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# Discussion on "Estimation of Hilbertian Varying Coefficient Models"

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This is discussion comment on "Estimation of Hilbertian Varying Coefficient Models" by Lee, Park, Hong and Kim.

Keywords and phrases: Hilbertian response, Varying coefficient model, Additive regression, Smooth backfitting, Compact operator.

I would like to congratulate the authors for this elegant piece of work on modeling a functional response against scalar predictors. Traditional models for this kind of functional data often assume varying coefficient functions on the domain of the response function, that is, the effects of predictor on the functional response are pointwise. Based on the framework of Bochner integrals, the authors consider predictor effects modeled by Hilbertian valued maps. The current work extends their earlier masterpiece, Jeon and Park (2020), from an additive regression model to a nonadditive one. This new approach of modeling functional data offers many new directions to explore further and I am particularly interested in hearing about the authors' insights from the following aspects.

This work clearly aims to introduce a notion of interaction effects into the original additive model in Jeon and Park (2020). The proposed interaction effects, in the form of $X_k \odot \mathbf{f}_{j,k}(X_j)$ with $\mathbf{f}_{j,k}(X_j)$ being the varying coefficient functions, extends the work for an Euclidean response in Lee, Mammen and Park (2012). A more general form of interaction effect would be $\mathbf{f}_{j,k}(X_j, X_k)$ where $\mathbf{f}_{j,k}$ would be a bivariate Hilbertian valued map rather than the univariate Hilbertian valued maps used in this paper and Jeon and Park (2020). Surely some constraints would be necessary for the model to be estimable. I am wondering whether such a bivariate Hilbertian valued map can be properly defined and if so whether Bochner integrals would be able to handle two-dimensional integration.

Another interesting extension for the proposed method is for regression models with a functional response and functional predictors. The function-on-function regression models have attacted quite some attention recently; see, e.g., Sun et al. (2018), Reimherr, Sriperumbudur and Taoufik (2018), and Cui, Lin and Lian (2020). An extension of the proposed method to this scenario may have to deal with Hilberttian valued maps whose domain is also a Hilbert space of functions. The Bochner integrals would need to be generalized too. Do the authors have any insights to share here?

My last interest is on the inference part of the method. The authors have established the asymptotic distributions of the proposed backfitting estimators. But the simulations have not assessed the empirical performance of the asymptotic distributions. It is common in nonparametric inference problems to have asymtotic distributions that may not perform well empirically and a bootstrapping kind of procedure is often required to make proper statistical inference in practice. Have the authors studied the empirical performance of the inference procedure based on the asymptotic distributions? Would any bootstrapping be required here too?

In summary, as a researcher in functional data analysis myself, I sincerely thank the authors for initiating such an novel direction in functional regression. This will undoubtedly lead to many interesting research results, which I really look forward to reading soon.

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