

Surveys in Differential Geometry

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Volume XIV

Surveys in
Differential Geometry

Geometry of Riemann surfaces
and their moduli spaces

edited by

Lizhen Ji, Scott A. Wolpert, and Shing-Tung Yau

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Geometry of Riemann surfaces and their moduli spaces

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Preface

The research of Pierre Deligne and David Mumford was reported on in *The irreducibility of the space of curves of given genus*, published in 1969, pages 75–109, tome 36 of Publications Mathématiques de L'I.H.É.S. In the subsequent Mathematical Review, Manfred Herrman wrote that the authors provide two separate proofs of the irreducibility of the moduli space of curves over algebraically closed fields of arbitrary characteristic. The main tool for the first proof is a stable reduction theorem for abelian varieties, due to Grothendieck and Mumford, applied to curves. The authors then follow ideas of Severi and Grothendieck for the conclusion and consider properties of the subscheme of tricanonically embedded stable curves in the Hilbert scheme. The second proof is based on Mumford's earlier research and the new notion of an algebraic stack. The authors apply concepts from the theory of schemes to algebraic stacks to show that the moduli stack over $\text{Spec } \mathbb{Z}$ is a proper, smooth and separated stack of finite type. From a generalization of the Enriques-Zariski connectedness theorem the authors conclude the irreducibility of the moduli space of curves, including the case of higher level moduli spaces.

In their introduction Deligne and Mumford cite a range of approaches to irreducibility. The approaches include a proof by Enriques-Chisini based on analyzing fixed degree coverings of \mathbb{P}^1 with fixed number of ordinary branch points, an analytic proof using Teichmüller's theorem that Teichmüller space is a topological cell and a proof by Grothendieck using the given stable reduction theorem and étale cohomology. Explanation is provided on the use of a *category* larger than the category of schemes. It is explained for the new *2-category*, that for objects X, Y then $\text{hom}(X, Y)$ is a category with all morphisms being isomorphisms. The objects of a 2-category are called algebraic stacks and the moduli space of curves is the *underlying coarse variety* of a moduli stack.

The present collection of papers is in honor of the fortieth anniversary of the research of Deligne and Mumford. Their work continues to the present as a fundamental contribution. Over the ensuing period, families of Riemann surfaces and algebraic curves have continued to arise in new and different areas of mathematics and physics. The range of approaches for studying

families of Riemann surfaces and algebraic curves has subsequently grown exponentially. A survey of current research on families and the moduli space of Riemann surfaces and algebraic curves would require a series of volumes. As editors, we set the goal of combining a collection of highly distinguished articles discussing a sampling of current research. A consideration for the goal was to include articles presenting algebraic, analytic and topological approaches. We also sought to combine articles representing the present state of techniques from the time of Deligne and Mumford, as well as articles representing approaches developed since that time.

There have been numerous influencing developments since the original research of Deligne and Mumford. The reader can find these developments and the continuing refinement of the existing methods of enumerative geometry, intersection theory and Brill-Noether theory throughout the contributions. The Maximal Rank Conjecture for the Hilbert function of a curve continues to guide considerations. Harer's calculation of the second homology group and stability of homology for the moduli space continues as a basic influence for topological considerations. More generally, Looijenga's conjecture for the moduli space of to be a union of $g - 1$ affine subsets influences current perspective. The Madsen-Weiss theorem that the stable cohomology of the mapping class groups is a polynomial algebra and the resolution of the Mumford conjecture on stable cohomology are continuing influences. Solutions of Witten's conjecture, the introduction of hierarchies and even combinatorial methods to study tautological intersection numbers are major themes for current research. The Harris and Mumford result that for large genus the moduli space of stable curves is of general type is a further continuing influence. The question of birationality for small genera moduli spaces continues to guide investigations. Algebraic developments have been matched by analytical and further topological approaches. Harmonic maps have developed as a major tool. An important development is the discovery that a harmonic map from a compact Kähler manifold to a hyperbolic Riemann surface induces a holomorphic foliation on the domain. More broadly, existence and comparison arguments for non linear PDEs have undergone dramatic development. The development is suggested in the Liu-Sun-Yau results on positivity and equivalence of the classical invariant metrics for Teichmüller space. Harer's results and the original proof of Witten's conjecture involve the action of the mapping class group on Teichmüller space. Developments have been accompanied by Thurston's foundational theory of measured foliations, measured geodesic laminations and the classification of elements of the mapping class group. Our understanding of the Teichmüller and Weil-Petersson metrics has undergone a renaissance in the ensuing period. Recent influencing results on the Teichmüller metric geometry include McMullen's proof in genus 2 that the closure of the $GL^+(2; \mathbb{R})$ action on the Hodge bundle over moduli space is an algebraic orbifold; the Sinai-type asymptotics of Eskin and Mirzakhani for counting pseudo Anosov

elements in the mapping class group and a precise picture of the Teichmüller metric in finite and infinite dimensions.

The volume represents a compilation of the time and effort of the authors. We would like to take this opportunity to thank each author for their contribution. We also take this opportunity to thank the referees for their important role. We invite the reader to study each of the articles.

Lizhen Ji
Scott A. Wolpert
Shing-Tung Yau
October, 2009

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