Statistics and Its Interface Volume 13 (2020) 91–101

Leverage effect in high-frequency data with market microstructure

Huiling Yuan, Yan Mu*, and Yong Zhou†

The leverage effect is an important explanation for volatility asymmetry, which has got extensively attention in the recent years. In this paper, we introduces a new estimator of leverage effect. The key feature of the proposed estimator is explored in the setting when the microstructure noise model is the parameter function of trading information. The proposed estimator shows good statistical performances via theorems and simulations study. Specially, the estimator has a convergence rate $n^{1/4}$. The QQ-Plots, Histogram plots and quartiles perform sufficient asymptotical normality compared with the exist estimated methods. An empirical study is carried out to demonstrate that the proposed estimator could present the efficient application value, and confirm that the leverage effect plays an important role in forecasting volatility.

AMS 2000 subject classifications: Primary 62M10, 62M20; secondary 62G05, 62G20.
Keywords and phrases: Quadratic covariation, Integral volatility, Microstructure noise, Consistency.

1. INTRODUCTION

One of the most important explanation for volatility asymmetry is leverage effect, which refers to the negative correlation between an asset return and its volatility changes. Typically, an increase in stock price is tied to a fall in volatility, and a fall in stock price is tied to an increase in volatility. Dating back the seminal papers of [5] and [9], this phenomenon is related to the so called "leverage effect": when the asset prices is declining, the companies' leverage (debt-to-equity ratio) become larger, so the stock becomes riskier since its volatility is increasing. Therefore, leverage effect implies a negative correlation structure between the analyzed asset return and its volatility. The early scholars studied mainly various financial theories of leverage effect or volatility asymmetry via the low-frequency data (see, e.g., [22]; [11]).

In recent years, high-frequency financial data that refer to intra-daily observations such as tick-by-tick stock prices or minute-by-minute exchange rates became available thanks to advances in information technology. High-frequency financial data have rich information due to the short observed time interval. Many later works focused on statistical and financial properties of the leverage effect via high-frequency data. [8] obtained the peak effect at the instantaneous correlation between return and volatility over fairly small time intervals. This is benefit to [24]'s definition of leverage effect as being instantaneous. By using of high-frequency five-minute S&P 500 futures, [6] found that there exists significantly negative correlation for several days between the absolute high-frequency returns and the current and past returns, and low correlations between the volatility and lagged return. [7] proposed a highly accurate discrete-time daily stochastic volatility model that distinguishes between the jump and continuous-time components of price movements, it is important to say that leverage effect works primarily in the continuous volatility component.

Leverage effect describes the negative correlation between the daily returns and the changes of daily volatility, therefore, it is intuitive and natural to view the correlation between the daily returns and daily volatility estimator as the estimator of leverage effect. In fact, several realized volatility estimation procedures have been proposed to estimate integrated volatility in high-frequency financial market. For instance, [26] proposed two-time scale realized volatility (TSRV) which is consistent estimator for integrated volatility in the presence of market microstructure noise. [25] improved TSRV to multi-scale realized volatility (MSRV) so that it can achieve the optimal convergence rate. Other forms of estimators that can achieve the optimal convergence rate are kernel realized volatility (KRV) ([4]), pre-averaging realized volatility (PRV) ([16]). [1] employed the integrated volatility estimators TSRV and PRV to study the estimators of leverage effect. However, these estimators of leverage effect cause shrinkage bias due to estimator error, resulting in the leverage effect puzzle.

[24] proposed the new nonparametric estimators of leverage effect (named $\hat{\varrho}_{WM}$) in the stochastic volatility model. They provided the related statistical properties of the proposed estimators in the cases both with and without microstructure noise. The estimator of the leverage effect has a convergence rate of $n^{1/4}$ in the absence of microstructure

*Corresponding author.
†Zhou's work was supported by the State Key Program of National Natural Science Foundation of China (71331006), the State Key Program in the Major Research Plan of National Natural Science Foundation of China (91546202).

noise. And the estimator has a convergence rate of $n^{1/8}$ under the setting when the microstructure noise is independent and identically distributed (i.i.d.), specially,

$$Y_{t_{n,j}} = X_{t_{n,j}} + \varepsilon_{t_{n,j}}, \quad 0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = 1, \tag{1}$$

where $\varepsilon_{t_{n,j}}$ are i.i.d. noise with mean zero and variance a^2 and independent of the $X_{t_{n,j}}$ process. $X_{t_{n,j}}$ is the latent efficient log-price, while $Y_{t_{n,j}}$ is the observed financial market log-price.

Actually, without the market microstructure noise, the volatility estimator would blow up as the sampling frequency increases, which was presented in many frequently traded stocks in [3]'s paper. The pattern is named "volatility signature plot pattern". And many empirical findings also suggest the market microstructure noise has complex structures, not only independent and identical. Studies on market microstructure noise can be traced back to the 1980s. [23] proposed a simple model for microstructure noise.

$$\varepsilon_{t_{n,j}} = \alpha I_{b/s}(t_{n,j}), \tag{2}$$

where $I_{b/s}(t_{n,j})$ denotes the trade type at time $t_{n,j}$, indicating if the trade is buyer-initiated $(+1)$ or seller initiated (-1). [12] showed that Roll's model can be extended by incorporating the trading volume. [14] noted that there were sources of noise other than just the bid-ask spread, and studied their effect on the Roll model. [2] obtained that the microstructure $\varepsilon_{t_{n,j}}$ can be viewed as a function of trade type and trading rate. [13] and [17] found that the i.i.d. assumption of the noise $\varepsilon_{t_{n,j}}$ is inconsistent with empirical findings.

In general, there are a diverse array of market microstructure noise, either informational or not: bid-ask spread bounces, differences in trade sizes, informational content of price changes, gradual response of prices to a block trade, the strategic component of the order flow, inventory control effects, discreteness of price changes, etc. All of these suggest that we should take advantage of the rich information available in the high-frequency market. Therefore, it is also important to provide the new estimator of the leverage effect to improve the exist estimators in the complex financial market.

Recently, [21] studied a more general microstructure noise model, which referred to the available trading information through a parametric function. The function can be linear or nonlinear.

$$Y_{t_{n,j}} = X_{t_{n,j}} + g\left(Z_{t_{n,j}};\theta_0\right), 0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = 1, \tag{3}$$

where $Y_{t_{n,j}}$ is the observed log-prices at time $t_{n,j}$, $X_{t_{n,j}}$ is the latent log-prices, and $Z_{t_{n,j}}$ is the information set, but not limited to trading volume, trade type, and bid-ask bounds, θ_0 is a finite-dimensional parameter, and $g(Z;\theta)$ is any parametric form of Z and θ.

Under those settings, they proposed a new estimator of integrated volatility called "estimated-price realized volatility" (ERV), and showed that ERV had a convergence rate of $\sqrt{n}$ (instead of $n^{1/4}$ under usual noisy settings), which demonstrated clearly that their ERV had some advantages in estimating volatility.

In this paper, we are interested in developing a new estimator of leverage effect (named $\hat{\varrho}_{yz}$) in model (3), which can allow us to estimate leverage effect in a complex microstructure noise model. The proposed estimator presents good consistency and asymptotic properties via both the theorems and simulations. Specially, in the general microstructure noise model (3), $\hat{\varrho}_{yz}$ provides a faster convergence rate $n^{1/4}$ than $\hat{\varrho}_{WM}$ based on the simple additive microstructure noise model (1). It is obvious that $\hat{\varrho}_{yz}$ has the same convergence rate as $\hat{\varrho}_{WM}$ in the absence of microstructure noise. Simulations provide Q-Q plots, Histogram plots, mean, quartiles Q_1, Q_2, Q_3, variances, biases and mean square error of the estimators to compare $\hat{\varrho}_{yz}$ with $\hat{\varrho}_{WM}$. In addition, a empirical study is carried out to demonstrate our proposed estimator that has extensive application in volatility forecasting.

The rest of this paper is organized as follows. Section 2.1 presents the data-generating mechanism. Sections 2.2 introduces the new estimator of leverage effect. Section 2.3 provides the related statistical theorems when the microstructure noise could be demonstrated by model (3). Section 2.4 studies the asymptotic variance. Simulation studies and empirical studies are carried out in Section 3 and Section 4, respectively. Section 5 concludes and discusses related issues. Proofs are in the Appendix.

2. ESTIMATED LEVERAGE EFFECT

2.1 Data-generating mechanism

We introduce a common filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$, which is a canonical space defined by means of 1-dimension independent Wiener process W_t and B_t on the time interval $[0,T]$. The latent log-price X_t and its spot volatility σ_t are defined as following.

$$\begin{cases} dX_t = \mu_t dt + \sigma_t dW_t, \quad X_0 = x_0, \\ d\sigma_t = a_t dt + b_t dW_t + g_t dB_t, \end{cases} \tag{4}$$

where W_t is a Wiener process, B_t is another Wiener process independent of W_t. μ_t, a_t, b_t, g_t, and σ_t are all adapted càdlàg locally bounded random processes, and defined in the probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$.

Formula (4) is the most popular model in econometrics and financial mathematics studies. The "integrated volatility (IV)" of latent log-price X_t is given as

$$\langle X, X \rangle_T = \int_0^T \sigma_t^2 dt, \tag{5}$$

without loss of generality, we set $T = 1$. The process (5) is the known-well quadratic variation of X_t, which plays an important role in high-frequency financial market.

Clearly, the stochastic processes X_t and σ_t have a common driving Wiener process W_t which accommodates the leverage effect. Then, the contemporaneous leverage effect could be studied by the quadratic covariation between X_t and $F(\sigma_t^2)$.

$$\varrho(T) = \langle X, F(\sigma_t^2)\rangle_T = \int_0^T 2F'\left(\sigma_t^2\right)\sigma_t^2 b_t dt. \tag{6}$$

For simplicity, $\varrho(T)$ is marked by ϱ.

Assumption 1. : $x \mapsto F(x)$ is twice continuously differentiable, monotone on $(0, \infty)$.

Actually, function $F(x)$ allows more flexibility forms such as $F(x) = x$, $F(x) = \frac{1}{2}\log(x)$ and so on, which depends on the practical purpose and empirical evidence. The paper is mainly to conduct the new estimator for leverage effect ϱ based on the complex noise model (3).

2.2 Estimated leverage effect with trading information

Here, we shall investigate the equidistant time case for the process X_t, specifically, it is observed every $\Delta t_{n,j} = \Delta t = \frac{T}{n}$ units of time, at times $0 = t_{n,0} < t_{n,1} \cdots < t_{n,n-1} = T$, where the time interval Δt is eventually goes to 0 as $n \to \infty$. We set $T = 1$ without general form. Moreover, we divide observed values into K_n blocks, with block size $M_n = [c\sqrt{n}]$ (except possibly for the first and last block, which does not matter for the asymptotic), for some constant c. The boundary points are on the grid $\mathcal{H} = 0 < \tau_{n,1} < \tau_{n,2} < \cdots < \tau_{n,K_n-1} \leq T$, where $K_n = \left[\frac{n}{M_n}\right]$.

Regarding noise model (3), we could obtain the estimator of latent log-prices $X_{t_{n,j}}$ at the time interval $[\tau_{n,i}, \tau_{n,i+1}]$ by

$$\widehat{X}_{t_{n,j}} := Y_{t_{n,j}} - g\left(Z_{t_{n,j}}; \hat{\theta}\right), \tag{7}$$

where $g(Z_{t_{n,j}}; \hat{\theta})$ is the estimator of $g(Z_{t_{n,j}}; \theta_0)$. The maximum likelihood estimator (MLE) $\hat{\theta}$ of θ_0 is given by

$$\widehat{\theta} = \arg\min Q_n(Y, Z, \theta),$$

where

$$Q_n(Y, Z, \theta) = \frac{1}{2} \sum_{t_{n,j} \in (\tau_{n,i}, \tau_{n,i+1}]} \left(\Delta Y_{t_{n,j}} - \Delta g\left(Z_{t_{n,j}}; \theta\right)\right)^2,$$

when $g(\cdot)$ is a linear function, $\hat{\theta}$ is given by

$$\widehat{\theta} = \left(\sum_{t_{n,j} \in (\tau_{n,i}, \tau_{n,i+1}]} \Delta Z_{t_{n,j}}{}^T \Delta Z_{t_{n,j}}\right)^{-1} \left(\sum_{t_{n,j} \in (\tau_{n,i}, \tau_{n,i+1}]} \Delta Z_{t_{n,j}}{}^T \Delta Y_{t_{n,j}}\right).$$

A natural estimator of leverage effect can be proposed by

$$\begin{aligned} \widehat{\langle X, F(\sigma^2)\rangle}_T = & 2\sum_{i=0}^{K_n-2} \left(\widehat{X}_{\tau_{n,i+1}} - \widehat{X}_{\tau_{n,i}}\right) \\ & \cdot \left(F\left(\widetilde{\sigma}^2_{\tau_{n,i+1}}\right) - F\left(\widetilde{\sigma}^2_{\tau_{n,i}}\right)\right), \end{aligned} \tag{8}$$

where

$$\widetilde{\sigma}^2_{\tau_{n,i}} = \frac{1}{M_n \times \triangle t} \sum_{t_{n,j} \in (\tau_{n,i}, \tau_{n,i+1}]} \left(\widehat{X}_{t_{n,j+1}} - \widehat{X}_{t_{n,j}}\right)^2. \tag{9}$$

For equation (9), $\sum_{t_{n,j} \in (\tau_{n,i}, \tau_{n,i+1}]} \left(\widehat{X}_{t_{n,j+1}} - \widehat{X}_{t_{n,j}}\right)^2 = \sum_{t_{n,j} \in (\tau_{n,i}, \tau_{n,i+1}]} \Delta \widehat{X}^2_{t_{n,j}}$ is the estimated-price realized volatility (ERV) (see: [21]) at time interval $[\tau_{n,i}, \tau_{n,i+1}]$. Obviously, all the statistical properties of ERV could be applied in the equation (9).

For simplicity, we name the new estimator $\widehat{\langle X, F(\sigma^2)\rangle}_T$ as $\hat{\varrho}_{yz}$. $\hat{\varrho}_{yz}$ is proposed in the general financial market microstructure noise setting, which improve the application of leverage effect estimator $\hat{\varrho}_{WM}$.

Remark 1. As discussed in [21], the factor 2 in $\hat{\varrho}_{yz}$ is crucial for the consistency of the estimator.

2.3 Asymptotic theory

This section establishes consistency and asymptotic distribution for proposed estimator $\hat{\varrho}_{yz}$. We first state the following assumptions. In real life, the latent log-price is often bounded, therefore, some assumption would be first given for $X_{t_{n,j}}$. Furthermore, maximum likelihood method is used for ERV, it would be better to give some assumptions for $Z_{t_{n,j}}$ and $g(z; \theta)$.

Assumption 2. (i) μ_t is locally bounded, and σ_t is locally bounded with $\inf_{t \in (0,1]} \sigma_t > 0$ almost surely.

(ii) For all $t_{n,j}$, $Z_{t_{n,j}}$ and $\Delta X_{t_{n,j}}$ are conditionally independent given $\mathcal{F}_{t_{n,j-1}}$.

(iii) $\max_{t_{n,j}} |Z_{t_{n,j}}|$ is bounded.

(iv) The parameter space Θ for θ is a compact set in $\mathbf{R}^p$ for some $p \in \mathbf{N}$, and $g(z; \theta)$ is twice continuously differentiable in θ in a neighborhood $\mathcal{N}(\theta_0) \subset \Theta$;

(v) For all $\theta \in \mathcal{N}(\theta_0)$, the first order and second order differential of $g(z; \theta)$ in θ is locally bounded;

(vi) For any $\varepsilon > 0$, as $n \to \infty$

$$\begin{aligned} & inf_{|\theta-\theta_0| \geq \varepsilon} \sum_{t_{n,j} \in (\tau_{n,i}, \tau_{n,i+1}]} \left|\Delta g(Z_{t_{n,j}}; \theta) - \Delta g(Z_{t_{n,j}}; \theta_0)\right|^2 \\ & \to \infty; \end{aligned}$$

almost surely;

(vii)

$$\left\| \left(\frac{1}{n} \sum_{t_{n,j} \in (\tau_{n,i}, \tau_{n,i+1}]} \Delta \frac{\partial g}{\partial \theta}(Z_{t_{n,j}}; \theta_0) \cdot \Delta \frac{\partial g}{\partial \theta^T}(Z_{t_{n,j}}; \theta_0) \right)^{-1} \right\| = O_p(1),$$

where $\|\cdot\|$ stands for its spectral norm.

Remark 2. (i)-(iii) in Assumption 2 are obviously about the process $X_{t_{n,j}}$ and $Z_{t_{n,j}}$, which are the usual assumptions in the continuous stochastic model and regression predictor. $X_{t_{n,j}}$ is clearly bounded when (i) is given. Specially, (ii) is analogous to the usual assumption in regression that the predictor and noise are independent, which is the same description as in [21], this is to say, the assumption amounts to assume that the immediate next trading depends solely on the market information up to the latest transaction under model (2)-(3). (iii) is the common assumption for information set $Z_{t_{n,j}}$. Moreover, (iv)-(vii) in Assumption 2 are referred to $g(z;\theta)$ and θ. Specially, (iv)-(vi) in Assumption 2 correspond to the identifiability condition in MLE, and (vii) in Assumption 2 is the invertibility condition of the Fisher information matrix.

The following theorems establish the statistical performances of $\widehat{X}_{t_{n,j}}$, $\widetilde{\sigma}^2_{\tau_{n,j}}$, and $\hat{\varrho}_{yz}$.

Theorem 2.1. *(A1). Under Assumption 1-2, as T fixed,*

$$\widehat{X}_{t_{n,j+1}} - \widehat{X}_{t_{n,j}} = X_{t_{n,j+1}} - X_{t_{n,j}} + O_p\left(n^{-1}\right), \tag{10}$$

$$\widetilde{\sigma}^2_{\tau_{n,j}} = \widehat{\sigma}^2_{\tau_{n,j}} + o_p\left(n^{-1/4}\right), \tag{11}$$

where $\widehat{X}_{t_{n,j}}$ and $\widetilde{\sigma}^2_{\tau_{n,j}}$ are defined in Eq. (7) and Eq. (9), respectively. $\widehat{\sigma}^2_{\tau_{n,i}}$ could be see in Section 2.3 in [24].

(A2). Under Assumption 1-2, as T fixed and $n \to \infty$,

$$\hat{\varrho}_{yz} = 2 \sum_{i=0}^{K_n-2} \left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right) \left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right). \tag{12}$$

In Theorem 2.1(A2), $2\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right) \cdot \left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right)$ is the actual estimator $\hat{\varrho}_{WM}$ when there is in the absence of market microstructure noise. From Theorem 2.1(A2), we can derive the asymptotic properties of $\hat{\varrho}_{yz}$ by $\hat{\varrho}_{WM}$ without microstructure noise.

Theorem 2.2. *Under Assumption 1-2, as T fixed and as $n \to \infty$, the estimator $\hat{\varrho}_{yz}$ have the following statistic property*

$$n^{1/4}\left(\hat{\varrho}_{yz} - \varrho\right) \xrightarrow{\mathcal{L}} Z \cdot \left(\frac{16}{c}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^6 dt + cT\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4 \left(\frac{44}{3}b_t^2 + \frac{32}{3}g_t^2\right) dt\right)^{1/2}, \tag{13}$$

stable in law, where Z is a standard normal random variable and independent of F_T, b_t and g_t as same as proposed in Itô process (4) are all locally bounded.

Remark 3. 1. "$\xrightarrow{\mathcal{L}}$" stands "stable in law", as follows the following definition.

2. Let Z_n be a sequence of χ-measurable variables, $\mathcal{F}_1 \subset \chi$. We say that Z_n converges $\mathcal{F}_1$-stable in law to Z as $n \to \infty$ if Z is measurable with respect to an extension of χ, so that for all $A \in \mathcal{F}_1$ and for all bounded continuous g, $E\left(I_A g(Z_n)\right) \to E\left(I_A g(Z)\right)$ as $n \to \infty$.

3. In order to properly choose c, one could minimize the limit variance in Theorem 2.2. The optimal value is

$$c^2 = \frac{16\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^6 dt}{T\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4 \left(\frac{44}{3}b_t^2 + \frac{32}{3}g_t^2\right) dt}. \tag{14}$$

2.4 Estimation of asymptotic variance

In this part, we give the estimation of asymptotic variance. Let

$$H_n^1 = 2n^{\frac{1}{2}} \sum_{i=0}^{K_n-2} \left(\widehat{X}_{\tau_{n,i+1}} - \widehat{X}_{\tau_{n,i}}\right)^2 \cdot \left(F(\widetilde{\sigma}^2_{\tau n,i+1}) - F(\widetilde{\sigma}^2_{\tau n,i})\right)^2,$$

and

$$H_n^2 = 2n^{\frac{1}{2}} M_n \Delta t \sum_{i=0}^{K_n-2} \widetilde{\sigma}^2_{\tau n,i}\left(F\left(\widetilde{\sigma}^2_{\tau n,i+1}\right) - F\left(\widetilde{\sigma}^2_{\tau n,i}\right)\right)^2.$$

By the simple deduction (see in Appendix A.3), we could obtain the following equations

$$H_n^1 = 2n^{\frac{1}{2}} \sum_{i=0}^{K_n-2} \left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right)^2 \cdot \left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right)^2 + O_p\left(n^{-1/2}\right),$$

and

$$H_n^2 = 2n^{\frac{1}{2}} M_n \Delta t \sum_{i=0}^{K_n-2} \widehat{\sigma}^2_{\tau n,i}\left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right)^2$$

$$+o_p\left(n^{-1/4}\right).$$

It is easy to see that the first term in H_n^1 is G_n^1 in formula (10) in [24]'s paper, and the first term in H_n^2 is G_n^2 in formula (10) in [24]'s paper via the formulas of H_n^1 and H_n^2. As $n \to \infty$, $H_n^1 + H_n^2$ and $G_n^1 + G_n^2$ are close so that we have the following converge in probability.

$$H_n^1 + H_n^2 \xrightarrow{\mathcal{P}} \frac{16}{c}\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^6 dt + cT\int_0^T \left(F'(\sigma_t^2)\right)^2 \sigma_t^4 \left(\frac{44}{3}b_t^2 + \frac{32}{3}g_t^2\right) dt.$$

By the above estimation of asymptotic variance, we could obtain the feasible version of the central limit distribution.

Theorem 2.3. *Under Assumption 1-2, as T fixed and as $n \to \infty$,*

$$\frac{n^{1/4}\left(\hat{\varrho}_{yz} - \varrho\right)}{\sqrt{H_n^1 + H_n^2}} \xrightarrow{\mathcal{L}} Z_1,$$

stable in law, where Z_1 is a standard normal random variable and independent of F_T.

3. SIMULATION STUDIES

In this section, we use simulation to verify the performance of our proposed estimators $\hat{\varrho}_{yz}$. In those simulations, we first check the normal property of $\hat{\varrho}_{yz}$ under Heston model via QQ-Plots, Histogram plots and quartiles. For comparison purpose, we also demonstrate the normal performances of $\hat{\varrho}_{WM}$. At last, we present the variances, biases and Mean Square Errors (MSE) of $\hat{\varrho}_{yz}$ and $\hat{\varrho}_{WM}$ to examine that our estimator $\hat{\varrho}_{yz}$ based on trade information noise model is efficient in high frequency financial market.

To present the statistical performance of the new estimator of leverage effect, we consider the stochastic volatility model of [15] for the latent log-price dynamics

$$dX_t = \left(\mu - \frac{\sigma_t^2}{2}\right)dt + \sigma_t dW_t, \tag{15}$$

$$d\sigma_t^2 = \kappa\left(\theta - \sigma_t^2\right)dt + \gamma\sigma_t\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right), \tag{16}$$

where W_t and B_t are independent, by Itô integral, we can get the following equation from the Eq. (16)

$$\begin{aligned} d\sigma_t &= \frac{1}{2}\left(\sigma_t^2\right)^{-\frac{1}{2}}d\sigma_t^2 + \frac{1}{2}\left[-\frac{1}{2}\times\frac{1}{2}(\sigma_t^2)^{-\frac{3}{2}}\right]d\sigma_t^2 d\sigma_t^2 \\ &= \left[\left(\frac{\kappa\theta}{2} - \frac{\gamma^2}{8}\right)\sigma_t^{-1} - \frac{\kappa}{2}\sigma_t\right]dt \\ &+ \frac{\gamma}{2}\rho dW_t + \frac{\gamma}{2}\sqrt{1-\rho^2}dB_t. \end{aligned}$$

Compared with Equation (4) and (6), we can get $b_t = \frac{\gamma}{2}\rho$, $g_t = \frac{\gamma}{2}\sqrt{1-\rho^2}$, and the contemporaneous leverage effect between X_t and σ_t^2 would be described by $\varrho = \langle X, \sigma_t^2\rangle_T = \int_0^T \sigma_t^2\gamma\rho dt$ when $F(x) = x$. We choose parameters as $\mu = 0.00015$, $\theta = 0.0003$, $\gamma = 0.01$, $\kappa = 0.2$, $\rho = -0.5$ over 1 trading day (they are consistent with parameter setting in [21]). We further set $X_0 = \log(30)$.

In this section, we choose two microstructure noise models including the trade information: $g_1(\cdot)$ and $g_2(\cdot)$.

$$\begin{aligned} g_1\left(V_{t_{n,j}}, I_{b/s}(t_{n,j}); \alpha, \beta\right) =& \alpha I_{b/s}(t_{n,j}) \\ &+ \beta I_{b/s}(t_{n,j}) V_{t_{n,j}}/\Delta t_{n,j}, \end{aligned} \tag{17}$$

$$g_2\left(V_{t_{n,j}}, I_{b/s}(t_{n,j}); \beta, \gamma\right) = I_{b/s}(t_{n,j})\log\left(\gamma + \beta V_{t_{n,j}}/\Delta t_{n,j}\right), \tag{18}$$

where $I_{b/s}(t_{n,j})$ denotes the trade type at time $t_{n,j}$, indicating if the trade is buyer-initiated (+1) or seller initiated (−1). $V_{t_{n,j}}$ denotes the trading volume at time $t_{n,j}$, $\Delta t_{n,j+1} := t_{n,j+1} - t_{n,j}$ denotes the duration between two consecutive transactions, and $V_{t_{n,j}}/\Delta t_{n,j}$ denotes the trading rate.

Function $g_1(\cdot)$ could be seen as a linear function of trade type and trading rate, which was proposed by [2]. And function $g_2(\cdot)$ is described that the market microstructure noise is concave for buys and convex for sells, which was proposed by [19]. When the trading rate is low, $g_2(\cdot)$ is very close to $g_1(\cdot)$.

Initial values for parameters are chosen to be $\alpha_0 = 1.875\times 10^{-4}$, $\beta_0 = 0.75\times 10^{-12}$, $\zeta_0 = 1 + \alpha_0 = 1.0001875$. And the trade type process $\{I_{b/s}(t_{n,j})\}$ could be generated by a Bernoulli process (±1 valued) with the success probability of $p = \frac{1}{2}$, while the trading volume process $\{V_{t_{n,j}}\}$ rounded by $\{|V^*_{t_{n,j}}|\}$ up to hundreds, where $\{V^*_{t_{n,j}}\}$ are simulated by independent Gamma variables with mean 400, and standard deviation 5,000 (see [21]).

Noted that the variance of $\hat{\varrho}_{WM}$ in the absence of noise, $\hat{\varrho}_{WM}$ with i.i.d noise and $\hat{\varrho}_{yz}$ with trade information noise model could be computed by the feasible form. In other word, $H_n^1 + H_n^2$ in Section 2.4 could be described as the variance of $\hat{\varrho}_{yz}$ when the microstructure noise is demonstrated by the trade information noise model, and the asymptotic variance of $\hat{\varrho}_{WM}$ in the absence of noise and $\hat{\varrho}_{WM}$ under i.i.d market noise model could be represented by $G_n^1 + G_n^2$ in Section 2.4 and Section 4.2 in [24]'s paper, respectively. Therefore, $\hat{\varrho}_{yz}$ and $\hat{\varrho}_{WM}$ could be standardized by Z_{yz} and Z_{WM}, respectively. We would give the form of Z_{yz}, and the form of Z_{WM} could seen in [24]'s paper.

$$Z_{yz} = \frac{n^{1/4}\left(\hat{\varrho}_{yz} - \varrho\right))}{\sqrt{H_n^1 + H_n^2}}. \tag{19}$$

We would check the distribution of this standardized statistics Z_{yz}. In order to compute easily, we choose $T = 1$. So if the estimator of the leverage effect is good, the distribution of Eq. (19) should be closed to the standard normal distribution. We choose the observed data at 1 second frequency, which corresponds to sample size 23400 under linear noisy model Eq. (17) and nonlinear noisy model Eq. (18), respectively. Compared with the proposed estimator $\hat{\varrho}_{yz}$, we also analysis the estimator $\hat{\varrho}_{WM}$ without any microstructure noise model and with i.i.d noise model. All the results are based on 2500 sample paths. Normal QQ-plots and histogram figures are showed in Figure 1–2, and the quartiles are presented in Table 1. The variances, biases and MSE of $\hat{\varrho}_{yz}$ and $\hat{\varrho}_{WM}$ are seen in Table 2.

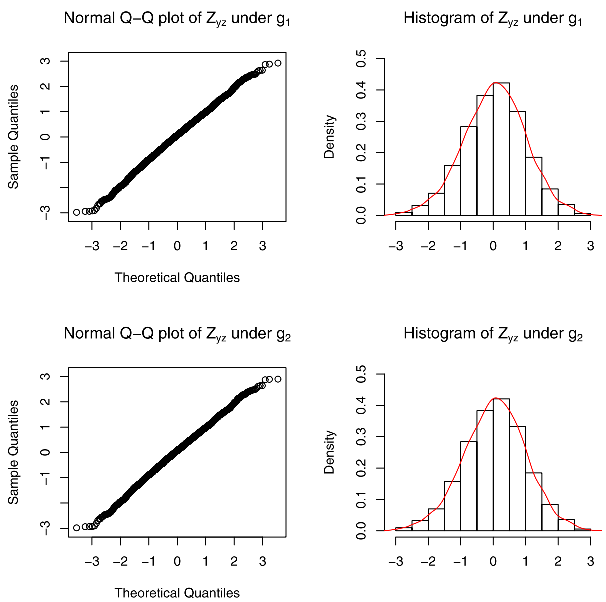

Figure 1. Normal Q-Q plots and histograms of Z_{yz} under the market microstructure noise g_1 and g_2, sample size 23400, replications 2500.

All the above Q-Q plots and Histogram plots present both Z_{yz} and Z_{WM} are approximate to the standard normal distribution, which shows our estimator $\hat{\varrho}_{yz}$ performs very well, because the proposed approaches for leverage effect under microstructure noise model are comparable to the benchmark estimator $\hat{\varrho}_{WM}$ without any financial microstructure noise.

According to Table 1, for sample size 23400, both mean, Q_1, Q_2, and Q_3 of Z_{yz} are close to the ones of Z_{WM} without noise model, which reflects that $\hat{\varrho}_{yz}$ has good normal performance. In another word, we demonstrate in this simulation study that $\hat{\varrho}_{yz}$ is efficient, although the estimator is proposed in the general trade information noise model.

Table 2 shows that our proposed estimator $\hat{\varrho}_{yz}$ have good finite sample performances, and as we expected, the biases, variances and MSE is close to the ones of $\hat{\varrho}_{WM}$ without noise model. However, $\hat{\varrho}_{WM}$ based on i.i.d noise model could provides smaller bias, variances and MSEs via comparing the first column, the second column with the forth column. It would be a good choice if we adapt model (3) with an

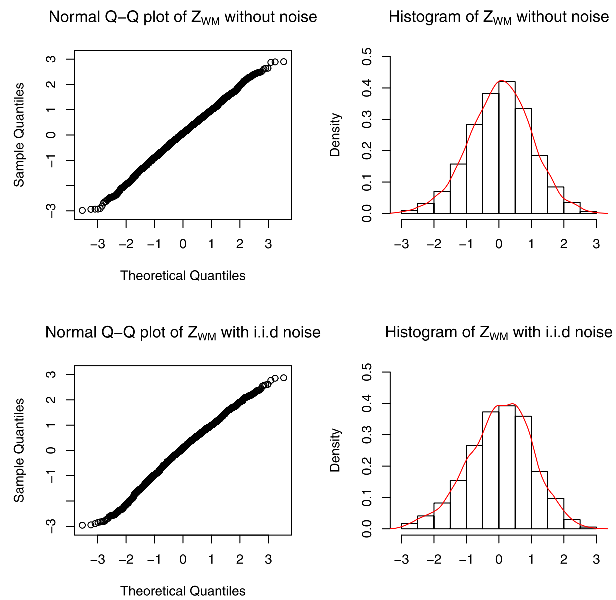

Figure 2. Normal Q-Q plots and histograms of Z_{WM} without noise model and with i.i.d noise model, sample size 23400, replications 2500.

additional i.i.d noise term, which will be also our further research work.

4. EMPIRICAL STUDIES

In the empirical study, we employ the minute-by-minute high-frequency trades data from RESSET (www.resset.cn). We select three stocks: Dong Feng Motor Corp, China Minsheng Bank, and Hisense Electric Corp from January 4th, 2007 to December 28th, 2007. Although the stocks are traded between 9:30 am-11:30 am and 1:00 pm-3:00 pm, we restrict our analysis to the time interval at 9:36 am-11:30 am and 1:00 pm-2:55 pm from Monday to Friday. The reason is that a great number of empirical studies show increased return volatility and trading volume at the open time and close time of the stock market. A 5-min cushion at the open and close may be a good choice in avoiding abnormal activities in the stock market. Therefore, the number of all analysed data is 53360 (232 days) for Dong Feng Motor Corp, 51980 (226 days) for China Minsheng Bank, and 45080 (196 days) for Hisense Electric Corp.

At RESSET database, "Trdirec" is used to classify whether a trade is buyer-initiated or seller-initiated. If "Trdirec" is "B", it is marked 1, if "Trdirec" is "S", it is marked -1, and if "Trdirec" is "F", it is marked 0. This classification scheme may be different from [10]), but it is reasonable. In additional, the number of "F" is ignorable relative to the total data so that mark 0 has no too much effect on analysing the data. Therefore, we describe the trade type $I_{b/s}(t_{n,j})$ at time $t_{n,j}$ by "Trdirec" in the empirical studies.

Table 1. The summary statistics do exhibit the target normality

	Mean	Q_1	Q_2	Q_3
$Z_{yz}(g_1)$	0.0459187	−0.58499171	0.0648180	0.6894439
$Z_{yz}(g_2)$	0.04570912	−0.58373175	0.06959636	0.68738653
Z_{WM} (without noise)	0.04573571	−0.58373175	0.06959636	0.68738653
$Z_{WM}(i.i.d)$	0.02767645	−0.60286276	0.08515147	0.72933781
Standard Normal Distribution	0	−0.674	0	0.674

Table 2. The Variances, Biases, Mean Square Errors for $\hat{\varrho}_{yz}$ and $\hat{\varrho}_{WM}$

$(\times 10^{-13})$	$\hat{\varrho}_{yz}(g_1)$	$\hat{\varrho}_{yz}(g_2)$	$\hat{\varrho}_{WM}(no)$	$\hat{\varrho}_{WM}(i.i.d)$
Variances	3.748044	3.749624	3.749545	2.59574
Biases	0.03697918	0.03660308	0.036636	0.008137843
Mean Square Errors (MSE)	3.785023	3.786227	3.786181	2.603878

Note that all prices data should be transformed to be log-price data. In order to deal with the price data conveniently, all log-prices data could be also multiplied to be $10\times\ln P_{n,d}$, $n=1,\dots,N, d=1,\dots,230$, $P_{n,d}$ is the dth 1-min stock price at nth trading day.

There are many models involved in volatility forecasting studies such as long memory ARFIMA model, GARCH models and stochastic models (see [20]). However, we do not try to find the best volatility forecasting model here. In order to investigate the volatility prediction conveniently, we use the following simple model.

(20)

$$\int_{t_i}^{t_{i+1}}\sigma_t^2dt-\int_{t_{i-1}}^{t_i}\sigma_t^2dt=\alpha_0+\alpha_1\left(\int_{t_{i-1}}^{t_i}\sigma_t^2dt-\int_{t_{i-2}}^{t_{i-1}}\sigma_t^2dt\right)$$
$$+\alpha_2\left(\int_{t_{i-2}}^{t_{i-1}}\sigma_t^2dt-\int_{t_{i-3}}^{t_{i-2}}\sigma_t^2dt\right)$$
$$+\alpha_3\Delta X_{t_i-}^2$$
$$+\alpha_4\int_{t_{i-1}}^{t_i}2b_tdt\times\Delta X_{t_i}+\epsilon_i.$$

- t_i denotes the ith day, and ΔX_{t_i-} denotes the overnight log return.
- The integrated volatility $\int_{t_i}^{t_{i+1}}\sigma_t^2dt$ describe the daily volatility here.
- $\int_{t_i}^{t_{i+1}}\sigma_t^2dt-\int_{t_{i-1}}^{t_i}\sigma_t^2dt$ describes the variability of daily volatility.
- $\int_{t_{i-1}}^{t_i}2b_tdt$ can be estimated by the leverage effect $\hat{\varrho}_{yz}$ by setting $F(x)=\frac{1}{2}\log(x)$.
- $\Delta X_{t_i}=X_{t_i}-X_{t_{i-1}}$ and ϵ_i is Gauss noise.

We will examine whether the leverage effect has effect on volatility prediction via the proposed estimator $\hat{\varrho}_{yz}$. Noted "$***$" represents that the parameter is significant under 0.001 confident level, "$**$" represents that the parameter is significant under 0.01 confident level, "$*$" represents that the parameter is significant under 0.05 confident level, and "." represents the parameter is significant under 0.1 confident level. All results could be summarized in the following Table 3–Table 8.

Table 3. Volatility prediction results for Dong Feng Motor Corp under g_1

	Estimate	Std. Error	t value	Pr(>|t|)	
α_0	0.0038	0.0052	0.74	0.4595	
α_1	−0.2101	0.0778	−2.70	0.0075	**
α_2	−0.1326	0.0619	−2.14	0.0331	*
α_3	−0.0121	0.0511	−0.24	0.8134	
α_4	0.0426	0.0062	6.85	0.0000	***

Table 4. Volatility prediction results for Dong Feng Motor Corp under g_2

	Estimate	Std. Error	t value	Pr(>|t|)	
α_0	0.5881	2.8824	0.20	0.8385	
α_1	−0.5997	0.0628	−9.55	0.0000	***
α_2	−0.3211	0.0629	−5.11	0.0000	***
α_3	−43.2171	28.3697	−1.52	0.1291	
α_4	−5.4679	2.7020	−2.02	0.0442	*

Table 5. Volatility prediction results for China Minsheng Bank under g_1

	Estimate	Std. Error	t value	Pr(>|t|)	
α_0	0.0023	0.0033	0.71	0.4758	
α_1	−0.2875	0.0758	−3.79	0.0002	***
α_2	−0.1618	0.0647	−2.50	0.0131	*
α_3	−0.0100	0.0185	−0.54	0.5879	
α_4	0.0593	0.0116	5.09	0.0000	***

These Tables display the following common features:

(i) According to Table 3 and Table 4, all P-values of α_1, α_2 and α_4 are less than 0.05, which show that variabilities of

Table 6. Volatility prediction results for China Minsheng Bank under g_2

	Estimate	Std. Error	t value	Pr(>|t|)	
α_0	−0.4730	2.4274	−0.19	0.8457	
α_1	−0.6800	0.0637	−10.67	0.0000	***
α_2	−0.3392	0.0642	−5.28	0.0000	***
α_3	−0.7133	13.9176	−0.05	0.9592	
α_4	−12.0321	7.0924	−1.70	0.0912	.

Table 7. Volatility prediction results for Hisense Electric Corp under g_1

	Estimate	Std. Error	t value	Pr(>|t|)	
α_0	0.0004	0.0039	0.10	0.9188	
α_1	−0.3291	0.0706	−4.66	0.0000	***
α_2	−0.2611	0.0707	−3.69	0.0003	***
α_3	−0.0196	0.0294	−0.67	0.5048	
α_4	−0.0050	0.0140	−0.36	0.7195	

Table 8. Volatility prediction results for Hisense Electric Corp under g_2

	Estimate	Std. Error	t value	Pr(>|t|)	
α_0	1.2822	2.8081	0.46	0.6485	
α_1	−0.6855	0.0678	−10.10	0.0000	***
α_2	−0.2993	0.0687	−4.36	0.0000	***
α_3	13.8445	21.6176	0.64	0.5227	
α_4	26.3286	10.2006	2.58	0.0106	*

the ahead 1 daily volatility, the variabilities of the ahead 2 daily volatility and leverage effect have significant effect on the next daily volatility under 0.05 confident level. On the other hand, P-values of α_3 are more than 0.1 in both Table 3 and Table 4, which describe that the overnight return does not play an important role in forecasting volatility.

(ii) Compared Table 5 with Table 6, P-values of α_1, α_2 and α_4 are less than 0.1, and P-values of α_3 are more than 0.1 for China Minsheng Bank. The results suggest that the leverage effect rather than the overnight return has an effect on the volatility forecasting.

(iii) From Table 7 and Table 8, P-values of α_1, α_2 are 0, and P-values of α_3 are more than 0.1 for Hisense Electric Corp. These results further examine that the variabilities of the ahead 1 daily volatility, the variabilities of the ahead 2 daily volatility have an impact on volatility prediction, but the overnight return has ignorable effect on volatility prediction. The leverage effect has an effect on volatility prediction based nonlinear function g_2, but no effect based on linear function g_1, because P-value of α_4 is 0.7195 in Table 7, and 0.0106 in Table 8. The reason is that the chosen forecasting volatility mode (20) is not good, which is reflected that Adjusted R-squared for the model is only 0.1148 based on linear function g_1.

These features suggest that the proposed estimator $\hat{\varrho}_{yz}$ could perform the sufficient application in high-frequency financial market. Specially, $\hat{\varrho}_{yz}$ check that the leverage effect plays an important roles in forecasting volatility.

5. CONCLUSION

In this paper, we propose a new estimator of leverage effect when the microstructure noise model includes the trading information. We have shown the proposed estimator $\hat{\varrho}_{yz}$ is capable of achieving a convergence rate $n^{1/4}$, and the convergence rate is faster than the one of $\hat{\varrho}_{WM}$ when the microstructure noise is independent and identically distributed (i.i.d.). Through a simulation study, we have demonstrated proposed estimators share good asymptotic behaviors, and the QQ-Plots, Histogram plots and quartiles perform the efficient normal properties compared with $\hat{\varrho}_{WM}$. And the variances, biases, and mean square errors of $\hat{\varrho}_{yz}$ also show the good performances based on both linear noise function g_1 and nonlinear noise function g_2. Finally, we have also demonstrated the ease of implementation for our proposed estimator $\hat{\varrho}_{yz}$ through an empirical study.

In addition, all our work is studied in the setting where the microstructure noise model is the parameter function of the trading information, however, there is rich microstructure information in high-frequency financial market, we would continue to explore the leverage effect estimator in the more complex microstructure noise model.

APPENDIX A

A.1 Proof of Theorem 2.1

We first proof equation (10) in Theorem 2.1(A1), recall that

$$\begin{aligned}\widehat{X}_{\tau_{n,i}} &= X_{t_{n,j}} + g\left(Z_{t_{n,j+1}};\theta_0\right) - g\left(Z_{t_{n,j+1}};\hat{\theta}\right),\\ \widehat{X}_{t_{n,j+1}} - \widehat{X}_{t_{n,j}} &= X_{t_{n,j+1}} - X_{t_{n,j}}\\ &\quad + \Delta g\left(Z_{t_{n,j+1}};\theta_0\right) - \Delta g\left(Z_{t_{n,j+1}};\hat{\theta}\right).\end{aligned}$$

Then, by Assumption 2(v), we have

$$\begin{aligned}\left|\Delta g(Z_{t_{n,i+1}};\hat{\theta}) - \Delta g(Z_{t_{n,i+1}};\theta_0)\right| &\leq \left(L_0(Z_{t_{n,i+1}}) + L_0(Z_{t_{n,i}})\right)\\ &\quad \cdot \left|\hat{\theta} - \theta_0\right|,\end{aligned}$$

where $L_0(Z_{t_{n,i}})$ and $L_0(Z_{t_{n,i+1}})$ are locally bounded.

Under Assumption 1-2, as $n \to \infty$, $\hat{\theta} - \theta_0 = O_p\left(n^{-1}\right)$ (see: [21]), therefore

$$\widehat{X}_{t_{n,j+1}} - \widehat{X}_{t_{n,j}} = X_{t_{n,j+1}} - X_{t_{n,j}} + O_p\left(n^{-1}\right).$$

Equation (10) is proved.

We next proof Equation (11) in Theorem 2.1(A1). Due to

$$\begin{aligned}\widehat{\sigma}^2_{\tau_{n,i}} =&\frac{1}{M_n \times \Delta t} \sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} \left(\widehat{X}_{t_{n,j+1}} - \widehat{X}_{t_{n,j}}\right)^2 \\ =&\frac{1}{M_n \times \Delta t} \sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} (\Delta X_{t_{n,j+1}} \\ &- \left(\Delta g\left(Z_{t_{n,j+1}};\hat{\theta}\right) - \Delta g\left(Z_{t_{n,j+1}};\theta_0\right)\right)\Big)^2 \\ =&\frac{1}{M_n \times \Delta t} \sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} \Delta X^2_{t_{n,j+1}} \\ &+\frac{1}{M_n \times \Delta t} \sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} \Big(\Delta g(Z_{t_{n,j+1}};\hat{\theta}) \\ &- \Delta g(Z_{t_{n,j+1}};\theta_0)\Big)^2 \\ &-\frac{2}{M_n \times \Delta t} \sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} \Big(\Delta g\left(Z_{t_{n,j+1}};\hat{\theta}\right) \\ &- \Delta g\left(Z_{t_{n,j+1}};\theta_0\right)\Big) \cdot \Delta X_{t_{n,j+1}} \\ =&:I_1 + I_2 - I_3.\end{aligned}$$

Note that term I_1 is just $\widehat{\sigma}^2_{\tau_{n,i}}$ in Eq. (5) of [24]. We will show that $I_2 - I_3 = o_p\left(n^{-1/4}\right)$.

By Assumption 2(v), we have

$$I_2 = \frac{M_n}{M_n \times \Delta t} O_p\left(n^{-2}\right) = O_p\left(n^{-1}\right).$$

It remains to show that $I_3 = o_p\left(n^{-1/4}\right)$. Note that since $\widehat{\theta}$ depends on the whole process (X_t, Z_{t_k}), then the term I_3 is not a martingale and hence the Burkholder-Davis-Gundy (BDG) inequality is not applicable. To overcome the issue, we set

$$\begin{aligned}F_n(\theta) =&\frac{1}{\sqrt{M_n \times \Delta t}} \sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} \Big(\Delta g\left(Z_{t_{n,j+1}};\theta\right) \\ &- \Delta g\left(Z_{t_{n,j+1}};\theta_0\right)\Big) \Delta X_{t_{n,j+1}}.\end{aligned}$$

Further define for any function $\phi : \mathcal{N}(\theta_0) \to \mathbf{R}$, the modulus of continuity as follows

$$\omega(\phi, h) := \sup\left\{|\phi(\theta_1) - \phi(\theta_2)| : |\theta_1 - \theta_2| \le h\right\},$$

for any $h \ge 0$.

Following the argument in the proof of Corollary 14.9 in [18] one obtains that for any $\ell > p/2$ and any $m \in \mathbf{N}$, we obtain

$$E\left(\omega\left(F_n, 2^{-m}\right)\right)^{2\ell} \le C 2^{-m(2\ell-p)}.$$

Taking m such that $2^{-m} \ge K/n > 2^{-m-1}$ yields

$$E\left(\omega\left(F_n, K/n\right)\right)^{2\ell} \le C(K/n)^{2\ell-p} = O\left(n^{-(2\ell-p)}\right).$$

Therefore, we could see that for all n such that $B(\theta_0, K/n) = (\{\theta : |\theta - \theta_0| \le K/n\} \subseteq \mathcal{N}(\theta_0))$, for all $\ell \in \mathbf{N}$ large enough,

$$E\left(\omega\left(F_n, K/n\right)\right)^{2\ell} = o\left(n^{-\ell}\right),$$

which clearly imply that for any $K > 0$

$$\begin{aligned}\sup_{|\theta-\theta_0|\le K/n} \Bigg| \frac{1}{\sqrt{M_n \times \Delta t}} \sum_{t_{n,j}\in(\tau_{n,i},\tau_{n,i+1}]} (\Delta g(Z_{t_{n,j+1}};\theta) \\ - \Delta g(Z_{t_{n,j+1}};\theta_0))\Delta X_{t_{n,j+1}}\Bigg| = o_p\left(n^{-1/2}\right).\end{aligned}$$

Therefore, $I_2 - I_3 = O_p(n^{-1}) - \frac{1}{\sqrt{M_n \times \Delta t}} o_p\left(n^{-1/2}\right) = o_p\left(n^{-1/4}\right)$. Equation (11) holds.

Then we proof Theorem 2.1(A2).

Using Assumption 2(v), it is easy to show that

$$\begin{aligned}&\left|\Delta g\left(Z_{\tau_{n,i+1}};\theta_0\right) - \Delta g\left(Z_{\tau_{n,i+1}};\hat{\theta}\right)\right| \\ \le& \left(L_0\left(Z_{\tau_{n,i+1}}\right) + L_0\left(Z_{\tau_{n,i}}\right)\right) \\ &\cdot \left|\theta_0 - \hat{\theta}\right|.\end{aligned}$$

By equation (10), we have

$$\begin{aligned}\hat{\varrho}_{yz} =&2\sum_{i=0}^{K_n-2} \left(\widehat{X}_{\tau_{n,i+1}} - \widehat{X}_{\tau_{n,i}}\right)\left(F\left(\widetilde{\sigma^2_{\tau n,i+1}}\right) - F\left(\widetilde{\sigma^2_{\tau n,i}}\right)\right) \\ =&2\sum_{i=0}^{K_n-2} \left(\widehat{X}_{\tau_{n,i+1}} - \widehat{X}_{\tau_{n,i}}\right)\Big(F\left(\widetilde{\sigma^2_{\tau n,i+1}}\right) - F\left(\widehat{\sigma^2_{\tau n,i+1}}\right) \\ &+ F\left(\widehat{\sigma^2_{\tau n,i+1}}\right) - F\left(\widehat{\sigma^2_{\tau n,i}}\right) + F\left(\widehat{\sigma^2_{\tau n,i}}\right) - F\left(\widetilde{\sigma^2_{\tau n,i}}\right)\Big) \\ =&2\left\{\sum_{i=0}^{K_n-2} \left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right)\right. \\ &\left.\cdot\left(F\left(\widehat{\sigma^2_{\tau n,i+1}}\right) - F\left(\widehat{\sigma^2_{\tau n,i}}\right)\right)\right\} \\ &+2\left\{\sum_{i=0}^{K_n-2} \left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right) F'(\sigma^2_{c1})\right. \\ &\left.\cdot\left(\widetilde{\sigma^2_{\tau n,i+1}} - \widehat{\sigma^2_{\tau n,i+1}}\right)\right\} \\ &-2\left\{\sum_{i=0}^{K_n-2} \left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right) F'(\sigma^2_{c2})\right. \\ &\left.\cdot\left(\widetilde{\sigma^2_{\tau n,i}} - \widehat{\sigma^2_{\tau n,i}}\right)\right\} \\ =&:I_4 + I_5 - I_6.\end{aligned}$$

Note that σ^2_{c1} is a constant between $\widehat{\sigma^2_{\tau n,i+1}}$ and $\widetilde{\sigma^2_{\tau n,i+1}}$. σ^2_{c2} is a constant between $\widehat{\sigma^2_{\tau n,i}}$ and $\widetilde{\sigma^2_{\tau n,i}}$. Because $F(\cdot)$ is twice continuously differentiable, $F'(\cdot)$ is bounded in closed intervals. It means that there exist some constant $M \ge 0$, such that

$$|F'(\cdot)| \le M.$$

We have proved $\widetilde{\sigma}^2_{\tau n,i} = \widehat{\sigma}^2_{\tau n,i} + o_p\left(n^{-1/4}\right)$, then

$$
\begin{aligned}
&|I_5 - I_6| \\
=&2\left|\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right)\right. \\
&\cdot F'(\sigma_{c1}^2) o_p\left(n^{-1/4}\right) \\
-&\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right) \\
&\left.\cdot F'(\sigma_{c2}^2) o_p\left(n^{-1/4}\right)\right| \\
\leq&2\left|\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right)\right. \\
&\left.\cdot \left\|F'(\sigma_{c1}^2)\right\| o_p\left(n^{-1/4}\right)\right| \\
+&2\left|\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right)\right. \\
&\left.\cdot \left\|F'(\sigma_{c2}^2)\right\| o_p\left(n^{-1/4}\right)\right| \\
\leq&4\left|\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right)\right| \cdot M \cdot o_p\left(n^{-1/4}\right) \\
=&4M \cdot o_p\left(n^{-1/4}\right)\left|X_{\tau_{n,K_n-1}} - X_{\tau_{n,0}} + O_p\left(n^{-1/2}\right)\right|.
\end{aligned}
$$

The last equality is due to $K_n = O_p\left(n^{1/2}\right)$. Moreover, $X_{\tau_{n,i+1}}$ comes from Eq. (4), is locally bounded. For some constant $M^{'} \geq 0$, we can get

$$
|I_5 - I_6| \leq 4M^{'} \cdot o_p\left(n^{-1/4}\right),
$$

when $n \to \infty$, $I_5 - I_6 = 0$.

Thus

$$
\begin{aligned}
\hat{\varrho}_{yz} =&2\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right) \\
&\cdot\left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right) \\
=&2\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right)\left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right) \\
+&2\sum_{i=0}^{K_n-2} O_p\left(n^{-1}\right)\left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right) \\
=&2\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right)\left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right) \\
+&O_p\left(n^{-1/2}\right).
\end{aligned}
$$

As $n \to \infty$, the proof of Theorem 2.1(A2) is completed.

A.2 Proof of Theorem 2.2

By Theorem 2.1(A2) in this paper. As $n \to \infty$, $\hat{\varrho}_{yz}$ and $\hat{\varrho}_{WM}$ in the absence of microstructure noise are very close so that we can derive the asymptotic properties of $\hat{\varrho}_{yz}$ by $\hat{\varrho}_{WM}$ in the absence of microstructure noise.

Theorem 2.2 in this paper can be easily obtain via the proof of Theorem 2.1 in [24]'s paper,

A.3 Proof of Theorem 2.3

In the former proof, we have known that $\widehat{X}_{\tau_{n,i+1}} - \widehat{X}_{\tau_{n,i}} = X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)$, and $\widetilde{\sigma}^2_{\tau_{n,i}} = \widehat{\sigma}^2_{\tau_{n,i}} + o_p\left(n^{-1/4}\right)$. Therefore, it is easy to obtain that

$$
\begin{aligned}
H_n^1 =&2n^{\frac{1}{2}}\sum_{i=0}^{K_n-2}\left(\widehat{X}_{\tau_{n,i+1}} - \widehat{X}_{\tau_{n,i}}\right)^2 \\
&\cdot\left(F\left(\widetilde{\sigma}^2_{\tau n,i+1}\right) - F\left(\widetilde{\sigma}^2_{\tau n,i}\right)\right)^2 \\
=&2n^{\frac{1}{2}}\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}} + O_p\left(n^{-1}\right)\right)^2 \\
&\cdot\left(F\left(\widetilde{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i+1}\right)\right. \\
&+ F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right) \\
&\left.+ F\left(\widehat{\sigma}^2_{\tau n,i}\right) - F\left(\widetilde{\sigma}^2_{\tau n,i}\right)\right)^2.
\end{aligned}
$$

By the simple deduction, we could obtain the following equation

$$
\begin{aligned}
H_n^1 =&2n^{\frac{1}{2}}\sum_{i=0}^{K_n-2}\left(X_{\tau_{n,i+1}} - X_{\tau_{n,i}}\right)^2 \\
&\cdot\left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right)^2 \\
&+O_p\left(n^{-1/2}\right).
\end{aligned}
$$

Noted the first term in H_n^1 is G_n^1 in [24]'s paper. Similarly, we have the following equation

$$
\begin{aligned}
H_n^2 =&2n^{\frac{1}{2}} M_n \Delta t \sum_{i=0}^{K_n-2}\widehat{\sigma}^2_{\tau n,i}\left(F\left(\widehat{\sigma}^2_{\tau n,i+1}\right) - F\left(\widehat{\sigma}^2_{\tau n,i}\right)\right)^2 \\
&+o_p\left(n^{-1/4}\right).
\end{aligned}
$$

As $n \to \infty$, we may obtain $H_n^2 = G_n^2$, G_n^2 could be seen in [24]'s paper. Therefore, Theorem 2.3 can be deduced.

Received 25 September 2018

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Huiling Yuan
School of Data Science
City University of Hong Kong
Hong Kong
China
E-mail address: yuanhuiling2013@163.com

Yan Mu
School of Economics
Nanjing University of Finance and Economics
China
E-mail address: muyan1111@126.com

Yong Zhou
Key Laboratory of Advanced Theory and Application in Statistics and Data Science
Ministry of Education
Academy of Statistics and Interdisciplinary Sciences
East China Normal University
China
E-mail address: yzhou@amss.ac.cn