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Volume XVI

Transformation Groups and Moduli Spaces of Curves

Editors: Lizhen Ji and Shing-Tung Yau

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Advanced Lectures in Mathematics

Preface

Transformation groups have played a fundamental role in many areas of mathematics such as differential geometry, geometric topology, algebraic topology, algebraic geometry, number theory. One of the basic reasons for their importance is that symmetries are described by groups (or rather group actions). Indeed, the existence of group actions makes the spaces under study more interesting, and properties of groups can also be understood better by studying their actions on suitable spaces.

Quotients of smooth manifolds by group actions are usually not smooth manifolds. On the other hand, if the actions of the groups are proper, then the quotients are orbifolds.

The notion of V-manifolds was first introduced by Satake in 1956 in the context of locally symmetric spaces and automorphic forms. V-manifolds were reintroduced and renamed orbifolds by Thurston near the end of 1978 in connection with the Thurston geometrization conjecture on the geometry of three dimensional manifolds. Basically, orbifolds are locally quotients of smooth manifolds by finite groups. Besides arising from transformation groups, many natural spaces in number theory and algebraic geometry are orbifolds.

An important example of such interaction is given by the action of the mapping class groups on the Teichmüller spaces, and the quotients give the moduli spaces of Riemann surfaces (or algebraic curves) and are orbifolds. One reason for the importance of this group action is that Riemann surfaces are fundamental objects in complex analysis, differential and complex geometry, low dimensional topology, algebraic geometry, number theory, mathematical physics etc., and the Teichmüller spaces are moduli spaces of marked Riemann surfaces. These moduli spaces and their variants have played a fundamental role in algebraic geometry and string theory. Properties of the moduli spaces can sometimes be understood more easily through this action on the Teichmüller spaces.

The moduli spaces of algebraic curves are noncompact and admit a well-known compactification, called the Deligne-Mumford compactification. An important fact is that the Deligne-Mumford compactification is also a compact orbifold.

The above discussions show that orbifolds arise naturally from different contexts. Recently, orbifolds have also found striking applications in algebraic geometry and string theory such as the McKay correspondence.

To introduce these basic and important concepts to the younger generation, two consecutive summer schools were organized at the Center of Mathematical Sciences, Zhejiang University: *Transformation Groups and Orbifolds* from June

30 to July 11, 2008, and *Geometry of Teichmüller Spaces and Moduli Spaces of Curves* from July 14 to July 20, 2008. Experts on topics related to transformation groups, orbifolds, Teichmüller spaces, mapping class groups, and moduli spaces of curves were invited to give expository lecture series. This book contains the expanded lecture notes of some of these lecture series.[1]

We would like to thank the speakers for their hard work in preparing the talks and writing up the lecture notes, and the referees for carefully reading the lecture notes and making valuable suggestions and comments. We hope that this book will convey the lively spirit and freshness of the lectures at the summer schools, and believe that it will be a valuable source for people who want to learn these beautiful topics.

Lizhen Ji
Shing-Tung Yau
January 22, 2010

[1]The last lecture of C.C. Liu is related to the paper Formulae of one-partition and two-partition Hodge integrals, *Geometry & Topology Monographs* 8 (2006) 105–128. We would like to thank the editors of the Geometry & Topology Monographs for their permission to allow us to reprint this paper here.

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