Some recent results on multiplier ideal sheaves

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Dedicated to 110th anniversary of S.S. Chern

Abstract. In this note, we give a survey on multiplier ideal sheaves, including the basic properties and some recent results.

This paper is grew out of my talk given at the conference in memory of Prof S.S. Chern on the occasion of his 110th anniversary held at Tsinghua, in which I gave a survey about some recent results on multiplier ideal sheaves.

1. Plurisubharmonic functions and multiplier ideal sheaves

Multiplier ideal sheaf deals with singularities of plurisubharmonic functions. Prof. S.T. Yau ([**Yau15**]) once said that the notion of multiplier ideal sheaf plays a central role in modern higher-dimensional algebraic geometry.

1.1. Singularity of plurisubharmonic function. A plurisubharmonic (psh) function is an upper semi-continuous function and subharmonic when the function restricts to any complex line. The singularity of a plurisubharmonic function φ is the set $\{\varphi = -\infty\}$ which is called a complete pluripolar set.

A typical example of a plurisubharmonic function is $\varphi = c\log(|f_1|^2 + \cdots + |f_k|^2)$, where $f_1, \cdots, f_k$ are holomorphic functions and $c > 0$. The complex analytic subset $f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) = \varphi^{-1}(-\infty)$ is an important example of the complete pluripolar set.

A psh function is called to be a plurisubharmonic function with (neat) analytic singularities if locally $\varphi = c\log(|f_1|^2 + \cdots + |f_k|^2)$ up to a bounded function (smooth function).

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One may define an equivalence among the singularities for plurisubharmonic functions. One says that φ is more singular than ψ, denoted by $\varphi \preccurlyeq \psi$ if $\varphi \leq \psi + O(1)$ and that φ and ψ have equivalent singularities if $\varphi \preccurlyeq \psi$ and $\psi \preccurlyeq \varphi$.

1.2. Holomorphic line bundle with singular metric. One can define a singular hermitian metric h on a holomorphic line bundle L over a complex manifold (X, ω) locally given by $e^{-\varphi}, \varphi \in L^1_{loc}$. Thus the Chern curvature current is well-defined as $\Theta = i\partial\bar{\partial}\varphi$ locally. A holomorphic line bundle is called pseudo-effective (*resp.* positive, or big) if φ is plurisubharmonic (*resp.* φ is smooth strictly plurisubharmonic, or $\Theta \geq \epsilon\omega$ for some $\epsilon > 0$).

1.3. Concept of multiplier ideal sheaf. Let us recall that the definition of multiplier ideal sheaf (m.i.s.) associated to a plurisubharmonic function φ, is an ideal subsheaf $\mathcal{I}(\varphi) \subset \mathcal{O}$ whose germs consist of holomorphic functions $f \in \mathcal{O}_x$ such that $|f|^2 e^{-\varphi}$ is locally integrable at x.

Similarly we can define L^p multiplier ideal sheaf if we replace 2 by p, i.e. $|f|^p e^{-\varphi}$ is locally integrable. Actually the origin of this notion goes back to Hörmander ([**H90**]), Bombieri ([**Bom70**]), Skoda ([**Sko72**], [**Sko77**]).

1.4. Invariants of singularity of plurisubharmonic function. It is well-known that there are several classical invariants for the singularities of plurisubharmonic functions and multiplier ideal sheaves:

(1) the Lelong number $v(\varphi, x) := \liminf\limits_{z \to x} \frac{\varphi(z)}{\log|z-x|}$;
(2) the complex singularity exponent (log canonical threshold)
$$c_x(\varphi) := \sup\{c \geq 0 : e^{-2c\varphi} \text{ is locally integrable at } x\};$$
(3) multiplier ideal sheaf, the zero variety $V(\mathcal{I}(\varphi)) := \mathrm{Supp}\ \mathcal{O}/\mathcal{I}(\varphi) = \{x | e^{-\varphi} \text{ not locally integrable at } x\}$ is analytic.

It is easy to see that if $\varphi(x) > -\infty$, then $\mathcal{I}(\varphi)_x = \mathcal{O}_x$, $v(\varphi, x) = 0$ and $c_x(\varphi) = +\infty$.

1.5. Basic properties of multiplier ideal sheaf. It's well-known that multiplier ideal sheaves satisfy some basic properties: coherence, integral closedness, the Nadel vanishing theorem and so on.

THEOREM 1 (Nadel vanishing theorem, [**Nad90**], [**Dem93**]). *Let $(L, e^{-\varphi})$ be a big line bundle on a weakly pseudoconvex Kähler manifold X. Then*
$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$$
for any $q \geq 1$.

The Nadel vanishing theorem could be used to give a unified treatment of Grauert's solution to Levi problem in several complex variables and Kodaira embedding theorem in complex algebraic geometry.

In addition, there are many well-known applications of multiplier ideal sheaves. You're referred to [**Siu05**], [**Siu02**], [**Dem07**], [**Ohs15**].

2. The strong openness conjecture and its applications

2.1. Solution of the strong openness conjecture. There are some conjectures related to the further properties of the multiplier ideal sheaves, e.g., the strong openness conjecture was posed in [**Dem01**].

CONJECTURE 2. *Given a psh function φ on a complex manifold, then*

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi),$$

where $\mathcal{I}_+(\varphi) := \bigcup_{\varepsilon>0} \mathcal{I}((1+\varepsilon)\varphi)$.

It follows from the Noetherian property of coherent sheaves that for given x there exists $\varepsilon_0 > 0$ such that $\bigcup_{\varepsilon>0} \mathcal{I}((1+\varepsilon)\varphi)_x = \mathcal{I}((1+\varepsilon_0)\varphi)_x$.

Actually the meaning of the conjecture is as follows if $|f|^2 e^{-\varphi}$ is locally integrable at x, then there exists an $\varepsilon_0 > 0$ such that $|f|^2 e^{-(1+\varepsilon_0)\varphi}$ is still locally integrable at x. In other words, $\{p \in \mathbb{R} : |f|^2 e^{-p\varphi}$ is locally integrable$\}$ is open.

Its origin comes from very fundamental property in calculus: $\{p \in \mathbb{R} : 1/|x|^p$ is locally integrable at the origin$\}$ is an open interval, which the name "openness" comes from. This conjecture was also posed by Y.T. Siu ([**Siu04**]), Demailly-Kollár ([**DK01**]) and many others.

There are other forms of the strong openness conjecture:

CONJECTURE 3.

$$\cup_j \mathcal{I}(\varphi_j) = \mathcal{I}(\varphi), \varphi_j \nearrow \varphi;$$
$$\cup_{\varepsilon>0} \mathcal{I}((\varphi + \varepsilon\psi)) = \mathcal{I}(\varphi),$$

where φ_j, φ, ψ are psh functions.

When the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is trivial, the conjecture was proved by Berndtsson ([**Ber13**]). As for the strong openness conjecture, when $\dim X = 2$, this was proved by Favre, Jonsson and Mustată ([**FJ05, JM12**]). Their methods are powerful. So probably because of this, strong openness conjecture "was thought to be rather inaccessible for $n > 2$" (Math. Reviews, MR3418526).

Finally Guan-Zhou ([**GZ13**]) proved the strong openness conjecture including the other forms by some unexpected observations together with untraditional use of the Ohsawa-Takegoshi L^2 extension theorem and the curve selection lemma. The proof in [**GZ13**] is quite different from the previous methods.

THEOREM 4 (Guan-Zhou, [**GZ13**]). *The strong openness conjecture and the other forms hold.*

2.2. Corollaries of the strong openness. Under the assumption that the strong openness conjecture holds, many mathematicians obtained important results. Let us recall several immediate consequences of the strong openness conjecture.

The first one is the general Kawamata-Viehweg-Nadel type vanishing theorem:

Theorem 5. *Let (L,φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n, and $nd(L,\varphi)$ be the numerical dimension of (L,φ), then*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

for any $q \geq n - nd(L,\varphi) + 1$.

- This was conjectured and proved for $\mathcal{I}_+(\varphi)$ by Junyan Cao in [**C14**].

When (L,φ) be a big line bundle, then the numerical dimension $nd(L,\varphi) = n$ and the theorem reduces to Nadel's vanishing theorem.

Another consequence is about the relation between multiplier ideal sheaves and the Lelong numbers. The first statement for multiplier ideal sheaf is that the following are equivalent

- $\mathcal{I}(c\varphi) = \mathcal{I}(c\psi)$, for any $c > 0$
- Lelong numbers up to proper modifications are the same: for all proper modifications $\pi : X \to \mathbb{C}^n$ above 0 and all points $p \in \pi^{-1}(0)$, we have $v(\varphi \circ \pi, p) = v(\psi \circ \pi, p)$

This is true by combining Boucksom-Favre-Jonsson's result [**BFJ08**] with the strong openness conjecture.

Another consequence is the following open problem listed in [**Laz17**]:

- For a big line bundle L, the equality $\mathcal{I}(||mL||) = \mathcal{I}(h^m_{min})$ holds for every integer $m > 0$.

Here asymptotic multiplier ideal $\mathcal{I}(||L||)$ is the maximal member of the family of the ideals $\{\mathcal{I}(1/k \cdot |kL|)\}$ for k large and h_{min} is the singular Hermitian metric with minimal singularities which always uniquely exists on pseudo-effective line bundles by Demailly ([**Dem12**]).

In addition, it follows from the strong openness for L^2 multiplier ideal sheaves and Hölder inequality that we can prove Siu's conjecture: L^p multiplier ideal sheaves for $0 < p < \infty$ satisfy the strong openness property and are also coherent (see Fornæss [**For15**]).

2.3. Nadel vanishing without Kähler assumption. Moreover, we can drop the Kähler condition in Nadel vanishing theorem using the strong openness. This is the joint work with my student Meng.

Theorem 6 (Meng-Zhou, [**MZ19**]). *Let (X,ω) be a compact Hermitian manifold, and let L be a big holomorphic line bundle. Then*

$$H^q(X, \mathcal{O}(K_X \otimes L) \otimes \mathcal{I}(h)) = 0 \quad for \quad q \geq 1.$$

2.4. Solutions of conjectures of Demailly-Kollár and Jonsson-Mustatӑ. In [**DK01**] Demailly-Kollár conjectured that:

Conjecture 7. *Let φ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and K be compact subset of Δ^n. If $c_K(\varphi) < +\infty$, then*

$$\frac{1}{r^{2c_K(\varphi)}}\mu(\{\varphi < \log r\})$$

has a uniform positive lower bound independent of $r \in (0,1)$, *where* $c_K(\varphi) = \sup\{c \geq 0 : e^{-2c\varphi}$ *is* L^1 *on a neighborhood of* $K\}$, *and* μ *is the Lebesgue measure on* $\mathbb{C}^n$.

And Jonsson-Mustată conjectured that in [**JM12**]:

CONJECTURE 8. *Let* I *be an ideal of* $\mathcal{O}_o$, *which is generated by* $\{f_j\}_{j=1,\cdots,l}$. *Let* ψ *be a plurisubharmonic function on* $\Delta^n \subset \mathbb{C}^n$. *If* $c_o^I(\psi) < +\infty$, *then*

$$\frac{1}{r^{2c_o^I(\psi)}}\mu(\{c_o^I(\psi)\psi - \log|I| < \log r\})$$

has a uniform positive lower bound independent of $r \in (0,1)$, *where*

$$\log|I| := \log \max_{1\leq j\leq l} |f_j|,$$

$c_o^I(\psi) = \sup\{c \geq 0 : |I|^2 e^{-2c\psi}$ *is* L^1 *on a neighborhood of* $o\}$ *is the jumping number, and* μ *is the Lebesgue measure on* $\mathbb{C}^n$.

Actually, the Demailly-Kollár conjecture is more precise than the openness conjecture and the Jonsson-Mustată conjecture is more precise than the strong openness conjecture. In [**GZ15b**], Guan-Zhou proved the above DK and JM conjectures and obtained an effectiveness result of the strong openness. In particular, Guan-Zhou in [**GZ15b**] gave actually different methods to prove the strong openness conjecture.

THEOREM 9 (Guan-Zhou). *Let* φ *be a plurisubharmonic function on* $\Delta^n \subset \mathbb{C}^n$. *Let* F *be a holomorphic function on* Δ^n. *Assume that* $|F|^2 e^{-\varphi}$ *is not locally integrable near* o. *Then*

$$\int_{\Delta^n} \mathbb{I}_{\{-(R+1)<\varphi<-R\}}|F|^2 e^{-\varphi} d\lambda_n$$

has a uniform positive lower bound independent of $R >> 0$. *Especially, if* $F = 1$, *then*

$$e^R \mu(\{-(R+1) < \varphi < -R\})$$

has a uniform positive lower bound independent of $R >> 0$.

2.5. Stability of multiplier ideal sheaf. Furthermore, based on [**GZ15b**], we established stability of the multiplier ideal sheaves:

THEOREM 10 (Guan-Li-Zhou, [**GLZ16**]). *Let* $(\varphi_j)_{j\in\mathbb{N}^+}$ *be a sequence of negative psh functions on* D, *which converges to* $\varphi \in Psh(D)$ *in Lebesgue measure, and* $\mathcal{I}(\varphi_j)_o \subset \mathcal{I}(\varphi)_o$. *Let* $F_j \in \mathcal{O}(D)$, $j \in \mathbb{N}^+$ *s.t.* $(F_j, o) \in \mathcal{I}(\varphi)_o$, *which compactly converges to* $F \in \mathcal{O}(D)$. *Then,* $|F_j|^2 e^{-\varphi_j}$ *converges to* $|F|^2 e^{-\varphi}$ *in the* L^1 *norm near* o. *In particular, there exists* $\varepsilon_0 > 0$ *such that* $\mathcal{I}(\varphi_j)_o = \mathcal{I}((1+\varepsilon_0)\varphi_j)_o = \mathcal{I}(\varphi)_o$ *for any large enough* j.

This also gives actually another different method to prove the strong openness conjecture. Immediately we can obtain that

COROLLARY 11 ([**DK01**], [**Hie14**]). *Semi-continuity of complex singularity exponents and weighted log canonical threshold holds.*

2.6. Local structure of m.i.s. with weight of Lelong number 1. It is well-known that Skoda [**Sko72**] proved that:

$$v(\varphi, 0) < 2 \Rightarrow \mathcal{I}(\varphi)_0 = \mathcal{O}_0.$$

There is a natural question: what happens when $v(\varphi, 0) = 2$?

Using our solution of the strong openness conjecture, we proved that

THEOREM 12 (Guan-Zhou, [**GZ15a**]). *If $v(\varphi, 0) = 2$, then either $\mathcal{I}(\varphi)_0 = \mathcal{O}_0$ or $\mathcal{I}(\varphi)_0 = \mathcal{I}(\log |h|)_0$ where h is the defining function of a germ of a regular analytic hypersurface through 0.*

3. Siu's lemma, L^2 extension and applications to twisted pluricanonical sheaves

3.1. Siu's lemma. Siu's lemma plays important roles in Siu's study of algebraic geometry problems and Phong-Sturm's study of holomorphic stability problem.

THEOREM 13 (Siu's lemma, [**PS00**]). *Let $\varphi(z)$ be a nonpositive plurisubharmonic function on $\mathbb{B}_r^1 \times \mathbb{B}_r^{n-1}$ such that*

$$I_\varphi := \int_{(z_2,\cdots,z_n)\in\mathbb{B}_r^{n-1}} e^{-\varphi(0,z_2,\cdots,z_n)} d\lambda_{n-1} < +\infty,$$

Assume that $r_1 \in (0, r)$. Then there exists a positive number C independent of φ, such that

$$\varliminf_{z_1\to 0} \int_{(z_2,\cdots,z_n)\in\mathbb{B}_{r_1}^{n-1}} e^{-\varphi(z_1,z_2,\cdots,z_n)} d\lambda_{n-1} \le C I_\varphi.$$

THEOREM 14 (Siu's semi-continuity of multiplier ideal sheaves). *The limit of the zero-sets of the multiplier ideal sheaves defined by a holomorphic family of multivalued holomorphic sections contains the zero-set of the limit.*

There is an equivalent version: let $\varphi(z', z'')$ be a plurisubharmonic function on $\Delta^n \times \Delta^m$, if $e^{-\varphi(z',z'')}$ is not integrable at $z' = 0$ for almost all $z'' \in \Delta^m \setminus 0$, then $e^{-\varphi(z',0)}$ is not integrable at $z' = 0$.

3.2. Siu's lemma with trivial multiplier ideal sheaves. In general, one couldn't expect to have

$$\lim_{z'\to 0} \int_{z''\in\mathbb{B}_r^{n-1}} e^{-\varphi(z',z'')} dV_{n-1} = \int_{z''\in\mathbb{B}_r^{n-1}} e^{-\varphi(0,z'')} dV_{n-1}$$

Using some ideas in the proof of strong openness, we get the following generalized Siu's lemma with trivial multiplier ideal sheaves.

THEOREM 15 ([**ZZ17**]). *Let $\varphi(z',z'')$ be a plurisubharmonic function, P be a nonnegative continuous function on $\mathbb{B}_r^m \times \mathbb{B}_r^{n-m}$ $(1 \le m \le n)$.*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\mu(\mathbb{B}_\varepsilon^m)} \int_{\mathbb{B}_\varepsilon^m \times \mathbb{B}_r^{n-m}} P(z',z'')e^{-\varphi(z',z'')}dV_n$$
$$= \int_{z''\in\mathbb{B}_r^{n-m}} P(0,z'')e^{-\varphi(0,z'')}dV_{n-m}$$

This version implies both Siu's lemma and Siu's semi-continuity lemma.

3.3. Siu's lemma with nontrivial multiplier ideal sheaves. Moreover, we can prove the generalized Siu's lemma with nontrivial multiplier ideal sheaves.

THEOREM 16. *Assume that*

$$I_{f,\varphi} := \int_{z''\in\mathbb{B}_r^{n-m}} |f(z'')|^2 e^{-\varphi(0,z'')}d\lambda_{n-m} < +\infty$$

Assume that $\varepsilon, r_1, r_2 \in (0,r)$ and $r_1 < r_2$.

Then there exists a holomorphic function $F(z',z'')$ on $\mathbb{B}_{r_2}^m \times \mathbb{B}_{r_2}^{n-m}$ s.t. $F(0,z'') = f(z'')$ on $\mathbb{B}_{r_2}^{n-m}$,

$$\int_{\mathbb{B}_{r_2}^m \times \mathbb{B}_{r_2}^{n-m}} |F(z',z'')|^2 e^{-\varphi(z',z'')}d\lambda_n < +\infty,$$

and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\lambda(\mathbb{B}_\varepsilon^m)} \int_{\mathbb{B}_\varepsilon^m \times \mathbb{B}_{r_1}^{n-m}} P(z',z'')|F(z',z'')|^2 e^{-\varphi(z',z'')}d\lambda_n$$
$$= \int_{z''\in\mathbb{B}_{r_1}^{n-m}} P(0,z'')|f(z'')|^2 e^{-\varphi(0,z'')}d\lambda_{n-m}$$

where P is only a nonnegative continuous function.

3.4. Siu's lemma with nontrivial L^p multiplier ideal sheaves. Using additionally the iteration method by Berndtsson-Paun, we obtain Siu's lemma with nontrivial L^p multiplier ideal sheaves, where $p \le 2$.

THEOREM 17 ([**ZZ20a**]). *Let $p \in (0,2]$. Let $\varphi(z',z'')$ be a plurisubharmonic function on $\mathbb{B}_r^k \times \mathbb{B}_r^{n-k}$ $(1 \le k \le n)$, $P(z',z'')$ be a nonnegative continuous function on $\mathbb{B}_r^k \times \mathbb{B}_r^{n-k}$, $M(z')$ be a bounded nonnegative measurable function on $\mathbb{C}^k$ with compact support, and $f(z'')$ be a holomorphic function on $\mathbb{B}_r^{n-k}$ satisfying*

$$\int_{z''\in\mathbb{B}_r^{n-k}} |f(z'')|^p e^{-\varphi(0,z'')}d\lambda_{n-k} < +\infty.$$

Assume that $r_1, r_2, r_3 \in (0,r)$ and $r_1 < r_2 < r_3$. Let β be a positive number such that

$$I_\beta := \int_{z''\in\mathbb{B}_{r_3}^{n-k}} |f(z'')|^p e^{-(1+\beta)\varphi(0,z'')}d\lambda_{n-k} < +\infty$$

(*the existence of* β *is guaranteed by the strong openness of multiplier ideal sheaf*), *and* $\alpha \in (1-\frac{p}{2k}\beta, 1)$ *be a nonnegative number.*

Then there exists a holomorphic function $F(z', z'')$ *on* $\mathbb{B}^k_r \times \mathbb{B}^{n-k}_{r_3}$ *s.t.* $F(0, z'') = f(z'')$ *on* $\mathbb{B}^{n-k}_{r_3}$,

$$\int_{(z',z'')\in\mathbb{B}^k_r\times\mathbb{B}^{n-k}_{r_3}} \frac{|F(z',z'')|^p e^{-(1+\beta)\varphi(z',z'')}}{|z'|^{2k\alpha}} d\lambda_n < +\infty$$

and

$$\lim_{\varepsilon\to 0^+} \int_{(z',z'')\in\mathbb{B}^k_r\times\mathbb{B}^{n-k}_{r_1}} \frac{1}{\varepsilon^{2k}} M\big(\frac{z'}{\varepsilon}\big) P(z',z'')|F(z',z'')|^p e^{-\varphi(z',z'')} d\lambda_n$$
$$= \int_{z'\in\mathbb{C}^k} M(z') d\lambda_k \int_{z''\in\mathbb{B}^{n-k}_{r_1}} P(0,z'')|f(z'')|^p e^{-\varphi(0,z'')} d\lambda_{n-k}.$$

3.5. Pseudoeffectivity of twisted relative pluri-canonical bundles. The result in the last section was used in the positivity of twisted relative pluri-canonical bundles in the Kähler setting. Now we consider that $\pi : X \longrightarrow Y$ is a surjective proper holomorphic map from a Kähler manifold (X, ω) to a complex manifold Y where π is not necessarily non-degenerate, and (L, h) be a pseudoeffective line bundle over X.

The projective algebraic setting was already studied by Berndtsson, Paun and Paun-Takayama. We adopt the same notations. We consider Y_0 is the set of all regular values of π in Y.

Let $X_y := \pi^{-1}(y)$, $L_y := L|_{X_y}$, $h_y := h|_{X_y}$, $Y_h := \{y \in Y_0;\ h_y \not\equiv +\infty\}$ and

$$Y_{m,\mathrm{ext}} := \big\{y \in Y_0;\ \dim H^0(X_y, mK_{X_y} + L_y) = \operatorname{rank} \pi_*(mK_{X/Y} + L)\big\}.$$

Assume that there exists a point $y' \in Y_h \cap Y_{m,\mathrm{ext}}$ such that

$$H^0\big(X_{y'}, (mK_{X_{y'}} + L_{y'}) \otimes \mathcal{I}_m(h_{y'}^{\frac{1}{m}})\big) \neq 0.$$

The relative m-Bergman kernel metric $B^o_{m,X/Y}$ at a point $x \in X_y$ is defined as

$$\sup\{u(x)\otimes\overline{u(x)};\ u \in H^0(X_y, mK_{X_y} + L_y), \int_{X_y} |u|^{\frac{2}{m}}_{h^{\otimes m}_{\omega_y}\otimes h_y} dV_{X_y} \le 1\}$$

Another singular metric $B_{m,y,2}(x)$ is defined by the same above formula but where u is further assumed to have a holomorphic extension to some open neighborhood of X_y in X.

In the projective algebraic case, Berndtsson-Paun-Takayama already proved that the relative m-Bergman kernel metric $(B^o_{m,X/Y})^{-1}$ on the twisted relative pluricanonical line bundle $(mK_{X/Y} + L)|_{X_{m,\mathrm{ext}}}$ is a singular Hermitian metric with semipositive curvature current, where $X_{m,\mathrm{ext}} := \pi^{-1}(Y_{m,\mathrm{ext}})$.

Moreover, $(B^o_{m,X/Y})^{-1}$ extends across $X \setminus X_{m,\text{ext}}$ uniquely to a singular Hermitian metric $(B_{m,X/Y})^{-1}$ on $mK_{X/Y} + L$ with semipositive curvature current on all of X. We extend Berndtsson-Paun-Takayama's result to Kähler case.

3.6. A question on comparison between fibre singular metrics. In addition, Berndtsson-Paun; Paun-Takayama mentioned the following conjecture:

Conjecture 18 (Berndtsson-Paun; Paun-Takayama).

$$(B_{m,X/Y})^{-1}|_{X_y} \leq (B_{m,y,2})^{-1} \quad (\forall\, y \in Y_0 \setminus Y_{m,\text{ext}})$$

Actually we prove that these two metrics are exactly equal to each other

Theorem 19 ([ZZ20a]). $(B_{m,X/Y})^{-1}|_{X_y} = (B_{m,y,2})^{-1}$ $(\forall\, y \in Y_0 \setminus Y_{m,\text{ext}})$.

For the projective case, you're referred to [BP10], [BP08], [PT18], [HPS18].

3.7. Subadditivity of the generalized Kodaira-Iitaka dimension. Using the above result in the subsection 3.5, we can consider the following result related to multiplier idea sheaf.

Let (L, h_L) be a holomorphic $\mathbb{Q}$-line bundle on X with a singular Hermitian metric h_L. Let k_0 be the smallest positive integer such that $k_0 L$ is a holomorphic line bundle. We introduce the generalized Kodaira-Iitaka dimension with multiplier ideal sheaves:

$$\begin{aligned} &\kappa(X, K_X + L, h_L) \\ := \quad &\sup\left\{ v \in \mathbb{Z};\ \varlimsup_{k\to+\infty} \frac{h^0\big(X, (kk_0K_X + kk_0L)\otimes \mathcal{I}_{kk_0}(h_L)\big)}{k^v} > 0 \right\} \end{aligned}$$

if $\varlimsup_{k\to+\infty} h^0\big(X, (kk_0K_X + kk_0L)\otimes \mathcal{I}_{kk_0}(h_L)\big) \neq 0$.

Otherwise, $\kappa(X, K_X + L, h_L)$ is defined to be $-\infty$.

As an application of our result on pseudoeffectivity of the twisted relative pluricanonical bundles, we obtain the subadditivity of generalized Kodaira-Iitaka dimensions with multiplier ideal sheaves for Kähler fibrations.

Let $\Pi : X^n \longrightarrow Y^m$ be a surjective holomorphic map with connected fibers from a compact Kähler manifold X to a compact connected complex manifold Y. Let (L, h_L) be a pseudoeffective holomorphic $\mathbb{Q}$-line bundle on X, i.e., the curvature current $\sqrt{-1}\Theta_{L,h_L} \geq 0$ on X.

Theorem 20 ([ZZ20b]). *Assume that the canonical bundle K_Y of Y possesses a singular Hermitian metric h such that*

(a) *(K_Y, h) is pseudoeffective,*
(b) *there exists an open subset U of Y and a continuous positive $(1,1)$-form γ on U such that $\sqrt{-1}\Theta_{K_Y,h} \geq \gamma$ on U in the sense of currents.*

Then

$$\kappa(X, K_X + L, h_L) \geq \kappa(Z, K_Z + L|_Z, h_L|_Z) + m,$$

where Z *denotes a general fiber of* Π.

When (L, h_L) is the trivial line bundle, since the assumption on K_Y in the above Theorem is equivalent to say that K_Y is big (due to Siu, Demailly, Ji-Shiffman, Boucksom, Popovici), the above Theorem reduces Kawamata-Viehweg's result:

$$\kappa(X) \geq \kappa(Z) + m.$$

When $\mathcal{I}(h_L) = \mathcal{O}_X$, the above Theorem becomes

$$\kappa(X, K_X + L) \geq \kappa(Z, K_Z + L|_Z) + m.$$

Such a kind of subadditivity in projective case was proved by Campana [**Cam04**] and Nakayama [**Nak04**].

4. Extension theorems of cohomology classes

4.1. Extension of cohomology classes. Now let us mention further development on the extension theorem of cohomology classes related to multiplier ideal sheaves.

Theorem 21 (Cao-Demailly-Matsumura [**CDM17**], Zhou-Zhu [**ZZ22**]). *Let* (X, ω) *be a holomorphically convex Kähler manifold,* ψ *be an* L^1_{loc} *function on* X *which is locally bounded above, and* (L, h) *be a singular hermitian line bundle over* X. *Assume that* $\alpha > 0$ *is a positive continuous function on* X*, and that the following two inequalities hold on* X *in the sense of currents:*

(i) $\sqrt{-1}\Theta_{L,h} + \sqrt{-1}\partial\bar{\partial}\psi \geq 0$,
(ii) $\sqrt{-1}\Theta_{L,h} + (1+\alpha)\sqrt{-1}\partial\bar{\partial}\psi \geq 0$.

Then the homomorphism induced by the natural inclusion $\mathcal{I}(he^{-\psi}) \longrightarrow \mathcal{I}(h)$,

$$H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi})\big) \longrightarrow H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

is injective for every $q \geq 0$.

In other words, induced by the natural sheaf surjection $\mathcal{I}(h) \longrightarrow \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$, the homomorphism

$$H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big) \longrightarrow H^q\big(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi})\big)$$

is surjective for every $q \geq 0$. This is the so-called extension of cohomology classes from a subvariety *supp*$(\mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$ in X.

The above theorem was proved by Cao-Demailly-Matsumura [**CDM17**] when ψ is a quasi-plurisubharmonic function with neat analytic singularities. In my paper with my former student Zhu (now a professor of Wuhan University), we can get this for the general case when ψ is a quasi-plurisubharmonic function with arbitrary singularities which was posed as a question in the same paper [**CDM17**].

4.2. Injectivity theorem. As an application of the above extension theorem of cohomology classes, we obtain the following injectivity theorem related to multiplier ideal sheaves.

THEOREM 22 ([**ZZ22**]). *Let X be a holomorphically convex Kähler manifold. Let (F, h_F) and (G, h_G) be two singular hermitian line bundles over X. Assume that the following two inequalities hold on X in the sense of currents:*

(i) $\sqrt{-1}\Theta_{F,h_F} \geq 0$,
(ii) $\sqrt{-1}\Theta_{F,h_F} \geq b\sqrt{-1}\Theta_{G,h_G}$ *for some* $b \in (0, +\infty)$.

Then, for a non-zero global holomorphic section s of G satisfying $\sup_{\Omega} |s|_{h_G} < +\infty$ for every $\Omega \subset\subset X$, the following map β induced by the tensor product with s

$$H^q\big(X, \mathcal{O}_X(K_X \otimes F) \otimes \mathcal{I}(h_F)\big) \xrightarrow{\beta} H^q\big(X, \mathcal{O}_X(K_X \otimes F \otimes G) \otimes \mathcal{I}(h_F h_G)\big)$$

is injective for every $q \geq 0$.

The above injectivity theorem on holomorphically convex Kähler manifolds unifies several recent injectivity theorems obtained by Matsumura [**Mat14, Mat15, Mat16**] Fujino-Matsumura [**FM21**], Gongyo-Matsumura [**GM17**]

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