Statistics and Its Interface Volume 15 (2022) 527–547

Multivariate Bernstein Fréchet copulas

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Finding joint copulas based on given bivariate margins is an interesting problem. It involves in obtaining the copula from the information of its bivariate marginal distributions. In this paper, we present a multivariate copula family called multivariate Bernstein Fréchet (BF) copulas. Each copula in the family is uniquely determined by its bivariate margins, the bivariate BF copulas. For this purpose, we first discuss properties of the bivariate BF copulas, including supermigrativity and TP_2 properties. The advantages of bivariate BF copula are identified by comparing it with the bivariate Gaussian copula and the bivariate Fréchet copula. We show that a multivariate BF copula is uniquely determined by its marginal bivariate BF copulas, and methods to construct the multivariate BF copula are discussed. Numerical studies are carried out for displaying the advantages of multivariate BF copulas.

Keywords and phrases: Copula construction, Multivariate Bernstein Fréchet copulas, Bivariate marginal copulas, Parametric estimation.

1. INTRODUCTION

Copulas are multivariate distribution functions with uniform margins on $[0, 1]$. The fundamental theory in copulas is Sklar's Theorem (Nelsen, 2006), which states that for each multivariate distribution function H whose marginal distributions are $F_1, \dots, F_n$, there exists an n-dimensional copula function C such that

$$H(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)),$$

and C is unique if the marginal distribution functions $F_1, \dots, F_n$ are all continuous.

Due to the need of modeling dependence structures, the construction of new copulas is an active research area. For instance, Bedford and Cooke (2002) defined the vine copula which can effectively model tail risks in financial studies. Sancetta and Satchell (2004) defined the Bernstein copula which can approximate any copula in a polynomial form. Molenberghs and Lesaffre (1994) proposed a construction method based on an extension of bivariate Plackett copula, and Hürlimann (2004) constructed a parametric family of multivariate distributions using bivariate linear Spearman copulas. An interesting problem is to develop a method for the construction of new copulas based on its higher dimensional marginal copulas instead of univariate margins, as introduced in Embrechts (2009).

In this paper, we focus on constructing copula families which can be uniquely determined by their bivariate margins. The most well-known copula family for this purpose is the Gaussian copula. Other copulas with this property include $C^{\mathcal{A},\mathcal{B}}$ copulas proposed by Yang, Qi and Wang (2009) and the multivariate copulas studied in Johnson and Kott (1975). To model dependence by these parametric copula families, one needs to estimate all $\binom{n}{2}$ bivariate marginal copulas with the multivariate copula uniquely determined by its bivariate margins.

Our new family of copulas is constructed by combining $C^{\mathcal{A},\mathcal{B}}$ copulas (Yang, Qi and Wang, 2009) and composite Bernstein copulas (Yang et al., 2015; Guo, Wang and Yang, 2017; Yang, Wang and Xie, 2020). We call it the multivariate Bernstein Fréchet (BF) copula. First we investigate the bivariate marginal copulas of the multivariate BF copulas, i.e., the bivariate BF copulas. The bivariate BF copulas contains many types of well known examples such as FGM copulas, bivariate copulas with cubic sections (Nelsen, Quesada-Molina and Rodríguez-Lallena, 1997), and the copulas studied in Baker (2008). The properties for dependence in the bivariate BF copulas, such as supermigrativity and TP_2, are discussed. By comparing the bivariate BF copula with the bivariate Gaussian copula and the bivariate Fréchet copula, the advantages of bivariate BF copula are identified. Then, we show that a multivariate BF copula can be uniquely determined by its bivariate margins under a loose condition. The probabilistic structure of the multivariate BF copula is also provided. It can be used for random simulation and portfolio risk modeling. Besides, the methods for the determination of multivariate BF copulas from their bivariate margins are identified. Finally, we develop estimation method and perform numerical studies to verify the advantages of the copula family.

The remainder of this paper is organized as follows. Section 2 introduces the definition of the bivariate BF copulas with properties of the copula functions. Section 3 investigates multivariate BF copulas with the proof of the main result of this paper. The methods to construct a multivariate BF copula based on its bivariate margins are also discussed in this section. Section 4 gives the estimation method

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for bivariate BF copulas, and simulation studies are carried out in this section. An empirical study with stock price data is presented in Section 5. Section 6 presents conclusions of the paper. Some proofs are put in Appendix A.

2. BIVARIATE BERNSTEIN FRÉCHET COPULAS

In this section we introduce the definition and properties of bivariate BF copulas, which is used to construct multivariate copulas in Section 3. We first introduce the composite Bernstein copula (Yang et al., 2015; Guo, Wang and Yang, 2017; Yang, Wang and Xie, 2020), and then the bivariate BF copula is defined as one bivariate copula family of composite Bernstein copulas. We show that some well-known copulas are included in the bivariate BF copulas, and important dependence properties such as TP_2 and supermigrativity can be derived based on a particular type of bivariate BF copulas. Finally, we analyze the advantages of bivariate BF copula by comparing it with bivariate Gaussian copula and bivariate Fréchet copula.

2.1 Brief introduction to Bernstein copulas and composite Bernstein copulas

Copulas are multivariate distribution functions with the common univariate $\mathcal{U}[0,1]$ marginal distributions. Let

$$M(u_1,\dots,u_n)=\min\{u_1,\dots,u_n\},$$

and

$$W(u_1,\dots,u_n)=\max\{u_1+\cdots+u_n-(n-1),0\},$$

where $(u_1,\dots,u_n)\in[0,1]^n$. Here $M(u_1,\dots,u_n)$ and $W(u_1,\dots,u_n)$ are called Fréchet-Hoeffding upper bound and Fréchet-Hoeffding lower bound respectively (Nelsen, 2006). Note that the function $M(u_1,\dots,u_n)$ is indeed a copula which models comonotonic dependence structure, and the function $W(u_1,\dots,u_n)$ is a copula only when $n=2$ (see Joe, 2015, pp. 48). Every copula $C(u_1,\dots,u_n)$ satisfies the Fréchet-Hoeffding inequality

$$W(u_1,\dots,u_n)\le C(u_1,\dots,u_n)\le M(u_1,\dots,u_n).$$

In addition, $\Pi(u_1,\dots,u_n)=\prod_{i=1}^n u_i, (u_1,\dots,u_n)\in[0,1]^n$ is a copula for modeling the independent structure between random variables.

In the two-dimensional case, the bivariate Fréchet copula is defined as

$$F(u,v)=\alpha M(u,v)+\beta\Pi(u,v)+\gamma W(u,v), \tag{1}$$

where $\alpha,\beta,\gamma\ge 0$ and $\alpha+\beta+\gamma=1$. This is a mixture form of comonotonicity, countermonotonicity, and independency, the three basic dependence structures (see Nelsen, 2006, pp. 32). Because of the simple form, one can use this copula to approximate other copulas locally (Zheng, Yang and Huang, 2011).

The construction of new copulas is an interesting research area. The well-known Bernstein copula (Sancetta and Satchell, 2004) is constructed from polynomial functions. With a given copula C and positive integers $m_i, i\le n$, the Bernstein copula is defined as

$$\begin{aligned}&C_{m_1,\dots,m_n}(u_1,\dots,u_n|C)\\&=\sum_{k_1=0}^{m_1}\cdots\sum_{k_n=0}^{m_n}C(\frac{k_1}{m_1},\dots,\frac{k_n}{m_n})p_{k_1,m_1}(u_1)\cdots p_{k_n,m_n}(u_n),\end{aligned}$$

where $p_{k_i,m_i}(u_i):=\binom{m_i}{k_i}u_i^{k_i}(1-u_i)^{m_i-k_i}$ is the probability mass function of the binomial distribution $\mathrm{Bin}(m_i,u_i)$. The Bernstein copulas are constructed for approximating complicated copulas in polynomial forms.

Given copulas C and D, Yang et al. (2015) presented a composite Bernstein copula (CBC) as follows,

$$\begin{aligned}&C_{m_1,\dots,m_n}(u_1,\dots,u_n|C,D)\\&=\mathbb{E}\left[C\left(\frac{F^{-1}_{Bin(m_1,u_1)}(U_1^D)}{m_1},\dots,\frac{F^{-1}_{Bin(m_n,u_n)}(U_n^D)}{m_n}\right)\right],\end{aligned}$$

where $(U_1^D,\dots,U_n^D)$ is a random vector with the distribution $\overline{D}$, the survival copula of D, and $F^{-1}_{Bin(m_i,u_i)}$ denotes the left-continuous inverse function of the binomial distribution function $F_{Bin(m_i,u_i)}$. The copulas C and D are called the target copula and the base copula of the CBC, respectively. The integers $m_1,\dots,m_n$ are called the smooth coefficients of the CBC. It is easy to verify that when $D=\Pi$, the CBC is actually a Bernstein copula, which means that the CBC can be regarded as a generalization of the Bernstein copula.

2.2 The definition of bivariate BF copulas

Consider a two-dimensional CBC. Let the target copula C be the bivariate Fréchet copula (1) and the base copula D be the independent copula Π. Then the bivariate BF copula is defined as

$$C_{m,m}(u,v;\alpha,\beta,\gamma):=C_{m,m}(u,v|F,\Pi), \tag{2}$$

where F is the bivariate Fréchet copula given by (1) and $m\ge 2$ is a fixed integer. By Proposition 2.3 in Yang et al. (2015), we have

$$\begin{aligned}&C_{m,m}(u,v;\alpha,\beta,\gamma)\\&=C_{m,m}(u,v|\alpha M+\beta\Pi+\gamma W,\Pi)\\&=\alpha C_{m,m}(u,v|M,\Pi)+\beta C_{m,m}(u,v|\Pi,\Pi)\\&\quad+\gamma C_{m,m}(u,v|W,\Pi).\end{aligned} \tag{3}$$

Using the expression of CBC (Guo, Wang and Yang, 2017, Theorem 2.1), we can get the explicit forms

$$C_{m,m}(u,v|M,\Pi)=\frac{1}{m}\sum_{k=1}^{m}O_{k,m}(u)O_{k,m}(v)$$

and

$$C_{m,m}(u,v|W,\Pi) = \frac{1}{m}\sum_{k=1}^{m} O_{k,m}(u)O_{m+1-k,m}(v),$$

where $O_{k,m}(x) = \sum_{j=k}^{m} p_{j,m}(x)$. Thus the explicit expression of (2) can be given as

$$\begin{aligned} C_{m,m}(u,v;\alpha,\beta,\gamma) &= \frac{\alpha}{m}\sum_{k=1}^{m} O_{k,m}(u)O_{k,m}(v) \\ &+ \frac{\gamma}{m}\sum_{k=1}^{m} O_{k,m}(u)O_{m+1-k,m}(v) + \beta uv. \end{aligned}$$

The correlation coefficient of the bivariate BF copula $C_{m,m}(u,v;\alpha,\beta,\gamma)$ can also be calculated. By detailed calculation we know that

$$\begin{aligned} \rho_+ :=& 12\int_0^1\int_0^1 uv\, dC_{m,m}(u,v|M,\Pi) - 3 \\ =& \frac{12}{m}\sum_{k=1}^{m}\int_0^1 udO_{k,m}(u)\int_0^1 vdO_{k,m}(v) - 3 \\ =& 12m\sum_{k=1}^{m}\int_0^1 up_{k-1,m-1}(u)du\int_0^1 vp_{k-1,m-1}(v)dv - 3 \\ =& 12m\sum_{k=1}^{m}\frac{k}{m(m+1)}\cdot\frac{k}{m(m+1)} - 3 = \frac{m-1}{m+1}, \end{aligned}$$

and

$$\rho_- := 12\int_0^1\int_0^1 uv\, dC_{m,m}(u,v|W,\Pi) - 3 = -\frac{m-1}{m+1}.$$

Therefore the correlation coefficient of the bivariate BF copula (2) is

$$\rho = \alpha\cdot\rho_+ + \beta\cdot 0 + \gamma\cdot\rho_- = (\alpha-\gamma)\frac{m-1}{m+1}.$$

From the expression, we find that the range of this correlation coefficient is $[-\frac{m-1}{m+1}, \frac{m-1}{m+1}]$.

Next we provide a proposition about the symmetry of parameters in bivariate BF copulas.

Proposition 2.1. For $u,v\in[0,1]$, we have

(a) $u - C_{m,m}(u,1-v;\alpha,\beta,\gamma) = C_{m,m}(u,v;\gamma,\beta,\alpha)$;
(b) $v - C_{m,m}(1-u,v;\alpha,\beta,\gamma) = C_{m,m}(u,v;\gamma,\beta,\alpha)$;
(c) $u+v-1+C_{m,m}(1-u,1-v;\alpha,\beta,\gamma) = C_{m,m}(u,v;\alpha,\beta,\gamma)$.

Proof. It is easy to check that

$$\begin{aligned} u - C_{m,m}(u,1-v|M,\Pi) &= C_{m,m}(u,v|W,\Pi), \\ u - C_{m,m}(u,1-v|W,\Pi) &= C_{m,m}(u,v|M,\Pi), \\ u - \Pi(u,1-v) &= \Pi(u,v). \end{aligned}$$

Then from (3) we have

$$\begin{aligned} & u - C_{m,m}(u,1-v;\alpha,\beta,\gamma) \\ &= \alpha(u - C_{m,m}(u,1-v|M,\Pi)) + \gamma(u - C_{m,m}(u,1-v|W,\Pi)) \\ &\quad + (1-\alpha-\gamma)uv \\ &= \alpha C_{m,m}(u,v|W,\Pi) + \gamma C_{m,m}(u,v|M,\Pi) + (1-\alpha-\gamma)uv \\ &= C_{m,m}(u,v;\gamma,\beta,\alpha). \end{aligned}$$

Thus (a) is proved.

Similarly, we can prove (b) and (c). □

We end this part with a proposition about the uniqueness of bivariate BF copulas.

Proposition 2.2. Let the copulas $C_{m,m}(u,v;\alpha,\beta,\gamma)$ and $C_{m,m}(u,v;\alpha',\beta',\gamma')$ be two bivariate BF copulas with $m\ge 3$. If $(\alpha,\beta,\gamma)\neq(\alpha',\beta',\gamma')$, the two copulas $C_{m,m}(u,v;\alpha,\beta,\gamma)$ and $C_{m,m}(u,v;\alpha',\beta',\gamma')$ are different.

Proof. Let two Fréchet copulas be

$$F = \alpha M + \beta\Pi + \gamma W, F' = \alpha' M + \beta'\Pi + \gamma' W,$$

where $\alpha,\beta,\gamma,\alpha',\beta',\gamma'\ge 0$ and

$$\alpha+\beta+\gamma = 1, \alpha'+\beta'+\gamma' = 1.$$

The assumption $(\alpha,\beta,\gamma)\neq(\alpha',\beta',\gamma')$ implies that $F\neq F'$.

We assume $C_{m,m}(u,v;\alpha,\beta,\gamma) = C_{m,m}(u,v;\alpha',\beta',\gamma')$, i.e.,

$$\begin{aligned} & C_{m,m}(u,v|\alpha M + \beta\Pi + \gamma W,\Pi) \\ &\quad = C_{m,m}(u,v|\alpha' M + \beta'\Pi + \gamma' W,\Pi). \end{aligned}$$

Note that

$$\begin{aligned} & C_{m,m}(u,v|\alpha M + \beta\Pi + \gamma W,\Pi) \\ &= \alpha C_{m,m}(u,v|M,\Pi) + (1-\alpha-\gamma)C_{m,m}(u,v|\Pi,\Pi) \\ &\quad + \gamma C_{m,m}(u,v|W,\Pi) \\ &= (1-\alpha-\gamma)uv + \frac{\alpha}{m}\sum_{k=1}^{m} O_{k,m}(u)O_{k,m}(v) \\ &\quad + \frac{\gamma}{m}\sum_{k=1}^{m} O_{k,m}(u)O_{m+1-k,m}(v). \end{aligned}$$

The correlation coefficient of this copula is $(\alpha-\gamma)\frac{m-1}{m+1}$ as we have calculated. Thus

$$\frac{m-1}{m+1}(\alpha-\gamma) = \frac{m-1}{m+1}(\alpha'-\gamma') \tag{4}$$

follows. Moreover, the density of this copula is

$$\begin{aligned} & \frac{\partial^2 C_{m,m}(u,v|\alpha M + (1-\alpha-\gamma)\Pi + \gamma W,\Pi)}{\partial u\partial v} \\ &= (1-\alpha-\gamma) + \alpha m\sum_{k=1}^{m} p_{k-1,m-1}(u)p_{k-1,m-1}(v) \end{aligned}$$

$$+\gamma m\sum_{k=1}^{m} p_{k-1,m-1}(u)p_{m-k,m-1}(v).$$

Then we can get the coefficient of term uv in copula $C_{m,m}(u,v;\alpha,\beta,\gamma)$. It is

$$\left.\frac{\partial^2 C_{m,m}(u,v|\alpha M+(1-\alpha-\gamma)\Pi+\gamma W,\Pi)}{\partial u\partial v}\right|_{(0+,0+)}$$
$$=(1-\alpha-\gamma)+m\alpha=1+(m-1)\alpha-\gamma.$$

Therefore we have another equation

$$1+(m-1)\alpha-\gamma=1+(m-1)\alpha'-\gamma'. \tag{5}$$

Because $m\geq 3$, from (4) and (5) we can get $\alpha=\alpha'$ and $\gamma=\gamma'$, thus $F=F'$ follows, contradicting to the fact that F and F' are two different copulas. □

From the above proposition, we can see that the three copulas $C_{m,m}(u,v|M,\Pi)$, $C_{m,m}(u,v|\Pi,\Pi)$, and $C_{m,m}(u,v|W,\Pi)$ are different.

2.3 Some classes of bivariate BF copulas

The bivariate BF copulas contain important classes by choosing suitable parameters α, β, γ, and m. In the following, we list some classes of bivariate BF copulas.

First, the family of bivariate BF copulas can be seen as an extension of FGM copulas. For the case $m=2$,

$$\begin{aligned}C_{2,2}(u,v;\alpha,\beta,\gamma)&=C_{2,2}(u,v|\alpha M+\beta\Pi+\gamma W,\Pi)\\&=uv[1+(\alpha-\gamma)(1-u)(1-v)],\end{aligned}$$

which becomes a FGM copula. Note that the FGM copula fails to capture higher correlation between random variables since its correlation coefficient is within the range of $[-\frac{1}{3},\frac{1}{3}]$. However, bivariate BF copulas with larger m extend this range. For instance, when $m=4$, the range is $[-\frac{3}{5},\frac{3}{5}]$, which is wide enough in many applications.

Second, some bivariate BF copulas belong to the copula family with cubic sections (Nelsen, Quesada-Molina and Rodríguez-Lallena, 1997). In the case when $m=3$, we have

$$C_{3,3}(u,v;q,1-q,0)=uv+u(1-u)[\alpha_1(v)(1-u)+\beta_1(v)u],$$

and

$$C_{3,3}(u,v;0,1-q,q)=uv+u(1-u)[\alpha_2(v)(1-u)+\beta_2(v)u],$$

where

$$\begin{aligned}&\alpha_1(v)=q(v^3-3v^2+2v), &&\beta_1(v)=q(-v^3+v),\\&\alpha_2(v)=q(v^3-v), &&\beta_2(v)=q(-v^3+3v^2-2v).\end{aligned}$$

The derivatives of above functions are as follows,

$$\begin{aligned}\alpha_1'(v)&=q(3v^2-6v+2)\in[-1,2],\\\beta_1'(v)&=q(-3v^2+1)\in[-2,1],\\\alpha_2'(v)&=q(3v^2-1)\in[-1,2],\\\beta_2'(v)&=q(-3v^2+6v-2)\in[-2,1].\end{aligned}$$

Thus, the two copulas belong to the copula family with cubic sections (see Nelsen, Quesada-Molina and Rodríguez-Lallena, 1997, Theorem 2.4).

Third, the copulas of Baker's bivariate distributions in Baker (2008) belong to the family of bivariate BF copulas. Let

$$H_+^{(m)}(x,y)=\frac{1}{m}\sum_{k=1}^{m}O_{k,m}(F(x))O_{k,m}(G(y))$$

and

$$H_-^{(m)}(x,y)=\frac{1}{m}\sum_{k=1}^{m}O_{k,m}(F(x))O_{m+1-k,m}(G(y)),$$

where $O_{k,m}(x)=\sum_{i=k}^{m}\binom{m}{i}x^i(1-x)^{m-i}$, $x\in[0,1]$, and $F(x)$ and $G(y)$ are two marginal distribution functions. Baker's bivariate distributions are defined as

$$H_{+,q}^{(m)}(x,y)=(1-q)F(x)G(y)+qH_+^{(m)}(x,y),$$

and

$$H_{-,q}^{(m)}(x,y)=(1-q)F(x)G(y)+qH_-^{(m)}(x,y),$$

where $q\in[0,1]$. Guo, Wang and Yang (2017) pointed out that the copulas of above two distributions can be expressed as

$$C_{+,q}^{(m)}(u,v)=(1-q)uv+\frac{q}{m}\sum_{k=1}^{m}O_{k,m}(u)O_{k,m}(v)$$

and

$$C_{-,q}^{(m)}(u,v)=(1-q)uv+\frac{q}{m}\sum_{k=1}^{m}O_{k,m}(u)O_{m+1-k,m}(v)$$

respectively. The two copulas $C_{+,q}^{(m)}$ and $C_{-,q}^{(m)}$ belong to the family of bivariate BF copulas because

$$\begin{aligned}C_{+,q}^{(m)}(u,v)&=(1-q)uv+\frac{q}{m}\sum_{k=1}^{m}O_{k,m}(u)O_{k,m}(v)\\&=C_{m,m}(u,v;q,1-q,0)\end{aligned} \tag{6}$$

and

$$\begin{aligned}C_{-,q}^{(m)}(u,v)&=(1-q)uv+\frac{q}{m}\sum_{k=1}^{m}O_{k,m}(u)O_{m+1-k,m}(v)\\&=C_{m,m}(u,v;0,1-q,q).\end{aligned} \tag{7}$$

The copulas given by (6) and (7) are bivariate BF copulas with γ or α being zero. Hereinafter, we use $C_{+,q}^{(m)}$ and

$C^{(m)}_{-,q}$ to denote the two copulas $C_{m,m}(u,v;q,1-q,0)$ and $C_{m,m}(u,v;0,1-q,q)$ for simplicity, and the two copulas $C^{(m)}_{+,q}$ and $C^{(m)}_{-,q}$ are called positive BF copula and negative BF copula, respectively.

Remark 2.1. *Baker (2008) proposed a multi-dimensional distribution version*

$$H(x_1,\dots,x_n)=(1-q)\prod_{i=1}^{n}F^{(i)}(x_i)+\frac{q}{m}\sum_{k=1}^{m}\prod_{i=1}^{n}O_{k,m}(F^{(i)}(x_i)),$$

where $F^{(i)}(x_i)$'s are univariate marginal distributions. The copula of this distribution is

$$C(u_1,\dots,u_n)=(1-q)\prod_{i=1}^{n}u_i+\frac{q}{m}\sum_{k=1}^{m}\prod_{i=1}^{n}O_{k,m}(u_i). \tag{8}$$

It is a higher-dimensional extension of (6). We will discuss this type in Section 3.

2.4 Supermigrativity of positive BF copulas

A symmetric bivariate copula C is called supermigrative if for any $0\le v\le u\le 1$ and $0\le\alpha\le 1$, the inequality $C(\alpha u,v)\ge C(u,\alpha v)$ holds, and C is called submigrative if the reversing inequality can be obtained. Supermigrativity is a positive dependence property implying the positive quadrant dependence (PQD) (Nelsen, 2006; Durante and Ricci, 2009). For more details about supermigrativity, please refer to Durante and Ghiselli-Ricci (2012).

Guo, Wang and Yang (2017) showed that positive BF copula $C^{(m)}_{+,q}$ is positive quadrant dependent and $C^{(m)}_{-,q}$ is negative quadrant dependent, i.e., $C^{(m)}_{+,q}(u,v)\ge\Pi(u,v)$ and $C^{(m)}_{-,q}(u,v)\le\Pi(u,v)$. In this subsection we show the supermigrativity of $C^{(m)}_{+,q}$. For this purpose, we need a lemma in Durante and Ghiselli-Ricci (2012).

Lemma 2.1 (Durante and Ghiselli-Ricci, 2012)**.** *Let C be a symmetric copula that has continuous first-order partial derivatives on $[0,1]^2$. Then C is supermigrative if and only if for all $(u,v)\in[0,1]^2$,*

$$(\ln(u)-\ln(v))\,(u\partial_u C(u,v)-v\partial_v C(u,v))\le 0.$$

By applying Lemma 2.1, we can obtain the following theorem.

Theorem 2.1. The positive BF copula $C^{(m)}_{+,q}$ is supermigrative.

Proof. Note that the copula (6) is a convex summation of an independent copula Π and the copula $C_{m,m}(u,v|M,\Pi)$. Thus it is sufficient to prove that $C_{m,m}(u,v|M,\Pi)$ is supermigrative since Π is supermigrative obviously.

We prove the supermigrativity of $C_{m,m}(u,v|M,\Pi)$ by applying Lemma 2.1. This needs to show that

$$u\frac{\partial C_{m,m}(u,v|M,\Pi)}{\partial u}>v\frac{\partial C_{m,m}(u,v|M,\Pi)}{\partial v}$$

under the case $0\le u<v\le 1$. The case of $0\le v<u\le 1$ can be proven in the same way due to the symmetry of $C_{m,m}(u,v|M,\Pi)$.

Using $O'_{k,m}(x)=mp_{k-1,m-1}(x)$, we can get

$$\begin{aligned}u\frac{\partial C_{m,m}(u,v|M,\Pi)}{\partial u}&=u\sum_{k=1}^{m}p_{k-1,m-1}(u)O_{k,m}(v)\\&=u\sum_{k=1}^{m}\binom{m-1}{k-1}u^{k-1}(1-u)^{m-k}\sum_{i=k}^{m}\binom{m}{i}v^i(1-v)^{m-i}\end{aligned}$$

and

$$\begin{aligned}v\frac{\partial C_{m,m}(u,v|M,\Pi)}{\partial v}&=v\sum_{k=1}^{m}p_{k-1,m-1}(v)O_{k,m}(u)\\&=v\sum_{k=1}^{m}\binom{m-1}{k-1}v^{k-1}(1-v)^{m-k}\sum_{i=k}^{m}\binom{m}{i}u^i(1-u)^{m-i}.\end{aligned}$$

Note that

$$\begin{aligned}&\binom{m-1}{k-1}u^k(1-u)^{m-k}\binom{m}{k}v^k(1-v)^{m-k}\\&\quad-\binom{m-1}{k-1}v^k(1-v)^{m-k}\binom{m}{k}u^k(1-u)^{m-k}=0.\end{aligned}$$

Then we have

$$\begin{aligned}&u\frac{\partial C_{m,m}(u,v|M,\Pi)}{\partial u}-v\frac{\partial C_{m,m}(u,v|M,\Pi)}{\partial v}\\&=\sum_{k=1}^{m}\sum_{i=k+1}^{m}\binom{m-1}{k-1}u^k(1-u)^{m-k}\binom{m}{i}v^i(1-v)^{m-i}\\&\qquad-\sum_{k=1}^{m}\sum_{i=k+1}^{m}\binom{m-1}{k-1}v^k(1-v)^{m-k}\binom{m}{i}u^i(1-u)^{m-i}\\&=\sum_{k=1}^{m}\sum_{i=k+1}^{m}\binom{m-1}{k-1}\binom{m}{i}(uv)^k((1-u)(1-v))^{m-k}\\&\qquad\times[v^{i-k}(1-v)^{k-i}-u^{i-k}(1-u)^{k-i}]\\&=\sum_{k=1}^{m}\sum_{i=k+1}^{m}\binom{m-1}{k-1}\binom{m}{i}(uv)^k((1-u)(1-v))^{m-k}\\&\qquad\times[(\frac{v}{1-v})^{i-k}-(\frac{u}{1-u})^{i-k}]>0,\end{aligned}$$

since function $f(x)=\frac{x}{1-x}$ is monotonically increasing on $(0,1)$. Therefore

$$u\frac{\partial C_{m,m}(u,v|M,\Pi)}{\partial u}>v\frac{\partial C_{m,m}(u,v|M,\Pi)}{\partial v}$$

holds for any $0\le u<v\le 1$. The case of $0\le v<u\le 1$ is similar. Thus by Lemma 2.1 the copula $C^{(m)}_{+,q}$ is supermigrative. □

Remark 2.2. *Durante and Ghiselli-Ricci (2012) pointed out that the bivariate Gaussian copula C_ρ is supermigrative* *if and only if $\rho \geq 0$. Besides, one can check that the bivariate Fréchet copula $F(u,v) = \alpha M(u,v) + \beta\Pi(u,v) + \gamma W(u,v)$ is supermigrative when $\gamma = 0$. From the above theorem, positive BF copulas are supermigrative.*

2.5 TP_2 property of positive BF copulas

The total positivity of order 2 (TP_2), defined by Karlin (1968), is a stronger positive dependence property than PQD. A nonnegative function $f(x,y)$ is called TP_2 if $f(x,y)f(x',y') \geq f(x',y)f(x,y')$ whenever $x \leq x'$, $y \leq y'$.

The TP_2 property is a strong property that implies all other quadrant dependence properties in Nelsen (2006). For more information about the TP_2 property and its applications, readers may refer to the monograph Gasca and Micchelli (1996). Note that the TP_2 property of the copula $C_{m,m}(u,v|M,\Pi)$ was proved in Dou, Kuriki and Lin (2013). In this part we are interested in investigating the TP_2 property of the positive BF copula $C^{(m)}_{+,q}$.

For a bivariate distribution H, it is known that if the density of H is TP_2, so is H itself (see Balakrishnan and Lai, 2009 and Theorem 9 in Lin, Dou and Kuriki, 2019). By applying this character, Dou, Kuriki and Lin (2013) proved the TP_2 property of $C_{m,m}(u,v|M,\Pi)$ by showing that its density is TP_2.

For the positive BF copula $C^{(m)}_{+,q}$, one can numerically test that TP_2 property does not hold for its density function. Next theorem states the TP_2 property of $C^{(m)}_{+,q}$. Its proof is given in Appendix A.

Theorem 2.2. The bivariate positive BF copula $C^{(m)}_{+,q}$ admits TP_2 property, i.e., for any $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$,

$$C^{(m)}_{+,q}(x_2,y_2)C^{(m)}_{+,q}(x_1,y_1) \geq C^{(m)}_{+,q}(x_1,y_2)C^{(m)}_{+,q}(x_2,y_1).$$

Remark 2.3. *Gupta, Kirmani and Srivastava (2010) showed that Gaussian copulas are TP_2 when the parameter $\rho \geq 0$. As for the bivariate Fréchet copula $F = \alpha M + \beta\Pi + \gamma W$, one can check that F is TP_2 when $\gamma = 0$. The above theorem states that the TP_2 property holds for the bivariate positive BF copula $C^{(m)}_{+,q}$.*

2.6 Comparisons with bivariate Gaussian copulas and bivariate Fréchet copulas

In this subsection we carry out some comparisons among bivariate BF copulas, bivariate Fréchet copulas, and bivariate Gaussian copulas. We find that these copulas share some similar properties as follows:

1. Supermigrativity: the Gaussian copula C_ρ is supermigrative when $\rho \geq 0$, the bivariate Fréchet copula $F(u,v) = \alpha M(u,v) + \beta\Pi(u,v) + \gamma W(u,v)$ is supermigrative when $\gamma = 0$, and the positive BF copula is supermigrative.
2. TP_2: the Gaussian copula C_ρ is TP_2 when the parameter $\rho \geq 0$, the bivariate Fréchet copula $F(u,v) = \alpha M + \beta\Pi + \gamma W$ is TP_2 when $\gamma = 0$, and the bivariate positive BF copulas are TP_2.

Comparing with bivariate Gaussian copulas and bivariate Fréchet copulas, the advantages of the bivariate BF copulas can be stated as follows:

1. Compared with the bivariate Fréchet copula, the bivariate BF copula admits density function. It can be treated as a continuous modification of the bivariate Fréchet copula. This makes the maximum likelihood estimation of the parameters in a bivariate BF copula feasible.
2. There are three parameters in a bivariate BF copula, while the bivariate Gaussian copula has only one parameter. When the correlation coefficient is fixed, there are more freedoms to adjust the parameters in a bivariate BF copula.
3. Compared with Gaussian copulas, bivariate BF copulas have explicit expression in polynomial forms and the copula functions are easily applied.

Figures 1-4 show the scatter plots of the three copula functions. Figure 1 is the scatter plot of a bivariate Gaussian copula with correlation parameter $\rho = 0.65$. Figure 2 displays the scatter plot of a positive BF copula (6) with $q = 0.65$ and $m = 20$. Figure 3 shows the scatter plot of a Fréchet copula (1) with $\alpha = 0.8, \beta = 0.05$, and $\gamma = 0.15$. In contrast, Figure 4 is the scatter plot of a bivariate BF copula with $m = 20$ and other parameters equal to those of the bivariate Fréchet copula in Figure 3.

The scatters drawn from bivariate BF copulas are more concentrated along the diagonal than those from Gaussian copulas. Actually, the level of concentration in the bivariate BF copula can be adjusted by the smooth coefficient m. The scatters get more concentrated while the value of m becomes larger.

Figure 5 and 6 show the scatter plots of bivariate distributions constructed by the copulas in Figure 1 and 4 with univariate standard normal marginal distributions. One can see that the second one provides two types of dependence, both the positive and the negative.

3. MULTIVARIATE BERNSTEIN FRÉCHET COPULAS

In this section, we introduce a multivariate copula, called the multivariate Bernstein Fréchet (BF) copula. The multivariate BF copula has bivariate BF copulas as its bivariate margins. We show that the copula function is uniquely determined by these bivariate margins.

To define the copula function, we need the copula $C^{\mathcal{A},\mathcal{B}}$ in Yang, Qi and Wang (2009). This copula is applied for defining multivariate BF copulas. Then the uniqueness and existence of multivariate BF copulas are studied. The methods

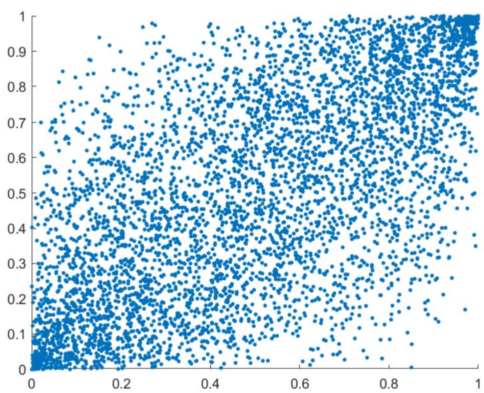

Figure 1. Scatter plot of bivariate Gaussian copula with $\rho = 0.65$.

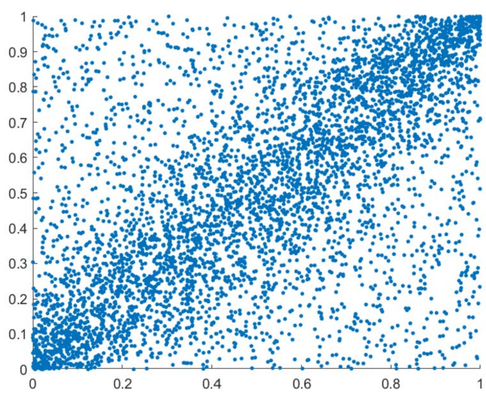

Figure 2. Scatter plot of positive BF copula with coefficients $q = 0.65$ *and* $m = 20$.

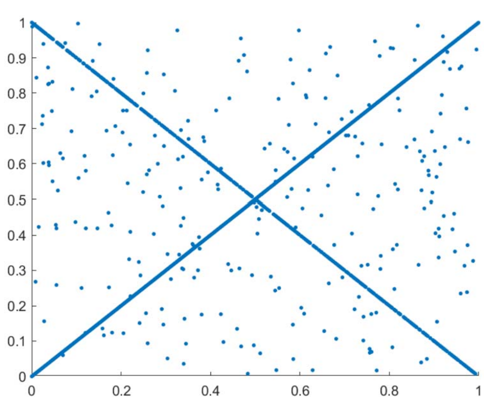

Figure 3. Scatter plot of bivariate Fréchet copula with coefficients $\alpha = 0.8, \gamma = 0.15$.

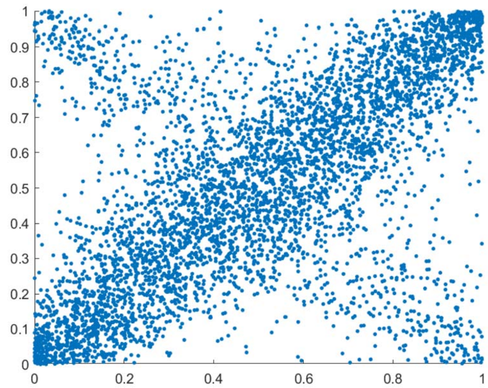

Figure 4. Scatter plot of a general bivariate BF copula with coefficients $\alpha = 0.8, \gamma = 0.15$ *and* $m = 20$.

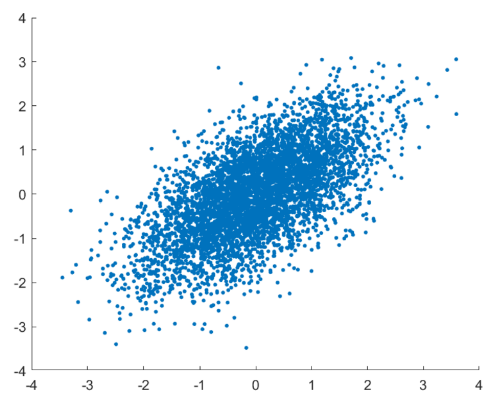

Figure 5. Scatter plots of bivariate distributions constructed by the copulas used in Figure 1, with standard normal marginal distributions.

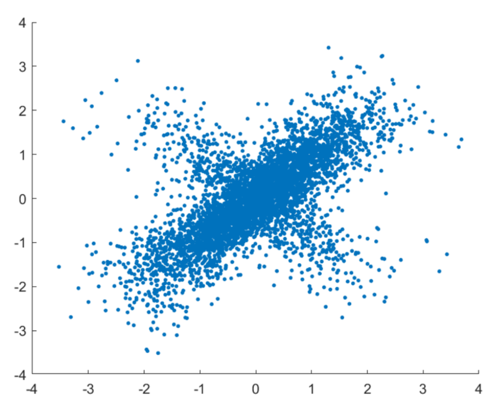

Figure 6. Scatter plots of bivariate distributions constructed by the copulas used in Figure 4, with standard normal marginal distributions.

of determining a multivariate BF copula from its bivariate margins are given. In addition, an application of multivariate BF copulas for modeling portfolio risk is discussed.

3.1 Brief introduction to $C^{\mathcal{A},\mathcal{B}}$ copulas

Yang, Qi and Wang (2009) introduced a family of multivariate copulas with all bivariate marginal copulas being bivariate Fréchet copulas, and each copula in the family is uniquely determined by its bivariate marginal copulas. In this subsection, the copula is introduced for the purpose of defining the multivariate BF copula.

Given $n+1$ random variables $U, U_1, U_2, \ldots, U_n$ with the uniform distribution $\mathcal{U}[0,1]$, we assume that $(U_1, \ldots, U_n)$ are conditionally independent on the common factor U and the copulas for (U, U_i), $i = 1, \ldots, n$ are

$$C(u, u_i) = a_{i,1}M(u, u_i) + a_{i,2}\Pi(u, u_i) + a_{i,3}W(u, u_i), \tag{9}$$

where $a_{i,1}, a_{i,2}, a_{i,3} \geq 0$ and $a_{i,1} + a_{i,2} + a_{i,3} = 1$. Yang, Qi and Wang (2009) proved that under above assumptions the copula of $(U_1, \cdots, U_n)$ has the form

$$\begin{aligned} &C^{\mathcal{A},\mathcal{B}}(u_1, \ldots, u_n) \\ &= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i} \right) C^{(j_1,\ldots,j_n)}(u_1, \ldots, u_n), \end{aligned} \tag{10}$$

where

$$\begin{aligned} &C^{(j_1,j_2,\ldots,j_n)}(u_1, u_2, \ldots, u_n) \\ &= W\left(\min_{i\leq n, j_i=1}\{u_i\}, \min_{i\leq n, j_i=3}\{u_i\} \right) \prod_{i\leq n, j_i=2} u_i, \end{aligned} \tag{11}$$

with convention that the minimum and product of empty set $\emptyset$ are defined to be 1. The copula (10) is named $C^{\mathcal{A},\mathcal{B}}$ copula. One desirable property of the copula is that its bivariate marginal copulas are bivariate Fréchet copulas

$$\begin{aligned} &C_{i,j}(u_i, u_j) \\ =&\alpha_{i,j}M(u_i, u_j) + \beta_{i,j}\Pi(u_i, u_j) + \gamma_{i,j}W(u_i, u_j), \end{aligned} \tag{12}$$

where $\alpha_{i,j} = a_{i,1}a_{j,1} + a_{i,3}a_{j,3}$, $\gamma_{i,j} = a_{i,1}a_{j,3} + a_{i,3}a_{j,1}$ and $\beta_{i,j} = 1 - \alpha_{i,j} - \gamma_{i,j}$.

Let the set of all copulas $C^{(j_1,j_2,\cdots,j_n)}, j_i = 1, 2, 3, i \leq n$ be $\mathcal{S}_n$, then there are $\frac{1}{2}(3^n - 2n + 1)$ different copulas in $\mathcal{S}_n$ (see Yang, Qi and Wang, 2009, Proposition 2.2). We present a proposition which shows the relationship between these copulas and their bivariate marginal copulas.

Proposition 3.1. The copula $C^{(j_1,j_2,\ldots,j_n)}$ is uniquely determined by its bivariate marginal copulas.

Proof. Suppose that the random vector $(U_1, \ldots, U_n)$ has copula $C^{(j_1,j_2,\ldots,j_n)}$. In the set $I = \{U_1, \ldots, U_n\}$, we write $U_i \sim U_j$ if $U_i = U_j$ with probability 1 (which actually means the bivariate copula of (U_i, U_j) is M). It is easy to verify that such defined $\sim$ is an equivalence relation on the set I. There exist at most two equivalence classes (including empty class) because the dependence structures between those variables can be comonotonic or countermonotonic excluding independence. Denote the two classes by I_1 and I_3, and let $I_2 = I - I_1 \cup I_3$. Then one can check

$$\begin{aligned} &C^{(j_1,j_2,\ldots,j_n)}(u_1, u_2, \ldots, u_n) \\ &\quad = W\left(\min\{u_i, i \in I_1\}, \min\{u_i, i \in I_3\}\right) \prod_{i\in I_2} u_i. \end{aligned}$$

Thus the copula $C^{(j_1,j_2,\ldots,j_n)}$ is uniquely determined by its bivariate marginal copulas. $\square$

Remark 3.1. *The above proposition shows that if two copulas $C^{(j_1,j_2,\ldots,j_n)}$ and $C^{(j'_1,j'_2,\ldots,j'_n)}$ share the same bivariate marginal copulas, they are the same.*

3.2 The definition of multivariate BF copulas

Starting from a CBC, by choosing $C^{\mathcal{A},\mathcal{B}}$ as its target copula and the independent copula Π as its base copula, the multivariate BF copula is defined as

$$C_{m,m,\ldots,m}\left(u_1, u_2, \ldots, u_n | C^{\mathcal{A},\mathcal{B}}, \Pi\right), \quad u_1, u_2, \ldots, u_n \in [0,1],$$

where $m \geq 2$ is a positive integer. Note that the multivariate BF copula can be written as

$$\begin{aligned} &C_{m,m,\ldots,m}\left(u_1, u_2, \ldots, u_n | C^{\mathcal{A},\mathcal{B}}, \Pi\right) \\ &= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i} \right) \\ &\quad \times C_{m,m,\ldots,m}\left(u_1, u_2, \ldots, u_n | C^{(j_1,j_2,\ldots,j_n)}, \Pi\right). \end{aligned} \tag{13}$$

This expression shows that the multivariate BF copula can be expressed as a linear combination of some copulas in the form of $C_{m,m,\ldots,m}\left(u_1, u_2, \ldots, u_n | C^{(j_1,j_2,\ldots,j_n)}, \Pi\right)$.

We now provide the explicit expression of the multivariate BF copula, and prove that for different basis copula $C_i \in \mathcal{S}_n$, the copula $C_{m,\cdots,m}(u_1, \ldots, u_n | C_i, \Pi)$ is different.

Theorem 3.1. (i) We have

$$\begin{aligned} &C_{m,m,\ldots,m}\left(u_1, u_2, \ldots, u_n | C^{(j_1,j_2,\ldots,j_n)}, \Pi\right) \\ =&\frac{1}{m} \prod_{i\leq n, j_i=2} u_i \sum_{k=1}^{m} \Bigg[\prod_{i\leq n, j_i=1} O_{k,m}(u_i) \\ &\times \prod_{i\leq n, j_i=3} O_{m+1-k,m}(u_i) \Bigg]. \end{aligned} \tag{14}$$

(ii) The multivariate BF copula can be represented as

$$
\begin{aligned}
&C_{m,m,\ldots,m}\left(u_1,u_2,\ldots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi\right)\\
&=\frac{1}{m}\sum_{j_1=1}^{3}\cdots\sum_{j_n=1}^{3}\left(\prod_{i=1}^{n}a_{i,j_i}\right)\\
&\quad\times\prod_{i\le n,j_i=2}u_i\sum_{k=1}^{m}\prod_{i\le n,j_i=1}O_{k,m}(u_i)\prod_{i\le n,j_i=3}O_{m+1-k,m}(u_i).
\end{aligned}
$$

(iii) Let C and C' be two different copulas in $\mathcal{S}_n$, then $C_{m,\ldots,m}(u_1,\ldots,u_n|C,\Pi)$ and $C_{m,\ldots,m}(u_1,\ldots,u_n|C',\Pi)$ are different.

Proof. (i) For simplicity, we consider the case $j_1=j_2=\cdots=j_l=1$, $j_{l+1}=\cdots=j_t=3$, $j_{t+1}=\cdots=j_n=2$ and $0\le l\le t\le n$. In this case,

$$
\begin{aligned}
&C^{(j_1,j_2,\cdots,j_n)}(u_1,u_2,\ldots,u_n)\\
&\quad=W\left(\min_{i\le l}\{u_i\},\min_{l+1\le i\le t}\{u_i\}\right)\cdot\prod_{t+1\le i\le n}u_i.
\end{aligned}
$$

Let $(V_1,\ldots,V_n)$ be a random vector with the distribution $C^{(j_1,j_2,\ldots,j_n)}$. Then the probability

$$
\begin{aligned}
&r_{k_1,k_2,\ldots,k_n}:=\mathbb{P}\left(V_i\in[\frac{k_i-1}{m},\frac{k_i}{m}),i=1,\ldots,n\right)\\
&=\left(\frac{1}{m}\right)^{\#\{i:j_i=2\}}\\
&\times\left(\frac{1}{m}\mathbb{I}\{\exists k,\forall i,j_i=1,k_i=k,\forall i,j_i=3,k_i=m+k-1\}\right)\\
&=\left(\frac{1}{m}\right)^{n-t}\\
&\times\left(\frac{1}{m}\mathbb{I}\{\exists k,\forall i\le l,k_i=k,\forall l+1\le i\le t,k_i=m+k-1\}\right),
\end{aligned}
$$

where $\mathbb{I}$ denotes the indicator function. By Theorem 2.1 in Guo, Wang and Yang (2017), the copula $C_{m,m,\ldots,m}(u_1,u_2,\ldots,u_n|C^{(j_1,j_2,\ldots,j_n)},\Pi)$ can be calculated as follows,

$$
\begin{aligned}
&C_{m,m,\ldots,m}\left(u_1,u_2,\ldots,u_n\middle|\right.\\
&\qquad\left.W(\min_{i\le l}\{u_i\},\min_{l+1\le i\le t}\{u_i\})\prod_{t+1\le i\le n}u_i,\Pi\right)\\
&=\sum_{k_1=1}^{m}\cdots\sum_{k_n=1}^{m}r_{k_1,k_2,\ldots,k_n}\cdot O_{k_1,m}(u_1)\cdots O_{k_n,m}(u_n)\\
&=(\frac{1}{m})^{n-t}\cdot\frac{1}{m}\sum_{k=1}^{m}\sum_{k_{t+1}=1}^{m}\cdots\sum_{k_n=1}^{m}O_{k,m}(u_1)\cdots O_{k,m}(u_l)\\
&\quad\cdot O_{m+1-k,m}(u_{l+1})\cdots O_{m+1-k,m}(u_t)\\
&\quad\cdot O_{k_{t+1},m}(u_{t+1})\cdots O_{k_n,m}(u_n)\\
&=(\frac{1}{m})^{n-t}\cdot\frac{1}{m}\sum_{k=1}^{m}O_{k,m}(u_1)\cdots O_{k,m}(u_l)\\
&\quad\cdot O_{m+1-k,m}(u_{l+1})\cdots O_{m+1-k,m}(u_t)\\
&\quad\cdot\sum_{k_{t+1}=1}^{m}O_{k_{t+1},m}(u_{t+1})\cdots\sum_{k_n=1}^{m}O_{k_n,m}(u_n)\\
&=(\frac{1}{m})^{n-t}\cdot\frac{1}{m}\sum_{k=1}^{m}O_{k,m}(u_1)\cdots O_{k,m}(u_l)\\
&\quad\cdot O_{m+1-k,m}(u_{l+1})\cdots O_{m+1-k,m}(u_t)\cdot mu_{t+1}\cdots mu_n\\
&=\frac{1}{m}\prod_{i=t+1}^{n}u_i\sum_{k=1}^{m}\prod_{i\le l}O_{k,m}(u_i)\prod_{l+1\le i\le t}O_{m+1-k,m}(u_i).
\end{aligned}
$$

The general unordered form (14) can be proved similarly.
(ii) This result can be directly deduced from (13) and (14).
(iii) Let $C,C'\in\mathcal{S}_n$, and $C\ne C'$. We assume

$$C_{m,\ldots,m}(u_1,\ldots,u_n|C,\Pi)=C_{m,\ldots,m}(u_1,\ldots,u_n|C',\Pi).$$

Then all bivariate marginal copulas of above two copulas must be the same, and these marginal copulas belong to $\{C_{m,m}(u,v|M,\Pi),C_{m,m}(u,v|\Pi,\Pi),C_{m,m}(u,v|W,\Pi)\}$, due to that the bivariate margins of C and C' belong to $\{M,\Pi,W\}$.

Since $C_{m,m}(u,v|M,\Pi)$, $C_{m,m}(u,v|\Pi,\Pi)$, and $C_{m,m}(u,v|W,\Pi)$ are three different copulas (see Proposition 2.2), C and C' must have the same bivariate marginal copulas to make sure that $C_{m,\ldots,m}(u_1,\ldots,u_n|C,\Pi)=C_{m,\ldots,m}(u_1,\ldots,u_n|C',\Pi)$. Then, by Proposition 3.1, $C=C'$, which is a contradiction. ∎

Since there are $\frac{1}{2}(3^n-2n+1)$ different copulas in $\mathcal{S}_n$, by this theorem we know the total number of the copulas in form of $C_{m,\ldots,m}\left(u_1,\ldots,u_n|C^{(j_1,j_2,\ldots,j_n)},\Pi\right)$ is also $\frac{1}{2}(3^n-2n+1)$. The multivariate copula (8) presented in Remark 2.1 is actually a convex summation of two special copulas among the $\frac{1}{2}(3^n-2n+1)$ copulas. It can be expressed in the following form

$$
\begin{aligned}
C(u_1,\ldots,u_n)&=(1-q)\cdot C_{m,\ldots,m}(u_1,\ldots,u_n|\Pi,\Pi)\\
&\quad+q\cdot C_{m,\ldots,m}(u_1,\ldots,u_n|M,\Pi).
\end{aligned}
$$

In the end of this subsection, we present the probability structure of multivariate BF copulas, which can be used to generate random numbers in practice. For a multivariate BF copula $C_{m,\cdots,m}(u_1,\ldots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$, the sample can be obtained by the following steps:

Algorithm 3.1.

1. *Generate a random vector $(V_1,\ldots,V_n)$ from the $C^{\mathcal{A},\mathcal{B}}$ copula;*
2. *Let $K_i=\lfloor mV_i\rfloor$;*
3. *Generate random numbers from Beta distribution $U_i\sim Beta(K_i+1,m-K_i)$ for $i=1,\ldots,n$.*

Then the copula of $(U_1,\ldots,U_n)$ is exactly the multivariate BF copula (Yang, Wang and Xie, 2020).

3.3 The uniqueness of multivariate BF copula with given bivariate marginal copulas

As we know, Gaussian copulas are uniquely determined by all the bivariate margins because the variance-covariance matrix of a Gaussian copula, is uniquely determined by the relationship between paired components. In the following, we discuss the uniqueness of multivariate BF copulas with given bivariate marginal copulas.

First we recall the marginal property of the CBC (see Yang et al., 2015, Proposition 2.1). Suppose $C_{m,\dots,m}(u_1,\dots,u_n|C,D)$ is a CBC. Then the marginal copulas of the CBC can be obtained by replacing the target copula and the base copula by their corresponding marginal copulas. More precisely, the marginal copula of the i-th and j-th variables is

$$\begin{aligned}&C_{m,\dots,m}(1,\dots,u_i,\dots,u_j,\dots,1|C,D)\\&\quad= C_{m,m}(u_i,u_j|C_{i,j},D_{i,j}),\end{aligned}$$

where $C_{i,j}(u_i,u_j) = C(1,\dots,u_i,\dots,u_j,\dots,1)$ and $D_{i,j}$ is defined similarly. Thus, for the multivariate BF copula $C_{m,\cdots,m}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$, if we denote the bivariate marginal copulas of the target copula $C^{\mathcal{A},\mathcal{B}}$ as

$$\begin{aligned}&C^{\mathcal{A},\mathcal{B}}_{i,j}(u_i,u_j)\\&= d^+_{i,j}M(u_i,u_j)+(1-d^+_{i,j}-d^-_{i,j})\Pi(u_i,u_j)+d^-_{i,j}W(u_i,u_j),\end{aligned}$$

the bivariate marginal copulas of the multivariate BF copula can be obtained as follows,

$$\begin{aligned}&C_{m,m}(u_i,u_j|C^{\mathcal{A},\mathcal{B}}_{i,j},\Pi)\\&\quad= C_{m,m}(u_i,u_j|d^+_{i,j}M+(1-d^+_{i,j}-d^-_{i,j})\Pi+d^-_{i,j}W,\Pi)\end{aligned}\tag{15}$$

where $1\le i<j\le n$.

For the multivariate BF copula $C_{2,\cdots,2}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$, the bivariate marginal copulas are all FGM copulas as we have shown in Section 2.3. Consider the three-dimensional BF copula $C_{2,2,2}(u_1,u_2,u_3|C^{\mathcal{A},\mathcal{B}},\Pi)$. Note that the parameters of bivariate marginal copulas of this copula is

$$\begin{aligned}\alpha_{i,j}-\gamma_{i,j} &= a_{i,1}a_{j,1}+a_{i,3}a_{j,3}-a_{i,1}a_{j,3}-a_{i,3}a_{j,1}\\&=(a_{i,1}-a_{i,3})(a_{j,1}-a_{j,3}),\end{aligned}$$

where the definitions of above variables are given in equations (9)-(12). Let the parameters for two $C^{\mathcal{A},\mathcal{B}}$ copulas be

$$\begin{pmatrix}a_{1,1}\,,a_{2,1}\,,a_{3,1}\\a_{1,3}\,,a_{2,3}\,,a_{3,3}\end{pmatrix}=\begin{pmatrix}0.7\,,0.5\,,0.1\\0.3\,,0.2\,,0.6\end{pmatrix},$$
$$\begin{pmatrix}a'_{1,1}\,,a'_{2,1}\,,a'_{3,1}\\a'_{1,3}\,,a'_{2,3}\,,a'_{3,3}\end{pmatrix}=\begin{pmatrix}0.5\,,0.4\,,0.2\\0.1\,,0.1\,,0.7\end{pmatrix}.$$

The two multivariate BF copulas are different since they have different values at $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$. One can check that they share the same bivariate marginal copulas. The above example shows that the multivariate BF copula $C_{2,\dots,2}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ can not be uniquely determined by its bivariate marginal copulas.

Next, we will show that for fixed parameter $m\ge 3$, the multivariate BF copula $C_{m,\dots,m}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ is uniquely determined by its bivariate margins. That is, from the parameters of the bivariate margins of $C_{m,\dots,m}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$, the multivariate BF copula can be obtained uniquely. The uniqueness property of the multivariate BF copula will allow us to deal with high-dimensional case by focusing on their two-dimensional margins.

Theorem 3.2. For fixed $m\ge 3$, the multivariate BF copula $C_{m,\dots,m}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ is uniquely determined by its bivariate marginal copulas.

Proof. To prove this theorem, we need to show that the parameters of the multivariate BF copula $C_{m,\dots,m}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ can be uniquely determined if its bivariate marginal copulas are given, i.e., parameters in these marginal copulas are known.

Suppose the bivariate marginal copulas of $C_{m,\dots,m}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ are given in the form of (15). The bivariate marginal copulas of the multivariate BF copula are all polynomial functions. For these copulas, we can calculate the correlation coefficient and the coefficient of the term u_iu_j. Denote them by $\rho_{i,j}$ and $c_{i,j}$ respectively.

Then, similar to the proof of Proposition 2.2, we can get an equation system

$$\begin{aligned}\frac{m-1}{m+1}(d^+_{i,j}-d^-_{i,j})&=\rho_{i,j},\\1+(m-1)d^+_{i,j}-d^-_{i,j}&=c_{i,j}.\end{aligned}\tag{16}$$

Because $m\ge 3$, this system is non-singular and we can solve all these numbers $\{d^+_{i,j},d^-_{i,j}|1\le i<j\le n\}$. These numbers are the marginal coefficients of the $C^{\mathcal{A},\mathcal{B}}$ copula in $C_{m,\dots,m}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$. Hence we can get the copula $C^{\mathcal{A},\mathcal{B}}$ by its uniqueness. Therefore, the copula $C_{m,\dots,m}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ is determined. ☐

Remark 3.2. *The non-singularity of system (16) is the key point of the proof. It explains why the condition $m\ge 3$ is important for uniqueness. Note that (16) is singular when $m=2$. This explains why $C_{2,\dots,2}(u_1,\dots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ can not be uniquely determined.*

We have shown that the value of the smooth parameter m can influence the uniqueness property of multivariate BF copulas. When $m=2$, the uniqueness theorem does not hold, but the case is still worth exploring because of its simple form and its connection with FGM copulas. We next state a theorem for this copula that its bivariate margins can be the same as FGM copulas under certain conditions.

Theorem 3.3. Given $\frac{n(n-1)}{2}$ FGM copulas

$$C_{i,j}(u_i,u_j)=u_iu_j[1+d_{i,j}(1-u_i)(1-u_j)], \tag{17}$$

where $d_{i,j}\in[-1,1]$ and $1\le i<j\le n$. If the following $\frac{n(n-1)}{2}$ equations can be solved for these $2n$ unknown variables $a_{i,1}$ and $a_{i,3}$ $(i=1,2,\ldots,n)$

$$(a_{i,1}-a_{i,3})(a_{j,1}-a_{j,3})=d_{i,j},$$

under the constraints $a_{i,1},a_{i,3}\in[0,1]$ and $1-a_{i,1}-a_{i,3}\ge 0$, then there exists a multivariate BF copula $C_{2,\ldots,2}(u_1,\ldots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ with the given marginal FGM copulas, and the $C^{\mathcal{A},\mathcal{B}}$ copula in $C_{2,\ldots,2}(u_1,\ldots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ is specified by these parameters $\{a_{i,1},a_{i,3}|i=1,2,\ldots,n\}$.

The theorem can be proven by substituting the $C^{\mathcal{A},\mathcal{B}}$ into the multivariate BF copula and directly calculating the parameters of its bivariate marginal copulas. So we omit the proof. The theorem discusses a special example of obtaining a multivariate BF copula from some given marginal copulas. The general methods for obtaining multivariate BF copulas with $m\ge 3$ are studied in Section 3.5.

Johnson and Kott (1975) proposed a multivariate copula

$$\begin{aligned}&C(u_1,\ldots,u_n)\\&=\prod_{i=1}^n u_i\left(1+\sum_{1\le k<j\le n}\alpha_{k,j}(1-u_k)(1-u_j)\right),\end{aligned} \tag{18}$$

whose bivariate marginal copulas are FGM copulas, similar to our multivariate BF copulas with the smooth coefficient $m=2$. Note that there are some constraints on the coefficients $\alpha_{k,j}$ of (18),

$$1+\sum_{k,j}\epsilon_k\epsilon_j\alpha_{k,j}\ge 0,\quad 1\le k<j\le n,$$

where $\epsilon_i\in\{1,-1\}$. In the following we give an example to show that there exists a copula $C_{2,\ldots,2}(u_1,\ldots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ which does not belong to the above family.

Example 3.1. *Given a three-dimensional random vector (U_1,U_2,U_3), assume that its bivariate marginal distributions are FGM copulas (17) with the parameters $d_{i,j}$, $1\le i<j\le 3$. Then if $d_{1,2}=d_{1,3}=d_{2,3}=\frac{1}{2}$, the copula of (U_1,U_2,U_3) in the form of (18) does not exist because the equation*

$$1-d_{1,2}-d_{1,3}-d_{2,3}\ge 0$$

does not hold. But the multivariate BF copula $C_{2,2,2}(u_1,u_2,u_3|C^{\mathcal{A},\mathcal{B}},\Pi)$ exists by letting all $a_{i,1}=\sqrt{\frac{1}{2}}$ and $a_{i,3}=0$, $i=1,2,3$.

3.4 On the existence of multivariate BF copulas with given bivariate marginal copulas

Given a family of $\binom{n}{2}$ bivariate copulas, it is natural to ask whether there exists a multivariate BF copula having the given copulas as its bivariate margins. In this subsection, the existence of multivariate BF copulas is discussed when the family of bivariate marginal copulas is given.

Note that for a multivariate BF copula, its bivariate margins are from the family of bivariate BF copulas. In the next, we provide one example to illustrate how to verify whether a bivariate copula is a bivariate BF copula.

Example 3.2. *Let $m=3$. Consider the bivariate BF copula*

$$\begin{aligned}&C_{3,3}(u,v;\alpha,1-\alpha-\gamma,\gamma)\\&\ =\alpha C_{3,3}(u,v|M,\Pi)+(1-\alpha-\gamma)C_{3,3}(u,v|\Pi,\Pi)\\&\quad\ +\gamma C_{3,3}(u,v|W,\Pi),\end{aligned}$$

where $\alpha,\gamma\ge 0$ and $\alpha+\gamma\le 1$. Note that the three copulas are explicitly expressed as followings:

$$C_{3,3}(u,v|M,\Pi)=\begin{pmatrix}u^3 & u^2 & u\end{pmatrix}\begin{pmatrix}2 & -3 & 1\\ -3 & 6 & -3\\ 1 & -3 & 3\end{pmatrix}\begin{pmatrix}v^3\\ v^2\\ v\end{pmatrix},$$

$$C_{3,3}(u,v|\Pi,\Pi)=\begin{pmatrix}u^3 & u^2 & u\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}v^3\\ v^2\\ v\end{pmatrix},$$

$$C_{3,3}(u,v|W,\Pi)=\begin{pmatrix}u^3 & u^2 & u\end{pmatrix}\begin{pmatrix}2 & -3 & 1\\ -3 & 3 & 0\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}v^3\\ v^2\\ v\end{pmatrix}.$$

Then we can obtain that

$$\begin{aligned}&C_{3,3}(u,v;\alpha,1-\alpha-\gamma,\gamma)\\&=\begin{pmatrix}u^3 & u^2 & u\end{pmatrix}\begin{pmatrix}2\alpha+2\gamma & -3\alpha-3\gamma & \alpha+\gamma\\ -3\alpha-3\gamma & 6\alpha+3\gamma & -3\alpha\\ \alpha+\gamma & -3\alpha & 1+2\alpha-\gamma\end{pmatrix}\begin{pmatrix}v^3\\ v^2\\ v\end{pmatrix}.\end{aligned}$$

Thus a bivariate copula C belongs to the family of bivariate BF copulas with $m=3$ if and only if there exists $\alpha^*,\gamma^*\ge 0,\alpha^*+\gamma^*\le 1$ such that

$$C(u,v)=C_{3,3}(u,v;\alpha^*,1-\alpha^*-\gamma^*,\gamma^*).$$

Remark 3.3. *Generally, for a copula $C(u,v)$ from the family of bivariate BF copulas, $C(u,v)$ must be a polynomial function and the integer m can be obtained from the highest power of the function $C(u,v)$. Once m is identified and the correlation coefficient ρ of the copula C is given, from the proofs of Proposition 2.2 and Theorem 3.2, we can get the following two equations for solving α,γ satisfying $\alpha+\gamma\le 1$*

and $\alpha, \gamma \geq 0$,

$$\frac{m-1}{m+1}(\alpha-\gamma)=\rho,$$
$$1+(m-1)\alpha-\gamma=\frac{\partial^2 C(0+,0+)}{\partial u \partial v}.$$

Finally, we have

$$C(u,v)=\alpha C_{m,m}(u,v|M,\Pi) + (1-\alpha-\gamma)C_{m,m}(u,v|\Pi,\Pi)+\gamma C_{m,m}(u,v|W,\Pi).$$

Next we focus on the family of bivariate BF copulas to state the existence of a multivariate BF copula with the bivariate marginal family.

Theorem 3.4. Give $\binom{n}{2}$ bivariate BF copulas

$$C_{i,j}(u,v)=C_{m_{i,j},m_{i,j}}(u,v|d^+_{i,j}M+(1-d^+_{i,j}-d^-_{i,j})\Pi+d^-_{i,j}W,\Pi),$$

$1\leq i<j\leq n$, where $d^+_{i,j}, d^-_{i,j}\geq 0$, and $d^+_{i,j}+d^-_{i,j}\leq 1$. There exists an n-dimensional BF copula C with the family of bivariate margins $\{C_{i,j}, i\leq n, j\leq n\}$ if and only if the following conditions hold:

(1) all $m_{i,j}$'s are the same;
(2) there exists non-negative constants $a_{i,k}$, $i\leq n$, $k=1,2,3$ satisfying $\sum_{k=1}^3 a_{i,k}=1$, and the following equations hold for all $1\leq i<j\leq n$:

$$\begin{cases} a_{i,1}a_{j,1}+a_{i,3}a_{j,3}=d^+_{i,j}, \\ a_{i,1}a_{j,3}+a_{i,3}a_{j,1}=d^-_{i,j}. \end{cases} \tag{19}$$

This theorem can be obtained from the existence theorem of $C^{\mathcal{A},\mathcal{B}}$ copulas (see Yang, Qi and Wang, 2009, Theorem 4.2), thus its proof is omitted.

Finally we discuss one example about the existence of multivariate BF copulas with given pair-wise correlation index, which will be applied in the simulation study of portfolio credit risk in Section 4.

Example 3.3. *For a copula with pair-wise correlation index* $\rho>0$*, it is known that such a dependence feature can be easily obtained by a unique Gaussian copula. However, the same correlation index can be be obtained by more than one multivariate BF copula.*

Suppose that the copula $C_{m,\ldots,m}(u_1,\ldots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ *has pair-wise correlation index* $\rho>0$*. Let the parameters* $a_{i,1}=\sqrt{q}$ *and* $a_{i,3}=0$ *for* $i=1,\ldots,n$*. Then one can check that the copula* $C^{\mathcal{A},\mathcal{B}}$ *have the common bivariate marginal copula* $qM(u,v)+(1-q)\Pi(u,v)$*. Therefore* $C_{m,\ldots,m}(u_1,\ldots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ *has the common bivariate marginal BF copula* $C_{m,m}(u,v;q,1-q,0)$*, whose correlation coefficient is* $\frac{m-1}{m+1}q$*. Thus we can choose* (m,q) *satisfying*

$$\frac{m-1}{m+1}q=\rho.$$

This makes the application of multivariate BF copulas more flexible than Gaussian copulas.

3.5 Methods for determining multivariate BF copulas

After showing that the multivariate BF copula with $m\geq 3$ can be uniquely determined by its bivariate margins, this subsection discusses the methods for obtaining a multivariate BF copulas from given bivariate marginal copulas.

Firstly, suppose that the bivariate marginal copulas of a multivariate BF copula $C_{m,\ldots,m}(u_1,\ldots,u_n|C^{\mathcal{A},\mathcal{B}},\Pi)$ are known, which have the form (15). In order to determine the multivariate BF copula, one needs to obtain the parameters of $C^{\mathcal{A},\mathcal{B}}$ in the multivariate BF copulas. All the parameters of a $C^{\mathcal{A},\mathcal{B}}$ copula can be represented in a matrix

$$A=\begin{pmatrix} a_{1,1},a_{2,1},\ldots,a_{n,1} \\ a_{1,3},a_{2,3},\ldots,a_{n,3}\end{pmatrix},$$

where $a_{i,1}$ and $a_{i,3}$ are defined in (9).

Consider another parameters matrix by exchanging the rows of A,

$$A'=\begin{pmatrix} 0\,1 \\ 1\,0\end{pmatrix}\begin{pmatrix} a_{1,1},a_{2,1},\ldots,a_{n,1} \\ a_{1,3},a_{2,3},\ldots,a_{n,3}\end{pmatrix}=\begin{pmatrix} a_{1,3},a_{2,3},\ldots,a_{n,3} \\ a_{1,1},a_{2,1},\ldots,a_{n,1}\end{pmatrix}.$$

This matrix corresponds to another $C^{\mathcal{A},\mathcal{B}}$. We denote the copula with parameters A by $C^{\mathcal{A},\mathcal{B}}_{(A)}$ and the latter one with parameters A' by $C^{\mathcal{A},\mathcal{B}}_{(A')}$. A proposition on the two copulas is given first.

Proposition 3.2. For any parameters matrix A, the two copulas $C^{\mathcal{A},\mathcal{B}}_{(A)}$ and $C^{\mathcal{A},\mathcal{B}}_{(A')}$ are identical.

We omit the proof because it can be directly proved by using the property

$$C^{(j_1,j_2,\ldots,j_n)}(u_1,u_2,\ldots,u_n) = C^{(4-j_1,4-j_2,\ldots,4-j_n)}(u_1,u_2,\ldots,u_n).$$

The determination method for $C^{\mathcal{A},\mathcal{B}}$ copula (i.e., for the matrix A) from marginal copulas (15) is given now. Note that the coefficients of (15) can be linked to A under the equation:

$$\begin{cases} d^+_{i,j}=a_{i,1}a_{j,1}+a_{i,3}a_{j,3}, \\ d^-_{i,j}=a_{i,1}a_{j,3}+a_{i,3}a_{j,1}. \end{cases} \tag{20}$$

This system has $n(n-1)$ equations (equal to the numbers of $d^{\pm}_{i,j}$) and only $2n$ unknown variables (namely $a_{i,1}$ and $a_{i,3}$, $i=1,\ldots,n$). The system is overdetermined since the number of equations is larger than that of unknown variables.

In the following, we focus on this overdetermined problem. Two methods are introduced to solve the matrix A. These methods aim to give an optimal approximation under the quadratic deviation.

To get an approximation of A, one needs to minimize the object function

$$f_1(A) = \sum_{1\le i<j\le n} \Big[(d^+_{i,j} - a_{i,1}a_{j,1} - a_{i,3}a_{j,3})^2 + (d^-_{i,j} - a_{i,1}a_{j,3} - a_{i,3}a_{j,1})^2\Big] \tag{21}$$

under the constraints that $a_{i,1}, a_{i,3} \ge 0$ and $a_{i,1} + a_{i,3} \le 1$. This is a non-convex optimization problem, and we call it the non-convex method in this paper. This method can not guarantee that the obtained minimum coincides with the global minimum (Boyd and Vandenberghe, 2004).

For ensuring to obtain the global minimum, rather than a local minimum, we introduce an alternative method, called the convex method in contrast, which transforms (20) to a linear system.

Note that by adding the two equations in (20) and then subtracting them, we can get

$$\begin{cases} d^+_{i,j} + d^-_{i,j} = (a_{i,1} + a_{i,3})(a_{j,1} + a_{j,3}), \\ d^+_{i,j} - d^-_{i,j} = (a_{i,1} - a_{i,3})(a_{j,1} - a_{j,3}). \end{cases} \tag{22}$$

Then we carry out a transformation of variables

$$\begin{cases} b^+_{i,j} = \log(d^+_{i,j} + d^-_{i,j}), \\ b^-_{i,j} = \log|d^+_{i,j} - d^-_{i,j}| \end{cases}$$

and

$$\begin{cases} x^+_i = \log(a_{i,1} + a_{i,3}), \\ x^-_i = \log|a_{i,1} - a_{i,3}|. \end{cases}$$

Through the above transformations, (22) becomes

$$\begin{cases} b^+_{i,m} = x^+_i + x^+_m, \\ b^-_{i,m} = x^-_i + x^-_m. \end{cases}$$

It's also an overdetermined system and the least square solution is still necessary. Let $X = (x^+_1, \dots, x^+_n, x^-_1, \dots, x^-_n)$. Then one needs to minimize the object function

$$f_2(X) = \sum_{1\le i<m\le n} \big[(b^+_{i,m} - x^+_i - x^+_m)^2 + (b^-_{i,m} - x^-_i - x^-_m)^2\big]$$

under the constraints

$$x^+_i \le 0,\ x^+_i - x^-_i \ge 0.$$

It can be verified directly that the object function $f_2(X)$ is a convex function and the constraint region of X is a convex set in $\mathbb{R}^{2n}$. Thus it is actually a convex optimization problem whose local minimum coincides with the global minimum (Boyd and Vandenberghe, 2004).

When a least square solution of $\{x^\pm_i | 1 \le i \le n\}$ is obtained, we can solve the matrix A through

$$\begin{cases} a_{i,1} \vee a_{i,3} = (e^{x^+_i} + e^{x^-_i})/2, \\ a_{i,1} \wedge a_{i,3} = (e^{x^+_i} - e^{x^-_i})/2 \end{cases}$$

if we ignore the order of $a_{i,1}$ and $a_{i,3}$ for the moment. Then one can exchange the values of $a_{i,1}$ and $a_{i,3}$ for some i's and preserve the values of others. There are totally 2^{n-1} combinations due to Proposition 3.2. Finally, based on these combinations, the one minimizing (21) among all 2^{n-1} matrices is the estimated matrix $\widehat{A}$.

The efficiency of the two methods will be discussed in Section 4.

3.6 Application for modeling portfolio credit risk

In this subsection we introduce an application of multivariate BF copulas in modeling the portfolio credit risk.

Let $Y_1, Y_2, \dots, Y_n$ be n random variables with marginal distributions F_i and $Y_i = F_i^{-1}(U_i)$, where the copula of $U_1, U_2, \dots, U_n$ is supposed to be a multivariate BF copula and F_i^{-1} denotes the left-continuous inverse function of F_i. Thus the copula of $(Y_1, Y_2, \dots, Y_n)$ is the multivariate BF copula, denoted by $C_{m,\dots,m}(u_1, \dots, u_n | C^{\mathcal{A},\mathcal{B}}, \Pi)$. Then one can calculate the expectation of a function $f(Y_1, Y_2, \dots, Y_n)$. For instance,

$$f(Y_1, \dots, Y_n) = e^{-rT} \max\{Y_1 + \dots + Y_n - M, 0\},\ T > 0,\ M > 0.$$

This expression often appears in the pricing of financial derivatives with multi-underlying assets, such as multi-stock options and collateralized debt obligations (CDOs).

In the following, we provide a proposition to simplify the calculation for a general non-linear function f.

Proposition 3.3. Let f be an n-variable function. Then

$$Ef(Y_1, \dots, Y_n) = \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i}\right) Ef(Y_1^{(j_1,\dots,j_n)}, \dots, Y_n^{(j_1,\dots,j_n)}).$$

Here for each $(j_1, j_2, \dots, j_n)$, the random vector $(Y_1^{(j_1,\dots,j_n)}, \dots, Y_n^{(j_1,\dots,j_n)})$ has marginal distributions $F_i, i \le n$ and the copula $C_{m,\dots,m}(u_1, \dots, u_n | C^{(j_1,\dots,j_n)}, \Pi)$, and a_{i,j_i}'s are the parameters of the $C^{\mathcal{A},\mathcal{B}}$ in the multivariate BF copula.

Proof. Using the linear addictive property of CBC, we have

$$Ef(Y_1, \dots, Y_n) = \int_0^1 \cdots \int_0^1 f(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n))\, C_{m,\dots,m}(du_1, \dots, du_n | C^{\mathcal{A},\mathcal{B}}, \Pi)$$

$$
\begin{aligned}
&= \int_0^1 \cdots \int_0^1 f(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)) \\
&\quad \times \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i} \right) \\
&\qquad C_{m,\ldots,m}(du_1, \ldots, du_n | C^{(j_1,\ldots,j_n)}, \Pi) \\
&= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i} \right) \\
&\quad \times \int_0^1 \cdots \int_0^1 f(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)) \\
&\quad C_{m,\ldots,m}(du_1, \ldots, du_n | C^{(j_1,\ldots,j_n)}, \Pi) \\
&= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i} \right) Ef(Y_1^{(j_1,\ldots,j_n)}, \ldots, Y_n^{(j_1,\ldots,j_n)}).
\end{aligned}
$$

This proposition is proved. □

From the proposition, we see that the calculation of the expectation $Ef(Y_1, \ldots, Y_n)$ can be divided into two steps. The first step is to calculate some basic expectations $Ef(Y_1^{(j_1,\ldots,j_n)}, \ldots, Y_n^{(j_1,\ldots,j_n)})$. The second step is to sum those expectations linearly with multipliers $\prod_{i=1}^n a_{i,j_i}$. Note that the multipliers are determined by the joint distribution of $(Y_1, \ldots, Y_n)$, while the expectations $Ef(Y_1^{(j_1,\ldots,j_n)}, \ldots, Y_n^{(j_1,\ldots,j_n)})$ are only related to the marginal distributions of $(Y_1, \ldots, Y_n)$. This means when modeling a portfolio consisting of large numbers of securities, using the multivariate BF copula as the dependence structure can make the pricing of this portfolio simple and flexible, because the above two steps of the calculation of $Ef(Y_1, \ldots, Y_n)$ can be carried out independently, which are determined by the marginal distributions and the copula respectively.

4. ESTIMATION METHOD AND SIMULATION STUDY

We provide an estimation method for multivariate BF copulas. Because the multivariate BF copula is uniquely determined by its bivariate marginal copulas, we shall first consider the estimation of bivariate marginal copulas and then use these bivariate copulas to obtain the corresponding multivariate BF copula. Finally, we perform simulations to validate the proposed estimation method.

4.1 Estimation method for positive BF copulas

On estimating coefficients of multivariate BF copulas, most real data shows positive correlated or negative correlated. Thus for simplicity, we focus our discussion on positive BF copulas. Note that the data with negative dependence can be transformed to positive dependence through transformation (see Proposition 2.1).

Coefficients q and m in (6) need to be estimated. We first calculate the density function of (6),

$$
c_{+,q}^{(m)}(u, v) = qm \sum_{k=1}^{m} p_{k-1,m-1}(u) p_{k-1,m-1}(v) + (1 - q).
$$

Denote the sample $(U_1, V_1), (U_2, V_2), \ldots, (U_N, V_N)$ with size N by $\mathbf{X}$, here we suppose that the sample is positively dependent. Then the likelihood function of $\mathbf{X}$ is

$$
\begin{aligned}
&f(q, m|\mathbf{X}) \\
&= c_{+,q}^{(m)}(U_1, V_1) c_{+,q}^{(m)}(U_2, V_2) \cdots c_{+,q}^{(m)}(U_N, V_N) \\
&= \prod_{i=1}^{N} \left[1 - q + q a(U_i, V_i) \right],
\end{aligned} \tag{23}
$$

where $a(U_i, V_i) = m \sum_{k=1}^m p_{k-1,m-1}(U_i) p_{k-1,m-1}(V_i)$ does not contain the variable q.

Here we use EM algorithm to carry out our estimation. The algorithm estimates q with a pre-given value of m, and the initial value of q is substituted by the sample correlation coefficient $\widehat{\rho}$ of $\mathbf{X}$.

Algorithm 4.1.

- ***Step 0:*** *set* $q = \widehat{\rho}$.
- ***Step 1: E-step***

$$
\widehat{\tau}_i := \frac{1 - q}{1 - q + a_i q}, \quad i = 1, 2, \ldots, N,
$$

where

$$
a_i = a(U_i, V_i) = m \sum_{k=1}^{m} p_{k-1,m-1}(U_i) p_{k-1,m-1}(V_i).
$$

- ***Step 2: M-step***

$$
q := 1 - \frac{1}{N} \sum_{i=1}^{N} \widehat{\tau}_i.
$$

Repeat Steps 1 and 2 until q converges.

In practice, the condition to quit the repeating loop of the two steps is chosen to be $|q_{\text{old}} - q_{\text{new}}| < 10^{-5}$ or the number of iteration goes larger than 200.

One can estimate q with a pre-given value of m by using Algorithm 4.1. Then, let the pre-given value vary in a range and estimate q for every value in this range (in practice we can choose the range to be all integers from 2 to 30). Finally by maximizing the likelihood function (23) among all (m, q)'s, the estimated coefficient is obtained.

As for the estimation of a multivariate BF copula, one can firstly estimate all its bivariate marginal copulas, and then use the methods introduced in Section 3.5 to determine the multivariate copula. In addition, the smooth parameter can be the average of smooth parameters of all the bivariate

Table 1. The estimated $(\widehat{m}, \widehat{q})$. Each array displays the estimated values of (m, q) with the parameters of the sample are the corresponding heads of the row and column

$m \diagdown q$	0.2	0.4	0.6	0.8	1.0
2	(2, 0.1495)	(2, 0.3972)	(2, 0.6086)	(2, 0.8004)	(2, 0.9704)
3	(3, 0.1976)	(3, 0.3759)	(3, 0.6013)	(3, 0.8035)	(3, 0.9920)
4	(5, 0.1939)	(4, 0.4071)	(4, 0.6054)	(4, 0.7949)	(4, 0.9981)
7	(7, 0.2066)	(7, 0.3992)	(7, 0.5972)	(7, 0.8000)	(7, 0.9993)
10	(10, 0.2074)	(10, 0.4062)	(10, 0.6017)	(10, 0.8035)	(10, 0.9998)
15	(15, 0.2026)	(16, 0.3909)	(15, 0.6020)	(16, 0.8021)	(15, 0.9999)
20	(19, 0.2036)	(20, 0.4031)	(20, 0.6020)	(20, 0.7936)	(20, 0.9999)
25	(26, 0.2021)	(25, 0.4033)	(25, 0.6022)	(25, 0.8032)	(25, 0.9999)

margins. In the next subsection, we will use above methods to estimate both positive BF copulas and multivariate BF copulas. Those methods are shown to be efficient through the simulations.

4.2 Simulation study

4.2.1 Estimation performance with simulation

In this subsection, we validate the estimation and determination methods discussed before. For bivariate case, we generate samples from positive BF copulas with given parameters, and then use Algorithm 4.1 to estimate the parameters. For multivariate case, we also generate samples from multivariate BF copulas with given parameters, estimate the parameters of their bivariate margins, and then determine the multivariate copulas from these bivariate margins.

We first exam the two-dimensional case. Samples are generated from copulas (6) with the parameter

$$(m, q) \in \{2, 3, 4, 7, 10, 15, 20, 25\} \times \{0.2, 0.4, 0.6, 0.8, 1.0\}.$$

Then Algorithm 4.1 is applied for estimating these samples.

For every combination of parameters, 10 independent samples are generated and repeated estimations are carried out. Table 1 displays the average estimated values of (m, q) while the parameters of the population are the corresponding heads of the row and the column.

From Table 1, one can see the estimation is more accurate when q is closer to 1 and m gets larger. When one of them becomes smaller, the correlation of the sample gets smaller, and the error grows bigger.

Next, we sample from a five-dimensional BF copula with the coefficients of $C^{\mathcal{A},\mathcal{B}}$

$$A = \begin{pmatrix} 0.8,\ 0,\ 0,\ 0.6,\ 0 \\ 0,\ 0.9,\ 0.5,\ 0,\ 0 \end{pmatrix}, \quad \text{and} \quad m = 15.$$

We estimate the bivariate marginal copulas and use them to obtain the coefficients matrix A and the smooth parameter m. Table 2 reports the average L^1-error, standard deviations and estimated $\widehat{m}$ of the two methods introduced in Section 3.5.

Table 2. Average L^1-errors and estimated m

Method	Non-convex	Convex
Average L^1-error	0.0490	0.0325
Standard deviations	0.0143	0.0123
estimated m	9	13

Table 3. Average L^1-errors and estimated m

Method	Non-convex	Convex
Average L^1-error	0.1388	0.0473
Standard deviations	0.0319	0.0183
estimated m	7	10

The corresponding estimated $A_{\text{Non-convex}}$ and A_{convex} are listed in the following as an example

$$A_{\text{Non-convex}} = \begin{pmatrix} 0.8165 & 0.0000 & 0.0000 & 0.6287 & 0.0000 \\ 0.0000 & 0.8824 & 0.5255 & 0.0000 & 0.0348 \end{pmatrix},$$
$$A_{\text{convex}} = \begin{pmatrix} 0.8116 & 0.0000 & 0.0000 & 0.6059 & 0.0000 \\ 0.0000 & 0.8937 & 0.4960 & 0.0000 & 0.0257 \end{pmatrix}.$$

The estimated $\widehat{m}$ are 9 and 13 respectively. From Table 2 one can conclude that the error of the convex method is more acceptable.

Another population with parameters

$$A = \begin{pmatrix} 0.4,\ 0.3,\ 0.6,\ 0,\ 0.8 \\ 0,\ 0,\ 0,\ 0.7,\ 0 \end{pmatrix}, \quad \text{and} \quad m = 10$$

is also carried out for simulation. The same error indexes follow in Table 3.

The two estimated parameters follow as an example,

$$A_{\text{Non-convex}} = \begin{pmatrix} 0.4456 & 0.3611 & 0.6242 & 0.0000 & 0.8083 \\ 0.0000 & 0.0000 & 0.0000 & 0.7286 & 0.0000 \end{pmatrix},$$
$$A_{\text{convex}} = \begin{pmatrix} 0.3916 & 0.3038 & 0.5958 & 0.0000 & 0.8063 \\ 0.0000 & 0.0000 & 0.0000 & 0.7365 & 0.0000 \end{pmatrix}.$$

One can see errors of the second method are smaller than those of the first. Therefore, the convex method is recommended in practice.

Table 4. Historical default probabilities (%) of rating AAA, AA, A, BBB, BB, and B with terms 5 year and 10 year. The source of these data is from Appendix 1 in Fitch (2020)

Term \ Rating	AAA	AA	A	BBB	BB	B
5	0.082	0.169	0.586	1.913	10.026	21.572
10	0.193	0.638	1.580	4.536	17.434	32.182

In this subsection both the estimation of bivariate BF copulas and the determination of multivariate BF copulas are discussed. From results of the simulations, we conclude that positive BF copulas are suitable for modeling highly positive correlation, and for the aim of determining multivariate BF copulas from margins, the convex method can obtain more accurate result than the non-convex method.

4.2.2 Simulation study of portfolio credit risk

For evaluating CDOs, the dependence of the default times plays a key role for modeling the credit risks. Copulas can be applied to model the dependence structure of the default times. Fitch (2020) provides historical data for analyzing CDOs and the Gaussian copula is applied to model the default correlation of the credit risks. Based on the historical data provided in Fitch (2020), we work on simulation to study the loss distribution of CDOs by comparing multivariate BF copulas with Gaussian copulas.

Consider a CDO of term T consisting of n defaultable underlying assets, where the credit exposures are all assumed to be 1 and the recovery rates are all assumed to be 0. The default times are denoted by $\tau_1, \ldots, \tau_n$ respectively, and let $Y_i := \mathbb{I}_{\{\tau_i \le T\}}, i \le n$. Then the loss ratio L of the credit portfolio can be expressed as

$$L = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n).$$

Note that the default probability of a single asset in the portfolio is simply a function of its rating and term.

For an AAA-rated CDO tranche with given term, its rating default rate (RDR) is defined as the loss percentage of the portfolio that is assumed to default in the respective rating scenario. Applying the percentile corresponding to the rating scenario and term (Fitch, 2020), RDR can be derived from the distribution of the portfolio loss ratio. Mathematically, the RDR of an AAA-rated CDO tranche with term T can be solved from the equation

$$\mathbb{P}(L > \mathrm{RDR}) = p_{CDO}(\mathrm{AAA}, T),$$

where $p_{CDO}(\mathrm{AAA}, T)$ is the CDO target default probability of an AAA-rated tranche with term T. Similarly, the RDRs of other tranches can be calculated in the same way.

In the following, we consider a CDO consisting of 300 homogeneous defaultable assets. The RDRs of the CDO tranches are calculated through simulation. As in Fitch (2020), we use the historical corporate default rate as the default probability of each underlying asset, and we also assume that each tranche's target default probability equals its corresponding historical corporate default rate. Some historical corporate default rates are listed in Table 4, and the complete version of the table can be found in Fitch (2020).

Table 5. Gaussian model's RDR levels (%) with 6.5% equal pair-wise correlation index

Tranche	**B**		**BB**		**BBB**		**A**	
	5	**10**	**5**	**10**	**5**	**10**	**5**	**10**
AAAsf	52.7	63.0	32.7	43.7	10.7	17.3	4.7	8.3
AAsf	50.0	58.7	30.7	39.3	9.7	14.7	4.3	6.7
Asf	45.7	55.0	26.7	35.7	8.0	12.7	3.3	5.7
BBBsf	40.7	49.7	23.0	31.0	6.3	10.3	2.3	4.3
BBsf	32.3	41.3	16.7	24.0	4.0	7.0	1.3	2.7
Bsf	27.7	36.3	13.7	20.0	2.7	5.3	1.0	2.0

In the credit portfolio, the copula of $(\tau_1, \cdots, \tau_{300})$ is chosen as a Gaussian copula and a multivariate BF copula in Example 3.3 respectively, with 6.5% equal pair-wise correlation index. The parameter ρ of the Gaussian copula needs to be set as 6.81% by solving $\frac{6}{\pi}\arcsin(\frac{\rho}{2}) = 6.5\%$, and the parameter (m, q) of the multivariate BF copula in Example 3.3 should satisfy $\frac{m-1}{m+1}q = 6.5\%$. Table 5 reports the RDRs of different rated tranches under the Gaussian copula model simulated for 1,000,000 times, and Table 6 and 7 shows the simulation results of the multivariate BF copulas with the parameter $m = 20$ and 10 respectively.

In these tables, each column head identifies the rating of the underlying assets in a credit portfolio. For example, the column with the head (BB, 5) means that the portfolio consists of 300 BB-rated assets with 5-year term, whose default probability of the underlying assets is assumed to be the historical default rate 10.026% in Table 4. Each row in this column displays the RDRs of the CDO tranches, from AAAsf to Bsf. To discriminate the ratings for the CDO tranches and the underlying assets in the portfolio, we add suffix -sf in the Tranche column to indicate that it is a rating for CDO structured products.

From these tables, we find that except for the underlying assets with the lowest rating B, the multivariate BF model's RDRs vary in a larger range than those of the Gaussian model. For example, for the CDO consisting of A-rated and 10-year-termed underlying assets, the RDR levels given by the multivariate BF copula with $m = 20$ range from 1.7 to

Table 6. Multivariate BF model's RDR levels (%) with $m = 20$ and 6.5% equal pair-wise correlation index

Tranche	B		BB		BBB		A	
	5	**10**	**5**	**10**	**5**	**10**	**5**	**10**
AAAsf	48.7	56.7	36.7	44.0	13.7	23.7	6.0	11.3
AAsf	48.0	55.3	35.7	42.3	13.3	22.0	5.3	10.3
Asf	46.3	54.0	34.0	40.7	12.0	20.7	4.7	9.3
BBBsf	44.3	52.0	31.7	38.0	10.3	16.3	3.7	6.3
BBsf	39.3	47.3	19.7	25.3	2.7	5.0	1.0	2.0
Bsf	28.7	36.7	10.3	16.0	2.0	4.0	0.7	1.7

Table 7. Multivariate BF model's RDR levels (%) with $m = 10$ and 6.5% equal pair-wise correlation index

Tranche	B		BB		BBB		A	
	5	**10**	**5**	**10**	**5**	**10**	**5**	**10**
AAAsf	48.0	56.7	31.7	42.3	10.0	18.0	4.3	8.0
AAsf	47.3	55.3	31.0	40.7	9.3	16.7	4.0	7.3
Asf	45.7	54.0	29.7	39.3	8.7	15.7	3.3	6.7
BBBsf	43.7	51.7	27.7	37.0	7.7	14.0	2.7	5.3
BBsf	37.7	46.3	19.3	26.0	3.7	5.3	1.3	2.0
Bsf	28.3	36.7	11.0	16.7	2.3	4.3	1.0	1.7

Table 8. The historical maximum and average loss (%), from Figure 23 in Fitch (2020)

Historical Loss	B		BB		BBB		A	
	5	**10**	**5**	**10**	**5**	**10**	**5**	**10**
Peak	38.7	49.5	19.7	29.7	4.5	9.3	1.61	4.02
Expected	21.6	32.2	10.1	17.5	1.9	4.5	0.6	1.6

11.3, while the range given by the Gaussian copula is from 2.0 to 8.3. For the lowest rating B, the results are reversed for the two models.

Comparing the results on $m = 20$ in Table 6 and the results on $m = 10$ in Table 7, we can see that the influence of the parameter m is mainly on high-rated tranches in the CDOs whose underlying assets are less likely to default. For example, in the credit portfolios whose underlying assets are rated higher than BB, the RDRs of tranches AAAsf, AAsf, Asf, and BBBsf get lower when m gets smaller.

Generally, rating agencies assume that model's RDRs for high-rated tranches, such as Asf and above, should cover the historical peak default rates. Table 8 provides the historical peak default rate and average default rate in Fitch (2020). By comparing Table 7 with the values in Table 8, one can find that even the RDR levels become lower, the results on $m = 10$ can still cover the peak losses for tranches from AAAsf to BBBsf. In conclusion, the application of multivariate BF copulas in evaluating CDOs is feasible and flexible.

5. EMPIRICAL STUDY

This section works on the empirical study of multivariate BF copulas on China's stock market. Through empirical

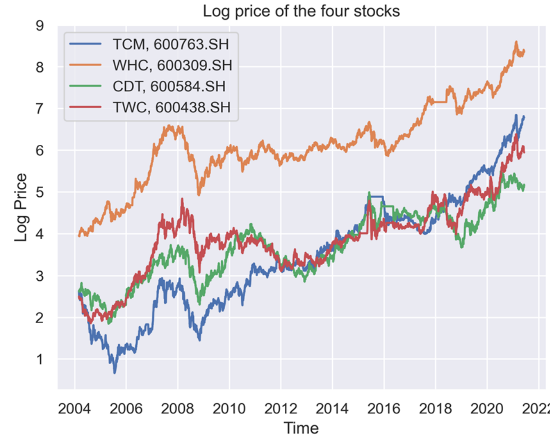

Figure 7. Log price of the four stocks. Significant positive correlation can be observed from the plot.

analysis, we show the advantages of multivariate BF copulas by comparing them with Gaussian copulas.

The data of four stocks, TCMedical (TCM, 600763.SH), WanHuaChem (WHC, 600309.SH), JCET CO. (JCET, 600584.SH), and TongWei CO. (TW, 600438.SH) are applied in our analysis. The data are from March 2, 2004 to June 8, 2021 with 4195 observations. The summary statistics of the log returns and the figure of the log prices are shown in Table 9 and Figure 7, respectively. The comparison of fitting the data with Gaussian copulas and multivariate BF copulas is carried out.

Table 9. Summary statistics of the four selected stocks' log returns

Summary	TCM	WHC	JCET	TW
Min.	-0.1065	-0.1056	-0.1059	-0.1061
1st Qu.	-0.0136	-0.0131	-0.0162	-0.0148
Median	0.0000	0.0000	0.0000	0.0000
Mean	0.0010	0.0010	0.0005	0.0008
3rd Qu.	0.0170	0.0157	0.0171	0.0163
Max.	0.0965	0.0957	0.0970	0.0960

For the log returns of each stock, we first fit its marginal distributions by ARMA-GARCH models. When a time series model is fitted, the standardized residuals of the model can be obtained. Then copula functions are applied to model the dependency of the residuals of the four stocks.

More concretely, we take the ARMA(1, 1)-GARCH(1, 1) model as an example to explain our method. Consider the log-return series $\{r_t\}$ of a stock. Assume that

$$\begin{cases} r_t = \mu + \phi_1 r_{t-1} + \psi_1 \epsilon_{t-1} + \epsilon_t, \\ \epsilon_t = \sigma_t \cdot Z_t\,, \quad Z_t \sim D(0,1) \text{ i.i.d.}\,, \\ \sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \end{cases} \tag{24}$$

where D is a certain error distribution, such as a normal distribution or a student-t distribution. After estimating the model, the standardized residuals $\tilde{\epsilon}_t = \epsilon_t/\sigma_t, t \geq 0$ are applied for fitting a copula function. Comparisons of the fitting accuracy between these copulas are presented in terms of the AIC, the Kendall's tau and the Spearman's rho.

Using R package `rugarch`, the fitted models for the four stock are MA(2), ARMA(2,2), MA(2), AR(2) respectively (all combined with GARCH(1,1)), here we use the student-t distribution as the error distribution. Then the four standard residual series are applied to fit Gaussian copulas and multivariate BF copulas respectively. For multivariate BF copulas, the bivariate marginal distributions are estimated by applying the method in Section 4.1, and the method in Section 3.5 is applied to get the parameters. The estimated parameters for the two copulas are as followings:

$$\hat{\Sigma} = \begin{pmatrix} 1 & 0.27 & 0.34 & 0.32 \\ 0.27 & 1 & 0.35 & 0.34 \\ 0.34 & 0.35 & 1 & 0.41 \\ 0.32 & 0.34 & 0.41 & 1 \end{pmatrix},$$
$$\hat{A} = \begin{pmatrix} 0.59 & 0.63 & 0.76 & 0.74 \\ 0.00 & 0.00 & 0.00 & 0.00 \end{pmatrix}, \hat{m} = 6,$$

where $\hat{\Sigma}$ is the estimated correlation matrix of the Gaussian copula, and $\hat{A}$ and $\hat{m}$ are estimated parameters of the multivariate BF copula.

The AICs of the fitted models are listed in Table 10. From the numerical results, we can see that the multivariate BF copula fits the data more accurately than the Gaussian copula in terms of AIC.

Table 10. AIC's of the two fitted copulas

Gaussian Copula	BF Copula
-2292.3	-2445.7

The comparisons of Kendall's tau and Spearman's rho are displayed in Table 11 and Table 12 respectively, where the values in brackets are deviations from the corresponding sample indexes (the first column in the table). In most cases, the correlation indexes obtained from multivariate BF copulas are closer to the sample rank correlation than those obtained from Gaussian copulas.

Table 11. Kendall's tau comparison table

Kendall's τ	Sample	Gaussian copula	BF copula
TCM & WHC	0.1891	0.1741(-0.0150)	0.1812(-0.0079)
TCM & CDT	0.2355	0.2210(-0.0145)	0.2212(-0.0143)
TCM & TWC	0.2154	0.2089(-0.0065)	0.2138(-0.0016)
WHC & CDT	0.2388	0.2243(-0.0145)	0.2366(-0.0022)
WHC & TWC	0.2399	0.2201(-0.0198)	0.2287(-0.0112)
CDT & TWC	0.2868	0.2664(-0.0204)	0.2795(-0.0073)

Table 12. Spearman's rho comparison table

Spearman's ρ	Sample	Gaussian copula	BF copula
TCM & WHC	0.2777	0.2586(-0.0191)	0.2643(-0.0134)
TCM & CDT	0.3441	0.3265(-0.0176)	0.3207(-0.0234)
TCM & TWC	0.3151	0.3091(-0.0060)	0.3104(-0.0047)
WHC & CDT	0.3479	0.3311(-0.0168)	0.3423(-0.0056)
WHC & TWC	0.3489	0.3252(-0.0237)	0.3312(-0.0177)
CDT & TWC	0.4122	0.3907(-0.0215)	0.4019(-0.0103)

6. CONCLUSION

Based on $C^{\mathcal{A},\mathcal{B}}$ copulas presented by Yang, Qi and Wang (2009) and composite Bernstein copulas studied in Yang et al. (2015), this paper introduced a copula family called the multivariate BF copula. The multivariate BF copula is uniquely determined by its bivariate marginal copulas, which are called bivariate BF copulas. The bivariate BF copula class contains many copulas such as FGM copulas, bivariate copulas with cubic sections (Nelsen, Quesada-Molina and Rodríguez-Lallena, 1997), and Baker's copulas studied in Baker (2008). Comparing bivariate BF copulas with Gaussian copulas and Fréchet copulas, bivariate BF copulas reveal advantages such as admitting density functions and showing more flexibility in modeling. For practical application, we investigated the estimation of positive BF copulas. Based on the estimation of the bivariate copulas, the method for obtaining the multivariate BF copulas was also discussed and validated. Moreover, simulation and empirical study were carried out for showing the feasibility and flexibility of multivariate BF copulas.

APPENDIX A. PROOF OF THEOREM 2.2

For ease of description, we denote $C_{m,m}(u, v|M, \Pi)$ by $C(u, v)$ through this proof. To prove that $C_{+,q}^{(m)}$ admits TP_2 property by definition, one needs to prove that for $0 \leq x_1 \leq x_2 \leq 1$, $0 \leq y_1 \leq y_2 \leq 1$,

$$C_{+,q}^{(m)}(x_2, y_2)C_{+,q}^{(m)}(x_1, y_1) - C_{+,q}^{(m)}(x_1, y_2)C_{+,q}^{(m)}(x_2, y_1) \geq 0.$$

We know that

$$\begin{aligned}
&C_{+,q}^{(m)}(x_2, y_2)C_{+,q}^{(m)}(x_1, y_1) - C_{+,q}^{(m)}(x_1, y_2)C_{+,q}^{(m)}(x_2, y_1) \\
&= [(1-q)x_2y_2 + qC(x_2, y_2)] \cdot [(1-q)x_1y_1 + qC(x_1, y_1)] \\
&\quad - [(1-q)x_1y_2 + qC(x_1, y_2)] \cdot [(1-q)x_2y_1 + qC(x_2, y_1)] \\
&= (1-q)qx_2y_2C(x_1, y_1) + (1-q)qx_1y_1C(x_2, y_2) \\
&\quad - (1-q)qx_1y_2C(x_2, y_1) - (1-q)qx_2y_1C(x_1, y_2) \\
&\quad + q^2C(x_2, y_2)C(x_1, y_1) - q^2C(x_1, y_2)C(x_2, y_1) \\
&= (1-q)q[x_2y_2C(x_1, y_1) + x_1y_1C(x_2, y_2) \\
&\qquad - x_1y_2C(x_2, y_1) - x_2y_1C(x_1, y_2)] \\
&\quad + q^2[C(x_2, y_2)C(x_1, y_1) - C(x_1, y_2)C(x_2, y_1)].
\end{aligned}$$

Dou, Kuriki and Lin (2013) proved that copula C has TP_2 property, thus the second term $q^2[C(x_2,y_2)C(x_1,y_1) - C(x_1,y_2)C(x_2,y_1)]$ in the last equation is non-negative. Therefore it is sufficient to show that the first term is non-negative. Since $(1-q)q$ is non-negative, we only need to prove that

$$\begin{aligned}A(x_1,x_2;y_1,y_2) := x_2y_2C(x_1,y_1) + x_1y_1C(x_2,y_2)\\ - x_1y_2C(x_2,y_1) - x_2y_1C(x_1,y_2)\end{aligned} \tag{25}$$

is non-negative for any $0 \le x_1 \le x_2 \le 1$ and $0 \le y_1 \le y_2 \le 1$.

Let $x_1 = x$, $x_2 = x + \Delta x$ and $y_1 = y$, $y_2 = y + \Delta y$. We first prove that $A(x_1,x_2;y_1,y_2) \ge 0$ locally, that is, $A(x_1,x_2;y_1,y_2) \ge 0$ when Δx and Δy are sufficiently small.

Calculate the Taylor expansion up to order 2 of $C(x+\Delta x, y+\Delta y)$,

$$\begin{aligned}&C(x+\Delta x,y+\Delta y)\\ &= C(x,y) + \frac{\partial C}{\partial x}(x,y)\Delta x + \frac{\partial C}{\partial y}(x,y)\Delta y + \frac{1}{2}\frac{\partial^2 C}{\partial x^2}(x,y)\Delta x^2\\ &\quad + \frac{\partial^2 C}{\partial x\partial y}(x,y)\Delta x\Delta y + \frac{1}{2}\frac{\partial^2 C}{\partial y^2}(x,y)\Delta y^2 + o(\Delta x\Delta y).\end{aligned}$$

Substituting the expansion into (25), one can finally get

$$\begin{aligned}A = \Delta x\Delta y\Big[&C(x,y) - x\frac{\partial C}{\partial x}(x,y) - y\frac{\partial C}{\partial y}(x,y)\\ &+ xy\frac{\partial^2 C}{\partial x\partial y}(x,y) + o(\Delta x\Delta y)\Big].\end{aligned}$$

Thus we need to prove that

$$C(x,y) - x\frac{\partial C}{\partial x}(x,y) - y\frac{\partial C}{\partial y}(x,y) + xy\frac{\partial^2 C}{\partial x\partial y}(x,y) > 0$$

for any $0 \le x \le 1, 0 \le y \le 1$. Because the inequalities at $x = 0,1$ or $y = 0,1$ can be verified directly, we will prove the inequality for $0 < x < 1, 0 < y < 1$ below.

Actually, note that $O'_{k,m}(x) = mp_{k-1,m-1}(x)$ and $xp_{k-1,m-1}(x) = C_{m-1}^{k-1}x^k(1-x)^{m-k} = \frac{k}{m}p_{k,m}(x)$, we have

$$\begin{aligned}&C(x,y) - x\frac{\partial C}{\partial x}(x,y) - y\frac{\partial C}{\partial y}(x,y) + xy\frac{\partial^2 C}{\partial x\partial y}(x,y)\\
=&\frac{1}{m}\sum_{k=1}^m O_{k,m}(x)O_{k,m}(y)\\
&-x\sum_{k=1}^m p_{k-1,m-1}(x)O_{k,m}(y) - y\sum_{k=1}^m p_{k-1,m-1}(y)O_{k,m}(x)\\
&+xym\sum_{k=1}^m p_{k-1,m-1}(x)p_{k-1,m-1}(y)\\
=&\frac{1}{m}\sum_{k=1}^m O_{k,m}(x)O_{k,m}(y) - \frac{1}{m}\sum_{k=1}^m kp_{k,m}(x)O_{k,m}(y)\\
&-\frac{1}{m}\sum_{k=1}^m kp_{k,m}(y)O_{k,m}(x) + \frac{1}{m}\sum_{k=1}^m k^2p_{k,m}(x)p_{k,m}(y)\\
=&\frac{1}{m}\sum_{k=1}^m\left(\sum_{i=k}^m p_{i,m}(x)\right)\left(\sum_{j=k}^m p_{j,m}(y)\right)\\
&-\frac{1}{m}\sum_{k=1}^m kp_{k,m}(x)\left(\sum_{j=k}^m p_{j,m}(y)\right)\\
&-\frac{1}{m}\sum_{k=1}^m kp_{k,m}(y)\left(\sum_{i=k}^m p_{i,m}(x)\right) + \frac{1}{m}\sum_{k=1}^m k^2p_{k,m}(x)p_{k,m}(y)\\
=&\frac{1}{m}\Bigg(\sum_{i=1}^m\sum_{j=1}^m i\wedge j\cdot p_{i,m}(x)p_{j,m}(y)\\
&-\sum_{i=1}^m\sum_{j=1}^m i\wedge j\cdot \mathbb{I}_{\{i\le j\}}p_{i,m}(x)p_{j,m}(y)\\
&-\sum_{i=1}^m\sum_{j=1}^m i\wedge j\cdot \mathbb{I}_{\{i\ge j\}}p_{i,m}(x)p_{j,m}(y)\\
&+\sum_{i=1}^m\sum_{j=1}^m i^2\mathbb{I}_{\{i=j\}}p_{i,m}(x)p_{j,m}(y)\Bigg)\\
=&\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^m (i^2-i)\mathbb{I}_{\{i=j\}}p_{i,m}(x)p_{j,m}(y)\\
=&\frac{1}{m}\sum_{k=1}^m (k^2-k)p_{k,m}(x)p_{k,m}(y) > 0.\end{aligned}$$

Therefore if Δx and Δy are sufficiently small such that the remainder term $o(\Delta x\Delta y)$ can be ignored, the inequality $A(x, x+\Delta x; y, y+\Delta y) \ge 0$ holds.

Up to now, the local non-negativeness of (25) is proved, i.e., for sufficiently close $x_1 \le x_2$ and $y_1 \le y_2$, $A(x_1,x_2;y_1,y_2) \ge 0$ holds. To finish the proof, we need to extend the conclusion from local to global, that is, to prove the inequality holds for any $0 \le x_1 \le x_2 \le 1$ and $0 \le y_1 \le y_2 \le 1$.

For this purpose, assume that for $0 \le x_1 \le x^* \le x_2 \le 1$ and $0 \le y_1 \le y^* \le y_2 \le 1$, A is non-negative in the four smaller squares $[x_1,x^*]\times[y_1,y^*]$, $[x^*,x_2]\times[y^*,y_2]$, $[x_1,x^*]\times[y^*,y_2]$, and $[x^*,x_2]\times[y_1,y^*]$, i.e., the following four inequalities holds,

$$\begin{aligned}A(x_1,x^*;y_1,y^*) =&x_1y_1C(x^*,y^*) + x^*y^*C(x_1,y_1)\\ &- x_1y^*C(x^*,y_1) - x^*y_1C(x_1,y^*) \ge 0,\\
A(x^*,x_2;y^*,y_2) =&x^*y^*C(x_2,y_2) + x_2y_2C(x^*,y^*)\\ &- x^*y_2C(x_2,y^*) - x_2y^*C(x^*,y_2) \ge 0,\\
A(x_1,x^*;y^*,y_2) =&x_1y^*C(x^*,y_2) + x^*y_2C(x_1,y^*)\\ &- x^*y^*C(x_1,y_2) - x_1y_2C(x^*,y^*) \ge 0,\\
A(x^*,x_2;y_1,y^*) =&x^*y_1C(x_2,y^*) + x_2y^*C(x^*,y_1)\\ &- x^*y^*C(x_2,y_1) - x_2y_1C(x^*,y^*) \ge 0.\end{aligned}$$

We show that A is non-negative in the bigger square $[x_1, x_2] \times [y_1, y_2]$ under above assumptions. Multiply the four inequalities by x_2y_2, x_1y_1, x_2y_1, and x_1y_2 respectively and then add them up together. We get

$$\begin{aligned}
&x_2y_2A(x_1, x^*; y_1, y^*) + x_1y_1A(x^*, x_2; y^*, y_2) \\
&\qquad + x_2y_1A(x_1, x^*; y^*, y_2) + x_1y_2A(x^*, x_2; y_1, y^*) \\
=&x_2y_2x_1y_1C(x^*, y^*) + x_2y_2x^*y^*C(x_1, y_1) \\
&- x_2y_2x_1y^*C(x^*, y_1) - x_2y_2x^*y_1C(x_1, y^*) \\
&+ x_1y_1x^*y^*C(x_2, y_2) + x_1y_1x_2y_2C(x^*, y^*) \\
&- x_1y_1x^*y_2C(x_2, y^*) - x_1y_1x_2y^*C(x^*, y_2) \\
&+ x_2y_1x_1y^*C(x^*, y_2) + x_2y_1x^*y_2C(x_1, y^*) \\
&- x_2y_1x^*y^*C(x_1, y_2) - x_2y_1x_1y_2C(x^*, y^*) \\
&+ x_1y_2x^*y_1C(x_2, y^*) + x_1y_2x_2y^*C(x^*, y_1) \\
&- x_1y_2x^*y^*C(x_2, y_1) - x_1y_2x_2y_1C(x^*, y^*) \\
=&x^*y^*\Big[x_1y_1C(x_2, y_2) + x_2y_2C(x_1, y_1) \\
&- x_2y_1C(x_1, y_2) - x_1y_2C(x_2, y_1)\Big] \geq 0.
\end{aligned}$$

Eliminate the non-negative term x^*y^* in the inequality. Finally one can get that

$$\begin{aligned}
A(x_1, x_2; y_1, y_2) =&x_1y_1C(x_2, y_2) + x_2y_2C(x_1, y_1) \\
&- x_2y_1C(x_1, y_2) - x_1y_2C(x_2, y_1) \geq 0.
\end{aligned}$$

The discussion above shows that when $(x_2 - x_1)$ and $(y_2 - y_1)$ are not sufficiently small, one can insert new points between them, and this insertion can be carried out repeatedly until every interval gets small enough to make sure $A \geq 0$ for every small square. This finishes the proof of TP_2 property of the positive BF copula $C_{+,q}^{(m)}$.

ACKNOWLEDGEMENTS

The authors thank one referee and an associate editor for their valuable comments and constructive suggestions that have led to improvements in the paper. The research of Xie and Yang were supported by the National Natural Science Foundation of China (Grant No. 12071016), and the research of Wang was supported by the National Natural Science Foundation of China (Grant No. 12171328) and Academy for Multidisciplinary Studies, Capital Normal University.

Received 13 December 2020

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