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Volume XXV

Surveys in Differential Geometry 2020

Surveys in 3-Manifold Topology and Geometry

Editors

Ian Agol · David Gabai

International Press
www.intlpress.com

Series Editor: Shing-Tung Yau

Surveys in Differential Geometry, Vol. 25 (2020)
Surveys in 3-Manifold Topology and Geometry

Volume Editors:
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Somerville, Massachusetts, U.S.A.
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ISBN: 978-1-57146-419-4

Printed in the United States of America.

26 25 24 23 22 1 2 3 4 5 6 7 8 9

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Surveys in 3-manifold topology

Ian Agol* and David Gabai†

In the last half-century, tremendous progress has been made in the study of 3-dimensional topology. Many revolutions in 3-manifold topology during this period have come from outside of the field, including Kleinian groups [**31**], minimal surfaces [**20**], foliations [**10**], von Neumann algebras [**12**], gauge theory [**9**], mathematical physics [**33**], 4-manifolds [**18**], symplectic topology [**25**], contact topology [**5**, **7**], Riemannian geometry and PDEs [**11**, **28**, **29**, **30**], number theory [**4**], dynamics [**13**], and geometric group theory [**1**, **32**]. The influx of ideas from neighboring fields has made the subject of 3-manifolds (and more generally low-dimensional topology) a very rich subject, creating subfields such as quantum topology. But this also means that there is a tremendous amount of background material for a novitiate in the subject to learn and master.

This volume is a collection of surveys meant to bring certain subfields of 3-manifold topology up-to-date. Many surveys and books on various topics already exist [**2**, **3**, **6**, **8**, **14**, **15**, **16**, **17**, **19**, **21**, **22**, **23**, **24**, **26**, **27**, **34**]. Thus we attempted to solicit contributions from experts in active topics that were not adequately covered by other surveys. Richard Bamler surveys results on Ricci flow-with-surgery on 3-manifolds stemming from Perelman's work on the geometrization theorem [**30**], including a discussion of his proof with Bruce Kleiner of the Generalized Smale Conjecture. Tobias Colding, David Gabai, and Daniel Ketover survey minimal surface techniques applied to the study of Heegaard splittings of 3-manifolds, including the resolution of the Pitts-Rubinstein conjecture. Vincent Colin and Ko Honda describe and develop the theory of foliations and contact structures on sutured 3-manifolds. John Etnyre and Lenhard Ng survey Legendrian contact homology of knots, comparing different perspectives on this topic. Sang-Hyun Kim and Genevieve Walsh describe hyperbolic groups with planar limit sets in relation to Kleinian groups. Marc Lackenby surveys algorithms in knot

*University of California, Berkeley. Supported by a Simons Investigator grant.

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theory and 3-manifold topology, including results on computational complexity. Yi Liu and Hongbin Sun survey topics related to the resolution of the virtual Haken conjecture, including subgroup separability, degree one maps between 3-manifolds, and torsion in the homology of covers. Mahan Mj describes Cannon-Thurston maps following his resolution of Question 14 from Thurston's problem list [**31**]. Jean-Marc Schlenker surveys the topic of renormalized volume of Kleinian groups and its relation to other notions of volume, which has received a surge of interest in recent years.

Most of these surveys demonstrate the inextricable link between geometry and topology in 3 dimensions. The study of geometric structures on 3-manifolds and their classification will continue to be an active area of study. Most of the surveys contain some open problems and questions which hopefully will inspire people to keep working on these topics.

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