Cartan proposed to consider, instead of a single 1-form, a collection $\mathcal{I}$ of 1-forms on a manifold M, and to find the conditions for finding the maximal submanifolds N of M to which all of the 1-forms in $\mathcal{I}$ pullback to be zero.

He found sufficient conditions for this, but had to use the Cauchy–Kovalewski theorem to solve a sequence of initial value problems to construct the maximal submanifolds, so his theory was only valid in the real-analytic category (which did not bother people that much at the time).

4.2. The equivalence problem. In modern terms, we would say that Cartan formulated his answer in terms of the algebra of the differential ideal on M generated by the collection of 1-forms $\mathcal{I}$. Cartan's version of this result sufficed for (nearly) all of Cartan's applications.

In 1933, Kähler found that Cartan's theory could be naturally generalized to the case of a differential ideal on M that was generated by forms of arbitrary degree (not just 1-forms), and he reformulated Cartan's "Test for Involutivity" to cover this more general case. That is what became known as the Cartan–Kähler Theorem.

The tools of Cartan–Kähler theory has deep influence on the works of Chern. His skill on constructing forms for the Gauss–Bonnet theorem and the characteristic forms cannot be surpassed by any geometer that I knew of.

It is also interesting to know the history of non-Abelian gauge theory, which are connections over vector bundles or principle bundles.

- In the beginning of 20-th century, Cartan recognized right away that the work of Levi-Civita and Jan Arnoldus Schouten (1883–1971) could be generalized to cover "covariant differentiation" of many different kinds of tensor fields on manifolds endowed with geometric structures.

In fact, he had already worked out, in his method of equivalence, a general method for computing curvature invariants and canonical parallelizations of what we now recognize as principal bundles by the time of his famous papers on pseudo-groups in the early 1900s.

Throughout the early 1920s, he published papers about intrinsic "connections" on manifolds endowed with (pseudo-)Riemannian, conformal, or projective structures, as well as many others (which he called "generalized spaces").

In his 1926 book on Riemannian geometry, he did talk about covariant differentiation of tensor fields.

- Of course, when Chern published the theory of Chern forms in 1946, he knew unitary connections on bundles. Both Ehresmann and Chern has written detailed survey paper for connections over general bundles in 1950. In fact, Chern gave the planetary talk on connections in 1950 in the international congress of mathematics in Harvard.

  Chern summarizes works about connections in general vector bundles in 1950 when Chern gave a planetary speech in the international congress of mathematics in 1950 in Harvard, where he explained the general theory in great detail.

  In fact, when Chern left China in late 1948 and arrived in Princeton in the new year of 1949, he gave a series of lectures in the Veblen Seminar in the Institute for Advanced Study. The lectures were written up in 1951 when Chern was in Chicago. The title was called: "*Topics in Differential Geometry*".

  He explained clearly the works of Cartan and himself on connections and characteristic classes for general vector bundles (The subject is called non abelian gauge theory by physicists and was founded and pioneered by Weyl in 1928. Weyl coined the term of gauge principle to explain the basic law behind matter).
- In 1954, C.-N. Yang (1922–) and Robert Mills (1927–1999) applied this theory to explain isospin in particle physics. But since they did not know how to quantize the theory, they did not know how to compute the mass as was pointed out by Wolfgang Pauli (1900–1958) who had also developed the non Abelian version of Weyl's gauge theory.

  Apparently, all of Pauli, Yang and Mills did not know the works of Cartan, Ehresmann and Chern had finished the theory of non-Abelian gauge theory. It may be interested to know that Yang took a course of geometry from Chern in China. He was also a student in Chicago and postdoctoral fellow in Princeton at the time when Chern was there. Yang's father was also a teacher of Chern.

> In a note written by Yang, he said that in 1948, he was in a meeting with Weil and Fermi. Weil told Fermi that the fact that Chern classes are integral would be used to quantize physical theories.

4.2.1. *Equivalence problem.* Let us now explain in more details of Chern's works in geometry. Most of the works of Chern are related to problem of equivalence, which dated back to Riemann.

In 1869, Christoffel and Lipschitz solved a special form of equivalence problem in Riemannian geometry. It was also called the form problem:

> *To decided when two* $\mathrm{d}s^2$ *differ by a change of coordinate, Christoffel introduced the covariant differentiation now called Levi-Civita connection.*

It was Cartan who formulated a more general form of equivalence problem which can be stated in the following:

> *Given two sets of linear differential forms* θ^i, θ^{*j} *in the coordinates* x^k, x^{*l} *respectively, where* $1 \le i, j, k, l \le n$ *both linearly independent. Given a Lie group* $G \subset \mathrm{GL}(n, \mathbb{R})$, *find the conditions that there are functions*
>
> $$x^{*l} = x^{*l}(x^1, \dots, x^n)$$
>
> *such that* θ^{*j}, *after the substitution of these functions, differ from* θ^i *by a transformation of* G.

The problem generally involves local invariants, and Cartan gave a procedure to generate such invariants.

4.2.2. *Chern (1932–1943).* Chern continued the tradition of Cartan and applied the Cartan–Kähler theory to solve various geometric questions related to equivalence problem.

For example, in projective differential geometry, he is interested in the following question:

> *Find a complete system of local invariants of a submanifold under the projective group and interpret them geometrically through osculation by simple geometrical figures.*

Chern studied web geometry, projective line geometry, invariants of contact pairs of submanifolds in projective space, transformations of surfaces (related to Bäcklund transform in soliton theory).

Another typical problem in projective differential geometry is to study the geometry of path structure by normal projective connections. Tresse (a student of Sophis Lie) studied paths defined by integral curves of

$$y'' = F(x, y, y')$$

by normal projective connections in space (x, y, y'). Chern generalized this to n-dimension: *Given* $2(n-1)$*-dimensional family of curves satisfying a differential system such that through any point and tangent to any direction at* *the point, there is exactly one such curve.* Chern defines a normal projective connection. He then extended it to families of submanifolds.

4.2.3. *Chern (1940–1942).* The first major work that Chern did was in 1939 where he studied integral geometry which was developed by Crofton, Blaschk.

Chern observed that such theory can be best understood in terms of two homogeneous spaces with the same Lie group G. Hence there are two subgroups H and K:

$$\begin{array}{ccc} & G & \\ \swarrow & & \searrow \\ G/H & & G/K. \end{array}$$

Two cosets aH and bK are incident to each other if they intersect in G. Important geometric quantities in G/H can be pulled back to G and then pushes forward to be interesting geometric quantities in G/K.

This work proceeded the important works of the Russian school led by Israel Gelfand (1913–2009) and the works of Shigeru Mukai (1953–). The transformations defined this way is sometimes called the Fourier–Mukai transformation.

In his work in integral geometry, Chern generalized several important formula of Crofton and much later, he uses this setting to generalize the kinematic formula of Poincaré, Santaló and Blaschke.

Weil commented on this work of Chern that it lifted the whole subject at one stroke to a higher plane than where Blaschke's school had lift it. I was impressed by the unusual talent and depth of understanding that shone through it.

4.2.4. *Chern's visit of Princeton (1943).* In 1943, Chern went from Kunming to Princeton, invited by Oswald Veblen (1880–1960) and Weyl. This was during the war time. It took him seven days to fly by military aircraft from Kumming to Miami via India, Africa and South America. He arrived at Princeton in August by train (It took him five years before he meet his new born son again).

Weyl was his hero. But it was Weil who suggested him to look into Fiber bundle theory of Cartan and Whitney. Weil pointed out that Stiefel–Whitney classes were only defined mod two. But there were works of Todd and Eger that constructed certain classes that are well-defined without mod two.

- John Arthur Todd (1908–1994) publishes his work: "*the geometric invariants of algebraic loci*" in 1937 on Todd class in Proceedings of the London Mathematical Society.
- Max Eger published his work "*Sur les systems canoniques d'une varietie algebrique a plusieurs dimensions*" in 1943 in Annales Scientifique de l'Ecole Normale Supérieure.

Weil just published his work on Gauss–Bonnet formula and told Chern the works of Todd and Eger on "canonical classes" in algebraic geometry (These works were done in the spirit of Italian geometers and rested on some unproved assumptions).

4.3. Chern–Gauss–Bonnet formula. Chern told everybody his best work was his intrinsic proof of Gauss–Bonnet formula. Here is a brief history of the formula:

- Carl Friedrich Gauss (1777–1855) did it for geodesic triangle (1827): "*Disquistiones Circa superficies Curvas*". He considered surface in $\mathbb{R}^3$ and used Gauss map.
- Pierre Ossian Bonnet (1819–1892) in 1948 generalized to any simply connected domain bounded by an arbitrary curve: "*Mémoire sur la théorie générale des surfaces*".
- Walther von Dyck (1856–1943) in 1888 generalized it to arbitrary genus: "*Beiträge zur analysis situs*".
- Hopf generalized the formula to codimension one hypersurfaces in $\mathbb{R}^n$.
- Carl B. Allendoerfer (1911–1974) in 1940 and Werner Fenchel (1905–1988) studied closed orientable Riemannian manifold which can be embedded in Euclidean space.
- C.B. Allendoerfer and Weil in 1943 extended the formula to closed Riemannian polyhedron and hence to general closed Riemannian manifold in"*The Gauss-Bonnet theorem for Riemannian polyhedra, Amer. Math. Soc.,* **53***(1943), 101–129.*"

But the proof of Allendoerfer–Weil depends on the possibility of isometric embedding of the manifold into Euclidean spaces. This was only established about 15 years by John Nash (1928–2015).

4.3.1. *The Chern–Gauss–Bonnet formula.* Weil in his comment in the introduction of Chern's selected works made the following comments:

- Weil: Following the footsteps of Weyl and other writers, the latter proof, resting on the consideration of "tubes", did depend (although this was not apparent at that time) on the construction of an sphere-bundle, but of a non-intrinsic one, viz. the transversal bundle for a given immersion.
- Weil: Chern's proof operated explicitly for the first time with an intrinsic bundle, the bundle of tangent vectors of length one, thus clarifying the whole subject once and for all.

4.3.2. *The proof of Chern–Gauss–Bonnet formula.* Let us explain what Chern did: in the simplest two dimensional case, he wrote, in terms of moving frame, the structure equation for a surface is

$$\begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_{12} \\ d\omega_{12} &= -K\, \omega_1 \wedge \omega_2 \end{aligned}$$

where ω_{12} is the connection form and K is the Gauss curvature.

If the unit vector e_1 is given by a globally defined vector field V by

$$e_1 = \frac{V}{\|V\|}$$

at points where $V \neq 0$, then we can apply Stokes' formula to obtain

$$- \int_M K\, \omega_1 \wedge \omega_2 = \sum_i \int_{\partial B(x_i)} \omega_{12} \tag{1}$$

where $B(x_i)$ is a small disk around x_i with $V(x_i) = 0$. Each term in the right hand side of (1) can be computed via the index of the vector field of V at x_i. According to the theorem of Hopf and Poincaré, summation of indices of a vector field is the Euler number. This proof of Chern is new even in two dimension. In higher-dimensional proof, the bundle is the unit tangent sphere bundle.

The curvature form Ω_{ij} is skew-symmetric. The Pfaffian is

$$\mathrm{Pf} = \sum \varepsilon_{i_1,\dots,i_{2n}} \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{2n-1} i_{2n}}.$$

The Gauss–Bonnet formula is

$$(-1)^n \frac{1}{2^{2n} \pi^n n!} \int_M \mathrm{Pf} = \chi_{\mathrm{top}}(M).$$

Chern managed to find, by tour de force, a canonical form Π on the unit sphere bundle so that $\mathrm{d}\Pi$ is the lift of Pf. This beautiful construction is called transgression and played an important role in topology theory of fiber bundle. This construction is very important. When it applies to the Pontryagin forms, it gives rises to the Chern–Simons forms, a joint work with Jim Simons (1938–) twenty some years later.

4.4. Invention on Chern classes. In the preface to Chern's selected works, Weil said that when Chern arrived in Princeton in 1943, both of them were deeply impressed by the works of Cartan and the masterly presentation by Kähler in the following paper:

> *Einfuhrung in die Theorie der Systeme von Differentialgleichungen.*

Both of them realized the importance of fiber bundles in geometry. Then Weil told Chern to look into the "canonical classes" in algebraic geometry introduced by Todd and Eger. Their work resembled the Stiefel–Whitney classes, but do not need to define mod 2. On the other hand, the works of these two authors were done in the spirit of Italian geometers and rested on some unproved assumptions.

Weil did not seem to realize that Chern was also influenced by two works of Lev Pontryagin (1908–1988)

1. *Characteristic cycles on manifolds, C.R.(Doklady) Acad. Sci. URSS, Vol* **35** *(1942), 34–37.*

2. *On some topological invariants of Riemannian manifolds, C.R.(Doklady) Acad. Sci. URSS(N.S.) Vol* **43***(1944), 91–94.*

These two papers were mentioned by Chern in the preface of his paper on Chern classes. In the second paper, Pontryagin has introduced closed forms defined by curvature form. He proved that the de Rham cohomology defined by the closed form is independent of the metric that defines the curvature forms. Apparently Pontryagin did not know how to integrate his curvature forms in Schubert cells to identify the cohomology classes they represented was the same classes that he defined topologically.

Chern attempted to solve this problem left in the work of Pontryagin, after he succeeded in giving the intrinsic proof of the Gauss–Bonnet formula. He did not succeed to carry out the calculation for the real Grassmannians, whose cell structure is more complicated. He did it for the complex Grassmanians. Chern said:

> *My introduction to characteristic class was through the Gauss–Bonnet formula, known to every student of surfaces theory. Long before 1943, when I gave an intrinsic proof of the n-dimensional Gauss–Bonnet formula, I know, by using orthonormal frames in surface theory, that the classical Gauss–Bonnet is but a global consequence of the Gauss formula which expresses the "Theorema Egregium".*
>
> *The algebraic aspect of this proof is the first instance of a construction later known as transgression, which is destined to play a fundamental role in the homology theory of fiber bundle, and in other problems.*

Cartan's work on frame bundles and de Rham's theorem have been always behind Chern's thinking. The history of fiber bundle can be briefed as follows:

- Stiefel in 1936 and Whitney in 1937 introduced Stiefel–Whitney classes. It is only defined mod two.
- acques Feldbau (1914–1945) in 1939, Ehresmann in 1941, 1942, 1943, Chern in 1944, 1945, and Norman Steenrod (1910–1971) in 1944 studied topology of fiber bundles.
- Pontrjagin in 1942 introduced Pontrjagin classes. He also associated topological invariants to curvature of Riemannian manifolds in 1944 (Doklady).

In the proof of Gauss–Bonnet formula, Chern uses one vector field and look at its set of zero to find the Euler characteristic of the manifold. If we replace a single vector field by k vector fields $s_1, \ldots, s_k$ in general position, they are linearly independent form a $(k-1)$-dimensional cycle whose homology class is independent of the choice of s_i. This was done by Stiefel in his thesis (1936).

Chern considered similar procedure for complex vector bundles. In the proof of Gauss–Bonnet formula, he used curvature forms to represent the

Euler class by zero set of vector field. It is therefore natural to do the same for the other Chern classes using set of degeneracy for k vector fields.

Whitney in 1937 considered sections for more general sphere bundles, beyond tangent bundles, and looked at it from the point of view of obstruction theory. He noticed the importance of the universal bundle over the Grassmannian $\mathrm{Gr}(q, N)$ of q planes in $\mathbb{R}^N$. He in 1937 showed that any rank q bundle over the manifold M can be induced by a map $f\colon M \to \mathrm{Gr}(q, N)$ from this bundle.

When N is large, Pontrjagin in 1942 and Steenrod in 1944 observed that the map f is defined up to homotopy. The characteristic classes of the bundle is given by

$$f^*\mathrm{H}^\bullet(\mathrm{Gr}(q, N)) \subset \mathrm{H}^\bullet(M).$$

The cohomology $\mathrm{H}^\bullet(\mathrm{Gr}(q, N))$ was studied by Ehresmann in 1936 and they are generated by Schubert cells.

In a recollection of his own works in the nineties, Chern said that it was a trivial observation and a stroke of luck, when I saw in 1944 that the situation for complex vector bundles is far simpler, because most of the classical complex spaces, such as the classical complex Grassmann manifolds, the complex Stiefel manifolds, etc. have no torsion.

For a complex vector bundle E, the Chern classes Chern defined are in three different ways: by obstruction theory, by Schubert cells and by curvature forms of a connection on the bundle. He proved their equivalences. Although the theory of Chern classes have a much bigger impact than his proof of Gauss–Bonnet theorem. Chern considered his proof of Gauss–Bonnet formula to be his best work. The formula was in fact carved in his Tomb stone in Nankai University.

I believe the reason is that he got some of his ideas on Chern classes from the Gauss–Bonnet theorem. Also in his proof of Gauss–Bonnet formula, he started to appreciate the power on study the geometry of forms on the intrinsic sphere bundle of tangent vectors with length one.

In the approach based on obstruction theory, it is parallel to the way that Stiefel generalized Hopf's vector field theory to Stiefel–Whitney classes by looking at them as an obstruction to multi vector fields that are linearly independent.

As for the curvature forms, the representations of Chern classes by curvature forms are clear analogue with Gauss–Bonnet formula. Therefore Chern did the Chern form for unitary connections. When Weil reported his work in Bourbaki Seminar, Weil formulated it so that it applies to connections based on connections with any compact Lie group.

According to Chern himself, he knew the formula for general G-connections. But he did not know the proof that the cohomology classes are independent of the choice of connections. In a way, this is surprising, because Weil simply forms a family of connections joining linearly two connections together, and then differentiates the characteristics forms defined by the connections in this family and obtains its transgression form.

This kind of idea was used by Kähler in 1933 to prove that the first Chern class as represented by the Ricci curvature form is independent of the Kähler metric. The same idea was also used by Pontryagin to prove similar statement for Pontryagin classes.

In 1945, Chern was invited to give a plenary address in the summer meeting of American mathematical society. His report appeared in 1946 in Bulletin of American Mathematical Society 52. It is titled: "*Some new view points in the differential geometry in the large*". In the mathematical review of this paper, Hopf wrote that Chern's work had ushered in a new era in global differential geometry.

Chern returned to China in April of 1946 where he became the deputy director of mathematics Institute for Academia Sinica in Nanking.

In this period and also in the period when he was teaching in Tsinghua University as part of Southwestern Associated University in Kuming, he trained a few young Chinese mathematicians that were influential in China. The most notable mathematicians were Hsien-Chung Wang (1918–1978), Kuo-Tsai Chen (1923–1987) and Wen-Tsun Wu (1919–2017). They made contributions to topology.

Chern also proved that the Chern classes of algebraic bundles are represented by algebraic cycles. This statement was known to Hodge for algebraic hypersurfaces.

When Hirzebruch was writing his paper "*Transferring some theorems of algebraic surfaces to complex manifolds of two complex dimension, J. Reine Angew. Math.* **191***(1953), 110–124.*", he noticed that some of the results of that paper could have been generalized to higher dimensions. But the so-called duality formula was not yet proved. This formula says that the total Chern class of the direct sum of two complex vector bundles equals the product of the total Chern classes of the summands.

Hirzebruch's paper has a remark written during proofreading that Chern and Kunihiko Kodaira (1915–1997) told Friedrich Hirzebruch (1927–2012) that the duality formula is proved in a forthcoming paper of Chern "*On the characteristic classes of complex spherebundles and algebraic varieties, Amer. J. Math.* **75***(1953), 565–597.*" Hirzebruch said:

> *My two years (1952–1954) at the Institute for Advanced Study were formative for my mathematical career. I had to study and develop fundamental properties of Chern classes, introduced the Chern character, which later (joint work with Michael Atiyah (1929–2019)) became a functor from K-theory to rational cohomology.*

4.4.1. *The fundamental paper of Chern (1946).* In the paper, "*Characteristic classes of Hermitian manifolds*", Chern also laid the foundation of

Hermitian geometry on complex manifolds. The concept of Hermitian connections was introduced.

If Ω is the curvature form of the vector bundle, one defines

$$\det\left(\mathrm{I}+\frac{\sqrt{-1}}{2\pi}\Omega\right)=\mathbf{1}+c_1(\Omega)+\cdots+c_q(\Omega).$$

The advantage of defining Chern classes by differential forms have tremendous importance in geometry and in modern physics.

An example is the concept of transgression created by Chern. Let ω be the connection form defined on the frame bundle associated to the vector bundle. Then the curvature form is computed via $\Omega = \mathrm{d}\omega - \omega\wedge\omega$ and hence

$$c_1(\Omega)=\frac{\sqrt{-1}}{2\pi}\mathrm{Tr}(\Omega)=\frac{\sqrt{-1}}{2\pi}\mathrm{d}(\mathrm{Tr}(\omega)).$$

Similarly, we have

$$\begin{aligned}\mathrm{Tr}(\Omega\wedge\Omega) &= \mathrm{d}\left(\mathrm{Tr}(\omega\wedge\omega)+\frac{1}{3}\mathrm{Tr}(\omega\wedge\omega\wedge\omega)\right)\\ &= \mathrm{d}(CS(\omega)).\end{aligned}$$

This term $CS(\omega)$ is called Chern–Simons form and has played fundamental role in three dimensional manifolds, in anomaly cancellation, in string theory and in solid state physics.

The idea of doing transgression on form level also gives rise to a secondary operation on homology, e.g. Massey product. It appeared in K.T. Chen's work on iterated integral.

When the manifold is a complex manifold, we can write $\mathrm{d} = \partial + \bar{\partial}$. In a fundamental paper, Raoul Bott (1923–2005) and Chern (1965) found: for each i there is a canonically constructed $(i-1, i-1)$-form $\tilde{T}c_i(\Omega)$ so that $c_i(\Omega) = \bar{\partial}\partial(\tilde{T}c_i(\Omega))$.

Chern made use of this theorem to generalize Nevanlinna theory of value distribution to holomorphic maps between higher dimensional complex manifolds. The form $\tilde{T}c_i(\Omega)$ plays a fundamental role in Arekelov theory.

Simon Donaldson (1957–) used the case $i = 2$ to prove the Donaldson–Uhlenbeck–Yau theorem for the existence of Hermitian Yang–Mills equations on algebraic surfaces. For $i = 1$,

$$c_1=\frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial\log\det(h_{i\bar{j}})$$

where $h_{i\bar{j}}$ is the hermitian metric. The right hand side is the Ricci tensor of the metric.

The simplicity of the first Chern form motivates the Calabi conjecture. The simplicity and beauty of geometry over complex number can not be exaggerated.

4.4.2. *Chern (IAS in 1949 and later in Chicago).* After the fundamental paper on Chern classes in 1946, he explored more details on the multiplicative structure of the characteristic classes.

In 1951, he had a paper with Edwin Spanier (1921–1996) on the Gysin sequence on fiber bundle. They proved the Thom isomorphism independently of René Thom (1923–2002).

4.4.3. *Splitting principle.* In the paper (1953), "*On the characteristic classes of complex sphere bundle and algebraic varieties*", Chern showed that by considering an associated bundle with the flag manifold as fibers the characteristic classes can be defined in terms of line bundles. As a consequence the dual homology class of a characteristic class of an algebraic manifold contains a representative of algebraic cycle.

This paper provides the splitting principle in K-theory and coupled with Thom isomorphism allows one to give the definition of Chern classes on the associated bundle as was done by Alexander Grothendieck (1928–2014) later.

Hodge has considered the problem of representing homology classes by algebraic cycles. He considered the above theorem of Chern and was only able to prove it when the manifold is complete intersection of nonsingular hypersurfaces in a projective space.

Chern's theorem is the first and the only general statement for the "Hodge conjecture". It also gives the first direct link between holomorphic K-theory and algebraic cycles. Chern's ability to create invariants for important geometric structure is unsurpassed by any mathematicians that I have ever known. His works on projective differential geometry, on affine geometry, on Chern–Moser invariants for pseudo-convex domains demonstrate his strength.

The intrinsic norm on cohomology of complex manifolds that he defined with Harold Levine (1922–2017) and Louis Nirenberg (1925–2020) has not been fully exploited yet. Before he died, a major program for him was to carry out Cartan–Kähler system for more general geometric situation.

In 1957, Chern wrote a paper called "*On a generalization of Kähler geometry.*" In effect, he was looking for geometry with special holonomic groups. But at that time, he could not find interesting examples, beyond Kähler Geometry.

On the other hand, in his review article on a book by André Lichnerowicz (1915–1998) in 1955 called "*Theorie globale des connexions et dea groupers d'holonomie.*" Chern pointed out that the classical works of Cartan pointed to the fact that the group concept is the basic underlying idea behind the work of Levi-Civita and Schouten on the theory of connections. He also wrote that people had confusion of Cartan's terminology. Cartan's "tangent space" is a fiber in the modern terminology and his space of moving frame is what is now called a principal fiber bundle.

In this review, he made a comment which did not come out as he thought. He said: *the holonomic group is a very natural notion in the theory of connections.* However, recent investigations by Marcel Berger (1927–2016) and

Isadore Singer (1924–) have shown that its possibilities are rather limited. Except for homogeneous spaces, It is perhaps not a strong invariant. Many years ago, Singer told me that both Warren Ambrose (1914–1995) and him attended the class taught by Chern in geometry in Chicago when they were graduate students together in Chicago. Afterwards, they managed to prove what we call Ambrose–Singer theorem which identifies the Lie algebra of the holonomic group by relating it with curvature tensors.

Berger in France developed this idea further and classified all possible Lie group that may appear as holonomic group in Riemannian geometry (a more conceptual proof was given by Simons later). Holonomic groups were introduced by Cartan in 1926. It gives rise to the concept of internal symmetry of the manifold and it gives geometric meaning of what modern physicists called supersymmetry.

Kähler manifolds are those whose holonomic group is a unitary group. Calabi–Yau manifolds are those with manifolds whose holonomic group is a special unitary group. To the contrary of what Chern expected, manifolds with special holonomy has been one of the most fascinating manifolds in modern geometry. The construction of such manifolds depend on nonlinear analysis which Chern was not very familiar with.

It may be interested to note that Chern gave a course on Hodge theory for Kähler manifolds in Chicago using potential theory after the works of Kodaria. But in late sixties, Chern wrote a booklet called "*complex manifold without potential theory*". For some reason, Chern gave up his interest in the direction in Kähler geometry pioneered by Kodaria starting on the proof of vanishing theorems.

In the late fifties, Chern studied to show interest in the old classical subject of minimal surfaces. His works largely followed the works of Jean Gaston Darboux (1842–1917), Cartan, and others, which was more local in nature. However he was immediately attracted by the works of Calabi in the global theory of minimal two spheres in higher-dimensional spheres. He observed that the Gauss map mapping minimal surfaces in higher-dimensional Euclidean spaces into the Grassmanian of two planes in higher-dimensional Euclidean space, is anti-holomorphic. Hence one can apply the theory of holomorphic curves to minimal surfaces theory to reprove the work of Bernstein–Osserman on minimal surfaces (Note that the Grassmanian of two planes has a natural complex structure).

His lectures on minimal surfaces in Berkeley influenced the important works of Simons on higher-dimensional minimal subvarities by making important contribution towards the stability questions on minimal cones which in turns solves some part of the Bernstein problem which gave better understanding of singularity of minimal subvarities. In particular, Simons made an important contribution towards the Bernstein problem in this theory which gave better understanding of singularity of minimal subvarities.

The last most important works that Chern did in the seventies were his work with Simons, now called Chern–Simons invariants, and his work with Jürgen Moser (1928–1999), now called Chern–Moser invariants for strongly pseudoconvex manifolds.

The first work was motivated by the idea of transgression started in his proof of Gauss–Bonnet formula. It has become a corner stone for works in theoretical physics and condensed matter. The last work continued the unfinished works of Cartan on construction of local invariants of domains invariant under biholomorphic transformations.

During the past forty years, the Chern-Simons form has grown in importance in theoretical physics. The developments can be briefly summarized as follows:

- In 1978, Albert Schwarz (1934–) introduced a topological quantum field theory including the Chern-Simons theory. His paper is titled "*The partition function of degenerate functional and Ray-Singer invariants*" (1978).
- In 1981, Roman Jackiw (1939–) and his student Stephen Templeton studied the three-dimensional QED of the Chern-Simons term; in 1982, he studied the non-commutative gauge field theory and three-dimensional Einstein gravity.
- In 1981, Laughlin (1950–) published a paper on the two-dimensional quantized Hall conduction; in 1983 he published a paper on the fractional quantum Hall effect, where the low energies can be described by the Chern-Simons term. Subsequent workers were Frank Wilczek (1951–), Anthony Zee (1945–), Alexander Markovich Polyakov (1945–), etc.
- Witten (1951–) developed the three-dimensional Chern-Simons theory into a quantum theory related to Jones polynomials. Witten's article set off explored the theory of knots, including the so-called volume conjecture for three-dimensional hyperbolic manifolds. The Chern-Simons theory and its extensions to condensed matter physics are too vast to be reviewed here.

The Chern-Simons theory is getting more and more powerful in fundamental physics. It may go beyond what Chern or Simons can imagine themselves.

4.5. Conclusion. When I was a student, Chern told me that he is interested in mathematics because it is fun and is the only thing he knew how to do. He feels that he can master very complicated calculation as was shown in his proof of Gauss–Bonnet theorem.

Despite of his tremendous influence in modern geometry, he said that he did not have a global vision as people would think that he is guided by it. He just followed his intuition to have fun. And he emphasized how important it was to him to have friends with brilliant minds.

Chern said

> *The importance of complex numbers in geometry is a mystery to me. It is well-organized and complete.*

Chern always regret that ancient Chinese mathematicians never discovered complex number. Chern's everlasting works in complex geometry make up the loss of Chinese mathematics for the last two thousand years.

At the last part of his life, Chern tried to promote Finsler Geometry. He wrote a book with David Bao in the subject. Since there is no concrete example of Finsler geometry to model, they had difficulty to develop their theory with great depth. In particular, they were not able to apply their theory to the concrete example of Finsler geometry appeared in Teichmüller space or in Kobayashi hyperbolic manifolds.

In Riemann's thesis, he thought about the possibility of replacing Riemannian metrics defined by quadratic differentials by quartic differentials, presumably to handle geometry of space which is far apart. It will be interested to know whether rich geometry can be developed based on quartic differentials. One has to solve the equivalence problem, i.e., to find complete invariants to determine whether two quartic differential are equal up to change of variables. Ironically, while Chern was a great admirer of Riemann, Cartan, Weyl and Weil, he did not think highly of Einstein and was slow in reaction to the ideas coming from theoretical physics.

He showed no interest in the part of geometry related to quantum field theory. The dream of Riemann to understand space of extremely small needs full understanding of quantum field theory and perhaps a new form of quantum geometry. But he is flexible in general. When I mentioned to him that I was working on Calabi conjecture, he did not think much of it until he realized that it could be used to solved problems that he wanted to solve in algebraic geometry. Since then, he realized the power of nonlinear analysis in geometry. This was reflected by the series of international conference called "conference on differential geometry and differential equations", organized by him after he returned to China.

There is no question that Chern is a great mathematician and will always be remembered in the history of mathematics, especially on his contributions to the theory of fiber bundle and its characteristic classes.

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