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Further study on testing the equality of response rates under Dallal's model

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Paired binary data naturally arises when paired body parts are investigated in clinical trials. In this paper, we will further study whether the response rates of g ($g \geq 2$) groups are equal under Dallal's model and propose eight test statistics (T_L^a, T_W^a, T_{SC}^a, T_R^a, T_L^b, T_W^b, T_{SC}^b and T_R^b). Some expressions of these tests are derived. The simulation results show that likelihood ratio and Wald-type tests are not robust with respect to empirical type I error rates (TIEs). The score and Ronser-type tests can produce satisfactory TIEs and power, and therefore are recommended. A real example is given to illustrate the proposed methods.

Keywords and phrases: Paired binary data, Dallal's model, Likelihood ratio test, Wald-type test, Score test, Ronser-type test.

1. INTRODUCTION

In the clinical trials of studying the diseases of body matching parts, paired binary data is usually collected according to multiple groups. For example, patients in ophthalmic studies are randomly assigned to $g(g \geq 2)$ treatment groups to check the patients' left and right eyes. The recorded results are often no, unilateral, or bilateral response(s). In this case, the two eyes of the same patient may be correlated. If their correlation is not considered, it will lead to the misleading results. For this reason, some probability models have been proposed to study such correlated paired data.

Ronser [1] proposed a possible model; that is, for one patient, the probability of the other eye's response in the case of one eye's response is R times of the unconditional probability. To clarify the idea, suppose that N patients are randomly assigned to g groups in an ophthalmic study. Let $m_i (i = 1, 2, \ldots, g)$ be the number of patients in the i-th group. Suppose $Z_{ijk} = 1$ if the k-th eye of the j-th patient has response in the i-th group for $i = 1, \ldots, g$, $j = 1, \ldots, m_i$, and $k = 1, 2$. Otherwise, $Z_{ijk} = 0$. For a patient, if the probability of an eye's response is π_i; that is, $\Pr(Z_{ijk} = 1) = \pi_i$, then $\Pr(Z_{ijk} = 1 | Z_{ij(3-k)} = 1) = R\pi_i$, where R is equal in g groups. Under Ronser's model, Tang et al. [2] proposed the exact and approximate unconditional procedures for testing the equality of proportions in such correlated data. Further, Tang et al. [3] provided eight procedures via the asymptotic and approximate unconditional methods. However, they only considered two groups. Ma [4] extended their results to $g(g \geq 2)$ groups, and proposed three statistics to test homogeneity of proportions.

Another widely used model assumes that g groups have a common intra-correlation coefficient (ICC) proposed by Donner [5], $\mathrm{ICC}(Z_{ijk}, Z_{ij(3-k)}) = \rho$. Pei et al. [6] considered the Pearson correlation coefficient $\mathrm{PCC}(Z_{ijk}, Z_{ij(3-k)}) = \rho$. Thus, we need to test whether the correlation coefficient is a constant before using Donner's model. Another important problem is to test if the correlation coefficient depends on the group component. For this reason, Pei et al. [6] proposed four kinds of statistical tests: likelihood ratio, Wald-type, Ronser-type and score tests for two groups. Liu and Ma [7] provided three statistics to test the equality of correlation coefficient for $g(g \geq 2)$ groups. Ma and Liu [8] considered the equality of response rates on the basis of common correlation coefficients in g groups.

However, Ronser's model may give a poor fit if the characteristic is almost certain to occur bilaterally with widely varying group-specific prevalence [7]. Dallal [9] proposed an alternative model assuming that the probability of the other eye's response in the case of one eye's response is γ_i for $g(g \geq 2)$ groups:

$$(1)\quad \Pr(Z_{ijk} = 1 | Z_{ij(3-k)} = 1) = \gamma_i, \quad i = 1, \ldots, g.$$

He derived likelihood ratio statistic to test the equality of response rates. For two groups, Tang et al. [10] used goodness-of-fit tests such as likelihood ratio, Pearson chi-square and an adjusted chi-square statistics to discuss model selections including Dallal's model with $\gamma_i = \gamma (i = 1, 2)$. They evaluated the performance of these statistics in terms of type I error rates (TIEs) and power. Liu and Ma [11] extended these tests to g groups ($g \geq 2$) for $\gamma_1 = \cdots = \gamma_g = \gamma$. MIan and Chen [12] proposed three objective Bayesian methods to study the risk difference, risk ratio, and the odds ratio for two groups with $\gamma_i = \gamma (i = 1, 2)$.

For Dallal's model, we observe that some of methods are inflated or conservative in TIEs for testing the equality of response rates for $g(g \geq 2)$ groups such as Pearson

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chi-square and adjusted chi-square statistics [11]. Score test performs better in terms of TIEs and power than likelihood ratio and Wald-type test in Ronser's and Donner's model [4, 7, 8], but it has not been studied in Dallal's model. Thus, this paper will further study statistical inference on Dallal's model and proposed eight test statistics according to two cases: (i) the parameters γ_i are different for each group, and (ii) all γ_i's are equal to a constant, that is, $\gamma_i = \gamma$ for $i = 1, 2, \ldots, g$. The rest of the paper is organized as follows. In Section 2, we review data structure, probability distribution, and establish two hypotheses under Dallal's model. In Section 3, the maximum likelihood estimations (MLEs) of unknown parameters are derived based on the different hypotheses. Eight test statistics are proposed in Section 4. In Section 5, simulation experiments are conducted to test the performance of these test statistics in terms of TIEs and power. In Section 6, a real example is provided to illustrate the proposed tests. A brief conclusion is given in Section 7.

2. PRELIMINARIES

Suppose N patients are randomly allocated to $g (g \geq 2)$ independent treatment groups. Let m_{li} be the number of patients in the i-th group with exact l response(s), $m_i (i = 1, 2, \ldots, g)$ be the number of patients in the i-th group, and $S_l (l = 0, 1, 2)$ be the number of patients with l response(s). Thus, $m_i = m_{0i} + m_{1i} + m_{2i}$, $S_l = \sum_{i=1}^{g} m_{li}$, and $N = \sum_{i=1}^{g} m_i = \sum_{l=0}^{2} S_l$. The frequencies of patients with l response(s) is shown in Table 1.

Let $p_{li} (l = 0, 1, 2, i = 1, \ldots, g)$ be the probabilities of the corresponding cells in Table 1. From the conditional probabilities (1), the response probabilities p_{li} for none, one, or both eyes are $p_{0i} = 1 - 2\pi_i + \pi_i\gamma_i$, $p_{1i} = 2\pi_i(1 - \gamma_i)$, and $p_{2i} = \pi_i\gamma_i$ for $i = 1, 2, \ldots, g$, where $p_{0i} + p_{1i} + p_{2i} = 1$. Denote $a = \max\{\pi_i, i = 1, \ldots, g\}$. Thus, $0 \leq \gamma_i \leq 1$ if $a \leq 1/2$, and $2 - 1/a \leq \gamma_i \leq 1$ if $a > 1/2$. Let $\mathbf{m}_i = (m_{0i}, m_{1i}, m_{2i})$. Obviously, the probability density function of the i-th group are written as follows:

$$f(\boldsymbol{m}_i|p_{0i}, p_{1i}, p_{2i}) = \frac{m_i!}{m_{0i}!m_{1i}!m_{2i}!} p_{0i}^{m_{0i}} p_{1i}^{m_{1i}} p_{2i}^{m_{2i}}$$

for $i = 1, \ldots, g$.

Table 1. The correlated data structure in g groups

Response (l)	Group (i)					Total
	1	2	3	$\cdots$	g	
0	m_{01}	m_{02}	m_{03}	$\cdots$	m_{0g}	S_0
1	m_{11}	m_{12}	m_{13}	$\cdots$	m_{1g}	S_1
2	m_{21}	m_{22}	m_{23}	$\cdots$	m_{2g}	S_2
Total	m_1	m_2	m_3	$\cdots$	m_g	N

Denote $\boldsymbol{m} = (m_{01}, m_{11}, m_{21}, \ldots, m_{0g}, m_{1g}, m_{2g})$, the likelihood function is expressed by

$$\begin{aligned} L(\boldsymbol{\pi}, \boldsymbol{\gamma}|\boldsymbol{m}) &= \prod_{i=1}^{g} f(\boldsymbol{m}_i|p_{0i}, p_{1i}, p_{2i}) \\ &= \prod_{i=1}^{g} \frac{m_i!}{m_{0i}!m_{1i}!m_{2i}!} p_{0i}^{m_{0i}} p_{1i}^{m_{1i}} p_{2i}^{m_{2i}}, \end{aligned}$$

where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_g)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_g)$. Thus, the log-likelihood can be obtained by

$$\begin{aligned} l(\boldsymbol{\pi}, \boldsymbol{\gamma}|\boldsymbol{m}) = \sum_{i=1}^{g} \Big[& m_{0i} \log(1 - 2\pi_i + \pi_i\gamma_i) \\ & + m_{1i} \log(2\pi_i(1 - \gamma_i)) + m_{2i} \log(\pi_i\gamma_i) \Big] \\ & + \log C, \end{aligned} \tag{2}$$

where $C = \prod_{i=1}^{g} \frac{m_i!}{m_{0i}!m_{1i}!m_{2i}!}$ is a constant.

Li and Ma [13] proposed several asymptotic methods for testing the equality of correlations for multiple bilateral data under Dallal's model. Based on their work, we are interested to study the prevalence of eye disease among patients with different treatments according to two cases as follows:

Case (i): If the correlations of g groups are not equal, our interest is to test the hypotheses: $H_{01} : \pi_1 = \cdots = \pi_g \triangleq \pi$, $\gamma_c \neq \gamma_d$ for at least one pair (c, d), versus $H_{1a} : \pi_r \neq \pi_s$ for at least one pair (r, s), $\gamma_c \neq \gamma_d$ for at least one pair (c, d).

Case (ii): If the correlations of g groups are equal, another hypotheses are given as $H_{02} : \pi_1 = \cdots = \pi_g \triangleq \pi$, $\gamma_1 = \cdots = \gamma_g \triangleq \gamma$, versus $H_{1b} : \pi_r \neq \pi_s$ for at least one pair (r, s), $\gamma_1 = \cdots = \gamma_g \triangleq \gamma$.

3. PARAMETER ESTIMATION

3.1 MLEs under H_{1a} and H_{1b}

For case (i), there exist $2g$ unknown parameters π_i and $\gamma_i (i = 1, \ldots, g)$. Let $\hat{\pi}_i^a$ and $\hat{\gamma}_i^a$ be the MLEs of π_i and γ_i under the alternative hypothesis H_{1a}, respectively. Differentiating l with respect to π_i and $\gamma_i (i = 1, \ldots, g)$ yields

$$\begin{cases} \dfrac{\partial l}{\partial \pi_i} = \dfrac{m_{0i}(\gamma_i - 2)}{1 - 2\pi_i + \pi_i\gamma_i} + \dfrac{m_{1i} + m_{2i}}{\pi_i} = 0, \\ \dfrac{\partial l}{\partial \gamma_i} = \dfrac{m_{0i}\pi_i}{1 - 2\pi_i + \pi_i\gamma_i} - \dfrac{m_{1i}}{1 - \gamma_i} + \dfrac{m_{2i}}{\gamma_i} = 0. \end{cases}$$

The MLEs $\hat{\pi}_i^a$ and $\hat{\gamma}_i^a$ are the solutions of the above equation as follows:

$$\begin{aligned} \hat{\pi}_i^a &= \frac{m_{1i} + 2m_{2i}}{2m_i}, \\ \hat{\gamma}_i^a &= \frac{2m_{2i}}{m_{1i} + 2m_{2i}} \text{ for } i = 1, \ldots, g. \end{aligned} \tag{3}$$

For case (ii), we have $\gamma_1 = \cdots = \gamma_g = \gamma$. Let $\hat{\pi}_i^b$ and $\hat{\gamma}^b$ be the MLEs of $\pi_i (i = 1, \ldots, g)$ and γ under the alternative hypothesis H_{1b}. Then, (2) is simplified as

$$l_1(\boldsymbol{\pi}, \gamma | \boldsymbol{m}) = \sum_{i=1}^{g} [m_{0i} \log(1 - 2\pi_i + \pi_i \gamma) + m_{1i} \log(2\pi_i(1-\gamma)) + m_{2i} \log(\pi_i \gamma)].$$

Similarly, differentiating l_1 with respect to $\pi_i (i = 1, \ldots, g)$ and γ, we obtain the following equations

$$\begin{cases} \dfrac{\partial l_1}{\partial \pi_i} = \dfrac{m_{0i}(\gamma - 2)}{1 - 2\pi_i + \pi_i \gamma} + \dfrac{m_{1i} + m_{2i}}{\pi_i} = 0, \\ \dfrac{\partial l_1}{\partial \gamma} = \displaystyle\sum_{i=1}^{g} \left[\dfrac{m_{0i} \pi_i}{1 - 2\pi_i + \pi_i \gamma} - \dfrac{m_{1i}}{1 - \gamma} + \dfrac{m_{2i}}{\gamma} \right] = 0. \end{cases}$$

Further, we solve the equations and obtain the MLEs $\hat{\pi}_i^b$ and $\hat{\gamma}^b$ as follows

$$\begin{aligned} \hat{\pi}_i^b &= \frac{(m_{1i} + m_{2i})(S_1 + 2S_2)}{2m_i(S_1 + S_2)} \text{ for } i = 1, \ldots, g, \\ \hat{\gamma}^b &= \frac{2S_2}{S_1 + 2S_2}, \end{aligned} \tag{4}$$

where $S_1 = \sum_{i=1}^{g} m_{1i}$ and $S_2 = \sum_{i=1}^{g} m_{2i}$.

3.2 MLEs under H_{01} and H_{02}

Under the null hypothesis H_{01}, that is, $\pi_1 = \cdots = \pi_g \triangleq \pi$, there are $g + 1$ unknown parameters for case (i). Let $\tilde{\pi}^a$ and $\tilde{\gamma}_i^a$ be the MLEs of π and $\gamma_i (i = 1, \ldots, g)$. The log-likelihood function is

$$l_0^a(\pi, \boldsymbol{\gamma} | \boldsymbol{m}) = \sum_{i=1}^{g} [m_{0i} \log(1 - 2\pi + \pi\gamma_i) + m_{1i} \log(2\pi(1-\gamma_i)) + m_{2i} \log(\pi \gamma_i)].$$

Differentiate l_0^a with respect to π and $\gamma_i (i = 1, \ldots, g)$, and set them equal to zero:

$$\begin{cases} \dfrac{\partial l_0^a}{\partial \pi} = \displaystyle\sum_{i=1}^{g} \left[\dfrac{m_{0i}(\gamma_i - 2)}{1 - 2\pi + \pi\gamma_i} + \dfrac{m_{1i} + m_{2i}}{\pi} \right] = 0, \\ \dfrac{\partial l_0^a}{\partial \gamma_i} = \dfrac{m_{0i}\pi}{1 - 2\pi + \pi\gamma_i} - \dfrac{m_{1i}}{1 - \gamma_i} + \dfrac{m_{2i}}{\gamma_i} = 0. \end{cases} \tag{5}$$

The MLEs $\tilde{\pi}^a$ and $\tilde{\gamma}_i^a$ are the solutions of the above equations. However, there are no closed-form solutions. Next we introduce two-step algorithm to obtain them. The process is described below.

(i) Take the initial values $\pi^{(0)} = (1/g) \sum_{i=1}^{g} \hat{\pi}_i^a$ and $\gamma_i^{(0)} = \hat{\gamma}_i^a (i = 1, \ldots, g)$. The first equation of (5) is simplified as a 2nd-order polynomial

$$m_i \pi \gamma_i^2 - [\pi(m_i + m_{1i} + 2m_{2i}) - (m_{1i} + m_{2i})]\gamma_i + m_{2i}(2\pi - 1) = 0.$$

The $(t+1)$-th update of $\gamma_i^{(t+1)} (i = 1, \ldots, g)$ can be obtained by the real root of the above equation.

(ii) The $(t+1)$-th update $\pi^{(t+1)}$ of π can be given by Newton-Raphson algorithm

$$\pi^{(t+1)} = \pi^{(t)} - \left(\frac{\partial^2 l_0^a}{\partial \pi^2} \right)^{-1} \frac{\partial l_0^a}{\partial \pi} \bigg|_{\gamma_i = \gamma_i^{(t)}, \pi = \pi^{(t)}},$$

where

$$\frac{\partial^2 l_0^a}{\partial \pi^2} = -\sum_{i=1}^{g} \left[\frac{m_{0i}(\gamma_i - 2)^2}{(\pi\gamma_i - 2\pi + 1)^2} + \frac{m_{1i} + m_{2i}}{\pi^2} \right].$$

Repeat the update process until convergence. By the two-step algorithm, we can obtain the MLEs $\tilde{\pi}^a$ and $\tilde{\gamma}_i^a (i = 1, \ldots, g)$.

For case (ii), there are only two unknown parameters π and γ under H_{02}. Let $\tilde{\pi}^b$ and $\tilde{\gamma}^b$ are the MLEs of π and γ under H_{02}. The log-likelihood function satisfies

$$l_0^b(\pi, \gamma | \boldsymbol{m}) = \sum_{i=1}^{g} [m_{0i} \log(1 - 2\pi + \pi\gamma) + m_{1i} \log(2\pi(1-\gamma)) + m_{2i} \log(\pi\gamma)].$$

Set the differentiation of l_0^b with respect to π and γ equal to zero:

$$\begin{cases} \dfrac{\partial l_0^b}{\partial \pi} = \displaystyle\sum_{i=1}^{g} \left[\dfrac{m_{0i}(\gamma - 2)}{1 - 2\pi + \pi\gamma} + \dfrac{m_{1i} + m_{2i}}{\pi} \right] = 0, \\ \dfrac{\partial l_0^b}{\partial \gamma} = \displaystyle\sum_{i=1}^{g} \left[\dfrac{m_{0i}\pi}{1 - 2\pi + \pi\gamma} - \dfrac{m_{1i}}{1 - \gamma} + \dfrac{m_{2i}}{\gamma} \right] = 0. \end{cases}$$

Thus, it is easy to get the MLEs $\tilde{\pi}^b$ and $\tilde{\gamma}^b$ below

$$\tilde{\pi}^b = \frac{S_1 + 2S_2}{2N}, \quad \tilde{\gamma}^b = \frac{2S_2}{S_1 + 2S_2}, \tag{6}$$

where $S_1 = \sum_{i=1}^{g} m_{1i}$ and $S_2 = \sum_{i=1}^{g} m_{2i}$.

4. THE PROPOSED METHODS

4.1 Likelihood ratio tests

Denote $\hat{\boldsymbol{\pi}}^a = (\hat{\pi}_1^a, \ldots, \hat{\pi}_g^a)$, $\hat{\boldsymbol{\gamma}}^a = (\hat{\gamma}_1^a, \ldots, \hat{\gamma}_g^a)$ and $\tilde{\boldsymbol{\gamma}}^a = (\tilde{\gamma}_1^a, \ldots, \tilde{\gamma}_g^a)$, where $\hat{\pi}_i^a$, $\hat{\gamma}_i^a$ and $\tilde{\gamma}_i^a (i = 1, \ldots, g)$ are the MLEs under H_{1a} and H_{01}, respectively. To test the hypothesis H_{01}, the likelihood ratio test can be given by

$$T_L^a = 2[l(\hat{\boldsymbol{\pi}}^a, \hat{\boldsymbol{\gamma}}^a | \boldsymbol{m}) - l_0^a(\tilde{\pi}^a, \tilde{\boldsymbol{\gamma}}^a | \boldsymbol{m})].$$

It can be simplified as

$$T_L^a = 2 \sum_{i=1}^{g} \left[m_{0i} \log \left(\frac{1 - 2\hat{\pi}_i^a + \hat{\pi}_i^a \hat{\gamma}_i^a}{1 - 2\tilde{\pi}^a + \tilde{\pi}^a \tilde{\gamma}_i^a} \right) + m_{1i} \log \left(\frac{\hat{\pi}_i^a (1 - \hat{\gamma}_i^a)}{\tilde{\pi}^a (1 - \tilde{\gamma}_i^a)} \right) + m_{2i} \log \left(\frac{\hat{\pi}_i^a \hat{\gamma}_i^a}{\tilde{\pi}^a \tilde{\gamma}_i^a} \right) \right].$$

Moreover, T_L^a is asymptotically distributed as a chi-square distribution with $g-1$ degrees of freedom under H_{01}.

Let $\hat{\boldsymbol{\pi}}^b = (\hat{\pi}_1^b, \ldots, \hat{\pi}_g^b)$, where $\hat{\pi}_i^b (i = 1, \ldots, g)$ are the MLEs under H_{1b}. For the hypothesis H_{02}, the likelihood ratio test is proposed by

$$T_L^b = 2[l_1(\hat{\boldsymbol{\pi}}^b, \hat{\gamma}^b | \boldsymbol{m}) - l_0^b(\tilde{\pi}^b, \tilde{\gamma}^b | \boldsymbol{m})],$$

where $\hat{\gamma}^b$, $\tilde{\pi}^b$ and $\tilde{\gamma}^b$ are the MLEs under H_{1b} and H_{02}, respectively. Note that the test T_L^b is the same as that of Dallal [9]. By simplifying T_L^b, we can get its explicit expression

$$T_L^b = 2\sum_{i=1}^{g}\left[m_{0i}\log\frac{Nm_{0i}}{m_i S_0} + (m_{1i}+m_{2i})\log\frac{N(m_{1i}+m_{2i})}{m_i(S_1+S_2)}\right].$$

Similarly, the asymptotical distribution of T_L^b is also a chi-square distribution with $g-1$ degrees of freedom under H_{02}.

4.2 Wald-type tests

Let $\boldsymbol{\beta}_1 = (\pi_1, \ldots, \pi_g, \gamma_1, \ldots, \gamma_g)$ and $\hat{\boldsymbol{\beta}}_1 = (\hat{\pi}_1^a, \ldots, \hat{\pi}_g^a, \hat{\gamma}_1^a, \ldots, \hat{\gamma}_g^a)$. Under H_{01}, we have $\pi_1 = \cdots = \pi_g$; that is, $\pi_1 - \pi_2 = \pi_2 - \pi_3 = \cdots = \pi_{g-1} - \pi_g$. The hypothesis H_{01} is equivalent to $\mathbf{C}_1\boldsymbol{\beta}_1^T = \mathbf{0}$ in matrix form, where

$$\mathbf{C}_1 = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & -1 & & & 0 & 0 & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

The Wald-type test statistic is given by

$$T_W^a = (\boldsymbol{\beta}_1\mathbf{C}_1^T)(\mathbf{C}_1 I_1^{-1}\mathbf{C}_1^T)^{-1}(\mathbf{C}_1\boldsymbol{\beta}_1^T)|_{\boldsymbol{\beta}_1=\hat{\boldsymbol{\beta}}_1},$$

where I_1 is the information matrix in the Appendix A.1. Let $\mathbf{A}_1 = \mathbf{C}_1 I_1^{-1}\mathbf{C}_1^T$, we obtain

$$\mathbf{A}_1 \triangleq \mathbf{C}_1 I_1^{-1}\mathbf{C}_1^T = \begin{pmatrix} \hat{u}_1+\hat{u}_2 & -\hat{u}_2 & 0 & & \cdots & 0 & 0 \\ -\hat{u}_2 & \hat{u}_2+\hat{u}_3 & -\hat{u}_3 & 0 & \cdots & 0 & 0 \\ 0 & -\hat{u}_3 & \hat{u}_3+\hat{u}_4 & -\hat{u}_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -\hat{u}_{g-1} & \hat{u}_{g-1}+\hat{u}_g \end{pmatrix},$$

where

$$\hat{u}_i = \frac{\hat{\gamma}_i^a(\hat{\gamma}_i^a - 1)}{\hat{\gamma}_i^a(\hat{\gamma}_i^a - 1) - (\hat{\gamma}_i^a - 2)(1 - 2\hat{\pi}_i^a + \hat{\gamma}_i^a)} - \frac{\hat{\pi}_i^a(1 - 2\hat{\pi}_i^a + \hat{\pi}_i^a\hat{\gamma}_i^a)}{m_i(\hat{\gamma}_i^a - 2)}.$$

Obviously, $\mathbf{A}_1$ is a symmetric tridiagonal matrix. Thus, the elements of its inverse matrix $\mathbf{A}_1^{-1}$ can be derived as follows

$$(\mathbf{A}_1^{-1})_{i,j} = \frac{(u_{i+1}\cdots u_j)(d_{j+1}\cdots d_j)}{\delta_i\cdots\delta_{g-1}},\ j > i, i, j = 1, \ldots, g-1,$$

$$(\mathbf{A}_1^{-1})_{i,i} = \frac{d_{i+1}\cdots d_{g-1}}{\delta_i\cdots\delta_{g-1}},\qquad i = 1, \ldots, g-1,$$

where $d_{g-1} = \hat{u}_{g-1}+\hat{u}_g$, $d_j = \hat{u}_j + \hat{u}_{j+1} - \frac{\hat{u}_{j+1}^2}{d_{j+1}} (j = 2, \ldots, g-1)$, and $\delta_1 = \hat{u}_1 + \hat{u}_2$, $\delta_i = \hat{u}_i + \hat{u}_{i+1} - \frac{\hat{u}_i^2}{\delta_{i-1}} (i = g-2, \ldots, 1)$. From the above calculation, the Wald-type statistic T_W^a can be simplified as

$$T_W^a = \sum_{i=1}^{g-1}\sum_{j=1}^{g-1}(\hat{\pi}_i^a - \hat{\pi}_{i+1}^a)(\hat{\pi}_j^a - \hat{\pi}_{j+1}^a)\mathbf{A}_1^{-1},$$

which is asymptotically distributed as a chi-square distribution with $g-1$ degrees of freedom under H_{01}.

Denote $\boldsymbol{\beta}_2 = (\pi_1, \ldots, \pi_g, \gamma)$ and $\hat{\boldsymbol{\beta}}_2 = (\hat{\pi}_1^b, \ldots, \hat{\pi}_g^b, \hat{\gamma}^b)$. The hypothesis H_{02} is equivalent to $\mathbf{C}_2\boldsymbol{\beta}_2^T = \mathbf{0}$, where

$$\mathbf{C}_2 = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & -1 & & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The Wald-type test statistic has the form

$$T_W^b = (\boldsymbol{\beta}_2\mathbf{C}_2^T)(\mathbf{C}_2 I_2^{-1}\mathbf{C}_2^T)^{-1}(\mathbf{C}_2\boldsymbol{\beta}_2^T)|_{\boldsymbol{\beta}_2=\hat{\boldsymbol{\beta}}_2},$$

where I_2 is the information matrix in the Appendix A.2. Similarly, we have

$$\mathbf{A}_2 \triangleq \mathbf{C}_2 I_2^{-1}\mathbf{C}_2^T = \begin{pmatrix} \hat{v}_1+\hat{v}_2 & -\hat{v}_2 & 0 & & \cdots & 0 & 0 \\ -\hat{v}_2 & \hat{v}_2+\hat{v}_3 & -\hat{v}_3 & 0 & \cdots & 0 & 0 \\ 0 & -\hat{v}_3 & \hat{v}_3+\hat{v}_4 & -\hat{v}_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -\hat{v}_{g-1} & \hat{v}_{g-1}+\hat{v}_g \end{pmatrix},$$

where

$$\hat{v}_i = \frac{(\hat{\pi}_i^b)^2}{(2-\hat{\gamma}^b)^2\sum_{i=1}^{g}\frac{2m_i\hat{\pi}_i((\hat{\gamma}^b)^2 - \hat{\pi}_i^b\hat{\gamma}^b - 4\hat{\pi}_i^b + 1)}{\hat{\gamma}^b(1-\hat{\gamma}^b)(2-\hat{\gamma}^b)(1-2\hat{\pi}_i^b+\hat{\pi}_i^b\hat{\gamma}^b)}} + \frac{\hat{\pi}_i^b(1-2\hat{\pi}_i^b) + \hat{\pi}_i^b\hat{\gamma}^b}{m_i(2-\hat{\gamma}^b)}.$$

Then, T_W^b has the simplified form

$$T_W^b = \sum_{i=1}^{g-1}\sum_{j=1}^{g-1}(\hat{\pi}_i^b - \hat{\pi}_{i+1}^b)(\hat{\pi}_j^b - \hat{\pi}_{j+1}^b)\mathbf{A}_2^{-1},$$

where $\mathbf{A}_2^{-1}$ is the inverse of $\mathbf{A}_2$, and its elements satisfy

$$(\mathbf{A}_2^{-1})_{i,j} = \frac{(v_{i+1}\cdots v_j)(q_{j+1}\cdots q_j)}{\eta_i\cdots\eta_{g-1}},\quad j > i,$$

$$(\mathbf{A}_2^{-1})_{i,i} = \frac{q_{i+1}\cdots q_{g-1}}{\eta_i\cdots\eta_{g-1}}$$

for $i, j = 1, \ldots, g-1$. Here, $q_{g-1} = \hat{v}_{g-1}+\hat{v}_g$, $q_j = \hat{v}_j + \hat{v}_{j+1} - \frac{\hat{v}_{j+1}^2}{b_{j+1}} (j = 2, \ldots, g-1)$, and $\eta_1 = \hat{v}_1 + \hat{v}_2$, $\eta_i = \hat{v}_i + \hat{v}_{i+1} - \frac{\hat{v}_i^2}{\hat{\eta}_{i-1}} (i = g-2, \ldots, 1)$. Under H_{02}, T_W^b has an asymptotic chi-square distribution with $g-1$ degrees of freedom.

4.3 Score tests

For the hypothesis $H_{01}: \pi_1 = \cdots = \pi_g \triangleq \pi$ in case (i), we know that $\pi_i (i = 1, \ldots, g)$ are the parameters of our interest, and $\gamma_i (i = 1, \ldots, g)$ are nuisance parameters. Define the score function

$$\mathbf{U}_1 = \left(\frac{\partial l}{\partial \pi_1}, \ldots, \frac{\partial l}{\partial \pi_g}, \frac{\partial l}{\partial \gamma_1}, \ldots, \frac{\partial l}{\partial \gamma_g} \right).$$

Denote $\tilde{\boldsymbol{\gamma}}^a = (\tilde{\gamma}_1^a, \ldots, \tilde{\gamma}_g^a)$. The score test statistic is expressed as

$$T_{SC}^a = \mathbf{U}_1 I_1^{-1} \mathbf{U}_1^T |_{\pi_1 = \cdots = \pi_g = \tilde{\pi}^a, \boldsymbol{\gamma} = \tilde{\boldsymbol{\gamma}}^a},$$

where $\tilde{\pi}^a$ is the MLE under H_{01}, and I_1 is the information matrix in the Appendix A.1. We can simplify the statistic

$$T_{SC}^a = \sum_{i=1}^{g} \frac{(m_{1i} + m_{2i} + m_i \tilde{\pi}^a (\tilde{\gamma}_i^a - 2))^2 (1 - 2\tilde{\pi}^a + \tilde{\gamma}_i^a)}{2\tilde{\pi}^a (1 - 2\tilde{\pi}^a + \tilde{\pi}^a \tilde{\gamma}_i^a)^2 m_i}.$$

Thus, T_{SC}^a has asymptotic chi-square distribution with $g-1$ degrees of freedom under H_{01}.

Define another score function $\mathbf{U}_2 = (\frac{\partial l}{\partial \pi_1}, \ldots, \frac{\partial l}{\partial \pi_g}, 0)$. For the null hypothesis H_{02} in case (ii), that is, $\pi_1 = \cdots = \pi_g \triangleq \pi$ and $\gamma_1 = \cdots = \gamma_g \triangleq \gamma$, a score test statistic is written as

$$T_{SC}^b = \mathbf{U}_2 I_2^{-1} \mathbf{U}_2^T |_{\pi = \tilde{\pi}^b, \gamma = \tilde{\gamma}^b},$$

where $\tilde{\pi}^b$, $\tilde{\gamma}^b$ are defined in (6) under H_{02}, and I_2 is the information matrix in the Appendix A.2. Moreover, it can be simplified as

$$T_{SC}^b = \sum_{i=1}^{g} \frac{(m_{0i}(S_1 + S_2) - S_0(m_{1i} + m_{2i}))^2}{S_0(S_1 + S_2) m_i}.$$

Under H_{02}, T_{SC}^b has asymptotic chi-square distribution with $g-1$ degrees of freedom.

4.4 Ronser-type tests

Ronser [1] proposed a statistic: $T = \sum_{i=1}^{g} (\hat{\pi}_i - \tilde{\pi})^2 / \widehat{\mathrm{Var}}(\hat{\pi}_i)$, where $\hat{\pi}_i$, $\tilde{\pi}$, and $\widehat{\mathrm{Var}}(\hat{\pi}_i)$ are the estimates of π_i, π, and $\mathrm{Var}(\hat{\pi}_i)$. Let $\tilde{\boldsymbol{\gamma}}^a = (\tilde{\gamma}_1^a, \ldots, \tilde{\gamma}_g^a)$ is the MLE under H_{01}. For the Dallal's model, under the null hypothesis H_{01}, Ronser-type statistic is:

$$T_R^a = \sum_{i=1}^{g} \frac{(\hat{\pi}_i^a - \tilde{\pi}^a)^2}{\widehat{\mathrm{Var}}(\hat{\pi}_i^a)},$$

where $\hat{\pi}_i^a$, $\tilde{\pi}^a$ are the MLEs under H_{1a} and H_{01}, respectively, and

$$\begin{aligned} \widehat{\mathrm{Var}}(\hat{\pi}_i^a) =& \mathrm{Var}(\hat{\pi}_i^a)\Big|_{\pi_i = \tilde{\pi}^a, \gamma_i = \tilde{\boldsymbol{\gamma}}^a} \\ =& \frac{\pi_i (1 - 2\pi_i + \gamma_i)}{2m_i}\Bigg|_{\pi_i = \tilde{\pi}^a, \gamma_i = \tilde{\boldsymbol{\gamma}}^a}. \end{aligned}$$

The detailed process is to see Appendix A.3. Thus, T_R^a can be simplified by

$$T_R^a = \sum_{i=1}^{g} \frac{(m_{1i} + 2m_{2i} - 2m_i \tilde{\pi}^a)^2}{2m_i \tilde{\pi}^a (1 - 2\tilde{\pi}^a + \tilde{\gamma}_i^a)}.$$

Under H_{01}, T_R^a has asymptotic chi-square distribution with $g-1$ degrees of freedom.

For the hypothesis H_{02}, the Ronser-type statistic

$$T_R^b = \sum_{i=1}^{g} \frac{(\hat{\pi}_i^b - \tilde{\pi}^b)^2}{\widehat{\mathrm{Var}}(\hat{\pi}_i^b)},$$

where $\hat{\pi}_i^b$ and $\tilde{\pi}^b$ are the MLEs under H_{1b} and H_{02}, respectively, and

$$\begin{aligned} \widehat{\mathrm{Var}}(\hat{\pi}_i^b) =& \mathrm{Var}(\hat{\pi}_i^b)|_{\pi_i = \tilde{\pi}^b, \gamma = \tilde{\gamma}^b} \\ =& \frac{\pi_i (2 - \gamma)(1 - 2\pi_i + \gamma)(S_1 + 2S_2)^2}{4m_i (S_1 + S_2)^2}\Bigg|_{\pi_i = \tilde{\pi}^b, \gamma = \tilde{\gamma}^b} \\ =& \frac{S_0 (S_1 + 2S_2)^2}{4N^2 (S_1 + S_2) m_i}. \end{aligned}$$

The derived process is to see Appendix A.4. Through simplification, we get the concrete expression

$$T_R^b = T_{SC}^b = \sum_{i=1}^{g} \frac{(m_{0i}(S_1 + S_2) - S_0(m_{1i} + m_{2i}))^2}{S_0(S_1 + S_2) m_i}.$$

Under H_{02}, T_R^b also has asymptotic chi-square distribution with $g-1$ degrees of freedom.

Let $\chi^2_{g-1,1-\alpha}$ be the $100(1-\alpha)$ percentile of chi-square distribution with $g-1$ degrees of freedom. For all tests $T_l^i (i = a, b, l = L, W, SC, R)$, the decision rule is to reject the null hypotheses H_{01} or H_{02} at nominal level α if T_l^i is more than $\chi^2_{g-1,1-\alpha}$, respectively.

5. COMPARISON WITH TEST METHODS

In this section, we will conduct simulation experiments to evaluate the performance of the eight test statistics T_l^a and $T_l^b (l = L, W, SC, R)$ in terms of TIEs and power.

We firstly consider the TIEs of these statistics under different parameter settings when $g = 2, 4, 6$, and sample sizes $m \triangleq m_1 = m_2 = \cdots = m_g = 30, 50, 70$. In the selection of parameters, π, γ and γ_i must satisfy the response probability(p_{li}) between 0.1 and 0.9 for each group of eyes, if the probability is too large or small, the element 0 may appear in the data table, which needs other methods to study. For each configuration, 10,000 replications are generated from H_{01} and H_{02}, respectively. The empirical TIEs is calculated by a number that the rejection times of null hypothesis are divided by 10,000 samples at the significance

Table 2. The empirical type I error rates under $H_{01}: \pi_1 = \cdots = \pi_g = \pi$

g	$(\gamma_1, \ldots, \gamma_g)$	π	$m = 30$				$m = 50$				$m = 70$			
			T_L^a	T_W^a	T_{SC}^a	T_R^a	T_L^a	T_W^a	T_{SC}^a	T_R^a	T_L^a	T_W^a	T_{SC}^a	T_R^a
2	(0.6,0.7)	0.3	0.058	**0.062**	0.056	0.056	0.048	0.053	0.046	0.046	0.054	0.055	0.052	0.052
		0.4	0.052	0.058	0.050	0.050	0.050	0.053	0.049	0.049	0.053	0.056	0.052	0.052
		0.5	0.058	**0.061**	0.054	0.054	0.049	0.052	0.049	0.049	0.054	0.056	0.053	0.053
	(0.7,0.8)	0.3	0.056	**0.060**	0.053	0.053	0.053	0.056	0.051	0.051	0.056	0.058	0.054	0.054
		0.4	0.053	0.059	0.051	0.051	0.054	0.057	0.052	0.052	0.050	0.054	0.049	0.049
		0.5	0.053	0.059	0.052	0.052	0.056	0.059	0.056	0.056	0.052	0.055	0.051	0.051
	(0.8,0.9)	0.3	0.056	**0.061**	0.054	0.054	0.049	0.052	0.048	0.048	0.050	0.052	0.049	0.049
		0.4	0.057	**0.063**	0.055	0.055	0.050	0.053	0.049	0.049	0.051	0.053	0.050	0.050
		0.5	0.051	0.058	0.049	0.049	0.053	0.057	0.052	0.052	0.050	0.052	0.050	0.050
4	(0.4,0.5,0.6,0.7)	0.3	0.056	**0.074**	0.049	0.049	0.052	**0.064**	0.048	0.048	0.056	**0.066**	0.053	0.053
		0.4	0.058	**0.073**	0.052	0.052	0.056	**0.065**	0.052	0.052	0.052	**0.060**	0.049	0.049
		0.5	0.056	**0.069**	0.047	0.048	0.057	**0.064**	0.052	0.052	0.054	**0.061**	0.051	0.051
	(0.7,0.8,0.7,0.8)	0.3	0.058	**0.079**	0.050	0.050	0.055	**0.069**	0.051	0.051	0.052	**0.060**	0.050	0.050
		0.4	0.052	**0.073**	0.047	0.047	0.054	**0.065**	0.051	0.051	0.051	0.059	0.048	0.048
		0.5	0.056	**0.073**	0.050	0.050	0.053	**0.066**	0.050	0.050	0.052	**0.060**	0.050	0.050
	(0.8,0.9,0.8,0.9)	0.3	0.057	**0.080**	0.049	0.049	0.055	**0.067**	0.052	0.052	0.049	0.058	0.047	0.047
		0.4	0.056	**0.073**	0.050	0.050	0.054	**0.065**	0.052	0.052	0.051	**0.060**	0.049	0.049
		0.5	0.056	**0.075**	0.052	0.052	0.057	**0.067**	0.053	0.053	0.050	0.056	0.049	0.049
6	(0.4,0.5,0.6,0.7,0.8,0.9)	0.3	0.058	**0.088**	0.051	0.051	0.056	**0.075**	0.051	0.051	0.054	**0.067**	0.051	0.051
		0.4	0.063	**0.088**	0.053	0.053	0.054	**0.070**	0.049	0.049	0.054	**0.065**	0.049	0.049
		0.5	0.057	**0.080**	0.047	0.047	0.054	**0.066**	0.050	0.050	0.056	**0.064**	0.053	0.053
	(0.7,0.8,0.7,0.8,0.7,0.8)	0.3	0.057	**0.091**	0.048	0.048	0.059	**0.075**	0.055	0.055	0.052	**0.064**	0.048	0.048
		0.4	0.056	**0.081**	0.050	0.050	0.050	**0.066**	0.045	0.045	0.053	**0.063**	0.051	0.051
		0.5	0.057	**0.080**	0.050	0.050	0.052	**0.066**	0.049	0.049	0.049	0.057	0.047	0.047
	(0.8,0.9,0.8,0.9,0.8,0.9)	0.3	0.057	**0.090**	0.049	0.049	0.055	**0.075**	0.050	0.050	0.052	**0.066**	0.049	0.049
		0.4	0.052	**0.085**	0.045	0.045	0.054	**0.069**	0.050	0.050	0.054	**0.066**	0.052	0.052
		0.5	0.056	**0.084**	0.051	0.051	0.051	**0.065**	0.047	0.047	0.052	**0.063**	0.049	0.049

level $\alpha = 0.05$, that is, *the number of rejections*/10000. Similar to [2], we define a test is liberal, or conservative if its TIE is greater than 0.06, or less than 0.04. Otherwise, it is robust.

Under $H_{01}: \pi = 0.3, 0.4, 0.5$, the parameter settings satisfy: (i) $(\gamma_1, \gamma_2) = (0.6, 0.7)$, $(0.7, 0.8)$, $(0.8, 0.9)$, (ii) $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (0.4, 0.5, 0.6, 0.7)$, $(0.7, 0.8, 0.7, 0.8)$, $(0.8, 0.9, 0.8, 0.9)$, and (iii) $(\gamma_1, \ldots, \gamma_6) = (0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$, $(0.7, 0.8, 0.7, 0.8, 0.7, 0.8)$, $(0.8, 0.9, 0.8, 0.9, 0.8, 0.9)$ for $g = 2, 4, 6$ and $m = 30, 50, 70$. Table 2 shows the empirical TIEs of T_L^a, T_W^a, T_{SC}^a and T_R^a. Note that T_R^a and T_{SC}^a are close to each other. We observe that T_L^a, T_{SC}^a and T_R^a are robust. However, T_W^a has the inflated TIEs. On the other hand, we take $g = 2, 4, 6$, $m = 30, 50, 70$, and $\gamma = 0.3, 0.5, 0.7$ under $H_{02}: \pi = 0.3, 0.4, 0.5$. Table 3 shows that T_{SC}^b $(= T_R^b)$ are more robust than T_L^b and T_W^b. In order to further compare these test statistics, we randomly generate 1,000 sets of parameters (π, γ_i) and (π, γ) according to the constrained ranges of parameters for $g = 2, 4, 6$ and $m = 30, 50, 70$ under H_{01} and H_{02}, respectively. For each of the 1,000 configurations, 10,000 replications are randomly generated to calculate the empirical TIEs. The box-plots of the empirical TIEs are shown in Figure 1. The results display that the score (T_{SC}^a, T_{SC}^b) and Ronser-type tests (T_R^a, T_R^b) work well under $H_{0i} (i = 1, 2)$. However, the likelihood ratio (T_L^a, T_L^b) and Wald-type tests (T_W^a, T_W^b) generally have the inflated TIEs.

Next we compare and summarize the empirical power of these test statistics. Under the hypotheses H_{01}: $\pi_1 = \cdots = \pi_g$ vs. H_{1a}: $(\pi_1, \ldots, \pi_g) = (\pi_{1a}, \ldots, \pi_{ga})$, some parameter settings are taken according to the following cases: (i) $(\gamma_1, \gamma_2) = (0.4, 0.5), (0.5, 0.6), (0.6, 0.7)$, and $(\pi_{1a}, \pi_{2a}) = (0.3, 0.4), (0.4, 0.5), (0.5, 0.6)$. (ii) $(\gamma_1, \ldots, \gamma_4) = (0.4, 0.5, 0.6, 0.7)$, $(0.5, 0.6, 0.5, 0.6)$, $(0.6, 0.7, 0.6, 0.7)$, and $(\pi_{1a}, \ldots, \pi_{4a}) = (0.3, 0.4, 0.3, 0.4)$, $(0.4, 0.5, 0.4, 0.5)$, $(0.5, 0.6, 0.5, 0.6)$. (iii) $(\gamma_1, \ldots, \gamma_6) = (0.4, 0.5, 0.6, 0.7, 0.8, 0.9)$, $(0.5, 0.6, 0.5, 0.6, 0.5, 0.6)$, $(0.6, 0.7, 0.6, 0.7, 0.6, 0.7)$, and $(\pi_{1a}, \ldots, \pi_{6a}) = (0.3, 0.4, 0.3, 0.4, 0.3, 0.4)$, $(0.4, 0.5, 0.4, 0.5, 0.4, 0.5)$, $(0.5, 0.6, 0.5, 0.6, 0.5, 0.6)$ for $g = 2, 4, 6$ and $m \triangleq m_1 = m_2 = \cdots = m_g = 30, 50, 70$. For another hypotheses $H_{02}: \pi_1 = \cdots = \pi_g$ and $H_{1b}: (\pi_1, \ldots, \pi_g) = (\pi_{1b}, \ldots, \pi_{gb})$, let $\gamma = 0.3, 0.4, 0.5$, and $(\pi_{1b}, \ldots, \pi_{gb}) = (\pi_{1a}, \ldots, \pi_{ga})$. Similarly, for each parameter setting, we randomly select 10,000 replications from the alternative hypotheses $H_{1j} (j = a, b)$,

Table 3. The empirical type I error rates under $H_{02}: \pi_1 = \cdots = \pi_g = \pi$

g	γ	π	$m=30$			$m=50$			$m=70$		
			T_L^b	T_W^b	$T_{SC}^b(T_R^b)$	T_L^b	T_W^b	$T_{SC}^b(T_R^b)$	T_L^b	T_W^b	$T_{SC}^b(T_R^b)$
2	0.3	0.3	0.052	0.052	0.051	0.056	0.057	0.056	0.054	0.054	0.054
		0.4	0.059	0.059	0.048	0.046	0.051	0.046	0.055	0.057	0.055
		0.5	0.058	0.059	0.052	0.054	0.055	0.053	0.049	0.049	0.049
	0.5	0.3	0.050	0.050	0.049	0.055	0.055	0.055	0.051	0.051	0.051
		0.4	0.051	0.051	0.049	0.050	0.050	0.050	0.049	0.051	0.049
		0.5	**0.061**	**0.061**	0.050	0.052	0.058	0.052	0.051	0.051	0.051
	0.7	0.3	0.049	0.049	0.046	0.047	0.048	0.047	0.049	0.051	0.049
		0.4	0.053	0.053	0.052	0.057	0.057	0.057	0.055	0.055	0.055
		0.5	0.057	0.057	0.050	0.050	0.052	0.050	0.048	0.052	0.048
4	0.3	0.3	0.053	**0.070**	0.051	0.054	**0.061**	0.053	0.049	0.058	0.048
		0.4	0.052	**0.074**	0.047	0.054	**0.065**	0.051	0.053	**0.062**	0.051
		0.5	0.048	**0.082**	0.049	0.055	**0.073**	0.049	0.053	**0.064**	0.050
	0.5	0.3	0.052	**0.068**	0.050	0.055	**0.064**	0.053	0.050	0.057	0.048
		0.4	0.053	**0.072**	0.050	0.049	0.059	0.047	0.052	0.059	0.050
		0.5	0.051	**0.074**	0.046	0.050	**0.064**	0.046	0.054	**0.061**	0.050
	0.7	0.3	0.051	**0.072**	0.047	0.053	**0.062**	0.051	0.048	0.053	0.047
		0.4	0.053	**0.067**	0.050	0.052	**0.061**	0.051	0.047	0.055	0.046
		0.5	0.055	**0.075**	0.049	0.052	**0.062**	0.051	0.049	0.057	0.047
6	0.3	0.3	0.057	**0.079**	0.051	0.052	**0.069**	0.049	0.052	**0.060**	0.049
		0.4	0.053	**0.082**	0.049	0.055	**0.071**	0.051	0.050	**0.062**	0.048
		0.5	0.054	**0.113**	0.041	0.059	**0.093**	0.051	0.056	**0.079**	0.050
	0.5	0.3	0.058	**0.081**	0.052	0.055	**0.065**	0.051	0.056	**0.065**	0.053
		0.4	0.054	**0.081**	0.049	0.052	**0.068**	0.048	0.050	**0.060**	0.048
		0.5	0.056	**0.091**	0.048	0.050	**0.070**	0.046	0.052	**0.066**	0.047
	0.7	0.3	0.057	**0.080**	0.052	0.053	**0.067**	0.050	0.048	0.058	0.046
		0.4	0.055	**0.080**	0.050	0.054	**0.067**	0.051	0.049	**0.060**	0.047
		0.5	0.055	**0.086**	0.049	0.056	**0.071**	0.052	0.054	**0.067**	0.052

T_L^b: Dallal's statistic [9].

respectively. The empirical power is the number of rejecting $H_{0i}(i = 1, 2)$ divided by 10,000 replications randomly generated from $H_{1j}(j = a, b)$. The simulated results are presented in Tables 4 and 5, respectively. From these tables, we observe that the empirical power will increase with the increase of sample size or response rate $\pi_{ia}(i = 1, \ldots, g)$ for the same configuration. Comparing with other statistics, Wald-type tests T_W^a and T_W^b have relatively higher powers. The power of all test statistics will close to each other when m is larger.

Figure 2 provides that the empirical powers of all the statistics with $m = 100$ for the configurations: (i) $g = 2$, $\pi_1 = 0.1(0.1)0.5$, and $\pi_2 = \pi_1 + 0.1$. (ii) $g = 4$, $\pi_1 = \pi_3 = 0.1(0.1)0.5$, and $\pi_2 = \pi_4 = \pi_1 + 0.1$. (iii) $g = 6$, $\pi_1 = \pi_3 = \pi_5 = 0.1(0.1)0.5$, $\pi_2 = \pi_4 = \pi_6 = \pi_1 + 0.1$. Figure 2 displays the simulation results. The results show that the power will increase as g is larger under H_{1a} and H_{1b}. Moreover, we observe that the power of T_L^j, T_W^j, T_{SC}^j and $T_R^j(j = a, b)$ are very close to each other when $g = 2$, and the Wald-type statistics (T_W^a, T_W^b) have the higher powers when $g = 4$ or $g = 6$.

6. A REAL EXAMPLE

In this section, we present an example from Rosner [1] to investigate the performance of the proposed methods. 216 patients aged 20–39 with retinitis pigmentosa (RP) at Massachusetts Eye and Ear Infirmary, were classified into four genetic-type groups: autosomal dominant RP (DOM), autosomal recessive RP (AR), sex-linked RP (SL), and isolate RP (ISO). The patients' Snellen visual acuity (VA) was collected. An eye was affected if VA was 20/50 or worse, and normal if VA was 20/40 or better. The data of 216 patients were presented in Table 6.

For such data, Li and Ma [13] applied likelihood ratio, score and Wald-type statistics to obtain the conclusion: the correlations of the four groups are equal, that is, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$. In this case, we are interested to test if the response rates of these four groups are equal, that is, $H_{02}: \pi_1 = \pi_2 = \pi_3 = \pi_4$. Table 7 lists the values of test statistics $T_L^b, T_W^b, T_{SC}^b, T_R^b$, and their p-values. All four test methods T_l^b $(l = L, W, SC, R)$ lead to the same conclusion that the null hypotheses H_{02} is rejected. It reveals that there are

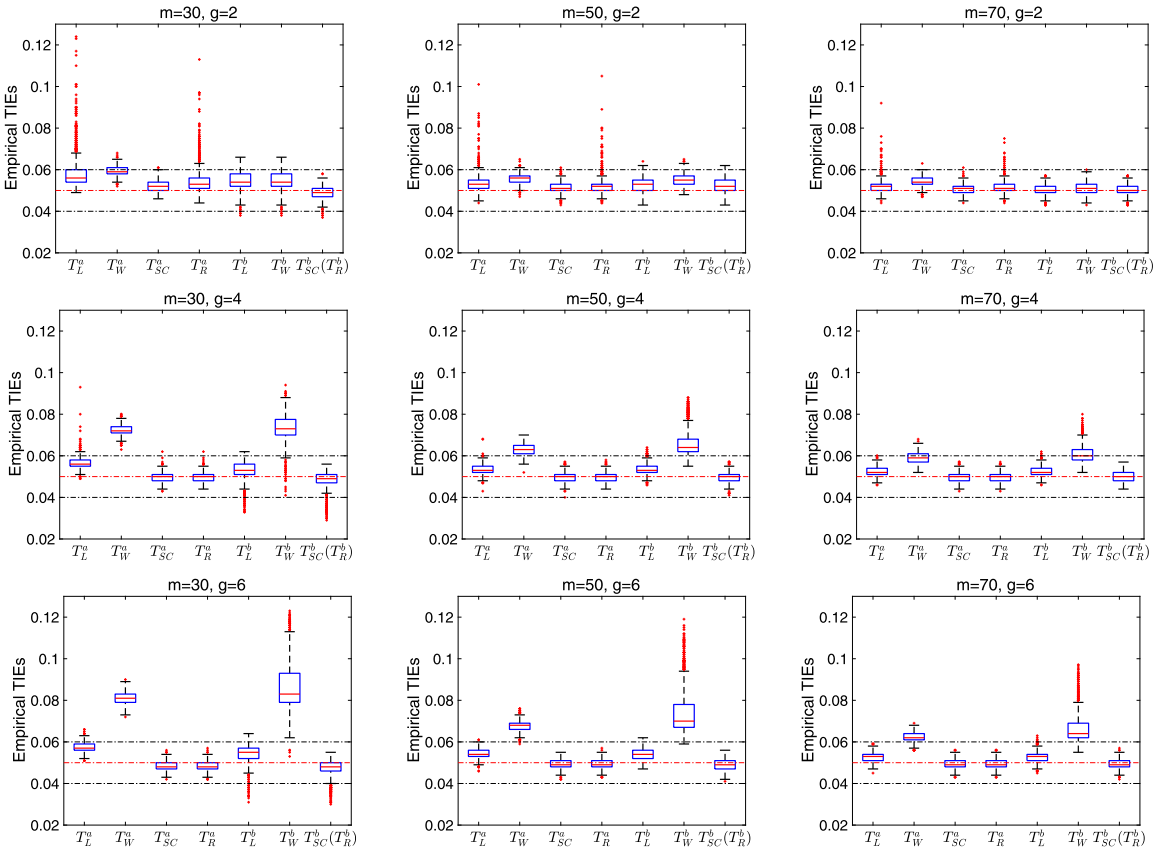

Figure 1. Box-plots of the empirical TIEs under 1,000 configurations.

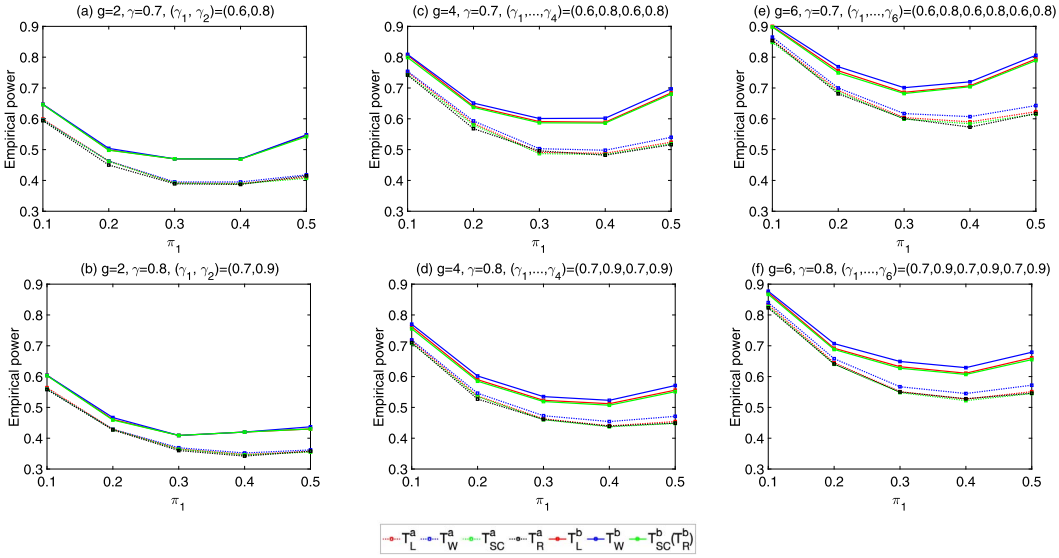

Figure 2. Power plots of the eight statistics under $m = 100$.

Table 4. The empirical power of tests under $H_{1a}: (\pi_1, \ldots, \pi_g) = (\pi_{1a}, \ldots, \pi_{ga})$

g	$(\gamma_1, \ldots, \gamma_g)$	$(\pi_{1a}, \ldots, \pi_{ga})$	$m = 30$				$m = 50$				$m = 70$			
			T_L^a	T_W^a	T_{SC}^a	T_R^a	T_L^a	T_W^a	T_{SC}^a	T_R^a	T_L^a	T_W^a	T_{SC}^a	T_R^a
2	(0.4,0.5)	(0.3,0.4)	0.194	0.203	0.189	0.189	0.281	0.289	0.277	0.277	0.380	0.386	0.376	0.376
		(0.4,0.5)	0.196	0.207	0.193	0.193	0.296	0.303	0.291	0.291	0.382	0.390	0.379	0.379
		(0.5,0.6)	0.351	0.251	0.245	0.272	0.443	0.373	0.360	0.390	0.537	0.487	0.468	0.500
	(0.5,0.6)	(0.3,0.4)	0.178	0.186	0.173	0.173	0.256	0.263	0.253	0.253	0.337	0.342	0.334	0.334
		(0.4,0.5)	0.185	0.195	0.179	0.179	0.262	0.270	0.258	0.258	0.336	0.343	0.334	0.334
		(0.5,0.6)	0.229	0.212	0.202	0.212	0.314	0.309	0.298	0.304	0.391	0.394	0.385	0.388
	(0.6,0.7)	(0.3,0.4)	0.164	0.174	0.161	0.161	0.244	0.251	0.240	0.240	0.299	0.305	0.297	0.297
		(0.4,0.5)	0.163	0.173	0.158	0.158	0.228	0.235	0.225	0.225	0.312	0.321	0.310	0.310
		(0.5,0.6)	0.182	0.185	0.169	0.174	0.252	0.261	0.248	0.248	0.339	0.348	0.336	0.336
4	(0.4,0.5,0.6,0.7)	(0.3,0.4,0.3,0.4)	0.206	0.235	0.191	0.191	0.313	0.336	0.303	0.303	0.434	0.454	0.426	0.426
		(0.4,0.5,0.4,0.5)	0.214	0.247	0.198	0.198	0.331	0.354	0.319	0.319	0.430	0.451	0.422	0.422
		(0.5,0.6,0.5,0.6)	0.277	0.279	0.234	0.242	0.394	0.405	0.372	0.377	0.520	0.532	0.505	0.507
	(0.5,0.6,0.5,0.6)	(0.3,0.4,0.3,0.4)	0.204	0.241	0.188	0.188	0.313	0.339	0.303	0.303	0.426	0.447	0.416	0.416
		(0.4,0.5,0.4,0.5)	0.204	0.238	0.188	0.188	0.315	0.342	0.305	0.305	0.422	0.441	0.414	0.414
		(0.5,0.6,0.5,0.6)	0.246	0.271	0.221	0.225	0.369	0.391	0.355	0.356	0.494	0.516	0.484	0.485
	(0.6,0.7,0.6,0.7)	(0.3,0.4,0.3,0.4)	0.180	0.216	0.169	0.169	0.283	0.304	0.275	0.275	0.380	0.399	0.373	0.373
		(0.4,0.5,0.4,0.5)	0.184	0.223	0.174	0.174	0.276	0.303	0.267	0.267	0.376	0.397	0.370	0.370
		(0.5,0.6,0.5,0.6)	0.202	0.233	0.186	0.187	0.305	0.331	0.294	0.295	0.416	0.436	0.409	0.409
6	(0.4,0.5,0.6,0.7,0.8,0.9)	(0.3,0.4,0.3,0.4,0.3,0.4)	0.223	0.275	0.202	0.202	0.354	0.392	0.339	0.339	0.473	0.503	0.464	0.464
		(0.4,0.5,0.4,0.5,0.4,0.5)	0.225	0.281	0.206	0.206	0.345	0.382	0.331	0.331	0.487	0.515	0.477	0.477
		(0.5,0.6,0.5,0.6,0.5,0.6)	0.267	0.305	0.237	0.236	0.405	0.440	0.385	0.386	0.550	0.576	0.539	0.539
	(0.5,0.6,0.5,0.6,0.5,0.6)	(0.3,0.4,0.3,0.4,0.3,0.4)	0.243	0.295	0.219	0.219	0.363	0.403	0.349	0.349	0.512	0.538	0.503	0.503
		(0.4,0.5,0.4,0.5,0.4,0.5)	0.244	0.290	0.225	0.225	0.379	0.419	0.366	0.366	0.517	0.542	0.507	0.507
		(0.5,0.6,0.5,0.6,0.5,0.6)	0.282	0.331	0.249	0.250	0.442	0.475	0.421	0.421	0.596	0.618	0.584	0.584
	(0.6,0.7,0.6,0.7,0.6,0.7)	(0.3,0.4,0.3,0.4,0.3,0.4)	0.210	0.263	0.193	0.193	0.330	0.366	0.316	0.316	0.463	0.489	0.453	0.453
		(0.4,0.5,0.4,0.5,0.4,0.5)	0.206	0.257	0.187	0.187	0.328	0.363	0.316	0.316	0.453	0.481	0.442	0.442
		(0.5,0.6,0.5,0.6,0.5,0.6)	0.228	0.284	0.205	0.205	0.364	0.398	0.348	0.348	0.503	0.533	0.492	0.492

significant differences of response rates among four groups in case (ii).

7. CONCLUSIONS

In this paper, we proposed eight test statistics to test two hypotheses of cases (i) and (ii) under Dallal's model. Under various hypotheses, the MLEs are obtained to construct several test statistics. For case (i), we derived four test statistics (T_L^a, T_W^a, T_{SC}^a, T_R^a) for testing the null hypothesis H_{01} and the alternative hypothesis H_{1a}. Similarly, we provided another four statistics T_L^b, T_W^b, T_{SC}^b and T_R^b to test H_{02} and H_{1b} in case (ii).

Simulation studies show that the score tests T_{SC}^a and T_{SC}^b are more robust than other tests, because the score tests only require the MLEs of parameters under the null hypothesis, and have satisfactory empirical power. The Ronser-type tests also have good performance in terms of TIEs and power. The likelihood ratio (T_L^a, T_L^b) and Wald-type (T_W^a, T_W^b) work well with respect to empirical power. However, they have the inflated TIEs when sample size is relatively small and g is larger. Therefore, the score and Ronser-type tests are recommended regardless of case (i) and (ii).

The methods proposed are all asymptotic, and do not work well when the sample size is relatively small such as 30 or less samples in each group in this paper. To overcome this problem, the exact methods are the direction of our future work.

APPENDIX

A.1 Derivation of information matrix I_1

If the parameters $\gamma_i (i = 1, \ldots, g)$ are different for various groups, then the second-order differential equations of l with respect to π_i and γ_i, $i = 1, \ldots, g$ are

$$\begin{aligned}
\frac{\partial^2 l}{\partial \pi_i^2} &= -\frac{m_{0i}(\gamma_i - 2)^2}{(\pi_i\gamma_i - 2\pi_i + 1)^2} - \frac{m_{1i} + m_{2i}}{\pi_i^2}, \\
\frac{\partial^2 l}{\partial \gamma_i^2} &= -\frac{m_{0i}\pi_i^2}{(\pi_i\gamma_i - 2\pi_i + 1)^2} - \frac{m_{1i}}{(\gamma_i - 1)^2} - \frac{m_{2i}}{\gamma_i^2}, \\
\frac{\partial^2 l}{\partial \pi_i \partial \gamma_i} &= \frac{m_{0i}}{(\pi_i\gamma_i - 2\pi_i + 1)^2}, \\
\frac{\partial^2 l}{\partial \pi_i \partial \pi_j} &= \frac{\partial^2 l}{\partial \gamma_i \partial \gamma_j} = \frac{\partial^2 l}{\partial \pi_i \partial \gamma_j} = 0, \quad i \neq j.
\end{aligned}$$

Table 5. The empirical power of tests under $H_{1b}: (\pi_1, \ldots, \pi_g) = (\pi_{1b}, \ldots, \pi_{gb})$

g	γ	$(\pi_{1b}, \ldots, \pi_{gb})$	$m = 30$			$m = 50$			$m = 70$		
			T_L^b	T_W^b	$T_{SC}^b(T_R^b)$	T_L^b	T_W^b	$T_{SC}^b(T_R^b)$	T_L^b	T_W^b	$T_{SC}^b(T_R^b)$
2	0.4	(0.3,0.4)	0.237	0.237	0.234	0.391	0.391	0.391	0.478	0.479	0.478
		(0.4,0.5)	0.301	0.301	0.276	0.429	0.449	0.429	0.568	0.571	0.568
		(0.5,0.6)	0.487	0.487	0.480	0.755	0.755	0.740	0.872	0.872	0.871
	0.5	(0.3,0.4)	0.219	0.219	0.219	0.346	0.346	0.346	0.437	0.437	0.437
		(0.4,0.5)	0.249	0.249	0.229	0.358	0.374	0.358	0.469	0.478	0.469
		(0.5,0.6)	0.362	0.362	0.330	0.532	0.542	0.531	0.663	0.663	0.663
	0.6	(0.3,0.4)	0.199	0.199	0.198	0.300	0.300	0.300	0.385	0.385	0.385
		(0.4,0.5)	0.210	0.210	0.196	0.310	0.317	0.310	0.407	0.415	0.407
		(0.5,0.6)	0.277	0.277	0.247	0.381	0.401	0.381	0.516	0.516	0.516
4	0.4	(0.3,0.4,0.3,0.4)	0.291	0.336	0.283	0.467	0.491	0.461	0.619	0.638	0.614
		(0.4,0.5,0.4,0.5)	0.354	0.399	0.343	0.554	0.578	0.542	0.710	0.727	0.703
		(0.5,0.6,0.5,0.6)	0.615	0.621	0.574	0.899	0.904	0.880	0.971	0.972	0.966
	0.5	(0.3,0.4,0.3,0.4)	0.254	0.295	0.248	0.411	0.432	0.404	0.540	0.564	0.537
		(0.4,0.5,0.4,0.5)	0.287	0.331	0.274	0.456	0.482	0.445	0.612	0.633	0.610
		(0.5,0.6,0.5,0.6)	0.436	0.493	0.400	0.665	0.687	0.649	0.821	0.829	0.811
	0.6	(0.3,0.4,0.3,0.4)	0.223	0.260	0.217	0.352	0.374	0.345	0.473	0.495	0.471
		(0.4,0.5,0.4,0.5)	0.241	0.281	0.229	0.373	0.401	0.366	0.520	0.540	0.516
		(0.5,0.6,0.5,0.6)	0.315	0.364	0.299	0.502	0.530	0.481	0.655	0.669	0.643
6	0.4	(0.3,0.4,0.3,0.4,0.3,0.4)	0.346	0.407	0.332	0.563	0.596	0.553	0.728	0.748	0.723
		(0.4,0.5,0.4,0.5,0.4,0.5)	0.434	0.496	0.411	0.665	0.695	0.649	0.830	0.846	0.824
		(0.5,0.6,0.5,0.6,0.5,0.6)	0.733	0.757	0.685	0.962	0.969	0.951	0.993	0.995	0.992
	0.5	(0.3,0.4,0.3,0.4,0.3,0.4)	0.308	0.364	0.296	0.488	0.523	0.478	0.654	0.681	0.649
		(0.4,0.5,0.4,0.5,0.4,0.5)	0.352	0.414	0.332	0.558	0.593	0.544	0.729	0.752	0.723
		(0.5,0.6,0.5,0.6,0.5,0.6)	0.524	0.600	0.477	0.784	0.808	0.768	0.915	0.924	0.910
	0.6	(0.3,0.4,0.3,0.4,0.3,0.4)	0.259	0.321	0.249	0.429	0.464	0.421	0.585	0.608	0.579
		(0.4,0.5,0.4,0.5,0.4,0.5)	0.289	0.346	0.273	0.468	0.507	0.455	0.624	0.650	0.618
		(0.5,0.6,0.5,0.6,0.5,0.6)	0.382	0.449	0.356	0.603	0.636	0.585	0.771	0.789	0.762

Table 6. The number of effected eyes for patients in genetic-type groups

Response (l)	Genetic type			
	DOM	AR	SL	ISO
0	15	7	3	67
1	6	5	2	24
2	7	9	14	57

Table 7. Statistic values and p-values

	T_L^b	T_W^b	T_{SC}^b	T_R^b
Statistic	8.8551	12.4018	8.1330	8.1330
p-value	0.0313	0.0061	0.0433	0.0433

Thus, the information matrix I_1 is given by

$$I_1 = -E\begin{pmatrix} \frac{\partial^2 l}{\partial \pi_1^2} & \cdots & 0 & \frac{\partial^2 l}{\partial \pi_1 \partial \gamma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^2 l}{\partial \pi_g^2} & 0 & \cdots & \frac{\partial^2 l}{\partial \pi_g \partial \gamma_g} \\ \frac{\partial^2 l}{\partial \pi_1 \partial \gamma_1} & \cdots & 0 & \frac{\partial^2 l}{\partial \gamma_1^2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^2 l}{\partial \pi_g \partial \gamma_g} & 0 & \cdots & \frac{\partial^2 l}{\partial \gamma_g^2} \end{pmatrix},$$

where

$$\begin{aligned} E\left(-\frac{\partial^2 l}{\partial \pi_i^2}\right) &= m_i\left[\frac{(\gamma_i - 2)^2}{1 - 2\pi_i + \pi_i\gamma_i} + \frac{2 - \gamma_i}{\pi_i}\right], \\ E\left(-\frac{\partial^2 l}{\partial \gamma_i^2}\right) &= m_i\left[\frac{\pi_i^2}{1 - 2\pi_i + \pi_i\gamma_i} + \frac{2\pi_i}{1 - \gamma_i} + \frac{\pi_i}{\gamma_i}\right], \\ E\left(-\frac{\partial^2 l}{\partial \pi_i \partial \gamma_i}\right) &= \frac{m_i}{1 - 2\pi_i + \pi_i\gamma_i} \end{aligned}$$

for $i = 1, \cdots, g$.

A.2 Derivation of information matrix I_2

If $\gamma_1 = \cdots = \gamma_g \triangleq \gamma$, then the second-order differential equations of l_1 with respect to $\pi_i (i = 1, \ldots, g)$ and γ are

$$\begin{aligned}
\frac{\partial^2 l_1}{\partial \pi_i^2} &= -\frac{m_{0i}(\gamma-2)^2}{(\pi_i\gamma - 2\pi_i + 1)^2} - \frac{m_{1i}+m_{2i}}{\pi_i^2}, \\
\frac{\partial^2 l_1}{\partial \gamma^2} &= \sum_{i=1}^{g}\left[-\frac{m_{0i}\pi_i^2}{(\pi_i\gamma - 2\pi_i + 1)^2} - \frac{m_{1i}}{(\gamma-1)^2} - \frac{m_{2i}}{\gamma^2}\right], \\
\frac{\partial^2 l_1}{\partial \pi_i \partial \gamma} &= \frac{m_{0i}}{(\pi_i\gamma - 2\pi_i + 1)^2},
\end{aligned}$$

and $\frac{\partial^2 l_1}{\partial \pi_i \partial \pi_j} = 0$ for $i \neq j$. Then, the information matrix I_2 is expressed by

$$I_2 = -E\begin{pmatrix}
\frac{\partial^2 l_1}{\partial \pi_1^2} & \cdots & 0 & \frac{\partial^2 l_1}{\partial \pi_1 \partial \gamma} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \frac{\partial^2 l_1}{\partial \pi_g^2} & \frac{\partial^2 l_1}{\partial \pi_g \partial \gamma} \\
\frac{\partial^2 l_1}{\partial \pi_1 \partial \gamma} & \cdots & \frac{\partial^2 l_1}{\partial \pi_g \partial \gamma} & \frac{\partial^2 l_1}{\partial \gamma^2}
\end{pmatrix},$$

where

$$\begin{aligned}
E\left(-\frac{\partial^2 l_1}{\partial \pi_i^2}\right) &= m_i\left[\frac{(\gamma-2)^2}{1-2\pi_i+\pi_i\gamma} + \frac{2-\gamma}{\pi_i}\right], \\
E\left(-\frac{\partial^2 l_1}{\partial \pi_i \partial \gamma}\right) &= \frac{m_i}{1-2\pi_i+\pi_i\gamma}, \\
E\left(-\frac{\partial^2 l_1}{\partial \gamma^2}\right) &= \sum_{i=1}^{g}\left[m_i\left(\frac{\pi_i^2}{1-2\pi_i+\pi_i\gamma} + \frac{2\pi_i}{1-\gamma} + \frac{\pi_i}{\gamma}\right)\right].
\end{aligned}$$

A.3 Derivation of $Var(\hat{\pi}_i^a)$ for Ronser-type statistic

For case (i), since $\hat{\pi}_i^a = \frac{m_{1i}+2m_{2i}}{2m_i}$, $m_{1i} \sim B(m_i, 2\pi_i(1-\gamma_i))$ and $m_{2i} \sim B(m_i, \pi_i\gamma_i)$, we can get the expectation and variance of m_{1i} and m_{2i} are $E(m_{1i}) = 2m_i\pi_i(1-\gamma_i)$, $E(m_{2i}) = m_i\pi_i\gamma_i$ and $Var(m_{1i}) = 2m_i\pi_i(1-\gamma_i)(1-\pi_i(1-\gamma_i))$, $Var(m_{2i}) = m_i\pi_i\gamma_i(1-\pi_i\gamma_i)$. According to the above conditions, the covariance of m_{1i} and m_{2i} can be obtained

$$\begin{aligned}
cov(m_{1i}, m_{2i}) &= E(m_{1i}m_{2i}) - E(m_{1i})E(m_{2i}) \\
&= 2m_i(m_i-1)\pi_i^2\gamma_i(1-\gamma_i) - 2m_i^2\pi_i^2\gamma_i(1-\gamma_i) \\
&= 2m_i\pi_i\gamma_i(\gamma_i-1).
\end{aligned}$$

Then

$$\begin{aligned}
Var(m_{1i}+2m_{2i}) &= Var(m_{1i}) + 4Var(m_{2i}) + 4cov(m_{1i}, m_{2i}) \\
&= 2m_i\pi_i(1-\gamma_i)(1-\pi_i(1-\gamma_i)) \\
&\quad + 4m_i\pi_i\gamma_i(1-\pi_i\gamma_i) + 4m_i\pi_i\gamma_i(\gamma_i-1) \\
&= 2m_i\pi_i(1-2\pi_i+\gamma_i).
\end{aligned}$$

The expression of $Var(\hat{\pi}_i^a)$ is

$$\begin{aligned}
Var(\hat{\pi}_i^a) &= Var(\frac{m_{1i}+2m_{2i}}{2m_i}) \\
&= \frac{Var(m_{1i}+2m_{2i})}{4m_i^2} \\
&= \frac{\pi_i(1-2\pi_i+\gamma_i)}{2m_i}.
\end{aligned}$$

A.4 Derivation of $Var(\hat{\pi}_i^b)$ for Ronser-type statistic

If $\gamma_1 = \cdots = \gamma_g \triangleq \gamma$, then $m_{1i} \sim B(m_i, 2\pi_i(1-\gamma))$ and $m_{2i} \sim B(m_i, \pi_i\gamma)$, we can get

$$\begin{aligned}
Var(m_{1i}+m_{2i}) &= Var(m_{1i}) + Var(m_{2i}) + 2cov(m_{1i}, m_{2i}) \\
&= 2m_i\pi_i(1-\gamma)(1-\pi_i(1-\gamma)) \\
&\quad + m_i\pi_i\gamma(1-\pi_i\gamma) + 4m_i\pi_i\gamma(\gamma-1) \\
&= m_i\pi_i(2-\gamma)(1-2\pi_i+\pi_i\gamma),
\end{aligned}$$

The expression of $Var(\hat{\pi}_i^b)$ is

$$\begin{aligned}
Var(\hat{\pi}_i^b) &= Var\left(\frac{(m_{1i}+m_{2i})(S_1+2S_2)}{2m_i(S_1+S_2)}\right) \\
&= \frac{(S_1+2S_2)^2}{(2m_i(S_1+S_2))^2}Var(m_{1i}+m_{2i}) \\
&= \frac{m_i\pi_i(2-\gamma)(1-2\pi_i+\pi_i\gamma)(S_1+2S_2)^2}{(2m_i(S_1+S_2))^2}.
\end{aligned}$$

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