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Asymptotic in a class of network models with a difference private degree sequence*

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The asymptotic properties of parameter estimators with a difference private degree sequence have been derived in β-model with common binary values, but the general asymptotic properties in network models are lacking. Therefor, we will establish the unified asymptotic result including the consistency and asymptotical normality of the parameter estimator in a class of network models with a difference private degree sequence. Simulations are provided to illustrate asymptotic results.

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1. INTRODUCTION

With the rapid development of Internet technology, a variety of Internet platforms (Alibaba, Tencent, Facebook, etc.) can collect and record users' online browsing records, consumption records or application usage records, which can help these enterprises improve user experience and optimize services. However, these data involve a lot of private information of users, it is necessary to protect the privacy of data. In order to protect the privacy and security of personal data in the network, it must be protected before publishing network information [Zhou, Pei and Luk (2008), Yuan, Lei and Yu (2011), Cutillo, Molva and Strufe (2010), Lu and Miklau (2014)]. In the analysis of this data privacy, a common method is to add noise into the original data Dwork and Smith (2006). However, the analysis of noisy data poses a challenge to statistical inference, especially the analysis of unstructured network data with noise [Fienberg (2012); Chang, Kolaczyk and Yao (2020)].

In order to protect data privacy information, many data privacy protection algorithms have been proposed in recent years [e.g. Dwork and Smith (2006), Sweeney (2002), Zhang et al. (2019), Machanavajjhala et al. (2007), Li, Li and Venkatasubramanian (2007)]. Sweeney (2002) provided a formal protection model named k-anonymity and a set of accompanying policies for deployment. However, k-anonymity cannot resist homogeneity attacks. In order to solve this problem, L-diversity and T-closeness methods are proposed to improve k-anonymity by Machanavajjhala et al. (2007) and Li, Li and Venkatasubramanian (2007). However, these methods are difficult to define all the possible background knowledge possessed by the attacker, and can not provide a strict degree of privacy protection. Dwork and Smith (2006) proposed a differential privacy technology to solve all possible background knowledge of attackers, and provided a quantitative evaluation method for privacy protection level. In the analysis of network data privacy, a common method is the perturbation algorithm based on the network graph structure, such as modifying the network graph structure by adding, deleting and exchanging operations, which makes the difference between the published data and the original data, thus playing the role of privacy protection. At the same time, it also keeps the original scale of the social network, and has higher data availability than the clustering method Yuan, Lei and Yu (2011).

Although many differentially private algorithms have been developed to publish network data or its aggregated network statistics safely, statistical inference of noisy data is still in its infancy. In many network models, how to use noise data to accurately estimate model parameters and analyze the asymptotic properties of the estimators are still unknown or not well studied. There are some new advances in statistical inference using differential privacy sequences of undirected graphs. The specific way is to add noise into the degree sequence, such as using Laplace Mechanism to release the sufficiency degree sequence of the graph, and to reduce the error between the true sequence and the release sequence Karwa and Slavković (2016). When noise is added to the degree sequence of network data, the estimation of node parameters in the network is inconsistent. And then Karwa and Slavković (2016) paid attention to the discrete Laplace distribution noise addition process through denoising method and obtained the asymptotic properties of the parameter estimator on the β-model to achieve valid inference. Because the observed network degree sequence contains noise, Pan and Yan (2019) used the moment equation to infer the degree parameters with the noisy random variables from the discrete Laplace distribution in undirected

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binary weighted network without de-noising process. Morever, Fan, Zhang and Yan (2020) derived the uniformly consistency and asymptotic normality of the parameters estimators for undirected finite weighted network. The asymptotic properties of parameter estimators for ordered networks with noise have also been proved by Luo and Qin (2021). In directed binary weighted network, Yan (2020) considered the discrete Laplace distribution noise addition process through de-noising method and without de-noising process, and obtained the asymptotic properties of the parameter estimator on the β-model.

In this paper, our main contribution is to provide the unified asymptotic theoretical framework of the parameter estimator in undirected networks with a differentially private degree sequence, which is different from the maximum likelihood estimator. First, we show that the estimator of the parameter in a class of undirected networks based on the moment equation in which the unobserved original sequence is directly replace by differentially private degree sequence. This is motivated by Yan (2020), who directly used the noisy degree sequence to estimate parameter and showed the consistency and asymptotic normality of the estimator. Second, the probability-mass or density function of the edge a_{ij} only depends on the sum of $\alpha_i + \alpha_j$, where α_i denotes the strength parameter of vertex i. And then the a_{ij} values are more general. In other words, a_{ij} can take finite discrete weighting, continuous weighting, infinite discrete weighting and so on. Finally, we consider a general probability mass function or density function for the edges, which is different from Luo et al. (2020) only study the finite discrete weighted networks.

For the rest of this article, we state as follows. In Section 2, we give the definitions of edge differential privacy and differential privacy degree sequence. In Section 3, we give a uniform asymptotic result which contains the consistency and asymptotic normality of the parameter estimator, when the number of parameters goes to infinity. In Section 4, we illustrate several applications on our main results. In Section 5, we carry out the simulation studies to evaluate the theoretical results. Proofs are in the Appendix.

2. ESTIMATION FROM A DIFFERENTIALLY PRIVATE DEGREE SEQUENCE

We consider an undirected random graph G_n with no self-loops on n vertices labeled by $1, 2, ..., n$. Let a_{ij} be the weight of edge (i, j). For convenience, let $a_{ii} = 0$, $i = 1, \ldots, n$, so there are no self-loops. We assume that $a_{ij}, 1 \le i < j \le n$ are mutually independent. Let $d_i = \sum_{j \ne i} a_{i,j}$. We have $\mathbf{d} = (d_1, ..., d_n)^T$ is the degree sequence of G_n. Consider a probability distribution $\mathbb{P}$ on the adjacency matrix $\mathbf{A} = (a_{ij})_{n\times n}$ of the undirected random graph G_n, with each edge a_{ij} having the form of the probability distribution

$$P(a_{ij} = a) = f((\alpha_i + \alpha_j)a), \tag{2.1}$$

or the density function $f((\alpha_i + \alpha_j)a)$, where $f(\cdot)$ where α_i is the strength parameter of vertex i. If edge only take two states "present" or "absent", then a is the dichotomous values "1" or "0". In this case, we can choose logistic function as f function. This model was coined the β-model by Chatterjee, Diaconis and Sly (2011). In communication networks, if edges denote the number of emails between two persons, then a takes values form the set N_0; and if edges denotes the calling time, then a take nonnegative real values. The maximum entropy models Hillar and Wibisono (2013) can be used as f function. Notice that $E(a_{ij})$ only depends on the sum $\alpha_i+\alpha_j$. Denote $\mu(\alpha_i+\alpha_j) = E(a_{ij})$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. According to Yan, Qin and Wang (2016), we have the following the moment equations

$$d_i = \sum_{j=1; j\ne i}^{n} \mu(\alpha_i + \alpha_j), \quad i = 1, \ldots, n. \tag{2.2}$$

If d_i contain sensitive information about individuals and their relationship (e.g., sexual relationships, email exchanges), directly publishing these sensitive data with anonymized or unanonymized nodes could cause severe privacy problems or even lead to legal actions. In this section, we first give a brief introduction to differential privacy. Then we release the degree sequence under edge differential privacy.

2.1 Differential privacy

Following Hay M. and D. (2009), we use edge differential privacy here. Let $\delta(G, G')$ be the number of edges on which G and G' differ. The formal definition of edge differential privacy is as follows.

Definition 2.1 (Edge differential privacy). Let $\epsilon > 0$ be a privacy parameter. Let G and G' be arbitrarily two neighboring graphs that differ in exactly one edge. A randomized mechanism $Q(\cdot|G)$ is ϵ-edge differentially private if

$$\sup_{G,G'\in\mathcal{G},\delta(G,G')=1} \sup_{S\in\mathcal{S}} \log \frac{Q(S|G)}{Q(S|G')} \le \epsilon,$$

where $\mathcal{G}$ is the set of all undirected graphs of interest on n nodes and $\mathcal{S}$ is the set of all possible outputs.

If the outputs are the network statistics, then a simple algorithm to guarantee edge differential privacy is the Laplace Mechanism [e.g., Dwork and Smith (2006)] that adds the Laplace noise. When $f(G)$ is integer, one can use a discrete Laplace random variable as the noise as in Karwa and Slavković (2016), where it has the probability mass function:

$$P(X = x) = \frac{1-\lambda}{1+\lambda}\lambda^{|x|}, \quad x \in \{0, \pm 1, \ldots\}, \lambda \in (0, 1). \tag{2.3}$$

To this end, Dwork and Smith (2006) introduced the definition of global sensitivity by the maximum l_1-norm among various dataset pairs (G, G').

Lemma 1 (Lemma 1 in Karwa and Slavković (2016)). *Let $f : \mathcal{G} \to R^k$. Let $e_1, \dots, e_k$ be independent and identically distributed discrete Laplace random variables with the parameter λ in (2.3). Then the discrete Laplace mechanism outputs $f(G) + (e_1, \dots, e_k)$ is ϵ-edge differentially private, where $\epsilon = -\Delta(f)\log\lambda$ and*

$$\Delta(f) = \max_{\delta(G,G')=1} \|f(G) - f(G')\|_1.$$

Furthermore, Lemma 1 still holds if the continuous Laplace distribution with density function $e^{-|x|/\lambda}/\lambda$ is replaced by the discrete version. Based on the definition of differential privacy, Dwork and Smith (2006) found that any function of a differentially private mechanism is also differentially private, as follow: Let f be an output of an ε-differentially private mechanism and g be any function. Then $g(f(G))$ is also ε- differentially private. This result indicates that any post-processing done on the noisy degree sequence obtained as an output of a differentially private mechanism is also differentially private.

2.2 Estimation

When network data containing sensitive individuals' and their relationships' information can not be directly made public due to privacy concerns. To guarantee the confidence information not be disclosed, they must be carefully treated before being made public. A simple method to deal with the privacy problem is the anonymization technique by removing identifiers from a network and releasing an anonymized isomorphic network, but it had been demonstrated that it is easy to attack [e.g. Narayanan and Shmatikov (2009), Wondracek et al. (2010)]. Dwork and Smith (2006) develop a rigorous definition of differential privacy(DP) to achieve privacy protection. Using the Laplace mechanism[Hay M. and D. (2009), Karwa and Slavković (2016)] to provide privacy protection, in which the independently and identically distributed Laplace random variables are added into the input data. This mechanism satisfies "differential privacy" Dwork and Smith (2006). Here, we release the degree sequences of undirected networks using the Laplace mechanism and use the moment equations for inferring the degree parameters.

We assume that random variables $\{e_i\}_{i=1}^n$ is mutually independent and distributed by discrete Laplace distributions with parameters $\lambda_n = \exp(-\epsilon_n/2)$ and $E(e_i) = 0 (i = 1, ..., n)$, where ϵ_n is privacy parameter. Then we observe the noisy sequence $\tilde{d}$ instead of d, where

$$\tilde{d}_i = d_i + e_i, \quad i = 1, \dots, n \tag{2.4}$$

Then, the moment equations are

$$\tilde{d}_i = \sum_{j=1; j\neq i}^{n} \mu(\alpha_i + \alpha_j), \quad i = 1, \dots, n. \tag{2.5}$$

We use the moment equations to estimate the degree parameter with the noisy sequence $\tilde{d}$ instead of d. Define a system of functions:

$$F_i(\boldsymbol{\alpha}) = \tilde{d}_i - E(d_i) = \tilde{d}_i - (\sum_{j=1; j\neq i}^{n} \mu(\alpha_i + \alpha_j)), \quad i = 1, \dots, n,$$
$$F(\boldsymbol{\alpha}) = (F_1(\boldsymbol{\alpha}), \dots, F_n(\boldsymbol{\alpha}))^\top.$$

Under this case, since adding or removing an edge change the degree at most two node, by 1 each, the global sensitivity for the degree sequence d is 2. Therefore, we have the privacy parameter $\epsilon_n := -\Delta_G(f)\log\lambda = -2\log\lambda$. The solution to $F(\boldsymbol{\alpha}) = 0$ is the estimator of $\boldsymbol{\alpha}$ induced by the moment equation $\tilde{\mathbf{d}} = E(\mathbf{d})$. Henceforth, we use $\widehat{\boldsymbol{\alpha}}$ to denote the estimator of $\boldsymbol{\alpha}$ satisfying $F(\widehat{\boldsymbol{\alpha}}) = 0$. Let $F'(\boldsymbol{\alpha})$ denote the Jacobin matrix of $F(\boldsymbol{\alpha})$ on $\boldsymbol{\alpha}$. We assume that the parameters of sub-exponential distributions are known. The Laplace random variables X with the density $f(x) = (2\lambda)^{-1}e^{(-|x|/\lambda)}$ is sub-exponential, where $\kappa = \lambda$. This is due to that $E|X|^p = \lambda^{p-1}\Gamma(p)$ and $\Gamma(n+s) < n^{1-s}\Gamma(n+1)$ [Gautschi (1959)] for $s \in (0,1)$ and arbitrarily positive integer n. The parameter κ_i measures how large a noise is. The larger the parameter κ_i is, the bigger the noise, providing more protection.

3. ASYMPTOTIC RESULTS

In this section, we will derive the asymptotic results for the estimator.

3.1 Notation

For a subset $\mathbf{C} \subset R^n$, let $\mathbf{C^0}$ and $\overline{\mathbf{C}}$ denote the interior and closure of $\mathbf{C}$, respectively. For a vector $\mathbf{x} = (x_1, \dots, x_n)^\top \in R^n$, denote by $\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$, the ℓ_∞-norm of x. For an $n \times n$ matrix $\mathbf{J} = (J_{i,j})$, let $\|\mathbf{J}\|_\infty$ denote the matrix norm induced by the ℓ_∞-norm on vectors in R^n, i.e.

$$\|\mathbf{J}\|_\infty = \max_{x\neq 0} \frac{\|\mathbf{J}\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} = \max_{1\le i\le n} \sum_{j=1}^{n} |J_{i,j}|. \tag{3.1}$$

Let D be an open convex subset of R^n. We say an $n \times n$ function matrix $G(\mathbf{x})$ whose elements $G_{ij}(\mathbf{x})$ are functions on vectors $\mathbf{x}$, is Lipschitz continuous on D if there exists a real number λ such that for any $\mathbf{v} \in R^n$ and any $\mathbf{x}, \mathbf{y} \in D$,

$$\|G(\mathbf{x})(\mathbf{v}) - G(\mathbf{y})(\mathbf{v})\|_\infty \le \lambda\|\mathbf{x} - \mathbf{y}\|_\infty\|\mathbf{v}\|_\infty, \tag{3.2}$$

where λ may depend on n but is independent of $\mathbf{x}$ and $\mathbf{y}$. For every fixed n, λ is a constant. Given $m, M > 0$, we say an $n \times n$ matrix $\mathbf{V} = (v_{ij})$ belongs to the matrix class $\mathbf{L}_n(m, M)$ if $\mathbf{V}$ is a diagonally balanced matrix with positive elements bounded by m and M,

$$\begin{array}{l} v_{ii} = \sum_{j=1, j\neq i}^{n} v_{ij}, \quad i = 1, \dots, n, \\ m \le v_{ij} \le M, \quad i, j = 1, \dots, n; i \neq j. \end{array} \tag{3.3}$$

We use $\mathbf{V}$ to denote the Jacobian matrix induced by the moment equations and show that it belongs to the matrix class $\mathbf{L}_n(m, M)$. We require the inverse of $\mathbf{V}$, which doesn't have a closed form. Yan and Xu (2013) proposed approximating the inverse $\mathbf{V}^{-1}$ of $\mathbf{V}$ by a matrix $\mathbf{S} = (s_{ij})$, where

$$s_{ij} = \frac{\delta_{ij}}{v_{ii}} \tag{3.4}$$

and δ_{ij} is the Kronecker delta function. We also use $\mathbf{S}$ to approximate $\mathbf{V}^{-1}$, whose approximate errors are given in Proposition 1 of the Yan, Zhao and Qin (2015).

3.2 Main results

Before presenting the asymptotic results, we first give the following proposition and assumptions, whose proof is given in Appendix.

Proposition 1. *Assume that*
(C1) $\mathbf{V} := Var(\mathbf{d}) \in \boldsymbol{L}_n(m, M)$;
(C2) $(d_i - E(d_i))/v_{ii}^{1/2}$ *are asymptotically standard normal as* $n \to \infty$.
If $\kappa^2 M/m^2 = o(n)$, *then for any fixed* k, *the first* k *elements of* $\mathbf{S}(\tilde{\mathbf{d}} - E(\mathbf{d}))$ *are asymptotically normal distribution with mean zero and covariance matrix given by the upper* $k \times k$ *submatrix of the diagonal matrix* $\mathbf{B} = diag(1/v_{11}, \ldots, 1/v_{nn})$, *where* $\mathbf{S}$ *is the approximate inverse of* $\mathbf{V}$ *defined at* (3.4).

Remark 1. Since d_i is the sum of n independent random variables, Yan, Qin and Wang (2016) have verified the result of the Proposition C2 without the noisy sequence by the central limit theorem for the bounded case which can be implied by Liapounov's central limit theorem [Chung (2001)]. Here, we have derived the central limit theorem for the bounded case with the noisy sequence and need the Propositions 1 to prove the Theorem 1.

Given $\boldsymbol{\alpha}$ with $q_n \le \alpha_i + \alpha_j \le Q_n$, assume $A \sim P_{\boldsymbol{\alpha}}$. We need the following conditions to prove the two Theorems 1 and 2:
(C3) If $F'(\boldsymbol{\alpha}) \in \mathbf{L}_n(m, M)$, $F'(\boldsymbol{\alpha})$ is Lipschitz continuous with $\vartheta = (n-1)\phi_1$, where $\phi_1 := \phi_1(q_n, Q_n)$.
(C4) With probability approaching one,

$$\max_{i=1,\ldots,n} |\tilde{d}_i - E(\tilde{d}_i)| \le \phi_2(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1)). \tag{3.5}$$

where $\phi_2 := \phi_2(q_n, Q_n)$.
(C5) $F'(\boldsymbol{\alpha}) \in \mathbf{L}_n(m, M)$ or $-F'(\boldsymbol{\alpha}) \in \mathbf{L}_n(m, M)$, where $m = m(q_n, Q_n)$ and $M = M(q_n, Q_n)$.
(C6) $|\mu''(\hat{\theta}_{ij})| = O_p(\phi_3)$, where $\hat{\theta}_{ij} = t(\alpha_i + \alpha_j) + (1-t)(\hat{\alpha}_i + \hat{\alpha}_j)$, $0 \le t \le 1$, where $\phi_3 := \phi_3(q_n, Q_n)$
(C7) Assume that $\kappa_0 = \log \frac{(\lambda^2 + 3\lambda)}{3\lambda + 1}$, where $\lambda \in (0, 1)$.

Remark 2. We use the Newton-Kantovorich Theorem, Lemma 5 in the Appendix, to prove the consistency of the moment estimator by constructing the Newton's iterative sequence. C3 requires that the Jacobin matrix $F'(\boldsymbol{\alpha})$ is Lipschitz continuous and C4 guarantees that the ℓ_∞ norm of $F_i(\boldsymbol{\alpha}) = \tilde{d}_i - E(d_i)(i = 1, ..., n)$ is bounded in the magnitude of $(n \log n)^{1/2}$, with probability approaching one. Condition C6 requires that the second derivative of $\mu(\hat{\theta}_{ij})$ is mainly determined by q_n and Q_n. Under condition C7, since adding or removing an edge change the degree at most two node, by 1 each, the global sensitivity for the degree sequence d is 2. Therefore, we have the privacy parameter $\epsilon_n := -\Delta_G(f) \log \lambda = -2 \log \lambda$.

Then, we have the following asymptotic results, whose proof is given in Appendix.

Theorem 1 (Consistency)**.** *Assume that (C3)-(C5) and (C7) hold and*

$$\frac{c_3 M^2 \phi_2}{m^3 (n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1)) = o(1) \tag{3.6}$$

$$\frac{M^4 \phi_1 \phi_2}{m^6 (n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1)) = o(1) \tag{3.7}$$

then as $n \to \infty$, *with probability approaching one, the estimator* $\hat{\alpha}$ *exists and satisfies*

$$\begin{aligned} &||\hat{\alpha} - \alpha||_\infty \\ &\quad = O_p\Big(\frac{c_3 M^2 \phi_2}{m^3(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))\Big) \\ &\quad = o_p(1) \end{aligned} \tag{3.8}$$

where $\|X\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^n |x_{ij}|$.

Remark 3. In the Theorem 1, We use the Newton-Kantovorich Theorem to prove the consistency of the estimator $\hat{\alpha}$ by constructing the Newton's iterative sequence. This indicates that the estimator of the parameter α involving noisy sequences is accurate under the non-denoised process.

Theorem 2 (Asymptotic normality)**.** *If inequality* (3.8) *and conditions (C1),(C2),(C6) and (C7), and if*

$$\frac{c_3 M^6 \phi_2 \phi_3}{m^9 (n-1)^2}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))^2 = o(1), \tag{3.9}$$

and $\kappa^2 M/m^2 = o(n)$, *then for any fixed* $k \ge 1$, *as* $n \to \infty$, *the vector consisting of the first* k *elements of* $(\mathbf{B}^{-1})^{1/2}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})$ *is asymptotically standard multivariate normal, where* $(\mathbf{B}^{-1})^{1/2} = diag(v_{11}^{1/2}, \ldots, v_{nn}^{1/2})$.

The proof of the theorem is in Appendix.

4. APPLICATIONS

In this section, we will apply the asymptotic result in Theorems 1 and 2 through different network models.

4.1 Generalized-β model

The beta model has been researched by many authors[e.g. Rinaldo et al. (2013); Yan and Xu (2013); Karwa and Slavković (2016); Chatterjee, Diaconis and Sly (2011)]. When using its discrete version to model weighted networks with finite discrete weights, Fan, Zhang and Yan (2020) have proved the asymptotic results for differentially private generalized β-models with parameters increasing. Assume that a_{ij}, $1 \leqslant i \neq j \leqslant n$, take values from the set $\Omega = \{0, 1, ...r_0 - 1\}$ with r_0 a fixed constant, and are distributed independently with

$$P(a_{ij} = a) = \frac{e^{a(\alpha_i+\alpha_j)}}{\sum_{k=0}^{r_0-1} e^{k(\alpha_i+\alpha_j)}}$$

Then, the moment equations are

$$\tilde{d}_i = \sum_{j=1;j\neq i}^{n} \sum_{a=0}^{r_0-1} \frac{e^{a(\hat{\alpha}_i+\hat{\alpha}_j)}}{\sum_{k=0}^{r_0-1} e^{k(\hat{\alpha}_i+\hat{\alpha}_j)}}, \quad i = 1, \ldots, n. \tag{4.1}$$

We use the moment equations to estimate the degree parameter with the noisy sequence $\tilde{d}$ instead of d. Correspondingly

$$\begin{aligned} F_i(\boldsymbol{\alpha}) &= \tilde{d} - E(d) \\ &= \tilde{d}_i - \sum_{j=1;j\neq i}^{n} \sum_{a=0}^{r_0-1} \frac{e^{a(\alpha_i+\alpha_j)}}{\sum_{k=0}^{r_0-1} e^{k(\alpha_i+\alpha_j)}}, \quad i = 1, \ldots, n, \end{aligned}$$

$$F(\boldsymbol{\alpha}) = (F_1(\boldsymbol{\alpha}), \ldots, F_n(\boldsymbol{\alpha}))^\top.$$

Let $F'(\alpha)$ be the Jacobin matrix of $F(\alpha)$ at α, then for $i, j = 1, \ldots n$,

$$\frac{\partial F_i}{\partial \alpha_i} = \sum_{j=1;j\neq i}^{n} \frac{\sum_{0\le k<l\le q-1}(k-l)^2 e^{(k+l)(\alpha_i+\alpha_j)}}{(\sum_{a=0}^{q-1} e^{a(\alpha_i+\alpha_j)})^2},$$

$$\frac{\partial F_i}{\partial \alpha_j} = \frac{\sum_{0\le k<l\le q-1}(k-l)^2 e^{(k+l)(\alpha_i+\alpha_j)}}{(\sum_{a=0}^{q-1} e^{a(\alpha_i+\alpha_j)})^2}, \quad j = 1, \ldots n; j \neq i.$$

From $L_n(m, M)$ in (3.3), we can see that $F'(\alpha) \in L_n(m, M)$ with $m = (2(1 + e^{Q_n}))^{-1}$ and $M = \frac{q_n^2}{2}$ if $q_n \leqslant \alpha_i + \alpha_j \leqslant Q_n, i \neq j$. Both theorem 1 and 2 can be applied. Fan, Zhang and Yan (2020) have derived that the estimator of the parameter is asymptotically consistent and normally distributed with a differentially private degree sequence and provided the numerical evaluations on asymptotic properties of the parameter estimator.

4.2 Weighted network with continuous weights

When using the maximum entropy distributions to model weighted graphs with continuous weights, Rinaldo et al. (2013) have derived the consistency of the MLE and the corresponding asymptotic normality has been done in Yan, Zhao and Qin (2015). They showed that $a_{ij}, 1 \le i \neq j \le n$, are mutually independent exponential random variables with density

$$f(a) = \frac{1}{(\alpha_i + \alpha_j)} e^{-(\alpha_i+\alpha_j)a}, \quad \alpha_i + \alpha_j > 0.$$

We use the moment equations to estimate the degree parameter with the noisy sequence $\tilde{d}$ instead of d. The moment estimating equations are

$$\tilde{d}_i = \sum_{j=1;j\neq i}^{n} \frac{1}{\hat{\alpha}_i + \hat{\alpha}_j}, \quad i = 1, \ldots, n. \tag{4.2}$$

Correspondingly

$$F_i(\boldsymbol{\alpha}) = \tilde{d} - E(d) = \tilde{d}_i - \sum_{j=1;j\neq i}^{n} \frac{1}{\alpha_i + \alpha_j}, \quad i = 1, \ldots, n,$$
$$F(\boldsymbol{\alpha}) = (F_1(\boldsymbol{\alpha}), \ldots, F_n(\boldsymbol{\alpha}))^\top.$$

Here, we consider the symmetric parameter space

$$D = \{\alpha \in R^n : q_n \le \alpha_i + \alpha_j \le Q_n, 1 \le i < j \le n, 0 < q_n < Q_n\}.$$

Let $F'(\alpha)$ be the Jacobin matrix of $F(\alpha)$ at α, then for $i, j = 1, \ldots n$,

$$\frac{\partial F_i}{\partial \alpha_i} = \sum_{j=1;j\neq i}^{n} \frac{1}{(\alpha_i + \alpha_j)^2},$$
$$\frac{\partial F_i}{\partial \alpha_j} = \frac{1}{(\alpha_i + \alpha_j)^2}, \quad j = 1, \ldots n; j \neq i.$$

Consequently, when $q_n \le \alpha_i + \alpha_j \le Q_n$ for any $i \neq j$,

$$\frac{1}{Q_n^2} \le |\frac{\partial F_i}{\partial \alpha_j}| \le \frac{1}{q_n^2}$$

Recall the definition of $\mathbf{L}_n(m, M)$, we can see that $F'(\alpha) \in \mathbf{L}_n(m, M)$, where $m = \frac{1}{Q_n^2}$ and $M = \frac{1}{q_n^2}$. Let

$$\mathbf{g}_{ij}(\alpha) = (\frac{\partial^2 F_i}{\partial \alpha_1 \partial \alpha_j}, \ldots, \frac{\partial^2 F_i}{\partial \alpha_n \partial \alpha_j})^T.$$

It is easy to verify that

$$\frac{\partial^2 F_i}{\partial \alpha_i^2} = -\sum_{j=1;j\neq i}^{n} \frac{2}{(\alpha_i + \alpha_j)^3} \quad \text{and} \quad \frac{\partial^2 F_i}{\partial \alpha_j \partial \alpha_i} = -\frac{2}{(\alpha_i + \alpha_j)^3}.$$

and

$$|\frac{\partial^2 F_i}{\partial \alpha_i^2}| \le (n-1) \times \frac{2}{q_n^3}, \quad |\frac{\partial^2 F_i}{\partial \alpha_j \partial \alpha_i}| \leqslant \frac{2}{q_n^3}. \tag{4.3}$$

This leads to that $\|\mathbf{g}_{ii}(\alpha)\|_1 \le (n-1)\frac{4}{q_n^3}$, where $\|\mathbf{x}\|_1 = \sum_i |x_i|$ for a general vector $\mathbf{x}$. Note that when $i \neq j$ and $k \neq i, j$, $\frac{\partial^2 F_i}{\partial \alpha_k \partial \alpha_j} = 0$.

Therefore, we have $||\mathbf{g}_{ij}(\alpha)||_1 \le \frac{2}{q_n^3}$, for $j \neq i$. By using the mean value theorem for vector-valued functions in (Lang (1993), p.341), for a vector $\mathbf{v}$, we have

$$\begin{aligned}
&||(F'(\mathbf{x}) - F'(\mathbf{y}))\mathbf{v}||_\infty \\
&= ||\mathbf{v}||_\infty \max_i \sum_j |\frac{\partial F_i}{\partial \alpha_j}(\mathbf{x}) - \frac{\partial F_i}{\partial \alpha_j}(\mathbf{y})| \\
&= ||\mathbf{v}||_\infty \max_i \sum_j |\int_0^1 \mathbf{g}_{ij}^T(t\mathbf{x} + (1-t)\mathbf{y})dt(\mathbf{x}-\mathbf{y})| \\
&\le ||\mathbf{v}||_\infty ||\mathbf{x}-\mathbf{y}||_\infty \max_i \sum_{j,k} |\int_0^1 \mathbf{g}_{kij}^T(t\mathbf{x} + (1-t)\mathbf{y})dt| \\
&\le \frac{8}{q_n^3}(n-1)||\mathbf{v}||_\infty ||\mathbf{x}-\mathbf{y}||_\infty.
\end{aligned}$$

Henceforth, $F'(\alpha)$ is Lipschitz continuous with the Lipschitz real number λ in terms of $\lambda = (n-1)\phi_1$, where λ is defined in $(C5)$. We can choose $\phi_1 = \frac{8}{q_n^3}$, then condition $(C3)$ holds.

Since a_{ij} is an exponential random variable with rate $\alpha_i + \alpha_j$, By Lemma A1[Yan, Leng and Zhu (2016)], $a_{ij} - 1/(\alpha_i + \alpha_j)$ is sub-exponential with parameter $2/(\alpha_i + \alpha_j) \leqslant 2/q_n$. Note that $a_{ij} - 1/(\alpha_i + \alpha_j), 1 \leqslant i \neq j \leqslant n$ are independent sub-exponential random variables. Therefore, we can apply the concentration inequality in Theorem 7 with $\kappa_1 = 2/q_n$ and

$$\epsilon_1 = (\frac{4 \log n}{\gamma_{11}(n-1)q_n^2})^{1/2}.$$

Assume n is sufficiently large such that $\epsilon_1/\kappa_1 = \sqrt{\log n/\gamma_{11}(n-1)} \leqslant 1$. According to Hillar and Wibisono (2013), for each $i = 1, ..., n$, we have

$$\max_i |d_i - E(d_i)| \le \sqrt{\frac{4}{\gamma_{11} q_n^2}(n-1)\log(n-1)}, \tag{4.4}$$

and by lemma 4, it yields that

$$\begin{aligned}
\max_{i=1,\dots,n} |\tilde{d}_i - E(d_i)| &\le \max_i |d_i - E(d_i)| + \max_i |e_i| \\
&\le O_p(\sqrt{\frac{1}{q_n^2}(n-1)\log(n-1)} + \kappa_0 \log(n+1)))
\end{aligned} \tag{4.5}$$

Thus condition C4 and C7 holds. We have

$$\begin{aligned}
&\frac{M^4\phi_1\phi_2}{m^6(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1)) \\
&= O(\frac{Q_n^6}{q_n^8(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1)))
\end{aligned}$$

If $\frac{Q_n}{q_n} = o(\frac{(n-1)}{(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))})^{1/8}$, then (3.7) is satisfied. By Theorem 1, the uniform consistency of $\hat{\alpha}$.

Corollary 3. *If* $\frac{Q_n}{q_n} = o(\frac{(n-1)}{(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))})^{1/8}$, *then as n goes to infinity, with probability approaching one $\hat{\alpha}$ exists and satisfies*

$$\begin{aligned}
&||\hat{\alpha} - \alpha||_\infty \\
&= O_p(\frac{c_3 M^2 \phi_2}{m^3(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))) \\
&= o_p(1)
\end{aligned} \tag{4.6}$$

Now $d_i = \sum_{i \neq j} a_{ij}$ is sums of $n-1$ independent exponential random variables. The covariance matrix $d - E(d)$ is $F'(\alpha)$ denote by V, Such that condition C1 holds. By the central limit theorem for the bounded case in (Loève (1977), p.289), we know that $v_{ii}^{-1/2}$ is asymptotically standard normal if v_{ii} diverges. Since

$$\frac{(n-1)}{Q_n^2} \le v_{ii} \le \frac{(n-1)}{q_n^2}, i = 1, ..., n$$

If $Q_n = o(n^{1/2})$, then $v_{ii} \to \infty$ and $\kappa_0^2 M/m^2 = o(n)$ such that conditions of Proposition 1 hold. By (4.3), we have $|\mu''(\theta_{ij})| \leqslant \frac{(n-1)}{q_n^2}$. We have

$$\frac{c_3 M^6 \phi_2 \phi_3}{m^9 (n-1)^2}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))^2 = o(1), \tag{4.7}$$

Corollary 4. *If* $\frac{Q_n}{q_n} = o(\frac{(n-1)}{(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))})^{1/18}$, *then for any fixed k, as $n \to \infty$ the vector consisting the first k elements of* $(\mathbf{B}^{-1})^{1/2}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})$ *is asymptotically standard multivariate normal, where* $(\mathbf{B}^{-1})^{1/2} = diag(v_{11}^{1/2}, \dots, v_{nn}^{1/2})$.

4.3 Poisson models

Assume that each a_{ij} is poisson distributed with parameter $\alpha_i + \alpha_j > 0$. Following Yan, Qin and Wang (2016), we showed that $a_{ij}, 1 \le i \neq j \le n$, are distributed independently with

$$P(a_{ij} = a) = \frac{e^{a(\alpha_i + \alpha_j)}}{a!}\exp(e^{\alpha_i + \alpha_j})$$

We use the moment equations to estimate the degree parameter with the noisy sequence $\tilde{d}$ instead of d. The moment estimating equations are

$$\tilde{d}_i = \sum_{j=1, j\neq i}^{n} e^{\widehat{\alpha}_i + \widehat{\alpha}_j}, \quad i = 1, \dots, n. \tag{4.8}$$

Here, we do not call $\widehat{\alpha}$ the moment estimator since equation (4.13) are not the true moment equations. In this subsection, we consider the parameter space

$$D = \{\alpha \in R^n : -Q_n \le \alpha_i + \alpha_j \le Q_n, \quad Q_n > 0, \quad 1 \le i < j \le n\}$$

The Jacobin matrix $F^{'}(\alpha)$ of $F(\alpha)$ can be calculated as follows. For $i, j = 1, ..., n$ Note the solution to the equation $\mathbf{F}(\boldsymbol{\alpha}) = 0$ is the estimator. Then the Jacobin $\mathbf{F}^{'}(\boldsymbol{\alpha})$ of $\mathbf{F}(\boldsymbol{\alpha})$ can be calculated as follows. For $i = 1, ..., n$

$$\frac{\partial F_i}{\partial \alpha_i} = -\sum_{j=1, j\neq i}^{n} e^{\alpha_i+\alpha_j},$$

$$\frac{\partial F_i}{\partial \alpha_j} = -e^{\alpha_i+\alpha_j}, j = 1, ..., n-1, j \neq i$$

Therefore, we have

$$\frac{1}{e^{Q_n}} \leq |\frac{\partial F_i}{\partial \alpha_i}| \leq e^{Q_n}$$

Recall the definition of $\mathcal{L}_n(m, M)$. We can see that $-\mathbf{F}^{'}(\boldsymbol{\alpha}) \in \mathcal{L}_n(m, M)$, where $m = \frac{1}{e^{Q_n}}$ and $M = e^{Q_n}$. Let

$$g_{ij}(\alpha) = (\frac{\partial^2 F_i}{\partial \alpha_1 \partial \alpha_j}, ..., \frac{\partial^2 F_i}{\partial \alpha_n \partial \alpha_j})$$

It is easy to verify that

$$\frac{\partial^2 F_i}{\partial \alpha_i^2} = -\sum_{j=1, j\neq i}^{n} e^{\alpha_i+\alpha_j}$$

and

$$\frac{\partial^2 F_i}{\partial \alpha_j^2} = -e^{\alpha_i+\alpha_j}$$

Let $g_{kij} = \partial^2 F_i/(\partial \alpha_k \alpha_j)$ and $g_{ij} = (g_{1ij}, ..., g_{nij})$. It is easy to derive that

$$g_{kij} = \begin{cases} -\sum_{l\neq i} e^{\alpha_i+\alpha_l}, & k = i = j \\ e^{\alpha_i+\alpha_k}, & k \neq i, i = j \\ e^{\alpha_i+\alpha_k}, & i \neq j, k = j \\ e^{\alpha_k+\alpha_j}, & k = i, i \neq j \\ 0, & otherwise \end{cases}$$

By the mean-value theorem for vector-value functions (Lang (1993), p.341) we have

$$\|(F'(x) - F'(y))v\|_\infty = \|v\|_\infty \max_i \sum_j |\frac{\partial F_i}{\partial \alpha_j}(x) - \frac{\partial F_i}{\partial \alpha_j}(y)|$$

$$= \|v\|_\infty \max_i \sum_{j=1}^{n} |\int_0^1 g_{ij}(tx + (1-t)y)dt|$$

$$\leq \|v\|_\infty \|x - y\|_\infty \max_i \sum_{j,k} |\int_0^1 g_{kij}(tx + (1-t)y)dt|$$

$$\leq 4e^{Q_n}(n-1)\|v\|_\infty \|x - y\|_\infty$$

It shows that $F^{'}(x)$ is Lipschitz continuous with the lipschitz coefficient $\lambda_0 = 4e^{Q_n}(n-1)$. The above calculation show that

$$m = 1, M = e^{Q_n}, \phi_1 = 4e^{Q_n}$$

Henceforth, $F'(\alpha)$ is Lipschitz continuous with the Lipschitz real number λ_0 in terms of $\lambda_0 = (n-1)\phi_1$, where λ_0 is defined in $(C5)$. We can choose $\phi_1 = 4e^{Q_n}$, then condition $(C3)$ holds.

To prove C4, we should introduce the following lemma.

Lemma 2. *With probability at least $1 - O(n^{-1})$, we have*

$$\max_{i=1,...,n} |d_i - E(d_i)| \leq O_p(\sqrt{(n-1)\log(n-1)})$$

Proof. Note that d_i is a sum of $n-1$ independent random variables. By Hoeffding's inequality [Hoeffding (1963)], we have

(4.9)
$$\mathbb{P}(|d_i - E(d_i)| \geq \sqrt{n \log n}) \leq 2\exp(\frac{-2n\log n}{(n-1)}) = O(\frac{1}{n^2})$$

Combining (4.9) and lemma 4, It yields that

(4.10)
$$\max_{i=1,...,n} |\tilde{d}_i - E(d_i)| \leq \max_i |d_i - E(d_i)| + \max_i |e_i|$$
$$= O_p(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))$$

This shows condition (C4) holds with ϕ_3 given in the above equation. ☐

Thus condition C4 and C7 holds. We have

$$\frac{M^4\phi_1\phi_2}{m^6(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0\log(n+1))$$
$$= O(\frac{e^{12Q_n}}{(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0\log(n+1)))$$

If $e^{Q_n} = o(\frac{(n-1)}{(\sqrt{(n-1)\log(n-1)} + \kappa_0\log(n+1))})^{1/12}$, then (3.7) is satisfied. By Theorem 1, the uniform consistency of $\hat{\alpha}$.

Corollary 5. *If $e^{Q_n} = o(\frac{(n-1)}{(\sqrt{(n-1)\log(n-1)} + \kappa_0\log(n+1))})^{1/12}$, then as n goes to infinity, with probability approaching one $\hat{\alpha}$ exists and satisfies*

(4.11)

$$||\hat{\alpha} - \alpha||_\infty = O_p(\frac{e^{7Q_n}}{(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0\log(n+1)))$$
$$= o_p(1)$$

Now $d_i = \sum_{i\neq j} a_{ij}$ is sums of $n-1$ independent exponential random variables. The covariance matrix $d - E(d)$ is $F'(\alpha)$ denote by V, Such that condition C1 holds. By the central limit theorem for the bounded case in (Loève (1977), p.289), we know that $v_{ii}^{-1/2}$ is asymptotically standard normal if v_{ii} diverges. Since

$$(n-1) \leq v_{ii} \leq (n-1)e^{Q_n}, i = 1, ..., n$$

If $e^{Q_n} = o(n^{1/2})$, then $v_{ii}\to\infty$ and $\kappa_0^2 M/m^2 = o(n)$ such that conditions of Proposition 1 hold. According to Yan, Qin and Wang (2016), we have $|\mu''(\theta_{ij})| \leqslant O_p(e^{Q_n \times e^{e^{7Q_n(\log n)^{1/2}n^{-1/2}}}})$. We have

(4.12)
$$\frac{c_3 M^6 \phi_2 \phi_3}{m^9 (n-1)^2}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))^2$$
$$= o(\frac{e^{18Q_n}}{(n-1)^2}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))^2),$$

Corollary 6. *If $e^{Q_n} = o(\frac{(n-1)}{(\sqrt{(n-1)\log(n-1)}+\kappa_0\log(n+1))})^{1/36}$, then for any fixed k, as $n\to\infty$ the vector consisting the first k elements of $(\mathbf{B}^{-1})^{1/2}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})$ is asymptotically standard multivariate normal, where $(\mathbf{B}^{-1})^{1/2} = diag(v_{11}^{1/2}, \dots, v_{nn}^{1/2})$.*

4.4 Weighted graphs with infinite discrete weights

When considering weighted graphs with discrete weights, Hillar and Wibisono (2013) has proved the consistency of MLEs. Yan, Zhao and Qin (2015) further establish the asymptotic normality of MLE with the discrete weights case, when the total number of parameter goes to infinity. Let $A = (a_{i,j})$ be the adjacency matrix of G_n, where $a_{i,j}$ is an indictor variable of the undirected edge from node i to node j. If there exists an undirected edge from i to j, let $a_{i,i} = 0$ for convenience. And a_{ij} have the following probability distribution:

$$P(a_{ij} = a) = (1 - e^{-(\alpha_i+\alpha_j)})e^{-(\alpha_i+\alpha_j)a}, \alpha_i + \alpha_j > 0$$

The estimate equation are
(4.13)
$$\tilde{d}_i = \sum_{j=1}^{n} \frac{e^{-(\alpha_i+\alpha_j)}}{1 - e^{-(\alpha_i+\alpha_j)}} = \sum_{j=1}^{n} \frac{1}{e^{(\alpha_i+\alpha_j)} - 1}, i = 1, ..., n$$

Here, we do not call $\widehat{\alpha}$ the moment estimator since equation (4.13) are not the true moment equations. In this subsection, we consider the symmetric parameter space

$$D = \{\alpha \in R^n : 0 < q_n \le \alpha_i + \alpha_j \le Q_n \;\; for \;\; 1 \le i < j \le n\}$$

The Jacobin matrix $F^{'}(\alpha)$ of $F(\alpha)$ can be calculated as follows. For $i, j = 1, ..., n$ Note the solution to the equation $\mathbf{F}(\boldsymbol{\alpha}) = 0$ is the estimator. Then the Jacobin $\mathbf{F}^{'}(\boldsymbol{\alpha})$ of $\mathbf{F}(\boldsymbol{\alpha})$ can be calculated as follows. For $i = 1, ..., n$

$$\frac{\partial F_i}{\partial \alpha_i} = \sum_{j\neq i}^{n} \frac{e^{(\alpha_i+\beta_j)}}{[e^{(\alpha_i+\beta_j)} - 1]^2},$$
$$\frac{\partial F_i}{\partial \alpha_j} = \frac{e^{(\alpha_i+\beta_j)}}{[e^{(\alpha_i+\beta_j)} - 1]^2}, j \neq i.$$

Therefore, when $q_n \le \alpha_i + \alpha_j \le Q_n$ for any $i \neq j$, we have

$$\frac{e^{Q_n}}{(e^{Q_n} - 1)^2} \le |\frac{\partial F_i}{\partial \alpha_i}| \le \frac{e^{q_n}}{(e^{q_n} - 1)^2}$$

Recall the definition of $\mathcal{L}_n(m, M)$. We can see that $-\mathbf{F}^{'}(\boldsymbol{\alpha}) \in \mathcal{L}_n(m, M)$, where $m = \frac{e^{Q_n}}{(e^{Q_n}-1)^2}$ and $M = \frac{e^{q_n}}{(e^{q_n}-1)^2}$. Both theorem 1 and 2 can be applied. Luo, Qin and Wang (2020) have derived that the estimator of the parameter is asymptotically consistent and normally distributed with a differentially private degree sequence. In this paper, we will provide the numerical evaluations on asymptotic properties of the parameter estimator.

4.5 Rayleigh distribution

In this subsection, we will consider a Rayleigh distribution for edges in which the moment equations are different from the maximum likelihood equations. The density for the Rayleigh distribution with the parameter $\sigma > 0$ is $f(x) = x\sigma^{-2}\exp(-x^2/(2\sigma^2))$ [Papoulis (1991), p.78]. The square root of the sum of the square of two independent normal random variables with mean zero and the same variance gives rise to the Rayleigh distribution. Assume a_{ij} is the Rayleigh distribution with the parameter $e^{(\alpha_i+\alpha_j)/2}$ such that $\alpha_i \in R$. Then the density of a_{ij} at the point a is

$$f(a) = \frac{a}{e^{\alpha_i+\alpha_j}} e^{-a^2/(2e^{\alpha_i+\alpha_j})}.$$

It is easily verified that $E(a_{ij}) = \sqrt{\pi/2}e^{(\alpha_i+\alpha_j)/2}$. We use the moment equations to estimate the degree parameter with the noisy sequence $\tilde{d}$ instead of d. The estimating equations are

$$\tilde{d}_i = \sum_{j\neq i; j=1}^{n} \sqrt{\frac{\pi}{2}} e^{(\hat{\alpha}_i+\hat{\alpha}_j)/2}, \quad i = 1, \dots, n. \qquad (4.14)$$

Since the equations (4.14) are similar to the moment equations (4.8) for the Poisson model, the arguments for deriving the asymptotic results of the moment estimator are also similar and we omit them here.

5. SIMULATION

In this section, we will evaluate the asymptotic results for different network model(weighted networks with continuous, infinite discrete weights and Possion weights) through numerical simulations. Following Yan, Zhao and Qin (2015) and Fan, Zhang and Yan (2020), the settings of parameter values take a linear form. Specifically, for the case with continuous weights, we set $\alpha = L + i * L^2/n, i = 1, ..., n$; for the case with infinite discrete weights, we set $\alpha = 0.1 + i * L/n, i = 1, ..., n$; for the case with Possion model, we set $\alpha = -0.3 + i * L/n, i = 1, ..., n$. A variety of L are chosen: $L = 1, \log(\log(n)), \log(n)^{1/2}$ for continuous weights; $L = 0, \log(\log(n)), \log(n)^{1/2}$ for

Table 1. Weighted networks with continuous weights: Estimated coverage probabilities of $\alpha_i - \alpha_j$ for pair (i, j) as well as the length of confidence intervals, and the P value of shapiro.test, and the probabilities that the MLE does not exist, multiplied by 100

n	(i,j)	$L = 1$	$L = log(log(n))$	$log(n)^{1/2}$
		$\epsilon = 2$		
100	(1,2)	94.76/1.95/0.26/0	94.56/3.63/0.46/0	$90.96/6.18/9.3e^{-7}/0$
	(50,51)	94.98/1.66/0.34/0	93.84/2.97//0.005/0	$93.24/4.86/3.4e^{-5}/0$
	(99,100)	94.68/1.39/0.29/0	94.76/2.31/0.63/0	$93.88/3.53/3.0e^{-3}/0$
200	(1,2)	94.58/0.96/0.47/0	94.82/1.79/0.41/0	94.44/2.69/0.67/0
	(100,101)	94.82/1.17/0.74/0	94.38/2.36/0.89/0	93.40/3.78/0.11/0
	(199,200)	95.02/1.37/0.37/0	94.16/2.91/0.005/0	92.68/4.85/0.001/0
		$\epsilon = \log(n)/n^{1/4}$		
100	(1,2)	94.90/1.38/0.44/0	94.82/2.31/0.77/0	$92.96/3.54/2e^{-4}/0$
	(50,51)	94.92/1.67/0.91/0	$93.40/2.97/6e^{-4}/0$	$91.76/4.78/1.0e^{-8}/0$
	(99,100)	94.32/1.95/0.14/0	92.70/3.64/0.03/0	$80.56/6.20/4e^{-15}/0$
200	(1,2)	94.54/0.97/0.59/0	94.72/1.79/0.28/0	93.24/2.69/0.32/0
	(100,101)	94.70/1.67/0.63/0	93.76/2.36/0.36/0	$92.58/3.78/2e^{-3}/0$
	(199,200)	94.96/1.36/0.11/0	$93.38/2.91/8e^{-4}/0$	$90.62/4.85/8e^{-8}/0$

infinite discrete weights; $L = 0, \log(\log(n)), \log(n)^{1/2}$; $L = -log(log(log(n)))^{1/2}, -log(log(n))^{2/3}, -log(log(n))^{1/2}$ for Possion model. We simulate two distinct value for ϵ: one is fixed ($\epsilon = 2$) and the other tend to zero with $n(\epsilon = \log(n)/n^{1/4})$, Here we discuss two values for n, $n = 100$ *and* 200. Each simulation is repeated 5000 times.

Note that by Theorem (2), $\widehat{\xi_{ij}} = (\hat{\alpha}_i + \hat{\alpha}_j - (\alpha_i + \alpha_j))/(1/\hat{v}_{ii}+1/\hat{v}_{ii})^{1/2}$ is asymptotically normal distribution, where $\hat{v}_{ii}$ is the estimator of v_{ii} by replacing α_i with $\hat{\alpha}_i$. The quantile-quantile(QQ) plots of ξ_{ij} are drawn. Further, we also record the coverage probability of the 95% confidence interval for $\alpha_i - \alpha_j$, the length of the confidence interval, and the frequency that the MLE did not exist.

For $\epsilon = 2, \log(n)/n^{1/4}$, the QQ-plots under $n = 100$ and 200 are similar. Thus, we here only show the QQ-plots for $\widehat{\xi_{ij}}$ under the case of $\epsilon = 2$ and $n = 100$ in Figure 1, Figure 2 and Figure 3. In this figure, the horizontal and vertical axes are the theoretical and empirical quantiles, respectively, and the straight lines correspond to the reference line $y = x$. In Figure 1, we can see that when the weights are continuous and $L = 1, \log(\log(n))$ and $\log(n)^{1/2}$, the empirical quantiles coincide with the theoretical ones very well. For infinite discrete weights in Figure 2, when $L = \log(n)^{1/2}$, the QQ-plots of pair $(n-1, n)$ has a little deviation. In Figure 3, when $L = -log(log(log(n)))^{1/2}, -log(log(n))^{1/2}$, for pairs $(n/2-1, n/2)$ $(n-1, n)$ have notable deviations. The coverage probability of the 95% confidence interval for $\alpha_i - \alpha_j$, the length of the confidence interval, and the frequency that the MLE did not exist, which are reported in Table 1, Table 2 and Table 3. We can see that the length of estimated confidence interval increases as L increases for fixed n, and decreases as n increases for fixed L.The coverage frequencies are lower than the nominal level 95%.

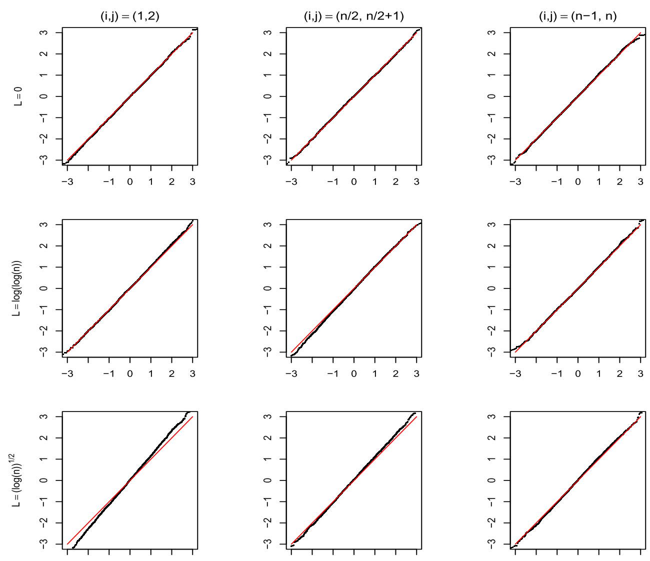

Figure 1. Weighted networks with continuous weights: The QQ plots of ξ_{ij} with red color for $\widehat{\xi}_{ij}$ ($n = 100, and \ \ \epsilon = 2$).

6. DISCUSSION

We have established a unified asymptotic theory for the moment estimator in a class of network models with a difference private degree sequence, and the edge weights are allowed to be binary, continuous or infinitely discrete and the number of vertices goes to infinity. In this class of models, we show that the estimator of the parameter in a class of undirected networks based on the moment equation in which the unobserved original sequence is directly replace by differentially private degree sequence. In our simulation studies show that when the privacy parameter ϵ is small, the private estimate fails to exist with positive frequencies according to

Table 2. Weighted networks with infinite discrete weights: Estimated coverage probabilities of $\alpha_i - \alpha_j$ for pair (i, j) as well as the length of confidence intervals, and the P value of shapiro.test, and the probabilities that the MLE does not exist, multiplied by 100

n	(i,j)	$L=1$	$L=log(log(n))$	$log(n)^{1/2}$
		$\epsilon = 2$		
100	(1,2)	95.76/0.11/0.05/0	95.48/0.37/$3e^{-3}$/1.4	95.56/0.46/e^{-3}/54.94
	(50,51)	95.02/0.11/0.08/0	95.42/0.99/e^{-3}/1.4	95.34/1.47/$3e^{-4}$/54.94
	(99,100)	95.26/0.11/0.49/0	94.77/1.76/$4e^{-8}$/1.4	98.71/2.95/$4e^{-14}$/54.94
200	(1,2)	95.24/0.08/0.09/0	95.12/0.26/0.09/0	94.70/0.31/0.32/17.24
	(100,101)	95.28/0.08/0.44/0	95.06/0.75/0.20/0	94.56/1.08/0.13/17.24
	(199,200)	95.10/0.08/0.11/0	94.44/1.35/$3e^{-3}$/0	95.60/2.33/$1.1e^{-15}$/17.24
		$\epsilon = \log(n)/n^{1/4}$		
100	(1,2)	95.76/0.11/0.06/0	95.40/0.37/$6e^{-3}$/3.84	96.50/0.50/0.09/74.32
	(50,51)	95.00/0.11/0.08/0	94.76/0.99/$5.2e^{-3}$/3.84	93.22/1.47/$1.5e^{-3}$/74.32
	(99,100)	95.30/0.11/0.48/0	93.53/1.78/$6.7e^{-10}$/3.84	97.27/2.97/$1.2e^{-9}$/74.32
200	(1,2)	95.22/0.08/0.08/0	95.09/0.26/0.12/0.26	94.46/0.31/0.12/38.60
	(100,101)	95.32/0.43/0.04/0	94.79/0.75/0.21/0.26	93.45/1.09/0.18/38.60
	(199,200)	95.10/0.11/0.01/0	93.10/1.36/$8e^{-3}$/0.26	94.33/2.36/$6.4e^{-15}$38.60

Table 3. Poisson model: Estimated coverage probabilities of $\alpha_i - \alpha_j$ for pair (i, j) as well as the length of confidence intervals, and the P value of shapiro.test, and the probabilities that the MLE does not exist, multiplied by 100

n	(i,j)	$L=-log(log(log(n)))^{1/2}$	$L=-log(log(n))^{2/3}$	$-log(log(n))^{1/2}$
		$\epsilon = 2$		
100	(1,2)	89.20/0.20/0.12/0	92.28/0.17/e^{-3}/9.62	92.63/0.18/$4e^{-3}$/17.48
	(50,51)	94.82/0.18/$8.6e^{-4}$/0	98.43/0.16/$1.2e^{-5}$/9.62	98.34/0.16/$1.2e^{-5}$/17.48
	(99,100)	97.98/0.18/$1.2e^{-5}$/0	99.95/0.16/$3.5e^{-10}$/9.62	99.87/0.16/$1.3e^{-9}$/17.48
200	(1,2)	86.80/0.14/0.21/0	91.88/0.13/0.09/0.10	92.11/0.13/0.03/0.46
	(100,101)	94.50/0.13/0.01/0	99.07/0.12/$3e^{-4}$/0.10	98.65/0.12/$3.1e^{-4}$/0.46
	(199,200)	96.32/0.13/0.03/0	99.97/0.11/$1.3e^{-7}$/0.10	99.96/0.11/$3.1e^{-6}$/0.46
		$\epsilon = \log(n)/n^{1/4}$		
100	(1,2)	88.30/0.20/$5.9e^{-3}$/0.18	91.35/0.17/0.02/22.42	91.47/0.18/0.09/34.58
	(50,51)	94.51/0.18/$5.2e^{-3}$/0.18	97.95/0.16/$4.9e^{-5}$/22.42	97.91/0.16/9.3^{-3}/34.58
	(99,100)	96.91/0.18/$7.8e^{-6}$/0.18	99.82/0.16/$2.4e{-}8$/22.42	99.72/0.16/$5.5e^{-8}$/34.58
200	(1,2)	89.34/0.14/0.02/0	92.48/0.12/0.08/1.18	92.65/0.12/0.01/4.02
	(100,101)	95.80/0.13/0.02/0	99.02/0.11/$8.5e^{-5}$/1.18	98.89/0.12/$9.3e^{-4}$/4.02
	(199,200)	98.22/0.12/$3.3e^{-3}$/0	99.87/0.11/$5.5e^{-8}$/1.18	99.89/0.11/$3.6e{-}7$/4.02

simulations, especially when the network dataset is sparse. And how to sample based on noise degree sequence of the networks is a problem for further study[Ai et al. (2021)]. The conditions in Theorems 1 and 2 induce an interesting trade-off between the private parameter measuring the magnitude of the noise and the growing rate of the parameter α. In particular, the conditions guaranteeing the asymptotic normality are stronger than those guaranteeing the consistency. If the parameter Q_n is large, α can be allowed to be relatively large. Simulation studies suggest that the conditions on ϵ might be relaxed. It can be noted that the asymptotic behavior of the parameter estimator depends not only on ϵ_n, but also on the configuration of all the parameters. We would like to investigate this problem in the future.

Interdependence is a common phenomenon in social networks. A more complex dependent case is that other network statistics, such as triangle measuring transitivity effect, are involved [Fienberg (2012)]. In this paper, we assume that the network edges are mutually independent. This assumption holds when we only consider the distribution of the vertex degrees. If edges are dependent, as long as the moment condition is correct, we should be able to obtain a consistent estimator since our method is driven by moment condition. However, without the mutual independence assumption, the resulting estimator's asymptotic distribution is not clear. We would like to investigate this problem in the future.

7. APPENDIX SECTION

7.1 Preliminaries

We present several results that we will use in this section.

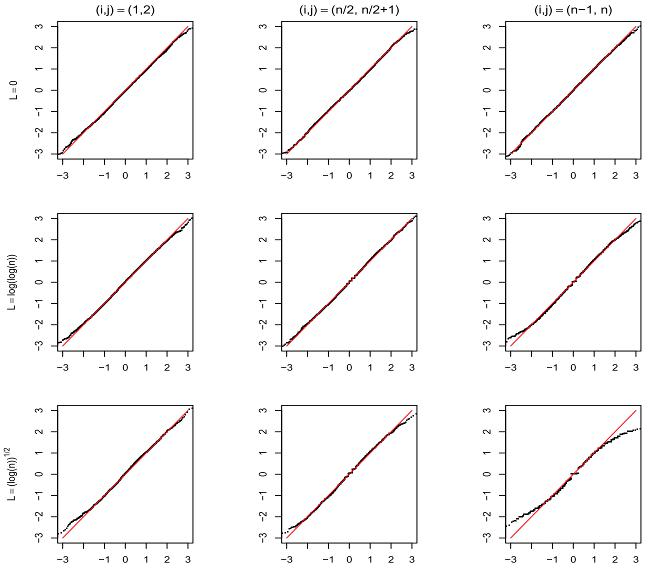

Figure 2. Weighted networks with infinite discrete weights: The QQ plots of ξ_{ij} with red color for $\widehat{\xi}_{ij}$ $(n = 100, and \quad \epsilon = 2)$.

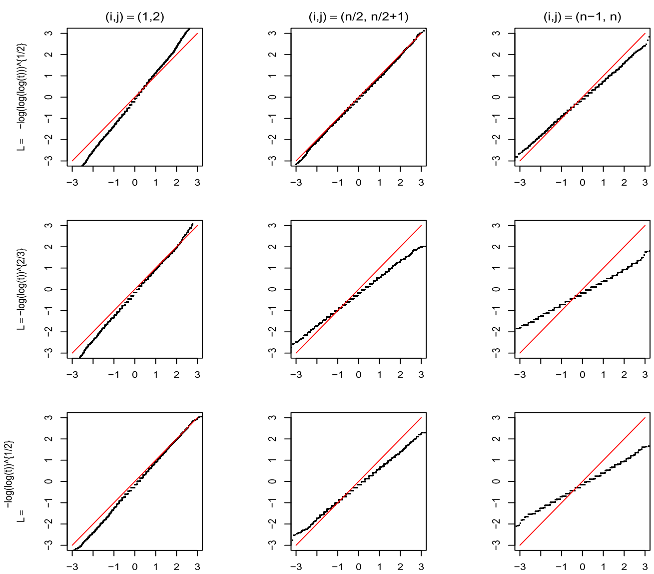

Figure 3. Weighted networks with Poisson weights: The QQ plots of ξ_{ij} with red color for $\widehat{\xi}_{ij}$ $(n = 100, and \quad \epsilon = 2)$.

7.1.1 Concentration inequality for sub-exponential random variables

A random variable X is *sub-exponential* with parameter $\kappa > 0$ [Vershynin (2012)] if

$$[E|X|^p]^{1/p} \le \kappa p \quad \text{for all } p \ge 1. \tag{7.1}$$

Sub-exponential random variables satisfy the following concentration inequality.

Theorem 7 (Corollary 5.17 in Vershynin (2012))**.** *Let $X_1, ..., X_n$ be independent centered random variables, and suppose each X_i is sub-exponential with parameter κ_i. Let $\kappa = \max_{i=1,..,n} \kappa_i$. Then for every $\epsilon \ge 0$,*

$$\mathbb{P}(\frac{1}{n}\sum_{i=1}^{n} X_i| \ge \epsilon) \le 2\exp[-\gamma n \cdot \min(\frac{\epsilon^2}{\kappa^2}, \frac{\epsilon}{\kappa})], \tag{7.2}$$

where $\gamma > 0$ is an absolute constant.

Note that if X is a κ-sub-exponential random variable with finite first moment, then the centered random variable $X - \mathbb{E}(X)$ is also sub-exponential with parameter 2κ. This follows from the triangle inequality applied to the p-norm, followed by Jensen's inequality for $p \ge 1$:

$$[\mathbb{E}|X - \mathbb{E}[X]|^p]^{1/p} \le [\mathbb{E}|X|^p]^{1/p} + |\mathbb{E}[X]| \le 2[\mathbb{E}|X|^p]^{1/p}$$

By Lemma 1 in Yan (2020), we derived that the discrete Laplace distributions is also sub-exponential with parameter $2(\log \frac{1}{\lambda})^{-1}$ for the discrete case. In other words, the discrete Laplace distributions is a special case of the sub-exponential case. In order to prove the $\max_i |e_i|$, we should give the maximal inequality for sub-exponential random variable.

Lemma 3. *For any sub-exponential random variable $\{X_i\}_{i=1}^n$, we have*

$$\begin{aligned} E(\max_{i=1,..,n}|X_i|) &\le \psi_1^{-1}(n)\max_{i=1,..,n}\|X_i\|_{\psi_1} \\ &= \log(1+n)\max_{i=1,..,n}\|X_i\|_{\psi_1}, \end{aligned} \tag{7.3}$$

where the definition of the sub-exponential norm is $\|X\|_{\psi_1} := \inf\{C > 0 : E(|X|/C) \leqslant 2\}$ [The definition is given in the Definition 4.2 of Zhang and Chen (2021).] for $\psi_1(x) = e^x - 1$.

Proof. Form Jenesen's inequality, for $C \in (0, \infty)$ and $\psi_1(x) = e^x - 1$, we get

$$\begin{aligned} \psi_1 E(\max_{i=1,..,n}|X_i|/C) &\le E(\max_{i=1,..,n}\psi_1(|X_i|/C)) \\ &\le \sum_{i=1}^{n} E\psi_1(|X_i|/C)) \le n \end{aligned} \tag{7.4}$$

where the last inequality is by the definition of sub-exponential norm: $E\psi_1(|X_i|/t) \leqslant 1$. Let $C = \max_{i=1,..,n}\|X\|_{\psi_1}$. Applying the non-decreasing property of $\psi_1(x)$ (so does its inverse function $\psi_1^{-1}(x) = \log(x+1)$), the (7.4) implies $E(\max_{i=1,..,n}|X_i|/C) \leqslant \psi_1^{-1}(n)$ by operating the map ψ_1^{-1}, and so we have (7.3). □

Note that $\{e_i\}_{i=1}^n$ are mutually independent and distributed by discrete Laplace distributions and have the probability function in (2.3). The following lemma gives the proof of the $\max_{i=1,..,n}|e_i|$.

Lemma 4. *Let $\{e_i\}_{i=1}^n$ are mutually independent and distributed by discrete Laplace distributions and have the probability function in (2.3). If $t < \log\lambda^{-1}$, then $\phi_{|e_i|}^{-1}(t) = \log\frac{(1+\lambda)t-2(1-\lambda)}{\lambda(1+\lambda)t}$, where the moment generating function of $|e_i|$ is $\phi_{|e_i|}(t) = \mathbb{E}e^{t|e_i|}$. Moreover, we have*

$$\max_{i=1,..,n}|e_i| \le O_p(\log(1+n)\log\frac{(\lambda^2+3\lambda)}{3\lambda+1}) = O_p(\kappa_0\log(1+n)) \tag{7.5}$$

Proof. In order to satisfy the condition of Lemma 4.1 in Zhang and Chen (2021), we first need to prove the existence of $\|e_i\|_{\psi_1}$. That is, the moment generating function $\phi_{|e_i|}(t) = \mathbb{E}e^{t|e_i|}$ of $|e_i|$ exists.

$$\begin{aligned}\phi_{|e_i|}(t) = \mathbb{E}e^{t|e_i|} &= \sum_{i=-\infty}^{+\infty}\frac{1-\lambda}{1+\lambda}\lambda^{|i|}e^{t|i|}\\ &= 2\sum_{i=0}^{+\infty}\frac{1-\lambda}{1+\lambda}\lambda^{|i|}e^{t|i|} - \frac{1-\lambda}{1+\lambda}\\ &= 2\times\frac{1-\lambda}{1+\lambda}\times\frac{1-\lim_{r\to\infty}\lambda^{r+1}e^{t(r+1)}}{1-\lambda e^t} - \frac{1-\lambda}{1+\lambda}\end{aligned}$$

If $t < \log\lambda^{-1}$, then $\lim_{r\to\infty}\lambda^{r+1}e^{t(r+1)} = 0$. Therefor, $\phi_{|e_i|}(t) = \frac{(1-\lambda)(1+\lambda e^t)}{(1+\lambda)(1-\lambda e^t)}$. And the inverse of function $\phi_{|e_i|}(t)$ is $\phi_{|e_i|}^{-1}(t)$ and equal to $\log\frac{(1+\lambda)t-(1-\lambda)}{\lambda(1+\lambda)t+\lambda(1-\lambda)}$. By Lemma 4.1 in Zhang and Chen (2021), we have $\|e_i\|_{\psi_1} = 1/\phi_{|e_i|}^{-1}(2) = \log\frac{(\lambda^2+3\lambda)}{3\lambda+1}$. And by Lemma 3, we have $E(\max_{i=1,..,n}|e_i|) \le \log(1+n)\max_{i=1,..,n}\|e_i\|_{\psi_1} = \log(1+n)\log\frac{(\lambda^2+3\lambda)}{3\lambda+1}$. By Theorem 11.14 of Severini (2005), we have $\max_{i=1,..,n}|e_i| \le O_p(\log(1+n)\log\frac{(\lambda^2+3\lambda)}{3\lambda+1}) = O_p(\kappa_0\log(1+n))$. □

7.1.2 Convergence rate for the Newton iterative sequence

For a subset $\mathbf{C}\subset R^n$, let $\mathbf{C}^0$ and $\overline{\mathbf{C}}$ denote the interior and closure of $\mathbf{C}$ in R^n, respectively. Let $\Omega(\mathbf{x}, r)$ denote the open ball $\{\mathbf{y}: \|\mathbf{x}-\mathbf{y}\| < r\}$, and $\overline{\Omega(\mathbf{x}, r)}$ be its closure. We use Newton's iterative sequence to prove the existence and consistency of the moment estimates relying on results of Gragg and Tapia (1974).

Lemma 5. *Let $F(\mathbf{x}) = (F_1(\mathbf{x}),\ldots,F_n(\mathbf{x}))^\top$ be a function vector on $\mathbf{x}\in R^n$. Assume that the Jacobian matrix $F'(\mathbf{x})$ is Lipschitz continuous on an open convex set D with the Lipschitz constant λ. Given $\mathbf{x}_0\in D$, assume that $[F'(\mathbf{x}_0)]^{-1}$ exists,*

$$\|[F'(\mathbf{x}_0)]^{-1}\|_\infty \le \aleph,\quad \|[F'(\mathbf{x}_0)]^{-1}F(\mathbf{x}_0)\|_\infty \le \delta,\quad h = 2\aleph\vartheta\delta \le 1,$$

$$\Omega(\mathbf{x}_0, t^*) \subset D^0,\quad t^* := \frac{2}{h}(1-\sqrt{1-h})\delta = \frac{2}{1+\sqrt{1-h}}\delta \le 2\delta,$$

where $\aleph$ and δ are positive constants that may depend on $\mathbf{x}_0$ and the dimension n of $\mathbf{x}_0$. Then the Newton iterates $\mathbf{x}_{k+1} = \mathbf{x}_k - [F'(\mathbf{x}_k)]^{-1}F(\mathbf{x}_k)$ exist and $\mathbf{x}_k\in\Omega(\mathbf{x}_0, t^*)\subset D^0$ for all $k\ge 0$; $\widehat{\mathbf{x}} = \lim\mathbf{x}_k$ exists, $\widehat{\mathbf{x}}\in\overline{\Omega(\mathbf{x}_0, t^*)}\subset D$ and $F(\widehat{\mathbf{x}}) = 0$. Thus if $t^*\to 0$, then $\|\widehat{\boldsymbol{x}} - \boldsymbol{x}_0\| = o(1)$.

7.1.3 Approximate inverse for the matrix V

Lemma 6 (Yan and Xu (2013))**.** *For a matrix $\mathbf{A} = (a_{ij})$, take $\|\mathbf{A}\| := \max_{i,j}|a_{ij}|$. If $\mathbf{V}\in\boldsymbol{L}_n(m, M)$ at (3.3), and n is large enough,*

$$\|\mathbf{V}^{-1} - \mathbf{S}\| \le \frac{c_1M^2}{m^3(n-1)^2},$$

where $\mathbf{S}$ is defined at (3.4) and c_1 is a constant that does not depend on M, m, and n.

Lemma 7 (Yan, Zhao and Qin (2015))**.** *If $\mathbf{V}\in\boldsymbol{L}_n(m, M)$, for large enough n,*

$$\begin{aligned}\|\mathbf{V}^{-1}\|_\infty &\le \|\mathbf{V}^{-1} - \mathbf{S}\|_\infty + \|\mathbf{S}\|_\infty\\ &\le \frac{c_1nM^2}{m^3(n-1)^2} + \frac{1}{m}(\frac{1}{n(n-1)} + \frac{1}{n-1})\\ &\le \frac{c_2M^2}{nm^3},\end{aligned}$$

where c_2 is a constant that does not depend on M, m, and n.

Before presenting the asymptotic results, we first prove the following proposition.

Proof of Proposition 1: By $\bar{d}_i = d_i + e_i$, we can analysis the asymptotic normality of the following proposition in two parts, i.e., $(d_i - E(d_i))/v_{ii}^{1/2}$ and $(e_i - E(e_i))/v_{ii}^{1/2}$. On the one hand, Yan, Qin and Wang (2016) have verified the result of the first part by the central limit theorem for the bounded case which can be implied by Liapounov's central limit theorem [Chung (2001)]. On the other hand, we can easily obtain the stochastic order of the second part by Chebyshev inequality.

Let $\bar{d}_i = d_i + e_i,\quad i = 1,\cdots,r$, then

$$\frac{\bar{d}_i - E(d_i)}{\sqrt{v_{ii}}} = \frac{d_i - E(d_i)}{\sqrt{v_{ii}}} + \frac{e_i}{\sqrt{v_{ii}}},\quad i = 1,\cdots,r. \tag{7.6}$$

Now, we only discuss the property of $\frac{e_i}{\sqrt{v_{ii}}}$. Note that $\{e_i\}_{i=1}^n$ is independently discrete Laplace random variables. In fact, by Chebyshev inequality and lemma 5.2 in Fan, Zhang and Yan (2020), for any constant $\tau > 0$, as n goes to infinity, we have

$$\begin{aligned}P(|\frac{e_i}{v_{ii}}| > \tau) &= P(|e_i| > \tau v_{ii})\\ &\le \frac{Var(e_i)}{\tau^2v_{ii}^2} \le \frac{4M}{\tau^2(n-1)m^2}\times\frac{2\lambda}{(1-\lambda)^2}\end{aligned}$$

If $M/m^2 = o(n)$, then

$$v_{ii}^{1/2}[\mathbf{S}(\tilde{d}_i - E(d_i))]_i = \frac{d_i - E(d_i)}{v_{ii}^{1/2}} + \frac{e_i}{v_{ii}} = \frac{d_i - E(d_i)}{v_{ii}^{1/2}} + o_p(1)$$

Therefore, for any fixed k, $\tilde{d}_i - E(d_i)/v_{ii}^{1/2}, i = 1, ..., k$, are asymptotically independent and standard normal distributions.

Proof. Proof of Theorem 1. To prove this theorem, it is sufficient to show that the Newton-Kantovorich conditions hold. We only give the proof in case $F'(\alpha) \in \mathbf{L}_n(m, M)$. The proof when $-F'(\alpha) \in \mathbf{L}_n(m, M)$ is similar, and we omit it. In Newton's iterative step, we take the true parameter vector α as the start point $\alpha^0 := \alpha$. The Jacobian matrix $F'(\alpha)$ of $F(\alpha)$ can be calculated as follows. For $i, j = 1, \dots, n$,

$$\frac{\partial F_i}{\partial \alpha_j} = \mu'(\alpha_i + \alpha_j) \quad \text{and} \quad \frac{\partial F_i}{\partial \alpha_i} = \sum_{j=1; j\neq i}^{n} \mu'(\alpha_i + \alpha_j).$$

and then $\mathbf{V} = F'(\alpha) \in \mathbf{L}_n(m, M)$ and $\mathbf{W} = \mathbf{V}^{-1} - \mathbf{S}$. By lemma 7, we have $\aleph = \frac{c_2 M^2}{nm^3}$. Note that $\mathbf{F}(\alpha) = \tilde{\mathbf{d}} - \mathbb{E}(\mathbf{d})$. By Hoeffding (1963) inequality, we have

$$\begin{aligned}
&P(|d_i - E(d_i)| \geq \sqrt{(n-1)\log(n-1)})\\
\leq & 2\exp\left(-\frac{2\left(\sqrt{(n-1)\log(n-1)}\right)^2}{(n-1)}\right)\\
\leq & \frac{2}{(n-1)^2}.
\end{aligned}$$

Since

$$P\left(\max_i |d_i - E(d_i)| \geq x\right) \leq \sum_i P(|d_i - E(d_i)| \geq x),$$

we have

$$\begin{aligned}
&P\left(\max_i |d_i - E(d_i)| \geq \sqrt{(n-1)\log(n-1)}\right)\\
&\leq \frac{2}{(n-1)} \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, with probability approaching one, we have

$$\max_i |d_i - E(d_i)| \leq \sqrt{(n-1)\log(n-1)}, \tag{7.7}$$

By lemma 4, for each $i = 1, ..., n$ we have

$$\max_{i=1,..,n} |e_i| \leq O_p(\kappa_0 \log(1+n)). \tag{7.8}$$

it yields that

(7.9)

$$\begin{aligned}
\max_{i=1,\dots,n} |\tilde{d}_i - E(d_i)| &\leq \max_i |d_i - E(d_i)| + \max_i |e_i|\\
&\leq O_p(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))
\end{aligned}$$

This shows condition (C4) holds. Assuming C4, by Lemma 6 we have

$$\begin{aligned}
&\|[F'(\alpha)]^{-1}F(\alpha)\|\\
&\leq n\|\mathbf{W}\|\|F(\alpha)\|_\infty + \max_i \frac{|F_i(\alpha)|}{v_{ii}}\\
&\leq \left(\frac{c_1 n M^2}{(n-1)^2 m^3} + \frac{1}{m(n-1)}\right)\\
&\times \phi_2(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))\\
&\leq \frac{c_3 M^2 \phi_2}{m^3(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))
\end{aligned}$$

where c_3 is a constant. Therefore, we can choose

$$\delta = \frac{c_3 M^2 \phi_2}{m^3(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))$$

If condition 4.11 holds, by C3

$$\begin{aligned}
h = 2\aleph\vartheta\delta &= \frac{c_2 M^2}{nm^3} \times (n-1)\phi_1\\
&\times \frac{c_3 M^2 \phi_2}{m^3(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))\\
&= \frac{M^4 \phi_1 \phi_2}{m^6(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))\\
&= o(1)
\end{aligned}$$

By Lemma 5, $||\hat{\alpha} - \alpha||_\infty = O_p(\frac{c_3 M^2 \phi_2}{m^3(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1)))$.By C5, (3.5) holds with probability approaching one such that (4.11) holds if (3.6) is satisfied.

□

Proof. Proof of Theorem 2. The aim of proving Theorem 2 is to establish the following equation

$$(\hat{\alpha} - \alpha)_i = [\mathbf{S}(\bar{\mathbf{d}} - E(\mathbf{d}))]_i + o_p(n^{-1/2}).$$

This will follow directly the from $[\mathbf{W}\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i = o_p(n^{-1/2}), n \to \infty$, and by proposition 2. To this end, we should prove the following conditions. Let $\overline{\mathbf{V}} = \text{cov}\{\bar{\mathbf{d}} - E(\mathbf{d})\}$, $\mathbf{V} = \text{cov}\{\mathbf{d} - E(\mathbf{d})\}$ and $\mathbf{E} = \text{cov}(\mathbf{e})$. For $1 \leq i \leq n$, the random variables d_i and e_i are mutually independent, then

$$\begin{aligned}
&\text{cov}(\bar{d}_i - E(d_i), \bar{d}_j - E(d_j))\\
&= \text{cov}(d_i + e_i - E(d_i), d_j + e_j - E(d_j))\\
&= \text{cov}(d_i - E(d_i), d_j + e_j - E(d_j)) + \text{cov}(e_i, d_j + e_j - E(d_j))\\
&= \text{cov}(d_i - E(d_i), d_j - E(d_j)) + \text{cov}(d_i - E(d_i), e_j)\\
&+ \text{cov}(e_i, d_j - E(d_j)) + \text{cov}(e_i, e_j)\\
&= \text{cov}(d_i - E(d_i), d_j - E(d_j)) + \text{cov}(e_i, e_j).
\end{aligned}$$

Two cases are discussed:
Case 1. If $i \neq j$, then $\text{cov}(\bar{d}_i - E(d_i), \bar{d}_j - E(d_j)) = \text{cov}(d_i - E(d_i), d_j - E(d_j))$;

Case 2. If $i = j$, then $\text{cov}(\bar{d}_i - E(d_i), \bar{d}_j - E(d_j)) = \text{Var}(d_i - E(d_i)) + \text{Var}(e_i)$.

Thus the elements of the matrix $\overline{\mathbf{V}}$ are denoted by $\bar{v}_{ij} = v_{ij}, \bar{v}_{ii} = v_{ii} + \text{Var}(e_i),\ i, j = 1, \cdots, n$. Let $\mathbf{U} = \text{cov}[\mathbf{W}\{\bar{\mathbf{d}} - E(\mathbf{d})\}]$ with $\mathbf{W} = \mathbf{V}^{-1} - \mathbf{S}$, then

$$\mathbf{U} = \mathbf{W}\overline{\mathbf{V}}\mathbf{W}^T = \mathbf{W}(\mathbf{V} + \mathbf{E})\mathbf{W}^T = \mathbf{W}\mathbf{V}\mathbf{W}^T + \mathbf{W}\mathbf{E}\mathbf{W}^T.$$

On the one hand, $\mathbf{W}\mathbf{V}\mathbf{W}^T = (\mathbf{V}^{-1} - \mathbf{S}) - \mathbf{S}(\mathbf{I} - \mathbf{V}\mathbf{S})$, where $n \times n$ matrix $\mathbf{I}$ is an identity matrix.

By (2.2), we obtain

$$|\{\mathbf{S}(\mathbf{I} - \mathbf{V}\mathbf{S})\}_{ij}| = |\frac{(\delta_{ij} - 1)v_{ij}}{v_{ii}v_{jj}}| \le \frac{2M}{m^2(n-1)^2}. \tag{7.10}$$

By Lemma 5.4 and (5.7), we have

$$\|\mathbf{W}\mathbf{V}\mathbf{W}^T\| \le \frac{c_1 M^2}{m^3(n-1)^2} + \frac{2M}{m^2(n-1)^2} \le O\left(\frac{M^2}{m^3 n^2}\right).$$

On the other hand,

$$\begin{aligned}\|(\mathbf{W}\mathbf{E}\mathbf{W}^T)_{ij}\| &= \|\sum_{k=1}^{n=1} w_{ik} e_k w_{kj}\| \le \max_k |e_k| \sum_{k=1}^{n} |w_{ik}||w_{kj}| \\ &\le n \max_k |e_k| \|\mathbf{W}\|^2 \le \frac{c_1 n M^4}{m^6(n-1)^4} \\ &\times 2\kappa_0 \frac{\log(n+1)}{\gamma}. \\ &\le O(\frac{\kappa_0 M^4 \log(n+1)}{m^6(n-1)^3})\end{aligned}$$

Hence, $\|\mathbf{U}\| \le O\left(\frac{M^2}{m^3 n^2} + \frac{\kappa_0 M^4 \log(n+1)}{m^6(n-1)^3}\right)$. Furthermore, by Chebyshev inequality, for any constant $a > 0$, we get

$$\begin{aligned}&P\left(\frac{[W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i}{n^{-1/2}} \ge a\right) \\ &\le P\left([W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i \ge a n^{-1/2}\right) \\ &\le \frac{n[\text{cov}\{W_n(\bar{\mathbf{d}} - E(\mathbf{d}))\}]_i}{a^2} \\ &\le O\left(\frac{M^2}{m^3 n} + \frac{\kappa_0 M^4 \log(n+1)}{m^6(n-1)^2}\right).\end{aligned}$$

If $M^2/m^3 = o(n)$, (3.6) and (3.7) hold, then

$$\|\mathbf{U}\| = o_p(n^{-1/2}) \tag{7.11}$$

□

Let $\hat{r}_{ij} = \hat{\alpha}_i + \hat{\alpha}_j - \alpha_i - \alpha_j$ and assume

$$\max_{i \ne j} |\hat{r}_{ij}| = O(\frac{c_3 M^2 \phi_2}{m^3(n-1)}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))).$$

For $i = 1, \cdots, n$, by the Taylor's expansion, we get

$$\begin{aligned}\bar{d}_i - E(d_i) &= \sum_{j \ne i}(\mu(\hat{\alpha}_i + \hat{\alpha}_j) - \mu(\alpha_i + \alpha_j)) \\ &= \sum_{j \ne i}[\mu'(\alpha_i + \alpha_j)((\hat{\alpha}_i + \hat{\alpha}_j) - (\alpha_i + \alpha_j))] + h_i,\end{aligned}$$

where $h_i = \frac{1}{2}\sum_{j \ne i} \mu''(\hat{r}_{ij})[((\hat{\alpha}_i + \hat{\alpha}_j) - (\alpha_i + \alpha_j))]^2$ and $\hat{r}_{ij} = t_{ij}(\alpha_i + \alpha_j) + (1 - t_{ij})(\hat{\alpha}_i + \hat{\alpha}_j), t_{ij} \in (0,1)$. Writing the above expressions into a matrix, we have

$$\bar{\mathbf{d}} - E(\mathbf{d}) = \mathbf{V}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) + \mathbf{h},$$

thus

$$\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} = V_n^{-1}(\bar{\mathbf{d}} - E(\mathbf{d})) + V_n^{-1}\mathbf{h},$$

where $\mathbf{h} = (h_1, \cdots, h_n)^T$. Assume that $\mu''(\hat{r}_{ij}) = O(\phi_3)$. Then

$$|h_i| \le \frac{1}{2}(n-1)\phi_3 \hat{r}_{ij}^2,$$

Therefore,

$$\begin{aligned}&|(\mathbf{V}^{-1}\mathbf{h})_i| = |(\mathbf{S}\mathbf{h})_i| + |(\mathbf{W}\mathbf{h})_i| \\ &\le \max_i \frac{|h_i|}{v_{ii}} + \|\mathbf{W}\| \sum_i |h_i| \\ &\le O\left(\frac{3\phi_3 \hat{r}_{ij}^2}{m^3} + \frac{c_2 M^2}{m^3(n-1)^2} \times \frac{1}{2} n(n-1)\phi_3 \hat{r}_{ij}^2\right) \\ &\le O\bigg(\frac{M^2\phi_3}{m^3} \times \frac{c_3 M^4 \phi_2}{m^6(n-1)^2}(\sqrt{(n-1)\log(n-1)} \\ &\qquad + \kappa_0 \log(n+1))^2\bigg) \\ &\le O\left(\frac{c_3 M^6 \phi_2 \phi_3}{m^9(n-1)^2}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))^2\right)\end{aligned}$$

If $\left(\frac{c_3 M^6 \phi_2 \phi_3}{m^9(n-1)^2}(\sqrt{(n-1)\log(n-1)} + \kappa_0 \log(n+1))^2\right) = o(n^{1/2})$, then $(\mathbf{V}^{-1}\mathbf{h})_i = o(n^{-1/2})$.

By Theorem 2, (3.8) holds with probability approaching 1. And by C6, $\mu''(\hat{r}_{ij}) = O(\phi_3)$. Consequently, by (7.11), we have

$$\begin{aligned}(\hat{\alpha} - \alpha)_i &= [\mathbf{S}(\bar{\mathbf{d}} - E(\mathbf{d}))]_i + o_p(n^{-1/2}) \\ &= \frac{\bar{d}_i - E(d_i)}{v_{ii}} + o_p(n^{-1/2}).\end{aligned}$$

Hence, Theorem 2.2 follows directly from Lemma 5.1. Finally, we conclude the proof by multiplying $\sqrt{v_{ii}}$ to left and right of the last display.

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