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Handbook of Group Actions V

edited by

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Preface

Groups are fundamental objects in mathematics and the other sciences for the basic reason that they are responsible for the symmetries of any object or system under consideration. The presence of symmetry enhances the problems addressed and often makes them more interesting and, as a general rule, easier to solve. In all cases, it is the group action that makes groups useful and important. This is not surprising since the notion of group was first introduced through group actions as permutation groups on roots of algebraic equations. On the other hand, the single notion of group action includes many different actions arising from various sources. This is amply illustrated by the collection of papers in this volume, the fifth of the Handbook of Group Actions. It covers topics from classical geometric groups, geometric group theory, diffeomorphism groups of manifolds, mapping class groups, three-dimensional topology, hyperbolic manifolds, automorphism groups of complex manifolds, dynamics, and number theory.

When we started the project of editing this Handbook of Group Actions, our goal was to acquaint the reader with the richness of the subject and to show how seemingly different topics are connected by the common theme of group actions. We hope that we have achieved this goal to a certain extent, but it should be clear to the knowledgeable readers that we are far away from exhausting all the interesting topics related to group actions.

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January 2019

Introduction

This is the fifth volume of the Handbook of Group Actions. It consists of eleven surveys, covering a large variety of situations where group actions play a fundamental role, including the theory of cellular automata, Coxeter groups, Artin groups, diffeomorphism groups of manifolds, mapping class groups of three-manifolds, groups acting on Λ trees, hyperbolic manifolds, geometric group theory, automorphism groups of complex manifolds, dynamics, and number theory.

The following is a more detailed overview of the content.

In Chapter 1, Guido Ahumada, Bernard Brighi, Nicolas Chevallier, and Augustin Fruchard consider the following general properties of fixed points for groups acting on a space: the group acts *with fixed points* if each element of the group fixes at least one point in the space; the group acts *with a common fixed point* if there is a point fixed by all the elements of the group; the group G is said to be *fixating* if any subgroup which acts with fixed points is a group with a common fixed point. They ask the question: *what groups are fixating?* This question is addressed in several contexts, and a great variety of examples are considered, including isometry groups in various geometries, and constructions arising from the existence of free subgroups in linear groups.

Chapter 2, by Tullio Ceccherini-Silberstein and Michel Coornaert, is a survey of the Garden of Eden theorem and its various generalizations. Without entering into the details of the theory, let us mention that this theorem, in its original form, is a result in the theory of cellular automata, obtained in the early 1960s by E. F. Moore and J. R. Myhill, stating that a cellular automaton is surjective if and only if it is pre-injective (a weak form of injectivity). Later, the question of what classes of groups satisfy a similar property ("surjectivity is equivalent to a form of injectivity") was addressed. The chapter provides a detailed exposition of the old and new work in this field, including a review of the necessary background on the theories of cellular automata over groups and cellular automata between subshifts, and with a survey of the long chain of applications of the Garden of Eden theorem in amenability, geometric and combinatorial group theory, the theory of algebraic group actions and in various dynamical settings.

Chapter 3, by Pallavi Dani, is concerned with Coxeter groups. These are finitely presented groups generated by involutions and defined by a special type of presentation, and they are classified by the so-called Coxeter diagrams. They constitute an important class of finitely presented groups that appear in geometry, where the defining involutions are reflections with respect to hyperplanes or faces of complexes. Coxeter groups first appeared in connection with symmetries of regular solids, and they are now studied in several contexts, including the theory of Lie groups and Lie algebras, combinatorics, and geometric group theory. Among Coxeter groups, the right-angled Coxeter groups constitute a special class in which the only relations between distinct generators are commuting relations. In Chapter 3, the author surveys the large-scale geometry of right-angled Coxeter groups, with an emphasis on the recent results on the actions of these groups on various non-positively curved spaces, in particular CAT(0) cube complexes, and on their visual boundaries. She also discusses the quasi-isometry and commensurability classification of these groups.

A 3-dimensional handlebody of genus g is a three-manifold with boundary obtained from the 3-ball by attaching to it g one-handles. The mapping class group of such a manifold is called the handlebody group of genus g. Chapter 4 of this volume, by Sebastian Hensel, is a survey of handlebody groups, with a focus on the similarities and differences between these groups, surface mapping class groups and outer automorphism groups of free groups. The chapter also contains applications of handlebody groups and open questions.

Chapter 5, by Olga Kharlampovich and Alina Vdovina, is a survey of generalizations of the theory of groups acting on simplicial trees (introduced by Serre). More precisely, the authors consider groups acting isometrically on $\mathbb{R}$-trees, and more generally on Λ-trees, the theory of complexes of groups, and the theory of lattices in products of trees. In the latter setting, they review several results on arithmetic groups acting on products of trees.

In Chapter 6, Thomas Koberda surveys the theory of right-angled Artin groups in the context of diffeomorphism groups of low dimensional manifolds. We recall that Artin groups are finitely presented groups defined by a special presentation. They generalize the braid groups, and they are also related to the Coxeter groups, studied in Chapter 3. In fact, each Artin group has a quotient group that is a Coxeter group. The class of right-angled Artin groups is a subclass which contains the classes of free groups of finite rank and free Abelian groups. On the other hand, the class of right-angled Artin groups is a subclass of the class of graph product of groups, and it is also a subclass of CAT(0) groups, thus, playing a central role in geometry.

Also in Chapter 6, after describing some of the subgroup structure of right-angled Artin groups, and after discussing the interplay between algebraic structure, compactness, and regularity for group actions, the author makes a detailed survey of the role of right-angled Artin groups in the study of diffeomorphism groups of manifolds, especially in dimension one. In particular, he shows the following: In the case of a compact 1-dimensional manifold, every right-angled Artin group acts faithfully by C^1 diffeomorphisms, but the class of right-angled Artin groups which act faithfully by C^2 diffeomorphisms on such a manifold is very restricted. On the other hand, every right-angled Artin group acts faithfully by C^∞ diffeomorphisms on the real line, but here, analytic actions are again restricted. In dimensions ≥ 2, every right-angled Artin group acts faithfully on every manifold by C^∞ diffeomorphisms.

Chapter 7, by Frank Kutzschebauch, is a review of holomorphic automorphism groups, and in particular, of the holomorphic automorphism group of affine space, in relation with the question of how to detect affine space among manifolds. At the same time, the author surveys the structure of the algebraic and holomorphic automorphism groups of affine space, he reviews the Andersén–Lempert Theorem and the Oka principle, and he addresses a density property and flexibility questions, including classical and new results on the holomorphic linearization problem.

Chapters 8 and 9 are two parts of the same long survey, *Topics in Geometric Group Theory: Part I and Part II*. Chapter 8, is written by Daniele Ettore Otera and Valentin Poénaru, and Chapter 9 is written by Valentin Poénaru. The general theme of the two chapters is the study of the asymptotic topological properties of finitely presented discrete groups, in relation with the topology of 3-manifolds. The topics discussed include the notions of quasi-simple filtration (a property due to Brick, Mihalik and Stallings, which generalizes of the notion of simple connectivity at infinity and which has its roots in the work of Max Dehn), geometric simple connectivity, topological inverse-representations and easy groups. A general theme in this context is the question of what finitely presented groups satisfy these conditions. At the same time, contributions of several people to the topology of 3-manifolds are revisited (Dehn, Stallings, Casson, Agol, Wise, etc.).

In Chapter 8, Otera and Poénaru present the background material, historical context, and motivation of the theory. The main notions discussed are those of topological inverse-representations—in some sense, a notion dual to the usual notion of group representations, tameness conditions at infinity for open manifolds, cell-complexes and discrete groups such as geometric simple connectivity, Dehn-exhaustibility, and the quasi-simple filtration property. The authors then survey recent joint work on almost convex groups, combing of groups and the so-called *Whitehead nightmare*, a phenomenon for representations which one encounters in the study of the Whitehead manifold, Casson handles, etc. In Chapter 9, Poénaru outlines the proof that all finitely presented groups satisfy the quasisimple filtration property.

Chapter 10, by Mark Pollicot, is a survey on the zeta functions of Selberg and Ruelle and of Poincaré series and their various uses in dynamics as counting functions for the distribution of orbits. The author surveys several results associated to orbits of various dynamical systems, in particular geodesic flows and Anosov flows. Key tools in this analysis are appropriate complex functions, such as the zeta functions of Selberg and Ruelle, and Poincaré series. The author places his study in a broader perspective by discussing at the same time several zeta functions which are defined as complex functions, such as the Riemann zeta function that appears in number theory, the Bowen–Lanford and the Ihara zeta functions for graphs and the Artin–Mazur zeta function for diffeomorphisms. The classical questions about these functions (their domain of definition, their zeros and poles, their values, etc.) are addressed from a geometric point of view.

Chapter 11, by José Seade, is a survey on the theory of Kleinian groups in high dimensions. In the classical terminology, a Kleinian group is defined in three equivalent manners: (1) a discrete group of Möbius transformations of the Riemann sphere (first introduced by Poincaré who studied them as monodromy groups of 2nd order differential equations in the complex plane); (2) a discrete group of isometries of the real 3-dimensional hyperbolic space (also studied by Poincaré); (3) a discrete group of automorphisms of the complex projective line. In higher dimensions, the three definitions are not equivalent, although the theories obtained are related. In this chapter, the author surveys the geometry and dynamics of discrete subgroups of $\mathrm{PSL}(n+1, \mathbb{C})$ acting on the n-dimensional complex projective space, with a stress on the analogue, in complex dimension two, of Sullivan's dictionary between Kleinian groups acting on the Riemann sphere and the theory of iteration of maps of one complex variable. The first step for developing such a dictionary is to have the right notion of a limit set of an action in this setting, and this is the main topic discussed in this survey.

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